

Title: PSI 17/18 - Condensed Matter - Lecture 12

Date: Jan 18, 2018 11:30 AM

URL: <http://pirsa.org/18010045>

Abstract:

Closed System

H Time independent
pure-point spectrum

$$H = \sum_n E_n |E_n\rangle\langle E_n| = \sum_n E_n P_n ; U_t = e^{-iHt}$$

$$\phi_t \equiv U_t \phi \equiv U_t \phi U_t^\dagger$$

$$A_t \equiv U_t A U_t^\dagger$$

Equilibration

$$\phi \rightsquigarrow \phi_t \rightsquigarrow \phi_{eq} : \mathcal{U}_t \phi_{eq} = \phi_{eq}$$

$$\begin{aligned} \lim_{t \rightarrow \infty} \phi_t = \phi_{eq} &\Rightarrow 0 = \lim_{t \rightarrow \infty} \|\phi_t - \phi_{eq}\| = \lim_{t \rightarrow \infty} \|\mathcal{U}_t \phi - \mathcal{U}_t \phi_{eq}\| \\ &= \lim_{t \rightarrow \infty} \|\mathcal{U}_t(\phi - \phi_{eq})\| = \|\phi - \phi_{eq}\| \\ &\Rightarrow \phi = \phi_{eq} \end{aligned}$$

Equilibration in QM

$$\phi \rightsquigarrow \phi_t \rightsquigarrow \phi_{eq} : \mathcal{U}_t \phi_{eq} = \phi_{eq}$$

$$\begin{aligned} \lim_{t \rightarrow \infty} \phi_t = \phi_{eq} &\Rightarrow 0 = \lim_{t \rightarrow \infty} \|\phi_t - \phi_{eq}\| = \lim_{t \rightarrow 0} \|\mathcal{U}_t \phi - \mathcal{U}_t \phi_{eq}\| \\ &= \lim_{t \rightarrow \infty} \|\mathcal{U}_t(\phi - \phi_{eq})\| = \|\phi - \phi_{eq}\| \\ &\Rightarrow \phi = \phi_{eq} \end{aligned}$$

$$\langle A(t) \rangle = \text{Tr}[A(t) \phi]$$

von Neumann

$\langle A(t) \rangle$ is quasi-periodic

$$\forall \epsilon > 0 : |\langle A(T) \rangle - \langle A \rangle| < \epsilon$$

$$\text{REMARK 1} \quad \phi_{eq} = \overline{\phi} := \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \phi_t dt = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \sum_{nn'} e^{-i(E_n - E_{n'})t} \phi_{nn'} |E_n\rangle \langle E_{n'}|$$

$$\overline{A} \equiv A_{eq} = \overline{A(t)} = \text{Tr}[A \overline{\phi}]$$

ASSUMPTION 1 H is non-degenerate

REMARK 1 $\phi_{eq} = \overline{\phi} := \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \phi_t dt = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \sum_{nn'} e^{-i(E_n - E_{n'})t} \phi_{nn'} |E_n\rangle \langle E_{n'}|$

$$\overline{A} \equiv A_{eq} = \overline{A(t)} = \frac{1}{T} \text{Tr}[A \overline{\phi}] = \sum_n \phi_{nn} \Pi_n = \sum_n \Pi_n \phi \Pi_n$$

ASSUMPTION 1 H is non-degenerate

$$|\phi\rangle = \sum_n a_n |E_n\rangle$$

$$\phi_{nn} = |a_n|^2$$

$$\phi = \begin{pmatrix} \phi_{11} & & \\ & \ddots & \\ & & \phi_{nn} \end{pmatrix}$$

completely
dephased
state

+ time
average $\overline{\phi} = \begin{pmatrix} \phi_{11} & & \\ & \ddots & \\ & & \phi_{nn} \end{pmatrix}$

$$= \int_0^T \sum_{nn'} e^{-i(E_n - E_{n'})t} \phi_{nn'} |E_n\rangle \langle E_{n'}|$$

