Title: PSI 17/18 - Gravitational Physics - Lecture 4

Date: Jan 08, 2018 10:15 AM

URL: http://pirsa.org/18010024

Abstract:

Lecture 4 Causal structure of black holes

A black hole is characterised by its event horizon - a causal boundary. Encode causal structure of a black hole in a Penrose-Carter diagram. $ds^2 = V dt^2 - \frac{dr^2}{V} - r^2 d\Omega_{(2)}^2$ $V = 1 - \frac{2GM}{r} = \frac{r - r_+}{r}$

$r_+ = 2GM$ – event horizon.

First take $r^* = \int \frac{dr}{V} = \int \frac{r dr}{r - r_+} = r + r_+ \ln \frac{(r - r_+)}{r_+}$

$r_+ = 2GM$ – event horizon.

First take $r^* = \int \frac{dr}{V} = \int \frac{r\,dr}{r - r_+} = r + r_+ \ln\left(\frac{r - r_+}{r_+}\right)$

$[ds^2 = V[dt^2 - dr^{*2}] - \cdots]$

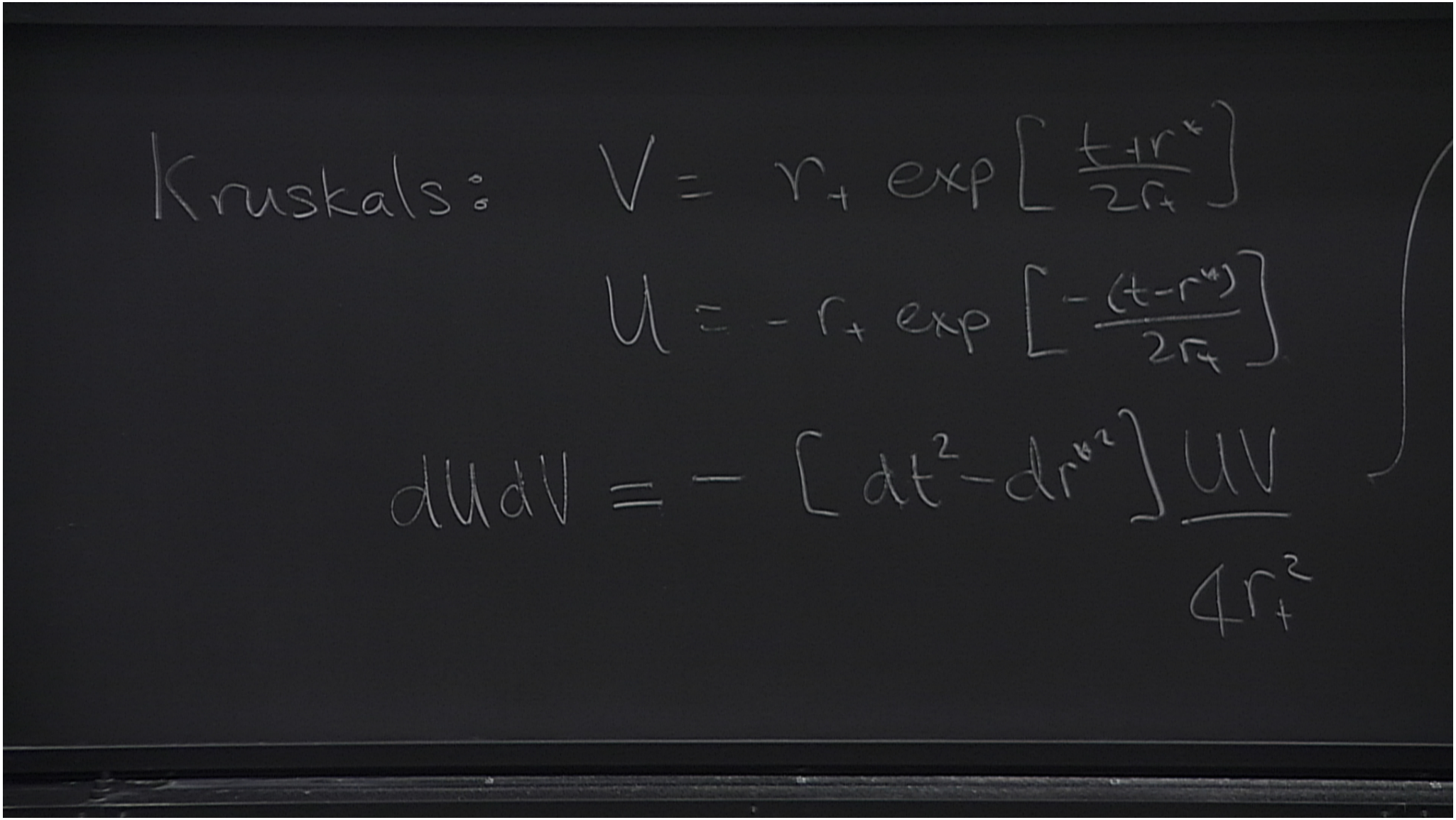
lightcones $t \pm r^* = \text{const}$

First take $r^* = \int \frac{dr}{V} = \int \frac{r\,dr}{r - r_+} = r + r_+ \ln\left(\frac{r - r_+}{r_+}\right)$

$ds^2 = V[dt^2 - dr^{*2}] = \cdots$

lightcones $t \pm r^* = \text{const}$

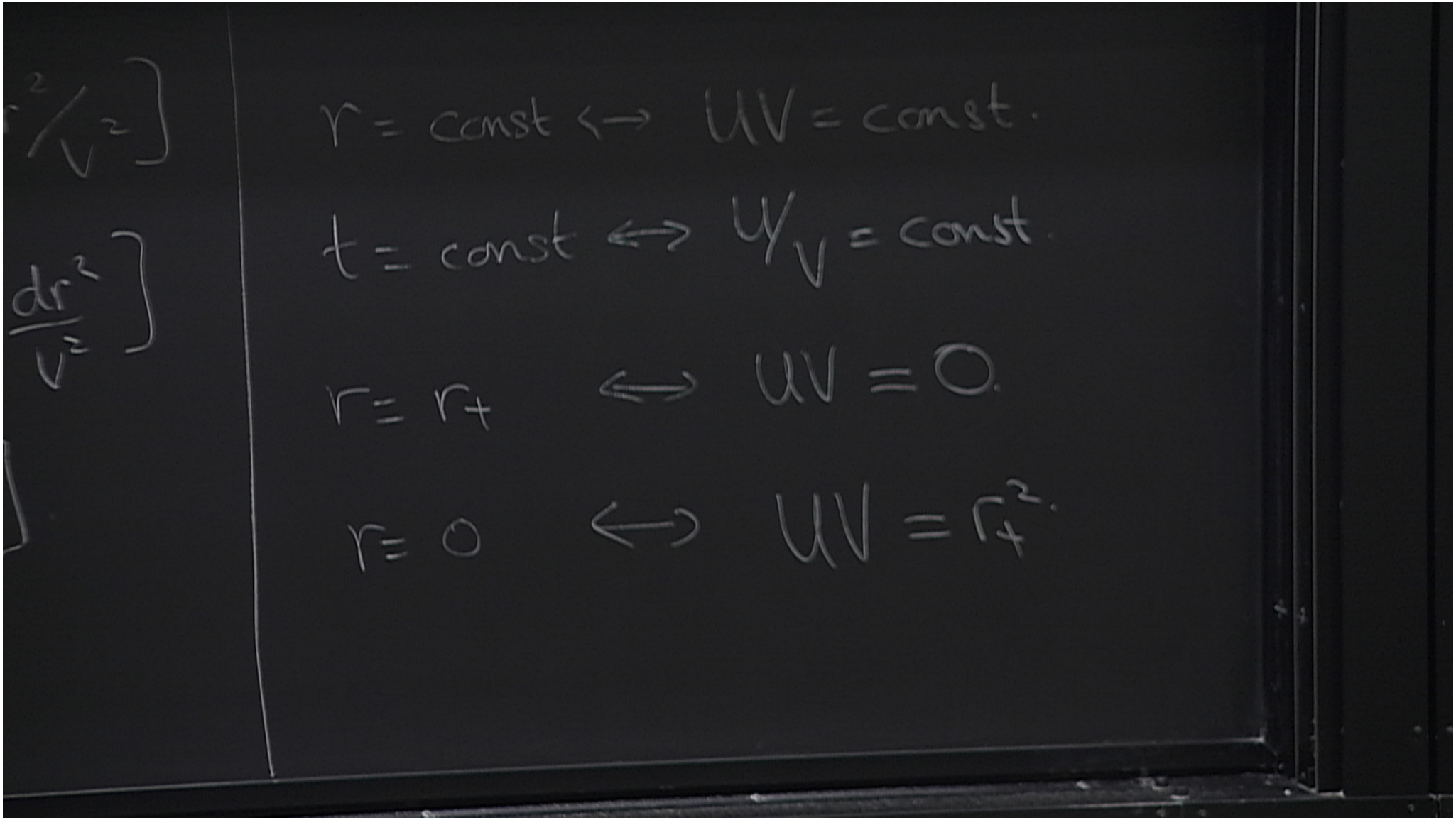
$v = t - r^*$
$u = t + r^*$
} Eddington Finkelstein adv/ret null coords.

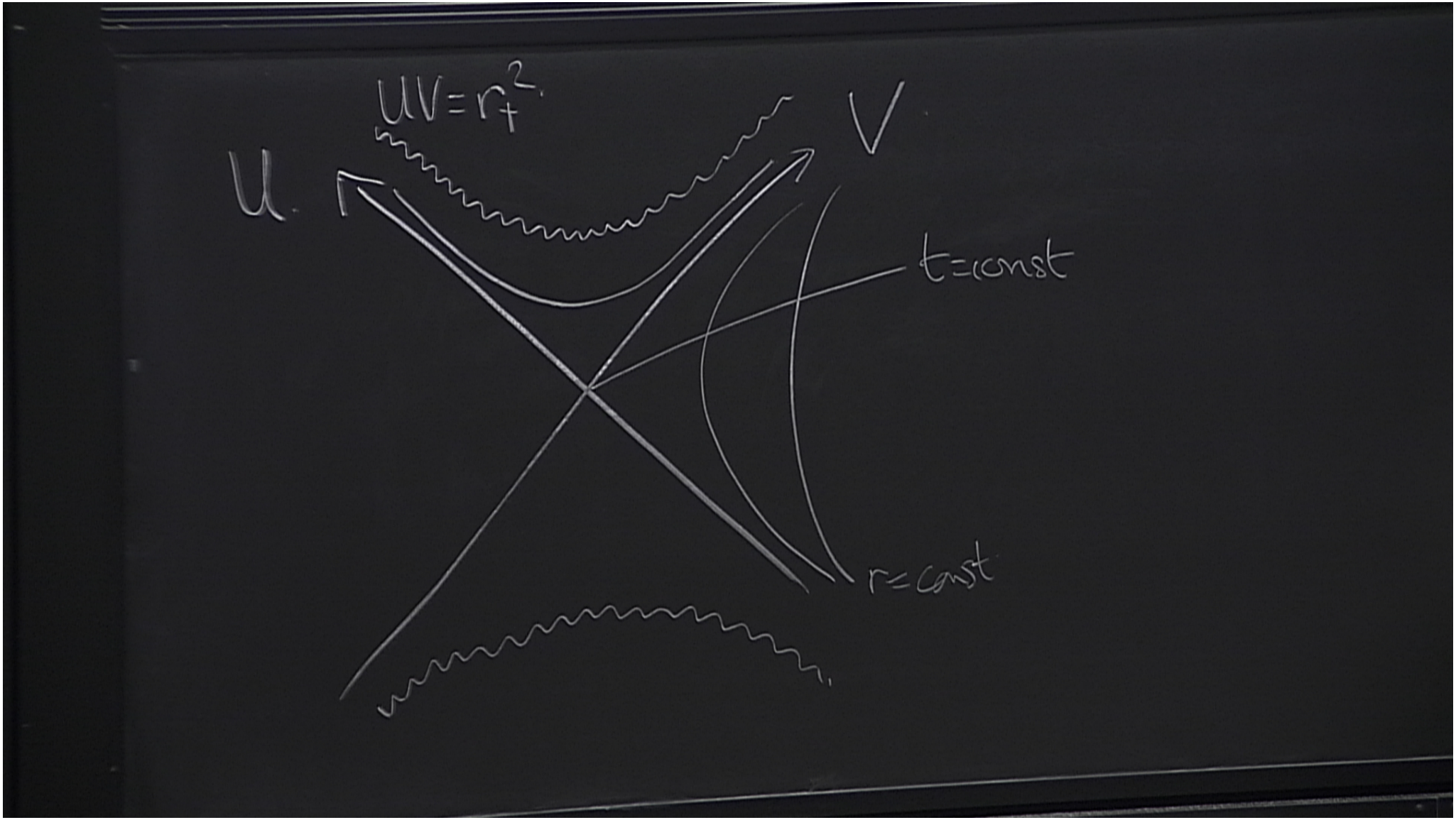

Kruskals: $V = r_+ \exp\left[\frac{t+r^*}{2r_+}\right]$
$U = -r_+ \exp\left[-\frac{(t-r^*)}{2r_+}\right]$
$dUdV = -[dt^2 - dr^{*2}]\frac{UV}{4r_+^2}$

