

Title: Quantum Field Theory for Cosmology (AMATH872/PHYS785) - Lecture 5

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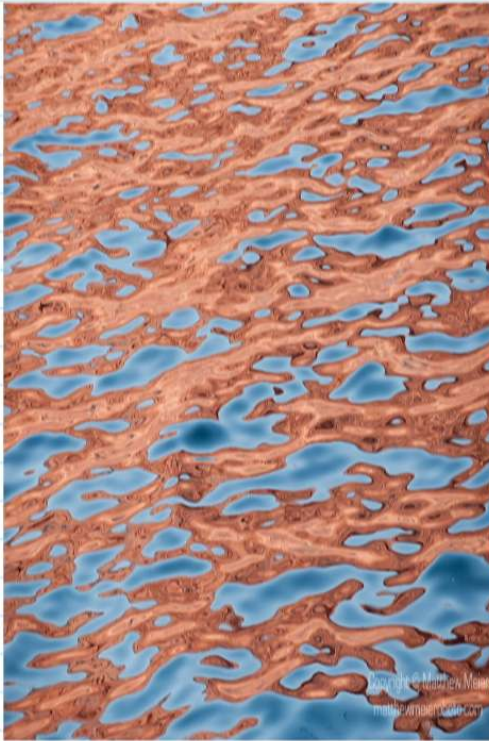
Abstract:

# QFT for Cosmology, Achim Kempf, Lecture 5

Note Title

## Particles in QFT

Back in the Heisenberg picture, to solve the QFT is to solve:



- The hermiticity conditions:

$$\hat{\phi}^\dagger(x, t) = \hat{\phi}(x, t), \quad \hat{\pi}^\dagger(x, t) = \hat{\pi}(x, t)$$

- The canonical commutation relations:

$$[\hat{\phi}(x, t), \hat{\pi}(x', t)] = i\delta(x-x')$$

- The equations of motion:

$$\dot{\hat{\pi}}(x, t) - \Delta \hat{\phi}(x, t) + m^2 \hat{\phi}(x, t) = 0$$

$$\dot{\hat{\pi}}(x, t) = \hat{\dot{\phi}}(x, t)$$

To simplify:  $\square$  Infrared regularization:

Box size  $L \times L \times L$  with  
periodic boundary conditions.

$\uparrow$  Project: uses Dirichlet boundary conditions.

$\square$  Then Fourier series expansion:

$$\hat{\phi}(x, t) = L^{-3/2} \sum_{\mathbf{k}} \hat{\phi}_{\mathbf{k}}(t) e^{i\mathbf{k}x}$$

$\uparrow \mathbf{k} = \frac{2\pi}{L}(n_1, n_2, n_3), n_i \in \mathbb{Z}$

Obtain:  $\ddot{\hat{\phi}}_{\mathbf{k}}(t) = -(k^2 + m^2) \hat{\phi}_{\mathbf{k}}(t)$  and  $[\hat{\phi}_{\mathbf{k}}, \hat{\phi}_{\mathbf{k}'}] = i \delta_{\mathbf{k}, -\mathbf{k}'}$ .

$$\hat{H} = \sum_{\mathbf{k}} \hat{H}_{\mathbf{k}} \quad \text{with} \quad \hat{H}_{\mathbf{k}} = \frac{1}{2} \hat{\pi}_{\mathbf{k}}^{\dagger} \hat{\pi}_{\mathbf{k}} + \frac{1}{2} \hat{\phi}_{\mathbf{k}}^{\dagger} (k^2 + m^2) \hat{\phi}_{\mathbf{k}}$$

i.e.: 
$$\hat{H} = \sum_{\mathbf{k}} \left( \frac{1}{2} \hat{\pi}_{\mathbf{k}}^{\dagger}(t) \hat{\pi}_{\mathbf{k}}(t) + \frac{1}{2} \hat{\phi}_{\mathbf{k}}^{\dagger}(t) (k^2 + m^2) \hat{\phi}_{\mathbf{k}}(t) \right)$$

## □ Crucial observations:

\* For each wave vector  $\mathbf{k} = (k_x, k_y, k_z)$  there is an independent harmonic oscillator with frequency  $\omega_{\mathbf{k}} = \sqrt{k^2 + m^2}$  and spectrum  $\text{spec}(H_{\mathbf{k}}) = \hbar \omega_{\mathbf{k}} \left\{ \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots \right\}$ .