$$\rho_{nn} \Pi_n = \sum_n \Pi_n \phi \Pi_n$$

totally
shared
state

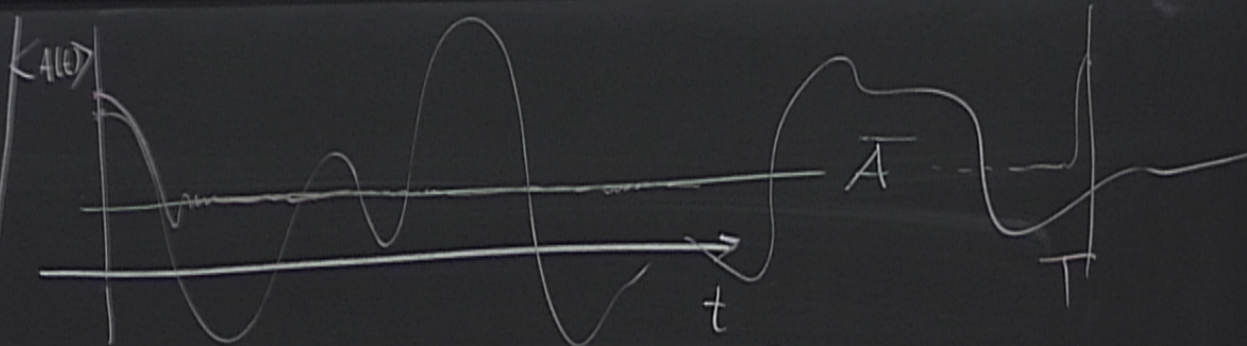
$$\bar{\phi} = \begin{pmatrix} \phi_{11} & & \\ & \ddots & \\ & & \phi_{nn} \end{pmatrix}$$

$$A_{eq} = \text{Tr}[A \bar{\phi}]$$

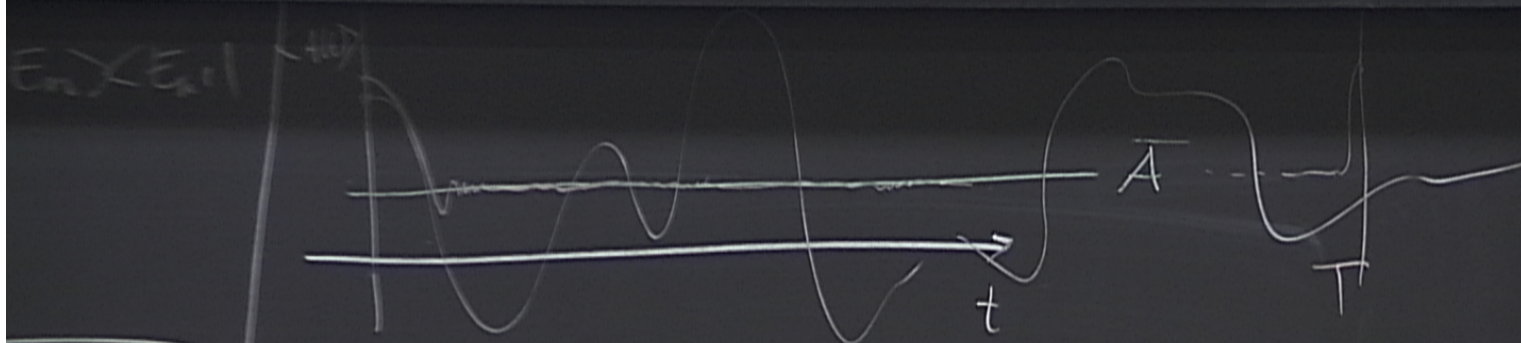
$$\phi_{\text{en}} |E_n\rangle \langle E_n|$$