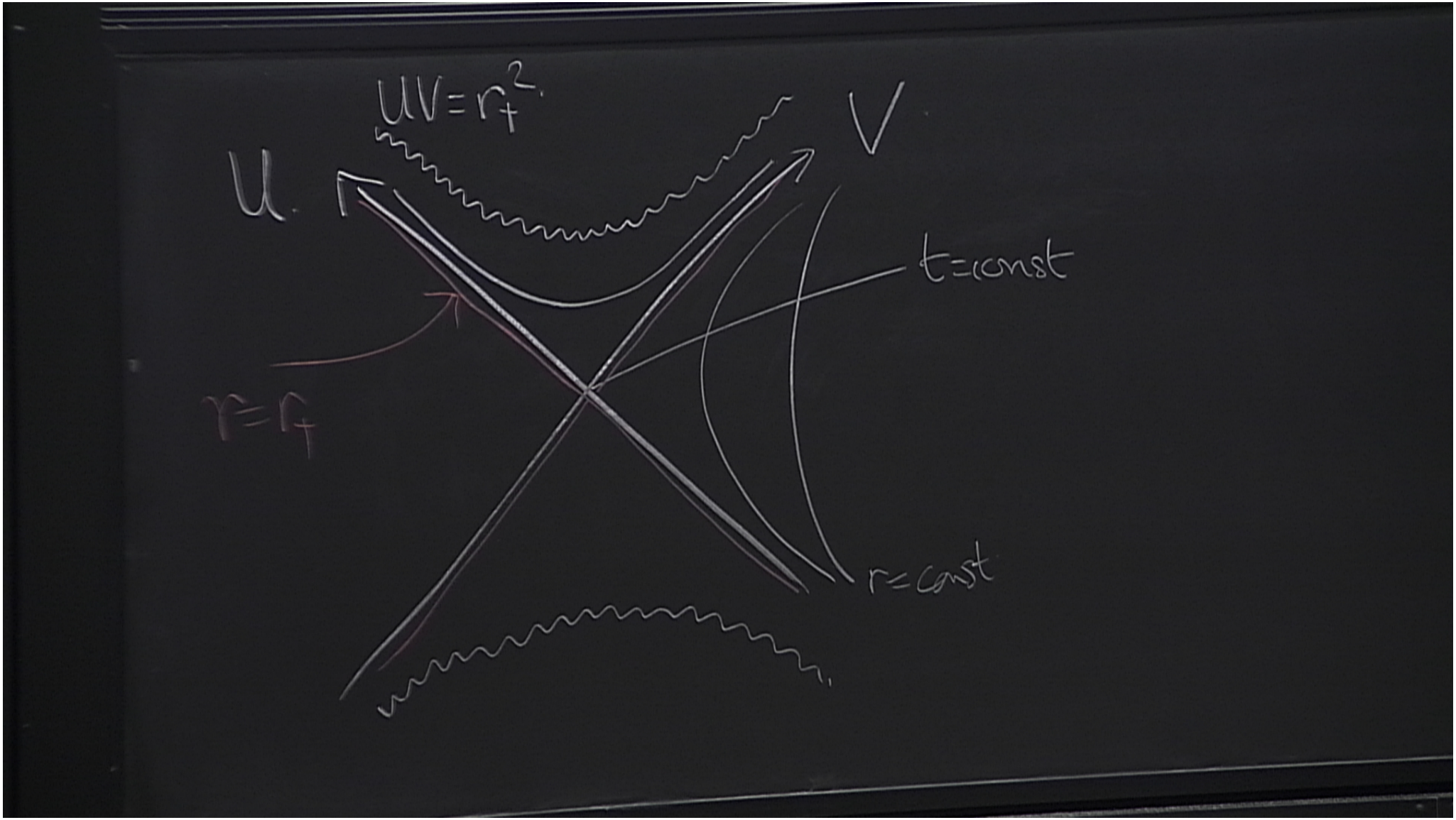
$$= \frac{1}{4} e^{r^*/r_+} \left[dt^2 - dr^2/V^2 \right]$$

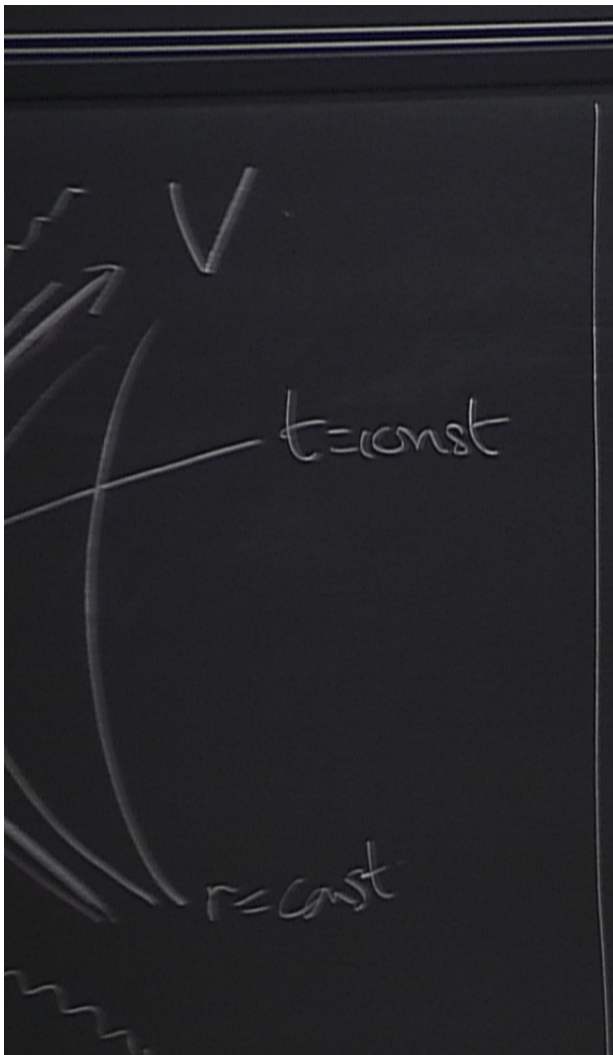
$$= \frac{1}{4} e^{r/r_+} \left(\frac{r - r_+}{r_+} \right) \left[dt^2 - \frac{dr^2}{V^2} \right]$$

$$= \frac{1}{4} \frac{r}{r_+} e^{r/r_+} \left[V dt^2 - \frac{dr^2}{V} \right]$$


$r = \text{const} \leftrightarrow UV = \text{const}.$
$t = \text{const} \leftrightarrow U/V = \text{const}.$
$r = r_+ \iff UV = 0.$
$r = 0 \iff UV = r_+^2.$


UV=r_f^2
U
V
t=const
r=const


UV=r_f^2
U
V
t=const
r=r_f
r=const



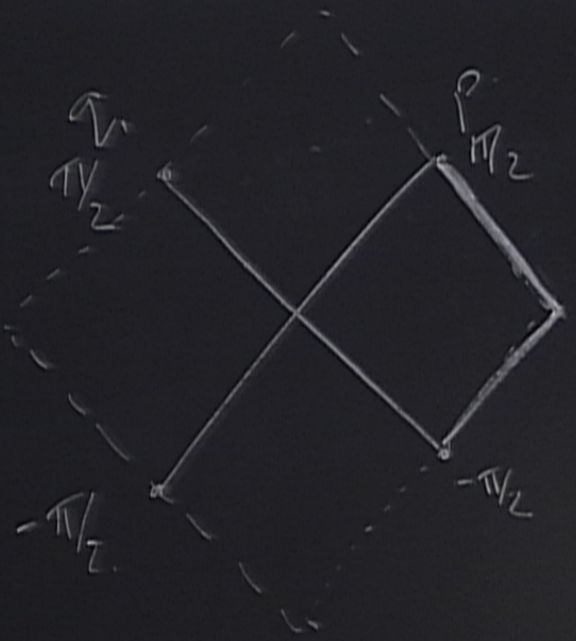
Now define $p = \arctan V/r_+$

$q = \arctan U/r_+$

e.g. $V \to \infty \leftrightarrow p \to \pi/2$

$V = r_+ \leftrightarrow p = \pi/4$

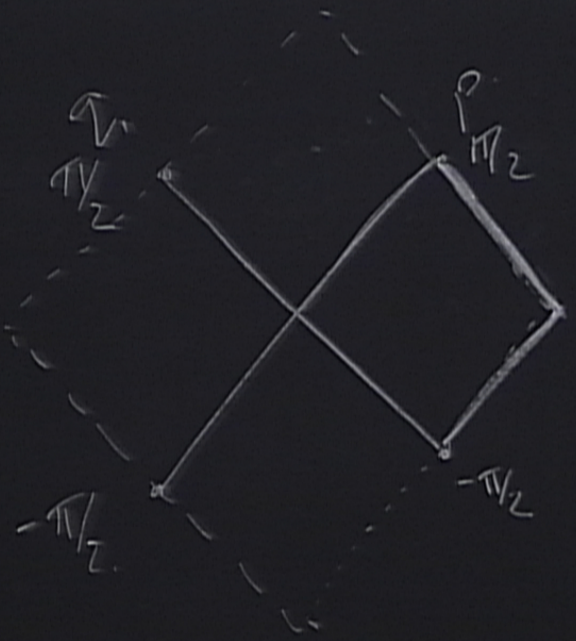
$V = 0 \leftrightarrow p = 0.$



Physical black hole

exterior: $V \in (0, \infty)$

$U \in (-\infty, 0)$

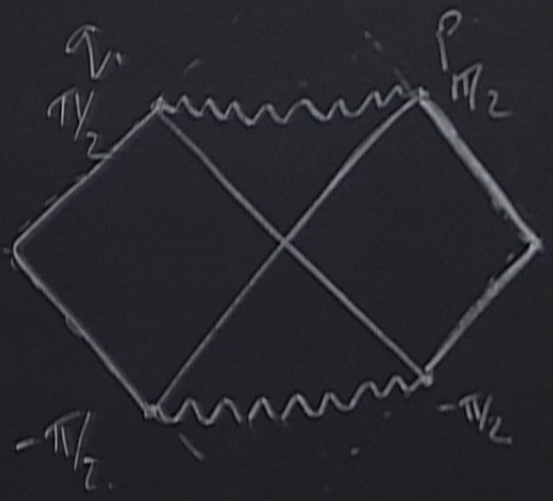


Physical black hole

exterior: $V \in (0, \infty)$

$U \in (-\infty, 0)$

? $r \to 0$.



Physical black hole

exterior: $V \in (0, \infty)$

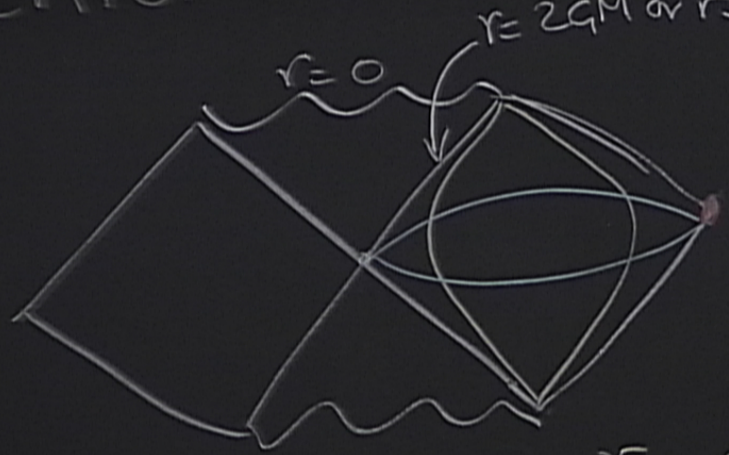
$U \in (-\infty, 0)$

? $r \to 0$. i.e. $UV = r_+^2 \Rightarrow \tan\rho \tan\eta = 1$

$\sin\rho \sin\eta = \cos\rho \cos\eta$

or $\cos(\rho+\eta) = 0$. $\rho+\eta = \pm\pi/2$.

Penrose - Carter diagram:

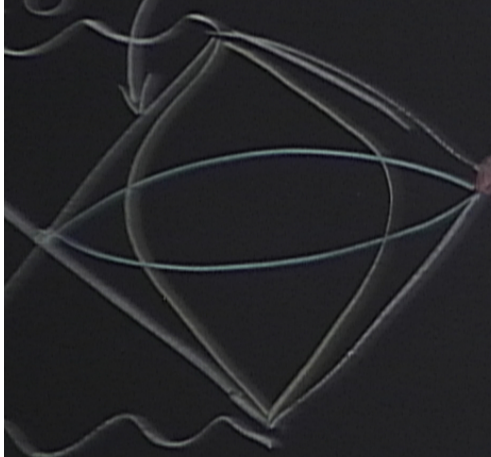


r = const

t = const.