⇒ The excitation levels of  $H_{\mathbf{k}}$  differ by the energy  $E = \hbar \omega_{\mathbf{k}} = \hbar \sqrt{k^2 + m^2}$  ( $\hbar = 1$ )

\* This is also the energy of a particle of momentum  $\mathbf{k}$ !

⇒ Hypothesis: Mode excitation = particle creation

Does this interpretation work?

Water:

$$\phi(x, t)$$

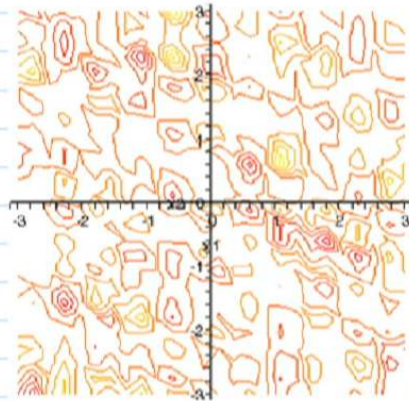


Probe amplitudes,  
e.g., with a cork:



Quantum field:

$$\hat{\phi}(x, t)$$



Probe amplitudes, e.g.,  
with atoms.



Use as a  
detector for  
the field's particles  
(e.g. photons for EM field)

One finds:

- Interpretation works but is acceleration and curvature dependent.
- Interpretation simple only in Minkowski space for inertial detectors.

Note: Conventional particle physics is based on that special case.

Then: Which is, e.g., the state  $|\psi\rangle$  in which we have

3 particles of momentum  $k_a$  and 7 particles of momentum  $k_b$ ?

$$\begin{aligned} |\psi\rangle &= |n_{k_a}=3, n_{k_b}=7, \text{ all other } n_k=0\rangle \\ &= |n_{k_a}=3\rangle \otimes |n_{k_b}=7\rangle \left( \bigotimes_{\substack{\text{all other} \\ k_c}} |n_{k_c}=0\rangle \right) \end{aligned}$$

$$\text{Energy: } \hat{H}_k |\psi\rangle = \begin{pmatrix} \hbar \omega_k \left(\frac{1}{2} + 3\right) & \text{if } k=k_a \\ \hbar \omega_k \left(\frac{1}{2} + 7\right) & \text{if } k=k_b \\ \hbar \omega_k \frac{1}{2} & \text{if } k \neq k_a, k_b \end{pmatrix} |\psi\rangle$$

$$\Rightarrow \hat{H} |\psi\rangle = \left( 3\omega_{k_a} + 7\omega_{k_b} + \sum_{\text{all } k} \frac{1}{2} \omega_k \right) |\psi\rangle$$

And one can have linear combinations:

Which is, e.g., the state  $|\psi\rangle$  in which we have

3 particles of momentum  $k_a$  or 7 particles of momentum  $k_b$ ,  
with probability amplitudes  $\alpha$ ,  $\beta = \sqrt{1-\alpha^2}$ ?

$$|\psi\rangle = \alpha |n_{k_a}=3, \text{ other } n_k=0\rangle + \beta |n_{k_b}=7, \text{ other } n_k=0\rangle$$

*Notice:* This is not a state of fixed particle number!

Remark: Some particle species have a number conservation law, e.g., leptons, i.e.  $e^-, \mu^-, \tau^-, \nu_e, \nu_\mu, \nu_\tau$ , (where the antiparticles count negatively).

"Superselection rule" then says we can't have such linear combinations: only number eigenstates allowed.

# Mechanisms for mode excitation/particle creation?

J.e.: What are mechanisms for exciting harmonic oscillators?

□ 2 types of mechanism: (here,  $\hat{q}(t)$  stands for  $\hat{\phi}_k(t)$ )

we'll begin with this effect → a.) A "driving force" shakes the oscillator:

$$\ddot{\hat{q}}(t) = -\omega^2 \hat{q}(t) + \hat{j}(t)$$

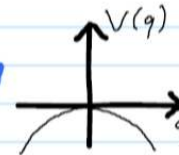


b.) A time dependence of  $\omega$  affects the oscillator:

$$\ddot{\hat{q}}(t) = -\omega^2(t) \hat{q}(t)$$



And,  $\omega^2(t)$  could even go negative!