Tr

$$= \text{Tr}[A \bar{\phi}]$$



$$\text{Pr} [|\langle A(t) \rangle - \bar{A}| > \epsilon] < \text{very small}$$



$$\Pr [|\langle A(t) \rangle - \bar{A}| > \epsilon] < \text{very small}$$

Equilibration in Probability

$$\Delta A_T^2 = \overline{(\langle A(t) \rangle - \bar{A})^2} = \overline{(\text{Tr}[A\phi_t] - \text{Tr}[A\bar{\phi}])^2}$$

$$\max_{\psi \in \mathcal{H}^+} \langle \psi | A | \psi \rangle$$

$$\tilde{\phi}_t = \phi_t - \bar{\phi} = \sum_{n \neq m} \phi_{mn} e^{i(E_n - E_m)t} |E_n\rangle \langle E_m|$$

$$\langle K | \tilde{\phi}_t | K \rangle = 0 \Rightarrow \text{Tr} \tilde{\phi}_t = 0$$

$$\mathcal{M}_+ = \text{supp}(\phi)$$

$$\max_{\psi \in \mathcal{H}^+} \langle \psi | A | \psi \rangle - \min_{\psi \in \mathcal{H}^+} \langle \psi | A | \psi \rangle = a_{\max} - a_{\min}$$

(En)

$$\max_{\psi \in \mathcal{H}^+} \langle \psi | A | \psi \rangle - \min_{\psi \in \mathcal{H}^+} \langle \psi | A | \psi \rangle = a_{\max} - a_{\min} \equiv S_A$$

(En)

$$\Delta^2 A_T = \overline{(\text{Tr}[A \tilde{\Phi}_t])^2}$$

$$\max_{\psi \in \mathcal{H}^+} \langle \psi | A | \psi \rangle - \min_{\psi \in \mathcal{H}^+} \langle \psi | A | \psi \rangle = a_{\max} - a_{\min} = S_A$$

$$\Delta^2 A_t = \overline{(\text{Tr}[A \tilde{\Phi}_t])^2} = \overline{(\text{Tr}[\tilde{A} \tilde{\Phi}_t])^2} = \sum_{\substack{j \neq k \\ m \neq n}} \tilde{A}_{jk} \phi_{kj} \tilde{A}_{mn} \phi_{nm} e^{\frac{i(E_j - E_k + E_m - E_n)t}{\hbar}}$$

$$A_T = U_T A = U_T^\dagger A U_T$$

REMARK 1 $\phi_{eq} = \bar{\phi} := \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \phi_t dt = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \sum_{nn'} e^{i(E_n - E_{n'})t} \phi_{nn'} |E_n\rangle \langle E_{n'}|$
 $\bar{A} \equiv A_{eq} = \overline{A(t)} = \frac{1}{T} \text{Tr}[A \bar{\phi}]$
 $= \sum_n \phi_{nn} \Pi_n = \sum_n \Pi_n \phi \Pi_n$

ASSUMPTION 1 H is non-degenerate

$$|\phi\rangle = \sum_n a_n |E_n\rangle$$

$$\phi_{nn} = |a_n|^2$$

ASSUMPTION 2

$$\phi = \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix}$$

completely
dephased
state
 $\bar{\phi} = \begin{pmatrix} \phi_{11} & 0 \\ 0 & \phi_{22} \end{pmatrix}$

$$A_{eq} = \text{Tr}[A \bar{\phi}]$$



$\text{Pr}[|\langle A(t) \rangle - \bar{A}| > \epsilon] < \text{very small}$
 Equilibration in Probability

$$\phi \rightarrow \phi_t \rightarrow \phi_{eq} = U_t \phi$$

$$\lim_{t \rightarrow \infty} \phi_t = \phi_{eq} \Rightarrow 0 = \lim_{t \rightarrow \infty} \frac{d}{dt} \phi_t = \lim_{t \rightarrow \infty} \frac{d}{dt} U_t \phi = \lim_{t \rightarrow \infty} \frac{1}{i} (H U_t \phi - U_t H \phi) = \lim_{t \rightarrow \infty} \frac{1}{i} (U_t H \phi - U_t H \phi) = 0$$