Carter diagram:

$r = 2M$ or r_+

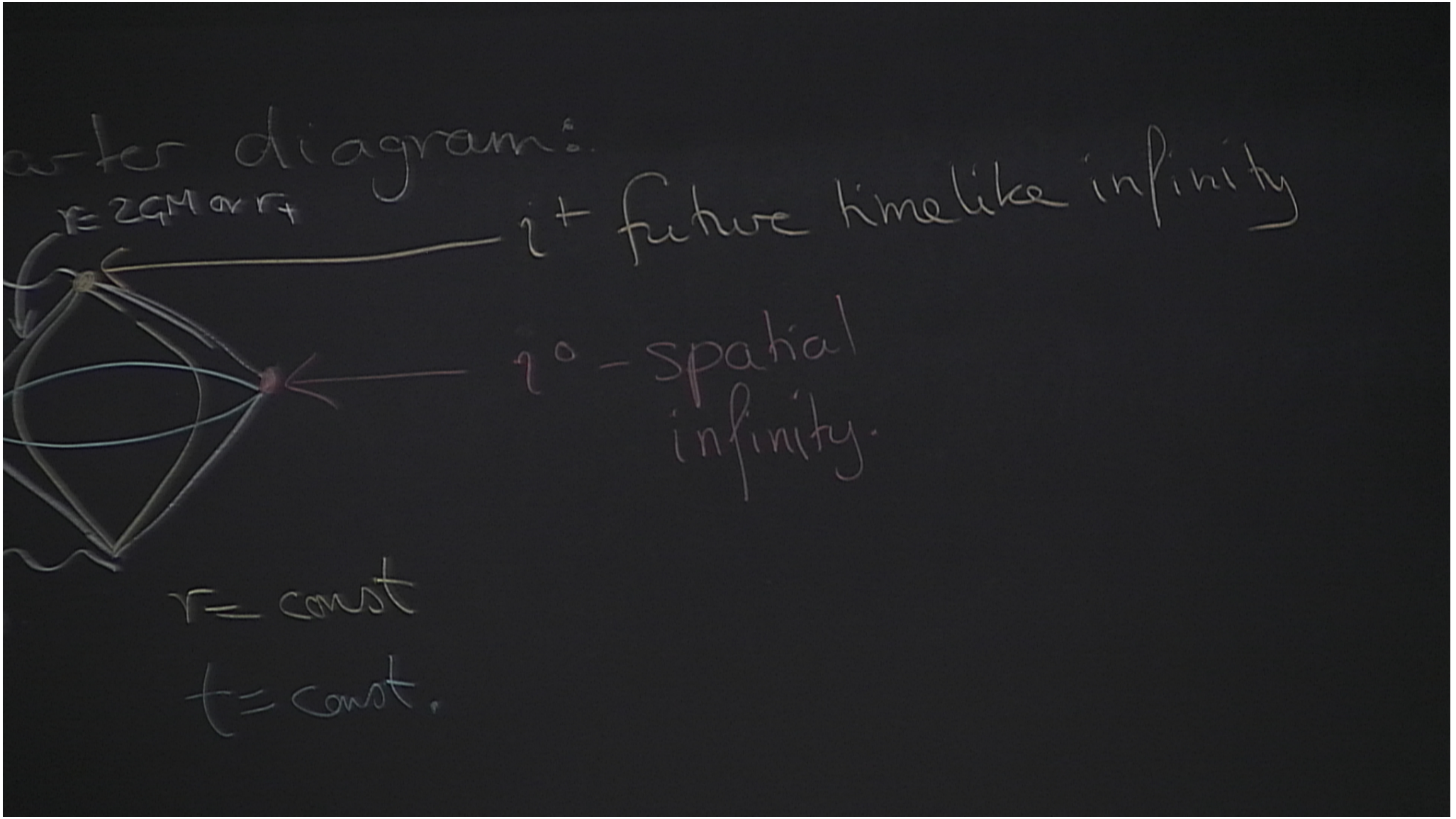


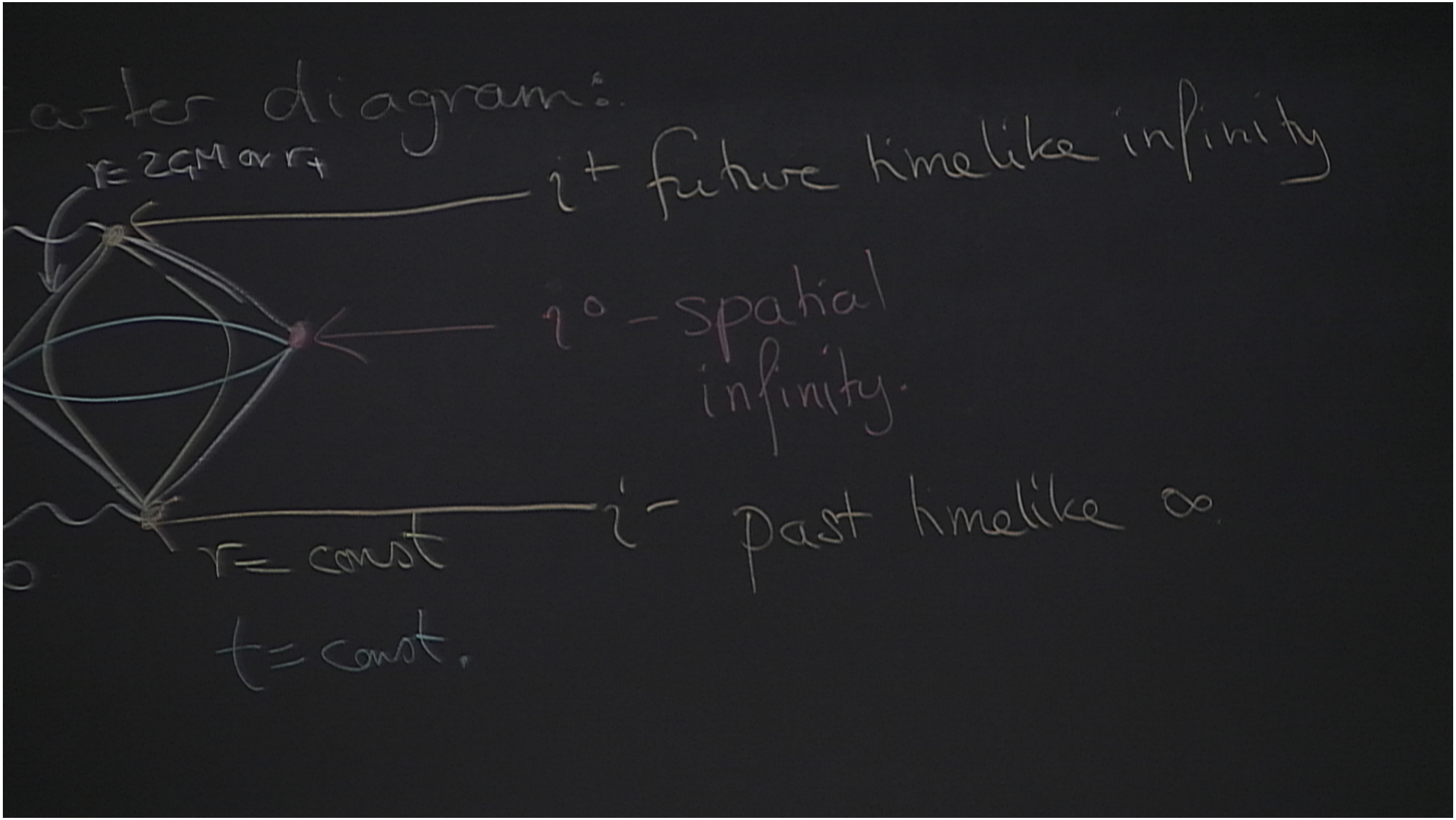
i^0 - spatial infinity.

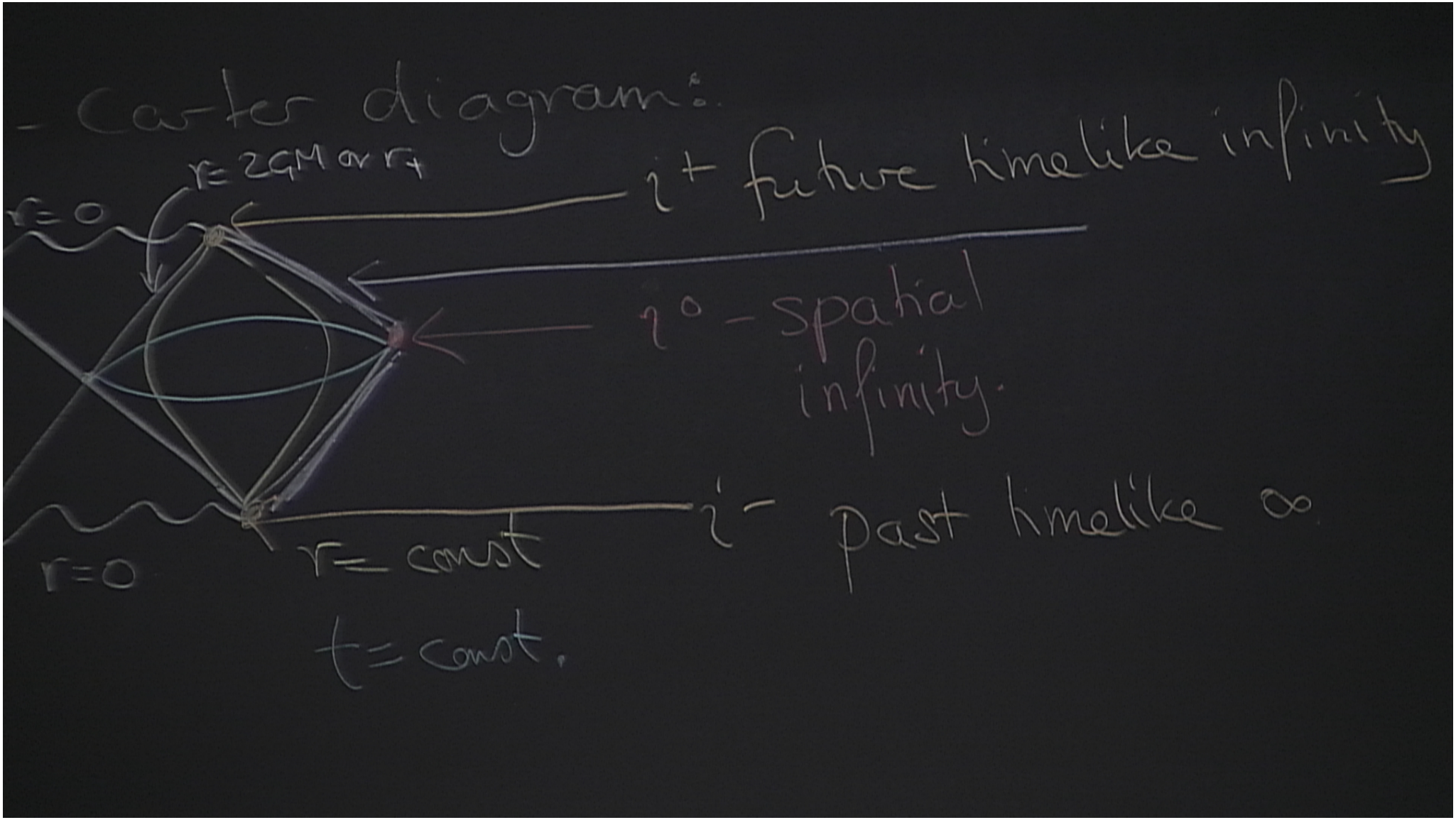
$r = 0$

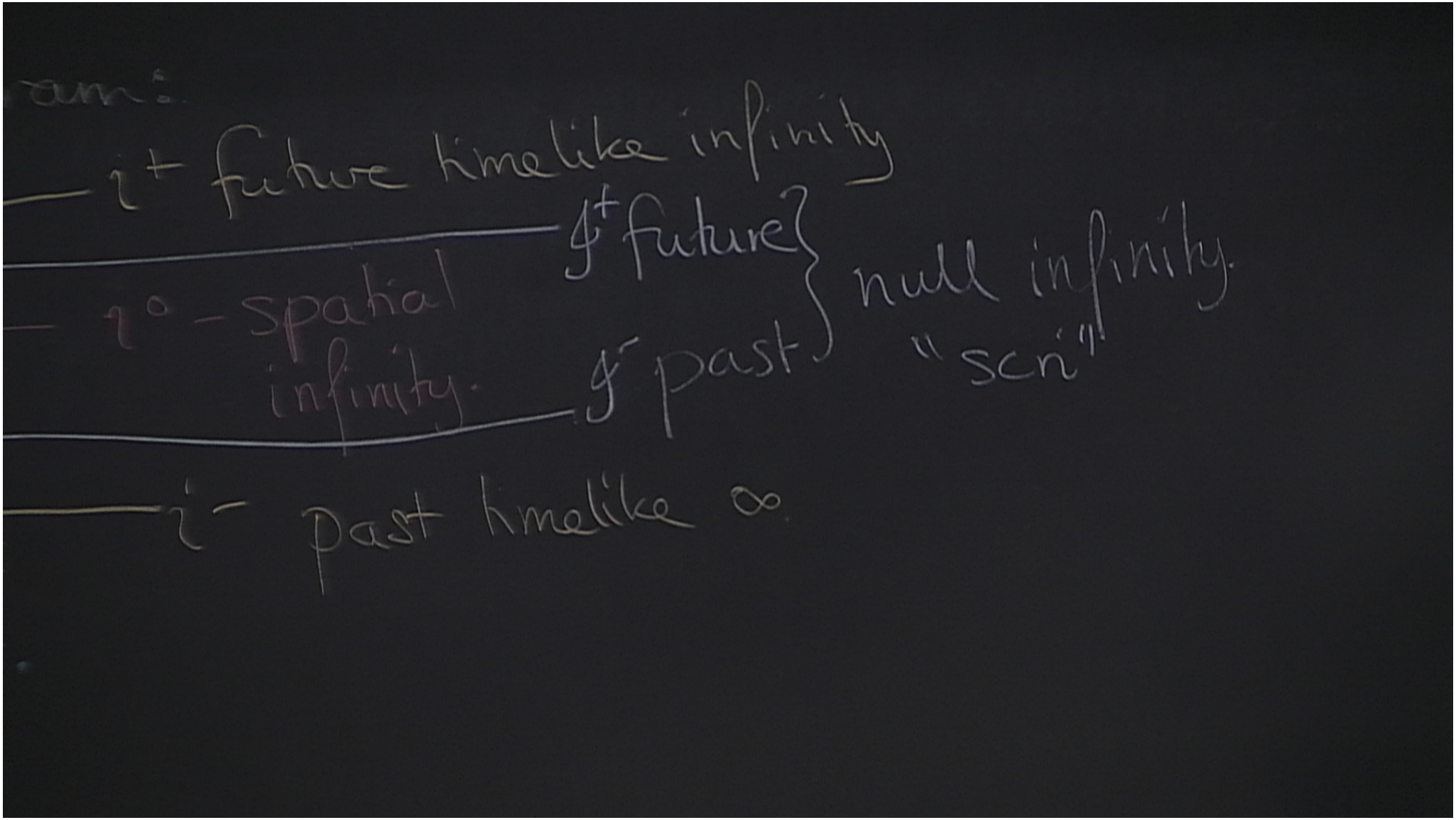
$r = \text{const}$

$t = \text{const}$.


r=2GM or r+
i+ future timelike infinity
i° - spatial infinity.
r= const
t= const.