## All occur in QFT:

A) Multiple fields enter into the Hamiltonian and into the eqns of motion. Thus, fields provide each other with  $J$  terms, e.g.:

$$H(\hat{\phi}, \hat{\psi}) = \hat{H}_1(\hat{\phi}) + \hat{H}_2(\hat{\psi}) + \int_{\mathbb{R}^3} \lambda \hat{\phi}(x,t) \hat{\psi}(x,t) d^3x$$

▣ Wave interpretation: Nontrivial interaction of waves of different types of fields

▣ Particle interpretation: The collision of particles happens when their mode oscillators drive another.

→ Collisions can create and annihilate particles.

▣ Strongest effects?

When oscillator "resonates" with driving force.

E.g.: JA takes high energy particles to make high energy particles

B) The presence of gravity can effectively influence the  $\omega_p(t)$ .

▢ Wave interpretation: \* E.g., cosmic expansion stretches the wavelength

⇒ expect  $\omega = \omega(t)$  decreases. True, and also:

\* if wavelength > horizon then  $\omega^2(t) < 0!$

⇒ runaway harmonic mode "oscillators" 

(then: field amplification but no particle interpretation)

▢ Particle interpretation:

Gravity can excite mode oscillators, i.e. it can create particles from the vacuum.

▢ Strongest effects?

When oscillator resonates with  $\omega(t)$ . This effect is called parametric resonance.

Case A: Particle creation through external driving of mode oscillators.

Example: Production of photons by an antenna:

□ We model the electromagnetic field as a Klein Gordon field.

(The fact that EM fields have polarization and have  $m=0$  is not important here)

□ Consider an arbitrary mode of the electromagnetic field:

$$\hat{\phi}_k(t)$$

should really be quantized too

□ We model the electric current as a given classical field  $j(x,t)$  whose modes are  $j_k(t)$ .

↓ should really be vector-valued



□ In a rough simplification, the EM  $\mathbf{k}$  mode obeys:

$$\hat{H}_k = \frac{1}{2} \hat{\pi}_k^\dagger(t) \hat{\pi}_k(t) + \frac{1}{2} \omega_k^2 \hat{\phi}_k^\dagger(t) \hat{\phi}_k(t) + \hat{\phi}_k(t) j_k(t)$$

⇒ If the current  $j(t)$  varies in time it can excite the mode oscillators, thus creating photons.

⇒ Need to study the quantized driven harmonic oscillator!

$$\hat{H}(t) = \frac{1}{2} \hat{p}(t)^2 + \frac{\omega^2}{2} \hat{q}(t)^2 - j(t) \hat{q}(t)$$

for  $\hat{H}_k(t)$       for  $\hat{\pi}_k(t)$       stands for a field mode  $\hat{\phi}_k(t)$

stands for a mode  $j_k(t)$  of another classical (or better quantum) field.

## I Preparation:

□ Recall that for all observables  $\hat{f}$ :

$$\bar{f}(t) = \langle \psi_0 | \hat{U}^\dagger(t) \hat{f}_0 \hat{U}(t) | \psi_0 \rangle$$

↙ state at initial time

↖ operator at initial time

with the time-evolution operator obeying:

$$\hat{U}(t_0) = 1, \quad i \frac{d}{dt} \hat{U}(t) = \hat{U}(t) \hat{H}(t)$$

↙ the original Hamiltonian

↖ "Heisenberg Hamiltonian"

□ Schrödinger picture? We write, equivalently: ↙ Exercise: check!

$$\begin{aligned} \bar{f}(t) &= \langle \psi_0 | \hat{U}^\dagger(t) \hat{f}_0 \underbrace{(\hat{U}(t) | \psi_0 \rangle)}_{= | \psi(t) \rangle} \\ &= \langle \psi(t) | \hat{f}_0 | \psi(t) \rangle \end{aligned}$$

Recall:  $\hat{H}_S(t) = \hat{H}(t)$  only if  $\frac{d}{dt} \hat{H}(t) = 0$

The dynamics is  $i \frac{d}{dt} |\psi(t)\rangle = \hat{H}_S(t) |\psi(t)\rangle$

with Schrödinger Hamiltonian:  $\hat{H}_S(t) = \hat{U}(t) \hat{H}(t) \hat{U}^\dagger(t)$