$$\Delta A_T^2 = \overline{(\langle A(t) \rangle - \bar{A})^2} = \overline{(\text{Tr}[A \phi_t] - \text{Tr}[A \bar{\phi}])^2}$$

$$\max_{\psi \in \mathcal{H}} \langle \psi | A | \psi \rangle - \min_{\psi \in \mathcal{H}} \langle \psi | A | \psi \rangle = \lambda_{\max} - \lambda_{\min} \equiv \Delta A$$

$$\tilde{\phi}_t = \phi_t - \bar{\phi} = \sum_{n \neq m} \phi_{nm} e^{i(E_n - E_m)t} |E_n\rangle \langle E_m|$$

$$\langle n | \tilde{\phi}_t | k \rangle = 0 \Rightarrow \text{Tr} \tilde{\phi}_t = 0$$

$$\mathcal{H}_t = \text{supp}(\tilde{\phi}_t)$$

$$\tilde{A} = A - A_{eq}$$

$$\Delta A_T^2 = \overline{(\text{Tr}[A \phi_t])^2} - \overline{(\text{Tr}[A \bar{\phi}])^2} = \sum_{j \neq k} \tilde{A}_{jk} \phi_{kj} \tilde{A}_{km} \phi_{mn} e^{i(E_j - E_k + E_m - E_n)t}$$

$$\max_{\psi \in \mathcal{H}^+} \langle \psi | A | \psi \rangle - \min_{\psi \in \mathcal{H}^+} \langle \psi | A | \psi \rangle = a_{\max} - a_{\min} \equiv \Delta A$$

$$\Delta A_t^2 = \overline{(\text{Tr}[A \tilde{\Phi}_t])^2} = \overline{(\text{Tr}[\tilde{A} \tilde{\Phi}_t])^2} = \sum_{\substack{j \neq k \\ m \neq n}} \tilde{A}_{jk} \phi_{kj} \tilde{A}_{mn} \phi_{nm} e^{i(E_j - E_k + E_m - E_n)t}$$

only
non-zero
terms

$$A_T = U_t A = U_t^\dagger A U_t$$

REMARK 1 $\phi_{eq} = \bar{\phi} := \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \phi_t dt = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \sum_{n,n'} e^{i(E_n - E_{n'})t} \phi_{nn'} |E_n\rangle \langle E_{n'}|$
 $\bar{A} \equiv A_{eq} = \overline{A(t)} = \frac{1}{T} \int_0^T A(t) dt = \sum_n \phi_{nn} \Pi_n = \sum_n \Pi_n \phi \Pi_n$

ASSUMPTION 1 H is non-degenerate

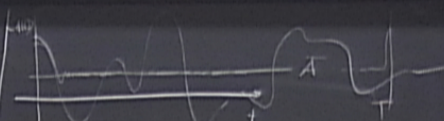
$$|\phi\rangle = \sum_n a_n |E_n\rangle$$

$$\phi_{nn} = |a_n|^2$$

$$\phi = \begin{pmatrix} \phi_{11} & \phi_{12} & \dots \\ \phi_{21} & \phi_{22} & \dots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

completely dephased state
time average $\bar{\phi} = \begin{pmatrix} \phi_{11} & 0 & \dots \\ 0 & \phi_{22} & \dots \\ \vdots & \vdots & \ddots \end{pmatrix}$

$$A_{eq} = \text{Tr}[A \bar{\phi}]$$



$\text{Pr}[|A(t) - \bar{A}| > \epsilon] < \text{very small}$
 Equilibration in Probability

$\phi \rightarrow \phi_t \rightarrow \phi_{eq} = U_t^\dagger \phi U_t$
 $\lim_{t \rightarrow \infty} \phi_t = \phi_{eq} \Rightarrow 0 = \lim_{t \rightarrow \infty} \frac{d}{dt} \phi_t = \lim_{t \rightarrow \infty} \frac{d}{dt} U_t^\dagger \phi U_t = 0$