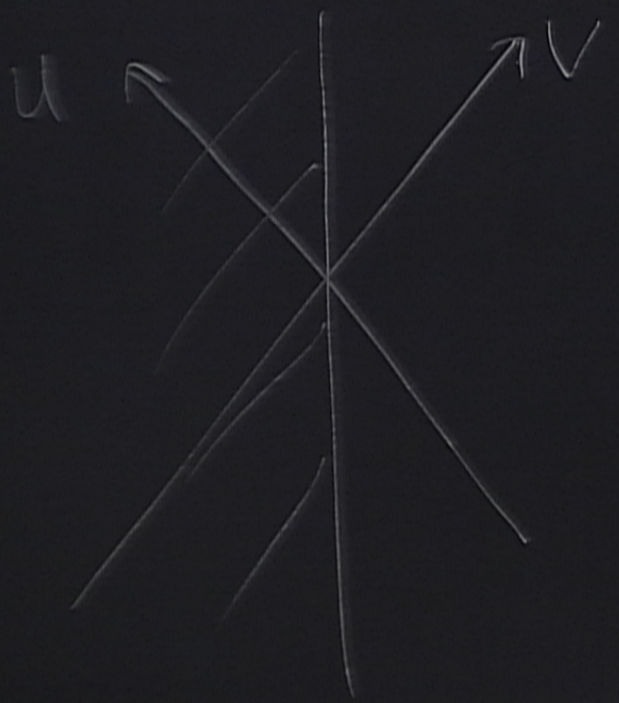

r=2GM or r+
i+ future timelike infinity
i° - spatial infinity.
i- past timelike ∞
r= const
t= const.


- Carter diagram:
r = 2GM or r+
r = 0
i+ future timelike infinity
i° - spatial infinity.
i- past timelike ∞
r = const
r = 0
t = const.

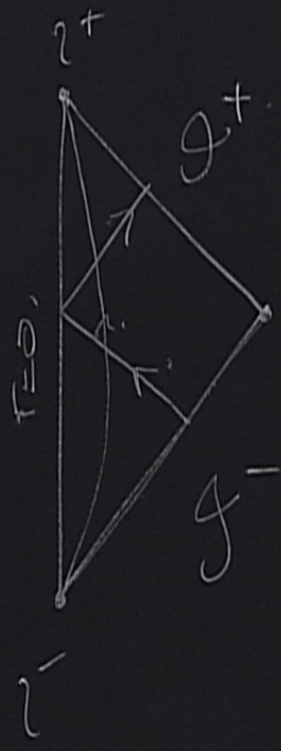

i+ future timelike infinity
I+ future
i0 - spatial infinity.
I- past
null infinity.
"scri"
i- past timelike ∞

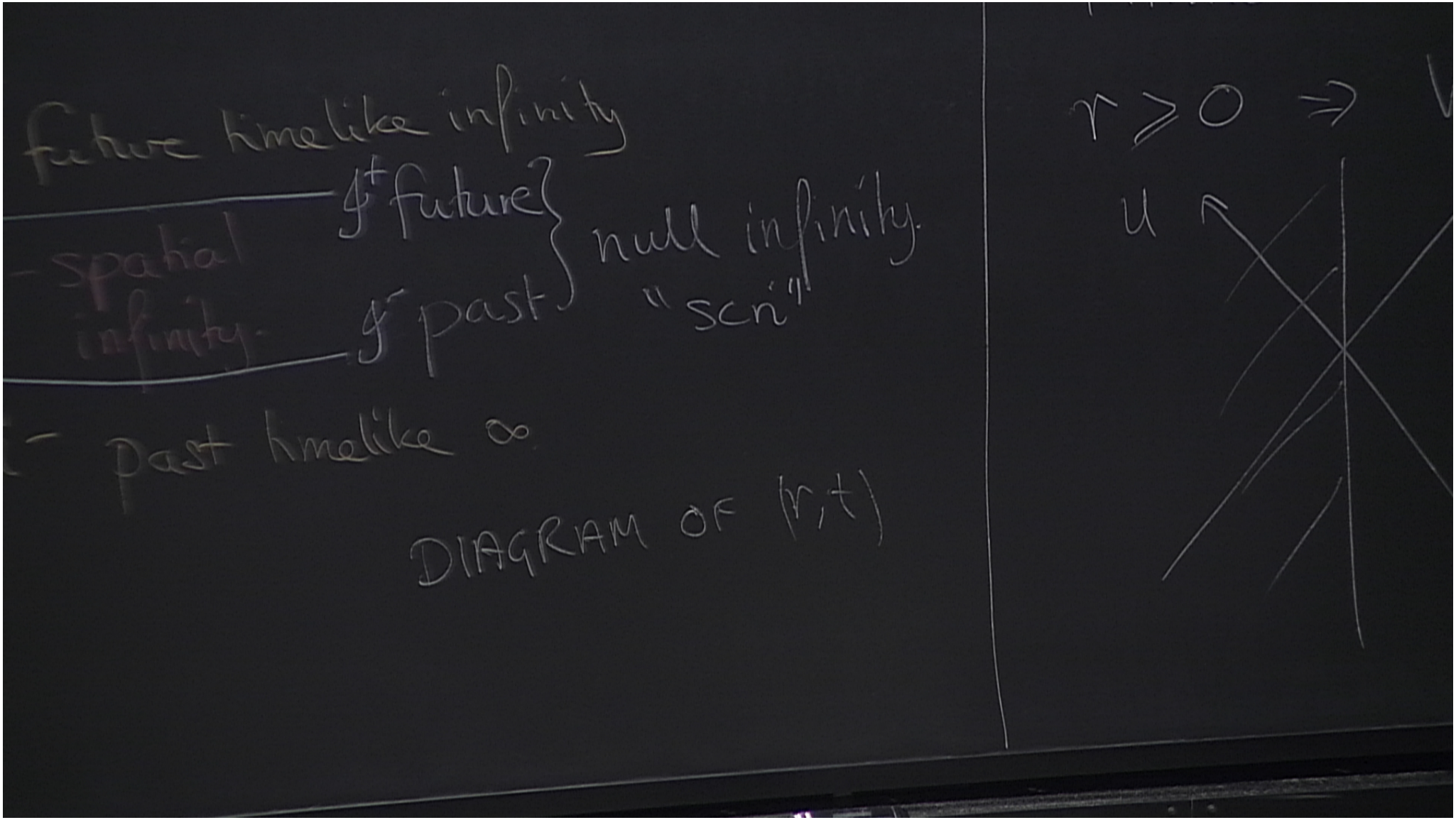
Minkowski: $ds^2 = dt^2 - dr^2 - r^2 d\Omega_{S^2}^2$ $V = t + r$, $U = t - r$.

$r \geq 0 \Rightarrow V \geq U$.



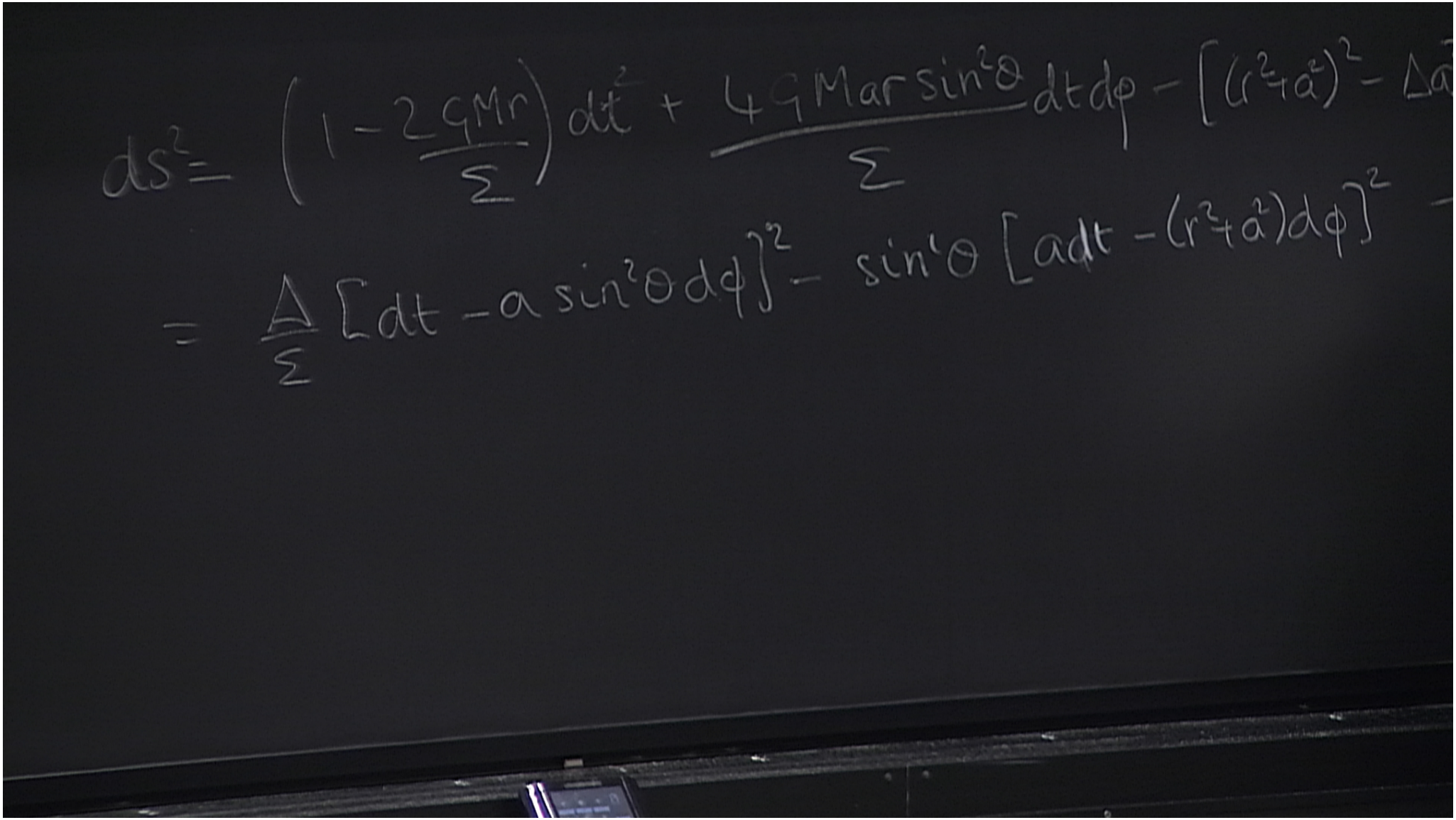
p, q.




future timelike infinity
$\mathcal{I}^+$ future
spatial infinity
$\mathcal{I}^-$ past
null infinity
"scri"
past timelike ∞
DIAGRAM OF (r,t)
r ≥ 0 ⇒
u

Kerr Black Hole

Astrophysical black holes rotate - described by Kerr soln.