Exercise: check

□ We will use, equivalently, the Heisenberg picture:

$$\begin{aligned} \bar{f}(t) &= \langle \psi_0 | \underbrace{\hat{U}^\dagger(t) \hat{f}_0 \hat{U}(t)}_{\hat{f}(t)} | \psi_0 \rangle \\ &= \langle \psi_0 | \hat{f}(t) | \psi_0 \rangle \end{aligned}$$

with dynamics:

$$i \frac{d}{dt} \hat{f}(t) = [\hat{f}(t), \hat{H}(t)]$$

## II Aspects of the Heisenberg picture:

- The **state** of the quantum system stays the same Hilbert space vector, say  $|\psi\rangle \in \mathcal{H}$  (from measurement to measurement).
- The **observables**, say  $\hat{H}(t)$ ,  $\hat{f}(t)$ , etc, are time-dependant operators in Hilbert space.
- Important implication:

The eigenbases and the eigenvalues of observables, such as  $\hat{H}(t)$  and any  $\hat{f}(t)$  depend on time!

$$\hat{f}(t) |f_n(t)\rangle = f_n(t) |f_n(t)\rangle$$

$$\hat{H}(t) |E_n(t)\rangle = E_n(t) |E_n(t)\rangle$$

Example: \* Assume the driven harmonic oscillator starts out at time  $t_1$  in  $n$ 'th energy state, say  $|\psi\rangle = |E_n(t_1)\rangle$ :

$$\hat{H}(t_1) |E_n(t_1)\rangle = E_n(t_1) |E_n(t_1)\rangle$$

\* State vector of the system stays  $|\psi\rangle$  for  $t > t_1$ .

\* But at later times, say  $t > t_1$ , the Hamiltonian and its eigenvectors and eigenvalues are

$$\hat{H}(t) |E_n(t)\rangle = E_n(t) |E_n(t)\rangle$$

and we generally have

$$E_n(t) \neq E_n(t_1), \quad |E_n(t)\rangle \neq |E_n(t_1)\rangle$$

⇒ At time  $t_2$  system is still in state  $|\gamma\rangle$  and still

$$|\gamma\rangle = |E_n(t_1)\rangle$$

but  $|\gamma\rangle$  is generally no longer with (or any other) energy eigenstate!

In particular:

\* Assume system starts out at  $t_1$  in lowest energy state (i.e., in vacuum):  $|\gamma\rangle = |E_0(t_1)\rangle$

\* Then if  $|\gamma\rangle = |E_0(t_1)\rangle \neq |E_0(t_2)\rangle$

⇒ At  $t_2$  the system's state  $|\gamma\rangle$  is not the ground state i.e. not the vacuum state, i.e. particles (e.g. photons) exist at time  $t_2$ .

### III Strategy for solving quantized driven harmonic oscillator

▢ Problem: \* CCR:  $[\hat{q}(t), \hat{p}(t)] = i\mathbb{1}$

\* Hermiticity:  $\hat{q}^\dagger(t) = \hat{q}(t), \quad \hat{p}^\dagger(t) = \hat{p}(t)$

\* Hamiltonian:  $\hat{H}(t) = \frac{1}{2} \hat{p}(t)^2 + \frac{\omega^2}{2} \hat{q}(t)^2 - J(t) \hat{q}(t)$

\* Heisenberg eqns  $i \dot{\hat{f}}(t) = [\hat{f}(t), \hat{H}(t)]$  yield:

$$\dot{\hat{q}}(t) = \hat{p}(t)$$

$$\dot{\hat{p}}(t) = -\omega^2 \hat{q}(t) + J(t)$$

This is a good strategy  
with and without a  
driving force

▢ Strategy: \* Combine

$a(t) := \alpha \hat{q}(t) + i\beta \hat{p}(t)$  (analogous to "real" & "imaginary" parts)  
↓ is operator even though no "hat".  
↑ Mukhanov calls it  $a^-(t)$

\* Choose  $\alpha, \beta$  so that  $\hat{H}(t)$  and eqn of motion simplify.