ASSUMPTION 2

Non degenerate gaps $j = n, k = m$

$$\Delta A_T^2 = \overline{(A(t) - \bar{A})^2} = \overline{(\text{Tr}[A \phi_t] - \text{Tr}[A \bar{\phi}])^2}$$

$$\max_{\psi \in \mathcal{H}} \langle \psi | A | \psi \rangle - \min_{\psi \in \mathcal{H}} \langle \psi | A | \psi \rangle = \lambda_{\max} - \lambda_{\min} \equiv \Delta A$$

$$\tilde{\phi}_t = \phi_t - \bar{\phi} = \sum_{n \neq m} \phi_{nm} e^{i(E_n - E_m)t} |E_n\rangle \langle E_m|$$

$$\langle n | \tilde{\phi}_t | k \rangle = 0 \Rightarrow \text{Tr} \tilde{\phi}_t = 0$$

$$\mathcal{H}_t = \text{supp}(\phi_t)$$

$$\tilde{A} = A - a_{min}$$

$$\Delta A_T^2 = \overline{(\text{Tr}[A \phi_t])^2} - \overline{(\text{Tr}[A \bar{\phi}])^2} = \sum_{j \neq k} \sum_{m \neq n} \phi_{kj} \tilde{A}_{nm} \phi_{nm} e^{i(E_j - E_k + E_m - E_n)t}$$

$E_j - E_k = E_n - E_m$ only terms

$$\max_{\psi \in \mathcal{H}} \langle \psi | A | \psi \rangle - \min_{\psi \in \mathcal{H}} \langle \psi | A | \psi \rangle = a_{\max} - a_{\min} = \Delta A$$

$$\Delta^2 A = \overline{(\text{Tr}[A \tilde{\Phi}_t])^2} = \overline{(\text{Tr}[\tilde{A} \tilde{\Phi}_t])^2} = \sum_{\substack{j \neq k \\ m \neq n}} \tilde{A}_{jk} \phi_{kj} \tilde{A}_{mn} \phi_{nm}$$

$$= \sum_{n \neq m} \tilde{A}_{nm} \phi_{mn} \tilde{A}_{mn} \phi_{nm}$$

$$e^{i(E_j - E_k + E_n - E_m)t}$$

$E_j - E_k = E_n - E_m$ only non-zero terms

$$\max_{\psi \in \mathcal{H}^+} \langle \psi | A | \psi \rangle - \min_{\psi \in \mathcal{H}^+} \langle \psi | A | \psi \rangle = \lambda_{\max} - \lambda_{\min} = \Delta A$$

$$\Delta^2 A = \overline{(\text{Tr}[A \tilde{\Phi}_t])^2} = \overline{(\text{Tr}[\tilde{A} \tilde{\Phi}_t])^2} = \sum_{\substack{j \neq k \\ m \neq n}} \tilde{A}_{jk} \phi_{kj} \tilde{A}_{mn} \phi_{nm} e^{i(E_j - E_k + E_m - E_n)t}$$

$$= \sum_{n \neq m} \tilde{A}_{nm} \phi_{mn} \tilde{A}_{mn} \phi_{nm} \leq \sum_{n \neq m} |\tilde{A}_{mn}|^2 |\phi_{mn}|^2$$