$ds^2 = \left(1 - \frac{2GMr}{\Sigma}\right) dt^2 + \frac{4GMar\sin^2\theta}{\Sigma} dt\,d\phi - \left[(r^2+a^2)^2 - \Delta a^2\right]$
$= \frac{\Delta}{\Sigma}\left[dt - a\sin^2\theta\, d\phi\right]^2 - \sin^2\theta\left[a\,dt - (r^2+a^2)d\phi\right]^2$

$$ds^2 = \left(1 - \frac{2GMr}{\Sigma}\right) dt^2 + \frac{4GMar\sin^2\theta}{\Sigma} dt\,d\phi - \left[(r^2+a^2)^2 - \right.$$

$$= \frac{\Delta}{\Sigma}\left[dt - a\sin^2\theta\, d\phi\right]^2 - \sin^2\theta\left[a\,dt - (r^2+a^2)d\phi\right]^2$$

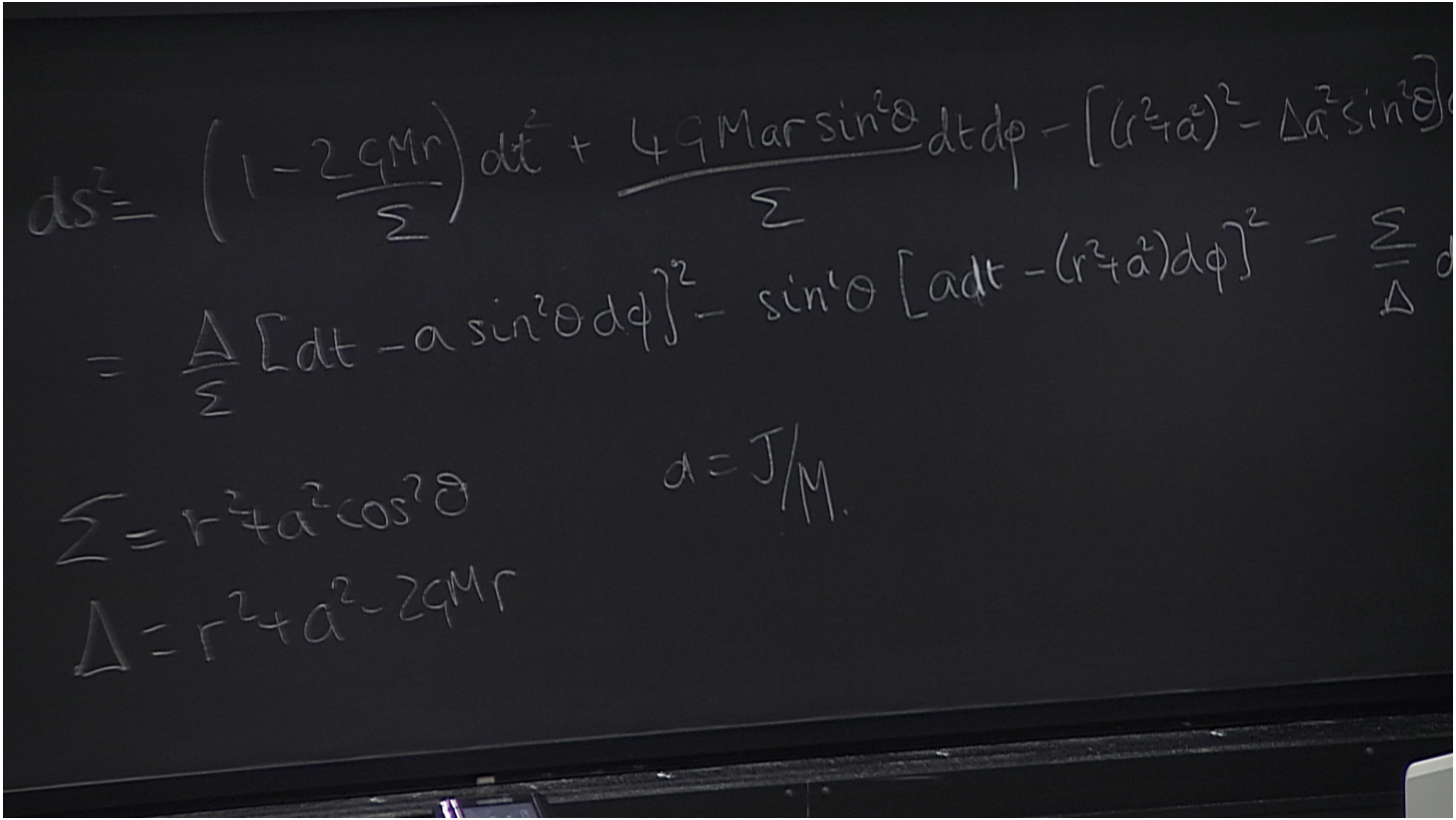
$$\Sigma = r^2 + a^2\cos^2\theta \qquad a = J/M.$$

$$\Delta = r^2 + a^2 - 2GMr$$

$$dt^2 + \frac{4GMar\sin^2\theta}{\Sigma} dt\,d\phi - \left[(r^2+a^2)^2 - \Delta a^2 \sin^2\theta\right]\frac{d\phi^2}{\Sigma} - \frac{\Sigma}{\Delta} dr^2 - \Sigma\, d\theta^2.$$

$$\sin^2\theta\, d\phi\Big]^2 - \sin^2\theta\,[a\,dt - (r^2+a^2)\,d\phi]^2 - \frac{\Sigma}{\Delta} dr^2 - \Sigma\, d\theta^2.$$

$$a = J/M.$$


$ds^2 = \left(1 - \frac{2GMr}{\Sigma}\right) dt^2 + \frac{4GMar\sin^2\theta}{\Sigma} dt\,d\phi - \left[(r^2+a^2)^2 - \Delta a^2 \sin^2\theta\right]$
$= \frac{\Delta}{\Sigma}\left[dt - a\sin^2\theta\, d\phi\right]^2 - \sin^2\theta\left[a\,dt - (r^2+a^2)\,d\phi\right]^2 - \frac{\Sigma}{\Delta}$
$\Sigma = r^2 + a^2\cos^2\theta$
$a = J/M.$
$\Delta = r^2 + a^2 - 2GMr$

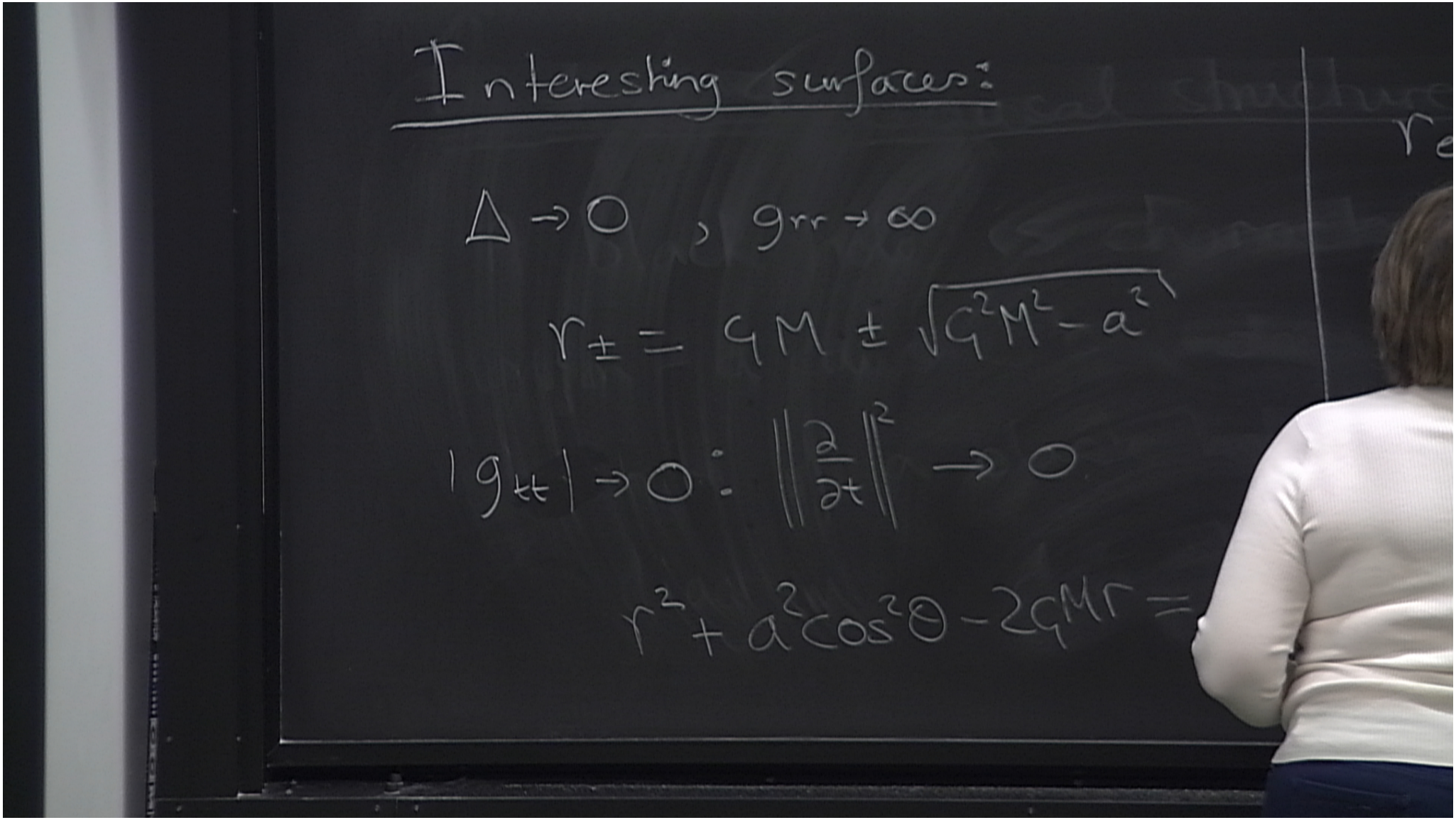
Interesting surfaces:

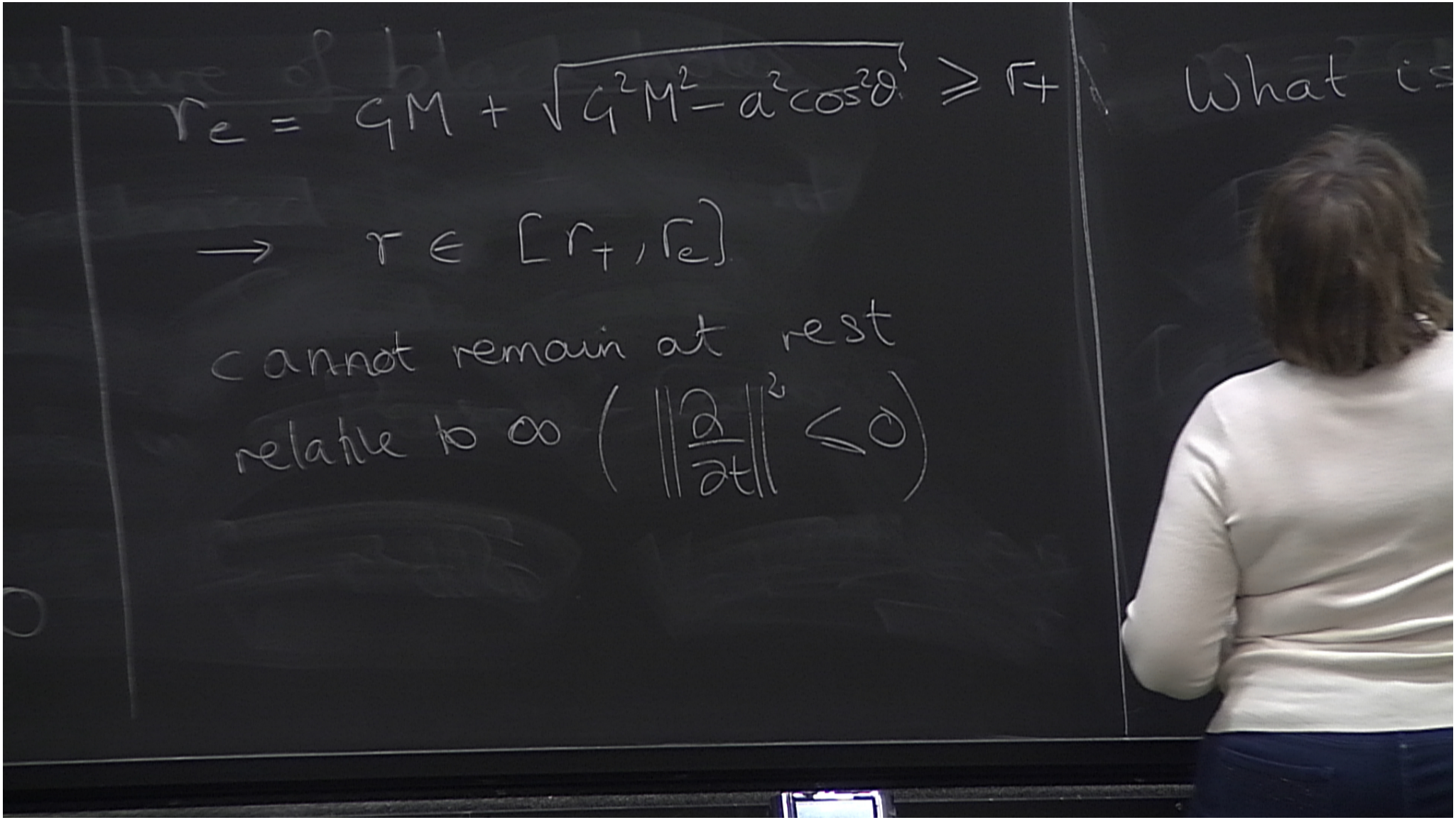
$\Delta \to 0$, $g_{rr} \to \infty$

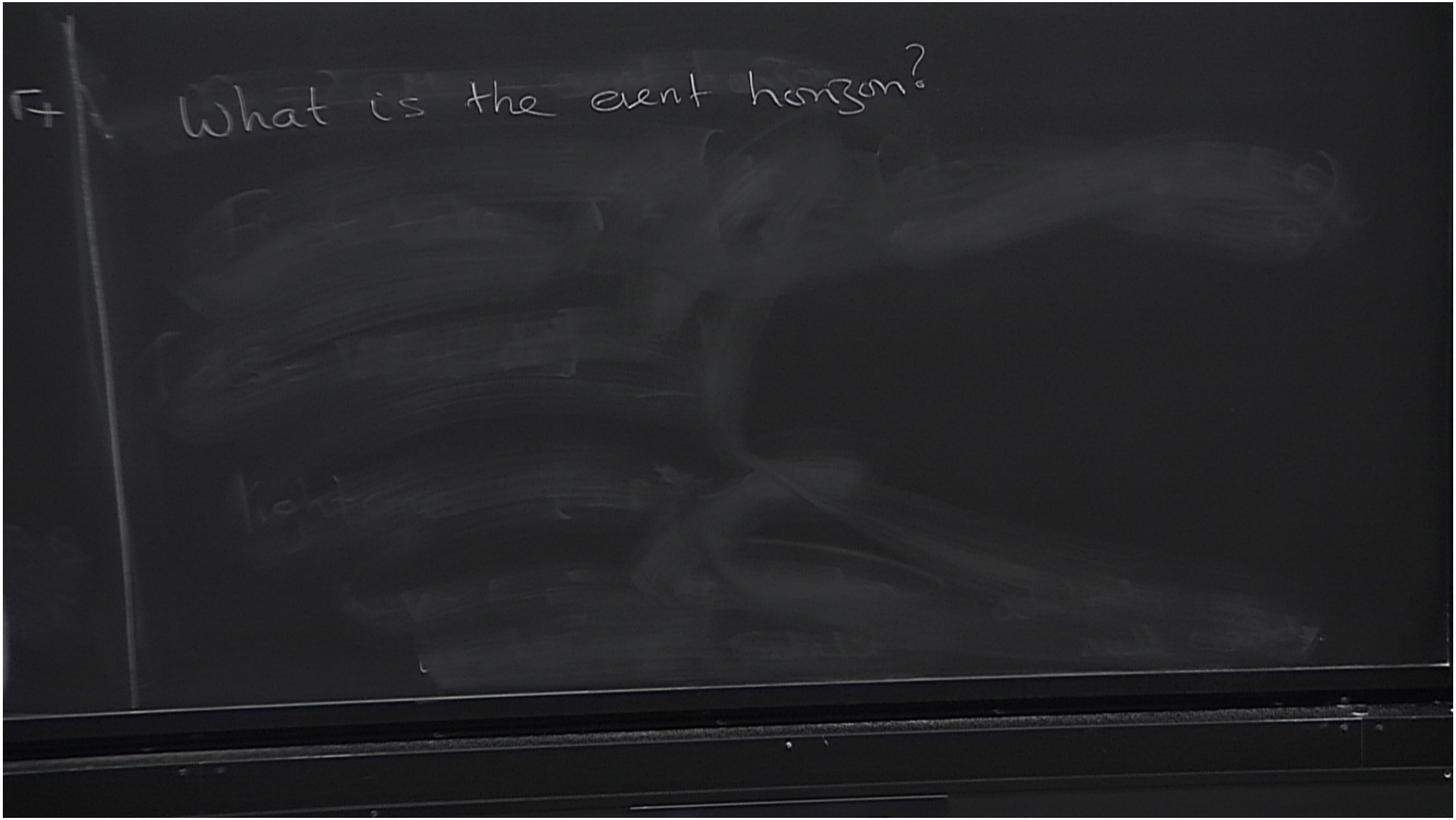
$$r_\pm = GM \pm \sqrt{G^2M^2 - a^2}$$

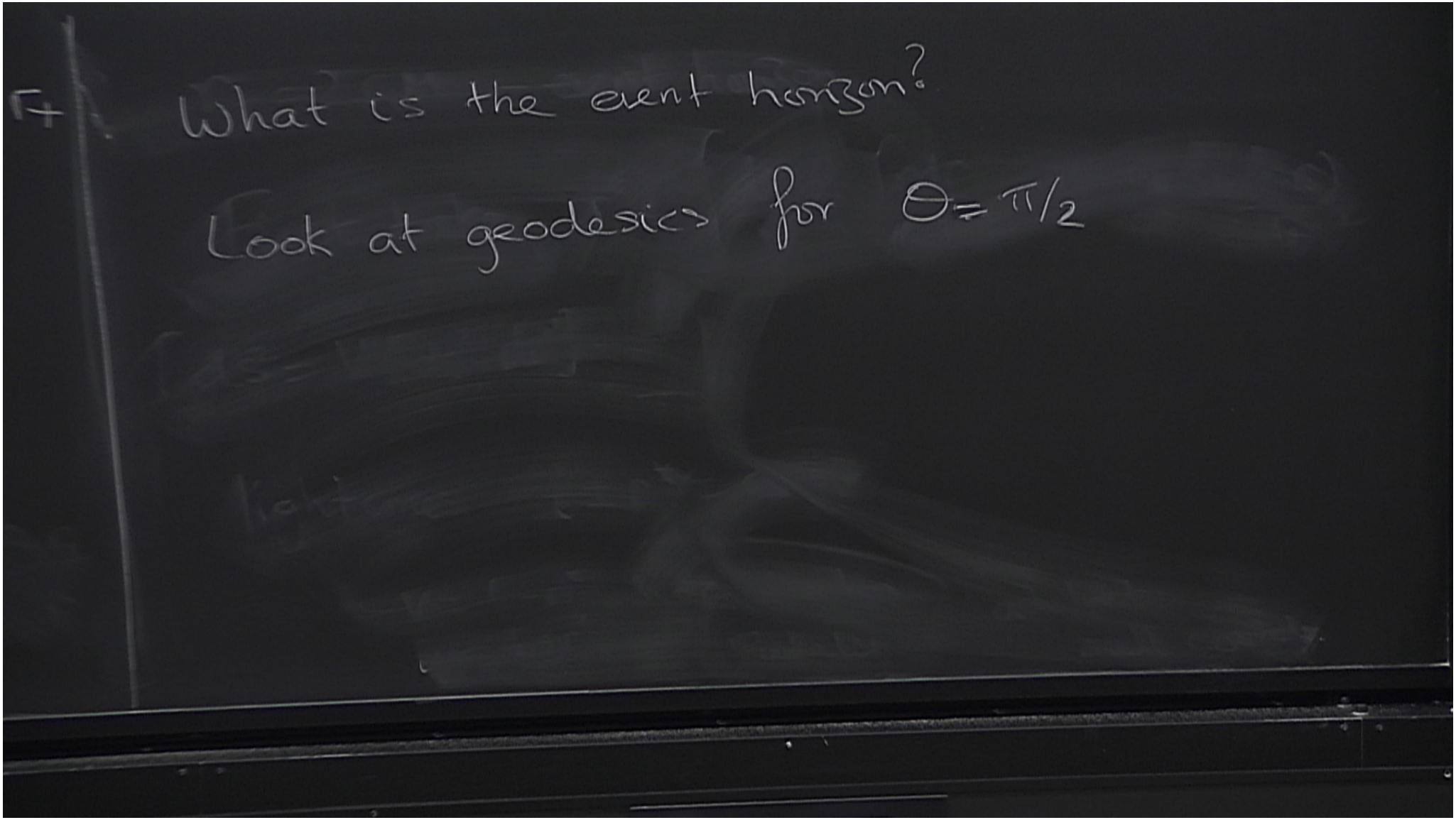
$|g_{tt}| \to 0$: $\left\| \frac{\partial}{\partial t} \right\|^2 \to 0$

$$r^2 + a^2\cos^2\theta - 2GMr = 0$$


Interesting surfaces:
$\Delta \to 0$, $g_{rr} \to \infty$
$r_\pm = GM \pm \sqrt{G^2M^2 - a^2}$
$|g_{tt}| \to 0 : \left\| \frac{\partial}{\partial t} \right\|^2 \to 0$
$r^2 + a^2\cos^2\theta - 2GMr =$


$r_e = GM + \sqrt{G^2M^2 - a^2\cos^2\theta} \geq r_+$
What is
$\rightarrow \quad r \in [r_+, r_e]$
cannot remain at rest
relative to ∞ $\left(\left\| \frac{\partial}{\partial t} \right\|^2 \leq 0 \right)$


What is the event horizon?


What is the event horizon?
Look at geodesics for $\theta = \pi/2$

What is the event horizon?

Look at geodesics for $\theta = \pi/2$

Recall, if k_μ is a Killing vector,

$$\frac{d}{d\tau}(k_\mu \dot{X}^\mu)$$

What is the event horizon?

Look at geodesics for $\theta = \pi/2$

Recall, if k_μ is a killing vector,

$$\frac{d}{d\tau}(k_\mu \dot{x}^\mu) = \dot{x}^\nu D_\nu (k_\mu \dot{x}^\mu)$$

$$= \dot{x}^\nu \dot{x}^\mu D_\nu k_\mu + k_\mu \dot{x}^\nu D_\nu \dot{x}^\mu$$

$D_\nu k_\mu$: 0 ; $D_\nu \dot{x}^\mu$: 0

Kerr has 2 Killing vectors, ∂_t & ∂_φ

Kerr has 2 Killing vectors, ∂_t & ∂_ϕ

- $k = \partial_t \Rightarrow \quad E = g_{t\mu} \dot{x}^\mu = \left(1 - \frac{2GMr}{\Sigma}\right)\dot{t} + \frac{2GMar\sin^2\theta}{\Sigma}\dot{\phi}$

$$\longrightarrow \left(1 - \frac{2GM}{r}\right)\dot{t} + \frac{2GMa}{r}\dot{\phi} \qquad (\theta = \pi/2)$$

Kerr has 2 Killing vectors, ∂_t & ∂_ϕ

- $k = \partial_t \Rightarrow \quad E = g_{t\mu}\dot{x}^\mu = \left(1 - \frac{2GMr}{\Sigma}\right)\dot{t} + \frac{2GMar\sin^2\theta}{\Sigma}\dot{\phi}$

$$\longrightarrow \left(1 - \frac{2GM}{r}\right)\dot{t} + \frac{2GMa}{r}\dot{\phi} \quad (\theta = \pi/2)$$

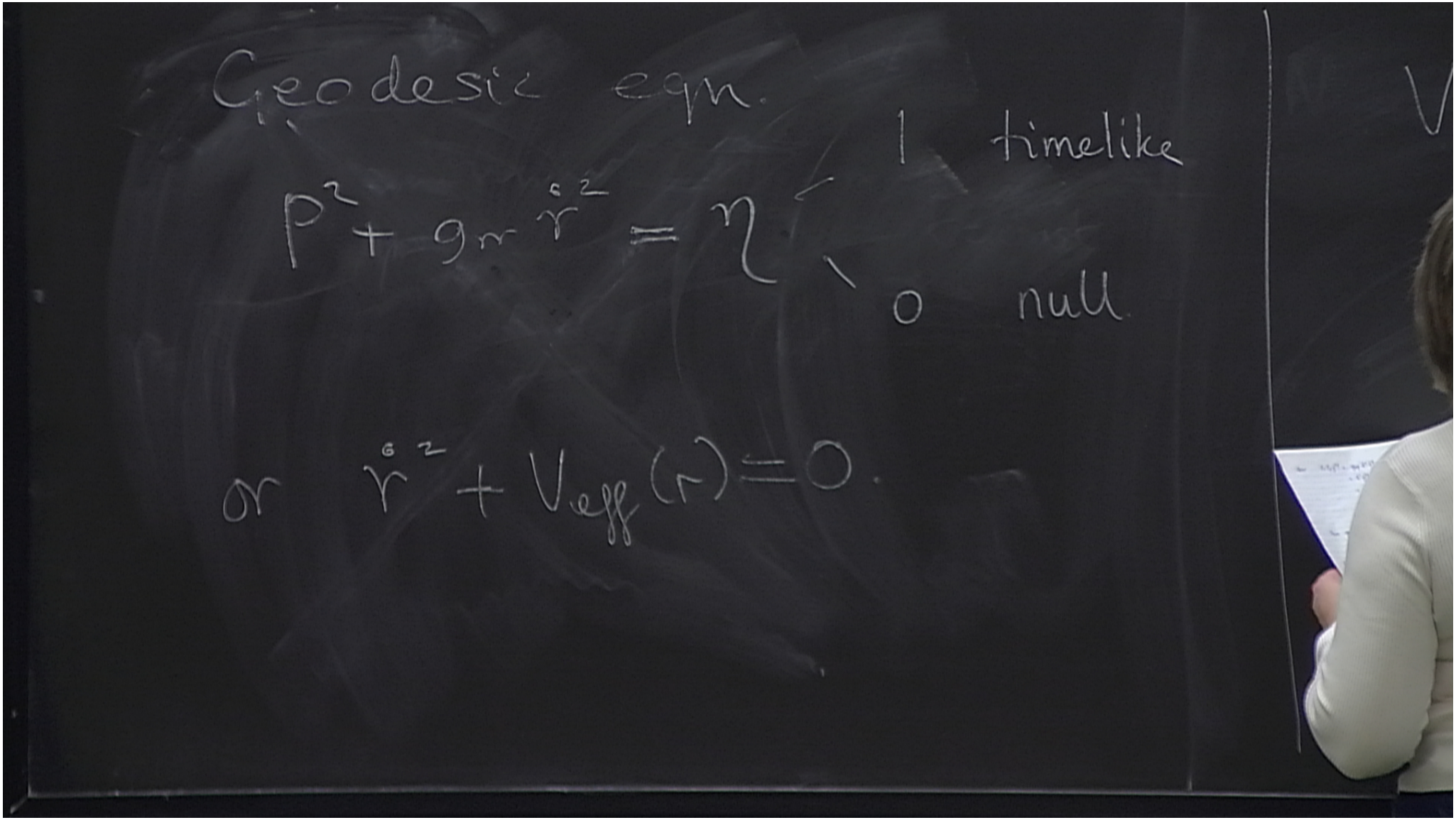
- $k = \partial_\phi \Rightarrow \quad h = -g_{t\mu}\dot{x}^\mu = \left(r^2 + a^2 + \frac{2GMa^2}{r}\right)\dot{\phi} - \frac{2GMa}{r}\dot{t}$