#### IV Determine $\alpha$ and $\beta$ :

□ Notice first that once we have  $a(t)$  we immediately obtain  $\hat{q}(t), \hat{p}(t)$ : Use of  $a^\dagger(t) = \alpha \hat{q}(t) - i\beta \hat{p}(t)$  yields:

$$\hat{q}(t) = \frac{1}{2\alpha} (a^\dagger(t) + a(t))$$

$$\hat{p}(t) = \frac{i}{2\beta} (a^\dagger(t) - a(t))$$

□ Use this to express  $[\hat{q}, \hat{p}] = i$  in terms of new variable  $a(t)$ :

$$\Rightarrow [a(t), a^\dagger(t)] = 2\alpha\beta$$

For simplicity, we choose  $\beta = \frac{1}{2\alpha}$  so that:

$$\boxed{[a(t), a^\dagger(t)] = 1}$$

□ Now express  $\hat{H}(t)$  in terms of new variable  $a(t)$ :

$$\hat{H}(t) = -\frac{1}{2}d^2 (a^+(t) - a(t))^2 + \frac{\omega^2}{2} \frac{1}{4d^2} (a^+(t) + a(t))^2 - J(t) \frac{1}{2d} (a^+(t) + a(t))$$

We notice that the terms  $\sim a^+(t)^2$  and  $\sim a(t)^2$  drop out if we choose:

$$-\frac{1}{2}d^2 + \frac{\omega^2}{2} \frac{1}{4d^2} = 0$$

Thus, we choose:  $d = \sqrt{\frac{\omega}{2}}$  and therefore  $\beta = \frac{1}{\sqrt{2\omega}}$

□ Thus, by definition:

$$a(t) := \sqrt{\frac{\omega}{2}} \hat{q}(t) + i \frac{1}{\sqrt{2\omega}} \hat{p}(t)$$

▫ The Hamiltonian simplifies to become:

$$\hat{H}(t) = \omega \left( a^\dagger(t) a(t) + \frac{1}{2} \right) - J(t) \frac{1}{\sqrt{2\omega}} (a^\dagger(t) + a(t))$$

VI Solve for  $a(t)$ :

▫ The Heisenberg equation  $i\dot{\hat{f}}(t) = [\hat{f}(t), \hat{H}(t)]$  reads for  $a(t)$ :

$$i\dot{a}(t) = \omega a(t) - \frac{1}{\sqrt{2\omega}} J(t)$$

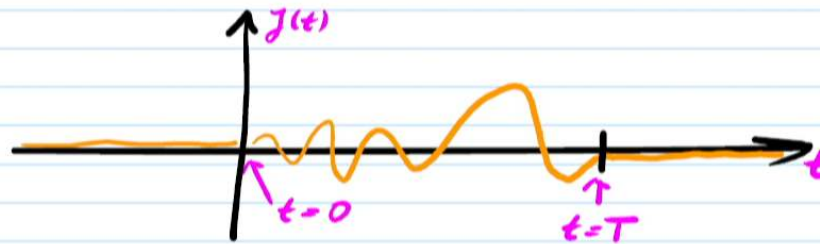
▫ Let us give  $a(t=0)$  a name:  $a_{in} = a(0)$ . Then:

Exercise:  
verify.

$$a(t) = a_{in} e^{-i\omega t} + \frac{i}{\sqrt{2\omega}} \int_0^t J(t') e^{i\omega(t'-t)} dt'$$

## VI Case of force of finite duration

□ Assume  $J(t) = 0$  for all  $t \notin [0, T]$



□ Define 
$$J_0 := \frac{i}{\sqrt{2\omega}} \int_0^T J(t') e^{i\omega t'} dt'$$

□ Then: 
$$\alpha(t) = \begin{cases} a_{in} e^{-i\omega t} & \text{for } t < 0 \\ \text{see above} & \text{for } t \in [0, T] \\ (a_{in} + J_0) e^{-i\omega t} & \text{for } t > T \end{cases}$$

$$J_0 := \frac{e}{\sqrt{2\omega}} \int_0^T J(t') e^{-i\omega t'} dt'$$

Then:

$$a(t) = \begin{cases} a_{in} e^{-i\omega t} & \text{for } t < 0 \\ \text{see above} & \text{for } t \in [0, T] \\ (a_{in} + J_0) e^{-i\omega t} & \text{for } t > T \end{cases}$$

Next:

Implications in terms of particle (e.g. photon) production?