$E_j - E_k = E_n - E_m$ only non-zero terms

$$\max_{\psi \in \mathcal{H}^+} \langle \psi | A | \psi \rangle - \min_{\psi \in \mathcal{H}^+} \langle \psi | A | \psi \rangle = a_{\max} - a_{\min} = \Delta A$$

$$\Delta^2 A = \overline{(\text{Tr}[A \tilde{\Phi}_t])^2} = \overline{(\text{Tr}[\tilde{A} \tilde{\Phi}_t])^2} = \sum_{\substack{j \neq k \\ m \neq n}} \tilde{A}_{jk} \phi_{kj} \tilde{A}_{mn} \phi_{nm} e^{i(E_j - E_k + E_m - E_n)t}$$

$$= \sum_{n \neq m} \tilde{A}_{nm} \phi_{mn} \tilde{A}_{mn} \phi_{nm} \leq \sum_{n \neq m} |\tilde{A}_{mn}|^2 |\phi_{mn}|^2$$

$E_j - E_k = E_n - E_m$ only non-terms
 $|\phi_{mn}|^2 \leq |\phi_{mm} \phi_{nn}|$

$$\begin{aligned}
&\leq \sum_{m,n} \tilde{A}_{mn} \phi_{mn} \tilde{A}_{nm} \phi_{nn} \\
&= \sum_{m,n} \langle E_m | \tilde{A} \tilde{\Phi} | E_n \rangle \underbrace{\langle E_n | \tilde{A} \tilde{\Phi} | E_m \rangle}_{=1} \\
&= \text{Tr} [\tilde{A} \tilde{\Phi}]
\end{aligned}$$

$$\begin{aligned}
&\leq \sum_{m,n} \tilde{A}_{mn} \phi_{mn} \tilde{A}_{nm} \phi_{nn} \\
&= \sum_{m,n} \langle E_m | \tilde{A} \bar{\phi} | E_n \rangle \underbrace{\langle E_n | \tilde{A} \bar{\phi} | E_m \rangle}_{11} \\
&= \text{Tr} [\tilde{A} \bar{\phi}]^2 \\
&\leq \delta_A^2 \text{Tr} \bar{\phi}^2
\end{aligned}$$

$$n|\tilde{A}\Phi|E_m\rangle$$

$$\text{Prob}(|x - \mu| > \epsilon) \leq \frac{\sigma^2}{\epsilon^2}$$

Chebyshev
Inequality

$$\text{Prob}\left[|\text{Tr}[A\phi_t] - \text{Tr}[A\bar{\Phi}]| > \epsilon\right] \leq \frac{\sqrt{4} \text{Tr}\bar{\Phi}^2}{\epsilon^2}$$