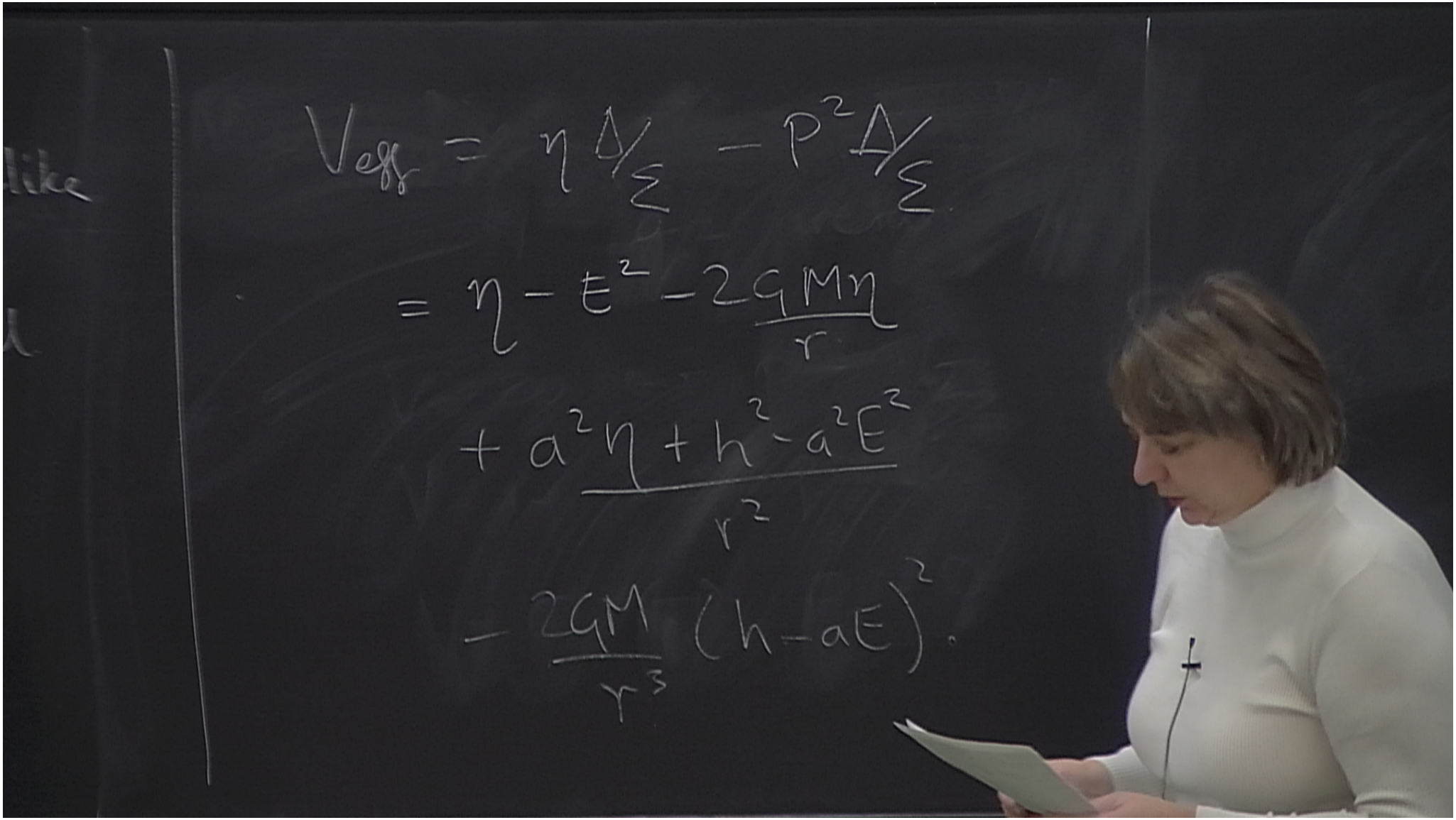
Define $P_\alpha = (E, -h) = g_{\alpha\beta}\dot{X}^\beta \quad \alpha,\beta = t,\phi$

$$P_\alpha P_\beta g^{\alpha\beta} = E^2 g^{tt} - 2Eh g^{t\phi} + h^2 g^{\phi\phi}$$

$$= \frac{1}{\Delta}\left[E^2(r^2+a^2) + 2\frac{GM}{r}(h-aE)^2 - h^2 \right]$$

$$= g_{\alpha\beta}\dot{X}^\alpha \dot{X}^\beta .$$


Geodesic eqn.
$p^2 + g_{rr}\dot{r}^2 = \eta$
1 timelike
0 null
or $\dot{r}^2 + V_{eff}(r) = 0$.

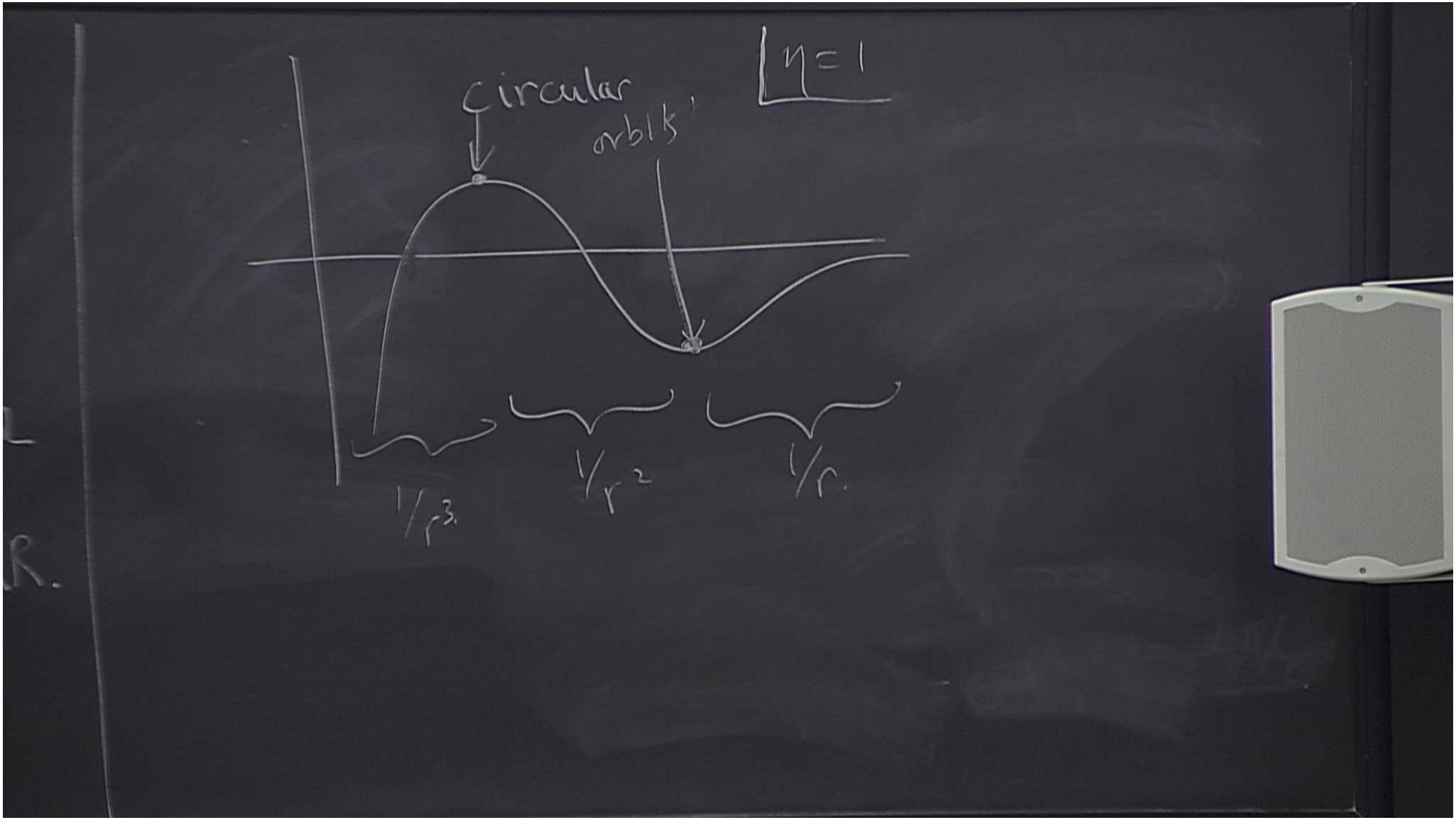

$V_{eff} = \eta \Delta/\Sigma - P^2 \Delta/\Sigma$
$= \eta - E^2 - \frac{2GMn}{r}$
$+ \frac{a^2\eta + h^2 - a^2E^2}{r^2}$
$- \frac{2GM}{r^3}(h - aE)^2$.

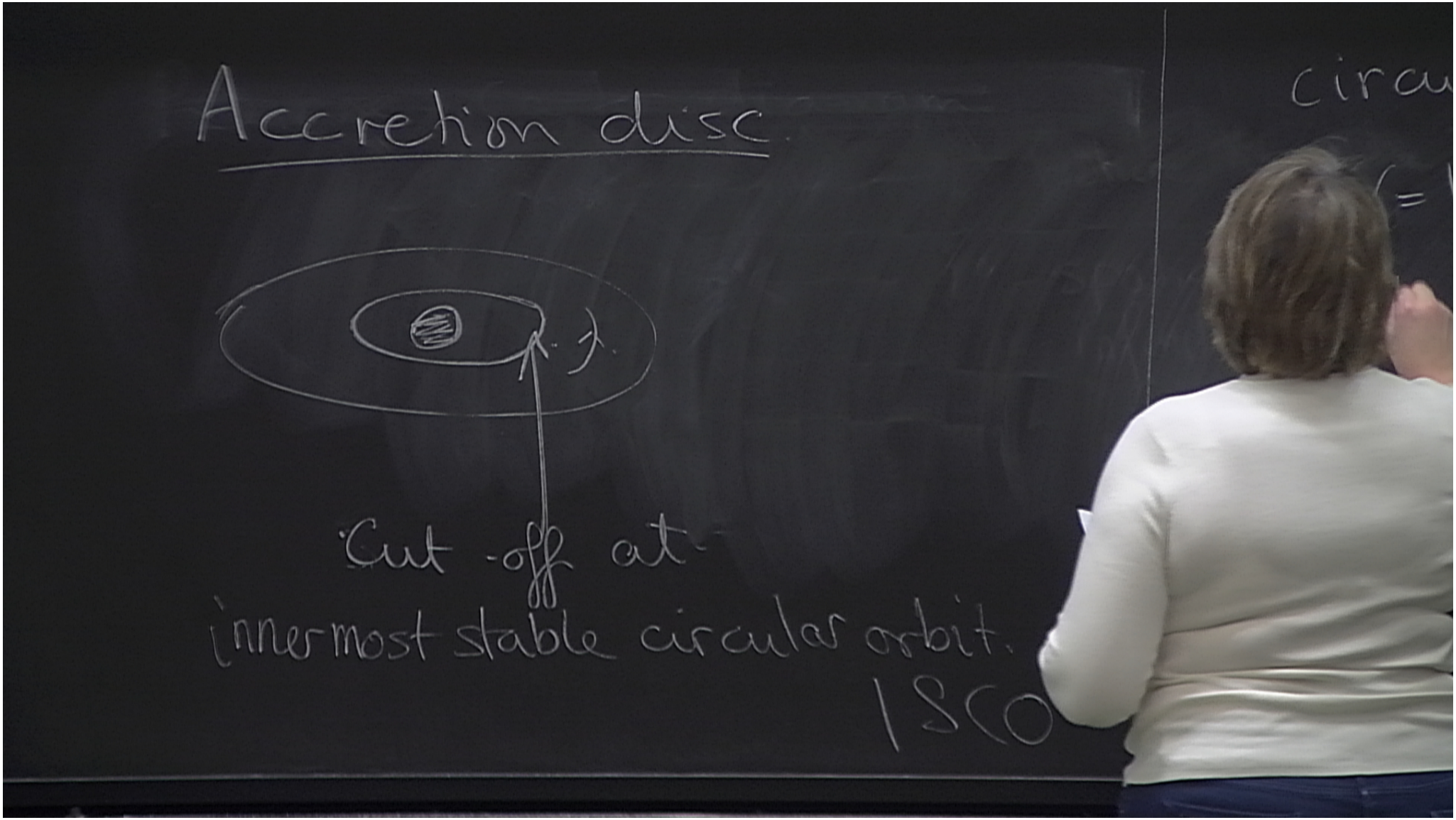
$$V_{eff} = \eta \Delta/\Sigma - P^2 \Delta/\Sigma$$

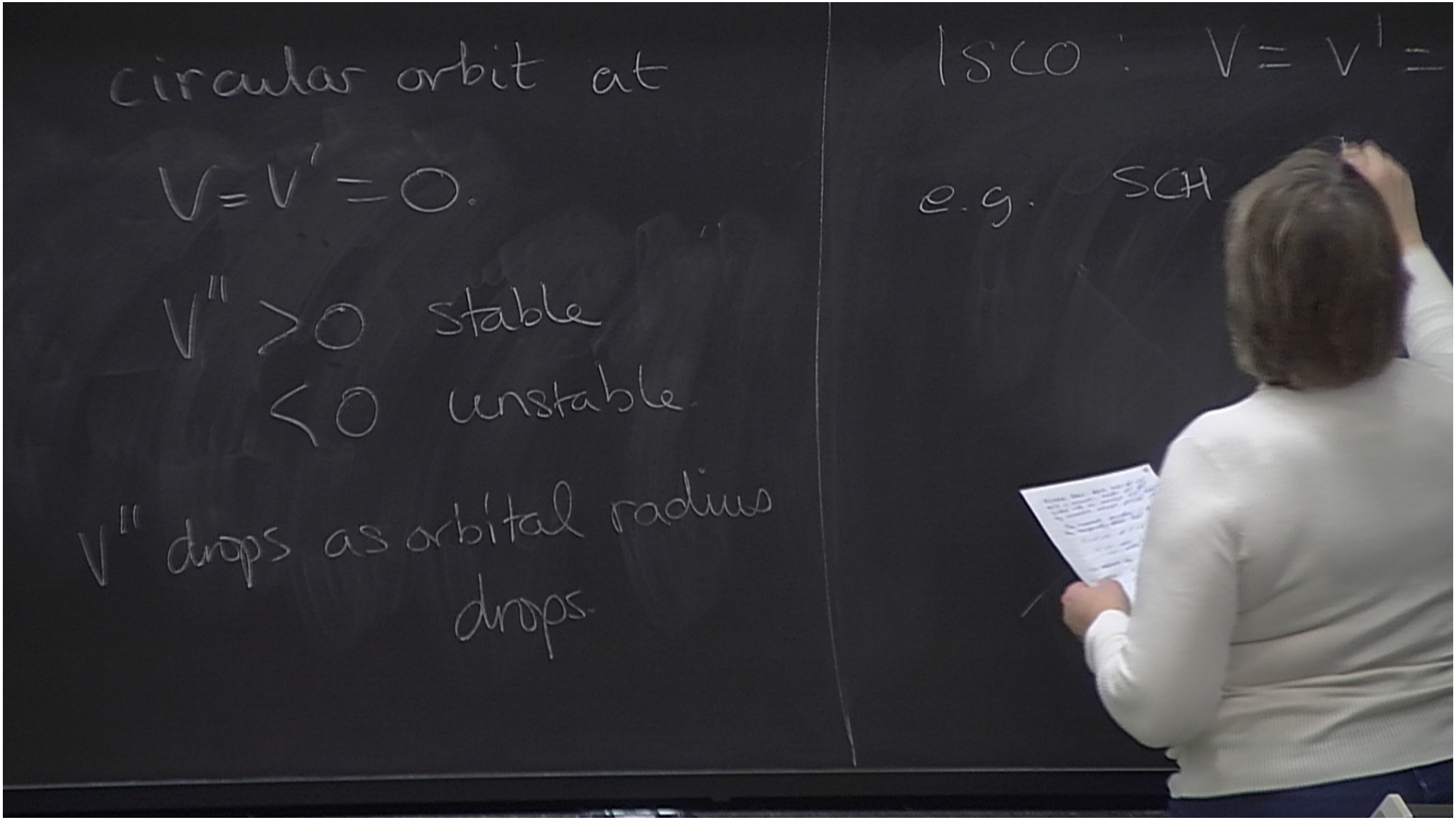
$$= \eta - E^2 - \frac{2GM\eta}{r} \leftarrow \text{NEWTON}$$

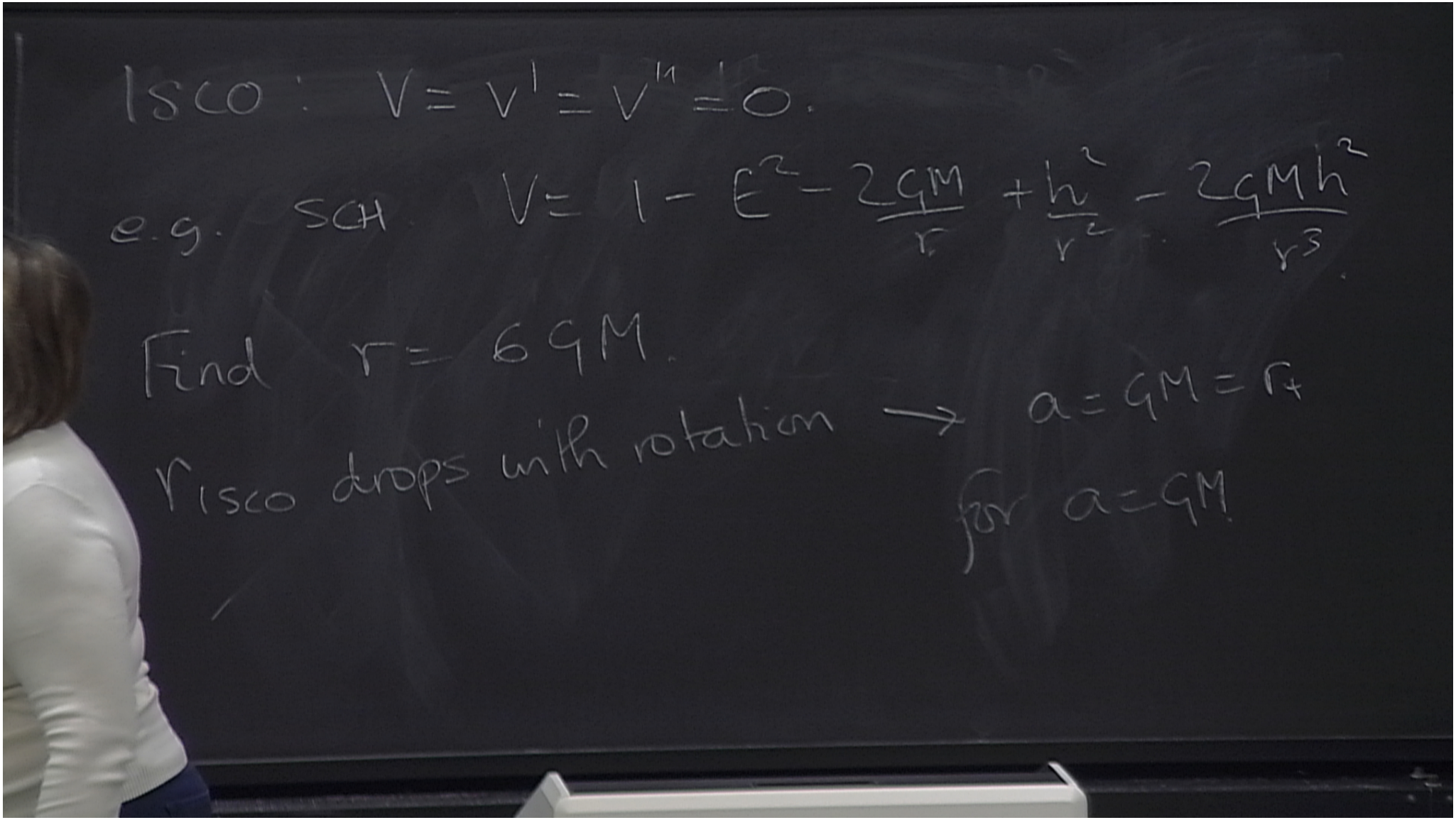
$$+ \frac{a^2\eta + h^2 - a^2E^2}{r^2} \leftarrow \text{CENTRIFUGAL}$$

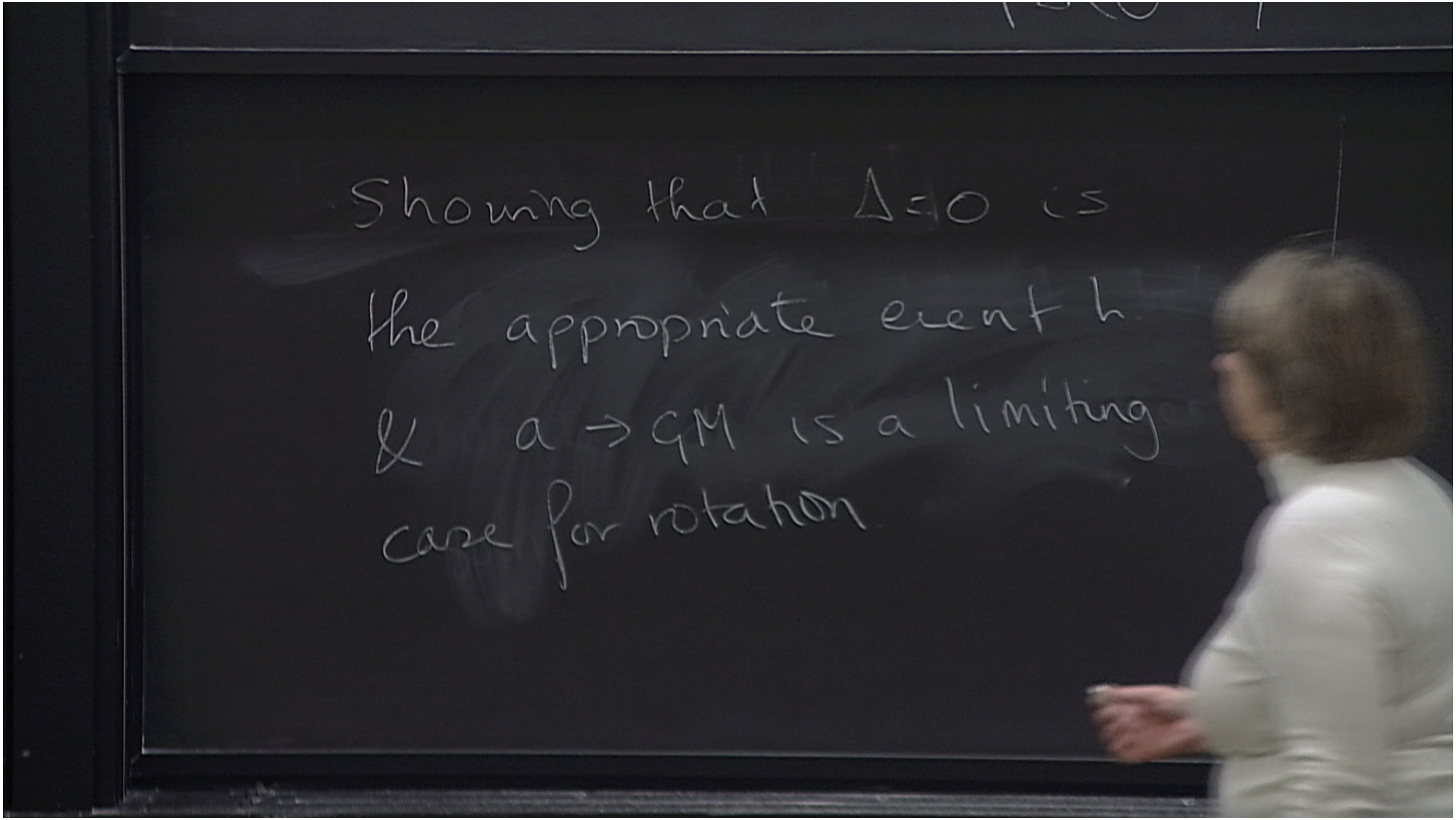
$$- \frac{2GM}{r^3}(h - aE)^2 \leftarrow \text{GR/KERR.}$$


circular orbits
$\eta = 1$
$1/r^3$
$1/r^2$
$1/r$


Accretion disc
circu
Cut-off at
innermost stable circular orbit.
ISCO


circular orbit at
V = V' = 0.
V'' > 0 stable
< 0 unstable.
V'' drops as orbital radius drops.
ISCO : V = V' =
e.g. SCH


ISCO: $V = V' = V'' = 0$.
e.g. SCH $V = 1 - E^2 - \frac{2GM}{r} + \frac{h^2}{r^2} - \frac{2GMh^2}{r^3}$.
Find $r = 6GM$.
r_{isco} drops with rotation $\rightarrow$ $a = GM = r_+$
for $a = GM$.


Showing that $\Delta = 0$ is
the appropriate event h
& $a \to GM$ is a limiting
case for rotation

Showing that $\Delta = 0$ is the appropriate event h & $a \to GM$ is a limiting case for rotation

EXTREMAL LIMIT

resting surfaces:

$\Delta = 0$, $g_{rr} \to \infty$

$r_\pm = GM \pm \sqrt{G^2M^2 - a^2}$

$\Delta \to 0$: $\left\| \frac{\partial}{\partial t} \right\|^2 \to 0$

$r^2 + a^2\cos^2\theta - 2GMr = 0$

$r_e = GM + \sqrt{G^2M^2 - a^2\cos^2\theta} \geq r_+$

$\to$ $r \in [r_+, r_e]$

cannot remain at rest

relative to ∞ $\left(\left\| \frac{\partial}{\partial t} \right\|^2 < 0 \right)$

ERGOSPHERE