$$\text{Prob}(|x - \mu| > \epsilon) \leq \frac{\sigma^2}{\epsilon^2}$$

Chebyshev
Inequality

$\epsilon_m >$

$$|\text{Tr}[A\phi_t] - \text{Tr}[A\bar{\Phi}]| > \epsilon \leq$$

$$\stackrel{O(1)}{=} \frac{\sum_4 \text{Tr} \bar{\Phi}^2}{\epsilon^2}$$

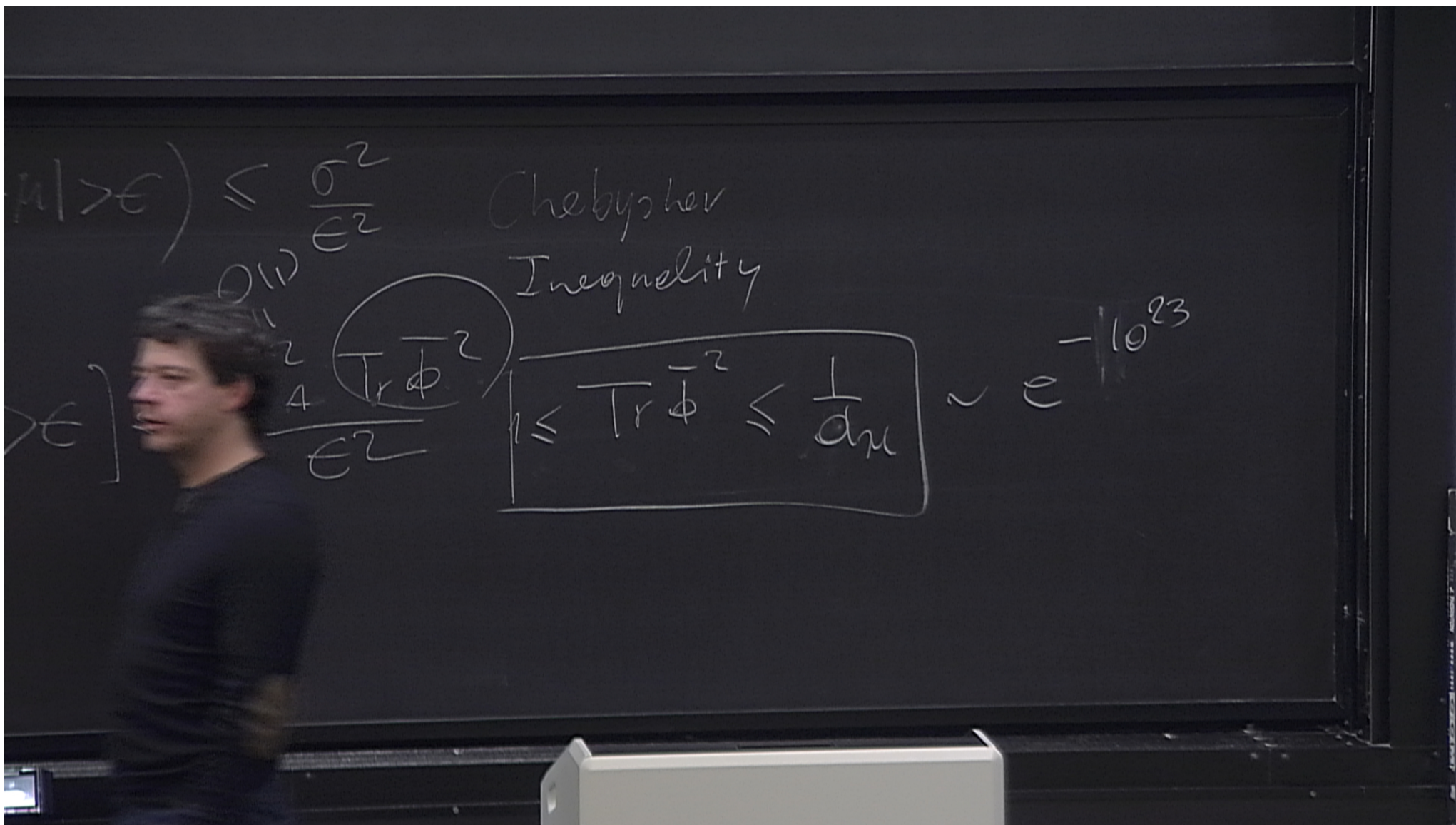
$$1 \leq \text{Tr} \bar{\Phi}^2 \leq \frac{1}{d_n}$$

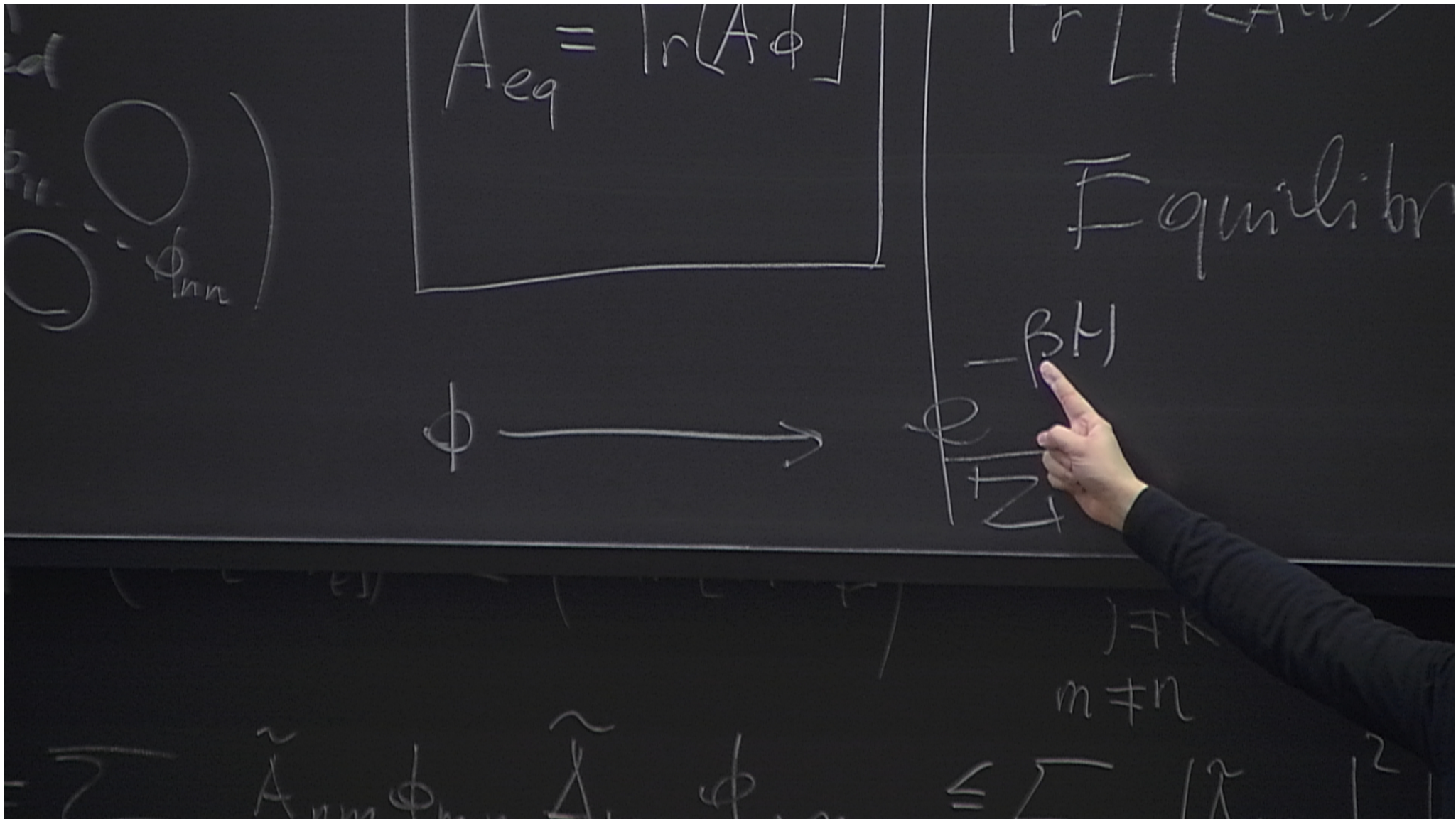
$$P(|x - \mu| > \epsilon) \leq \frac{\sigma^2}{\epsilon^2}$$

Chebyshev
Inequality

$$P([A\bar{\Phi}] > \epsilon) \leq \frac{\overset{O(1)}{\sqrt{4}} \sqrt{\text{Tr } \bar{\Phi}^2}}{\epsilon^2}$$

$$1 \leq \text{Tr } \bar{\Phi}^2 \leq \frac{1}{d\mu} \sim e^{-N}$$





$$\Pr[|\langle A(t) \rangle - \bar{A}| > \epsilon] < \text{very small}$$

Equilibration in Probability

$$\bar{A} = \frac{\text{Tr}[A e^{-\beta H}]}{\text{Tr}[e^{-\beta H}]} = \text{Tr}[A \Phi]$$

$$= \sum_{j \neq k} A_{jk} \Phi_{kj} = \sum_{m \neq n} A_{mn} \Phi_{nm} e^{-\beta(E_n - E_m)} \quad \text{only } E_n - E_m$$