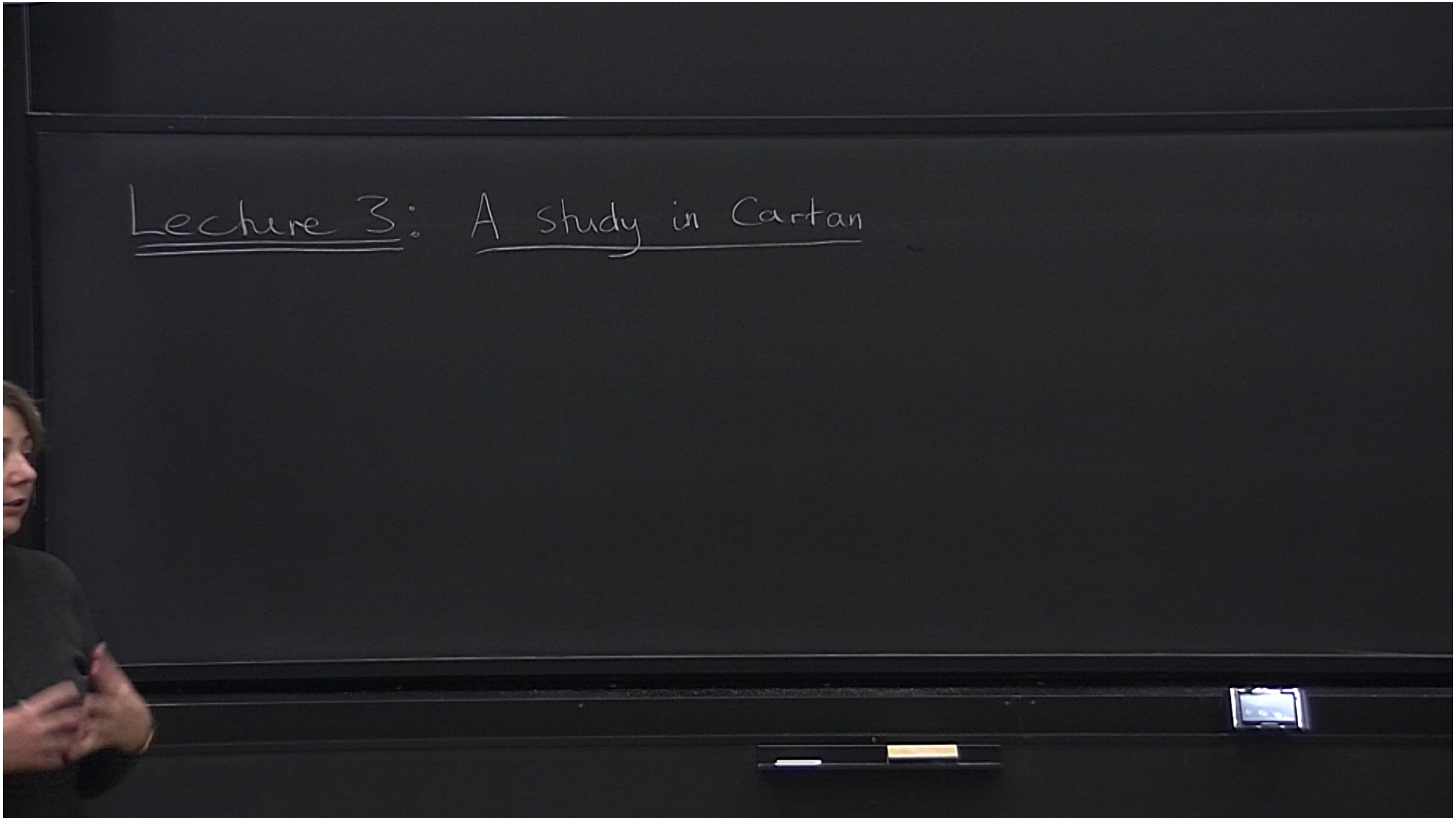
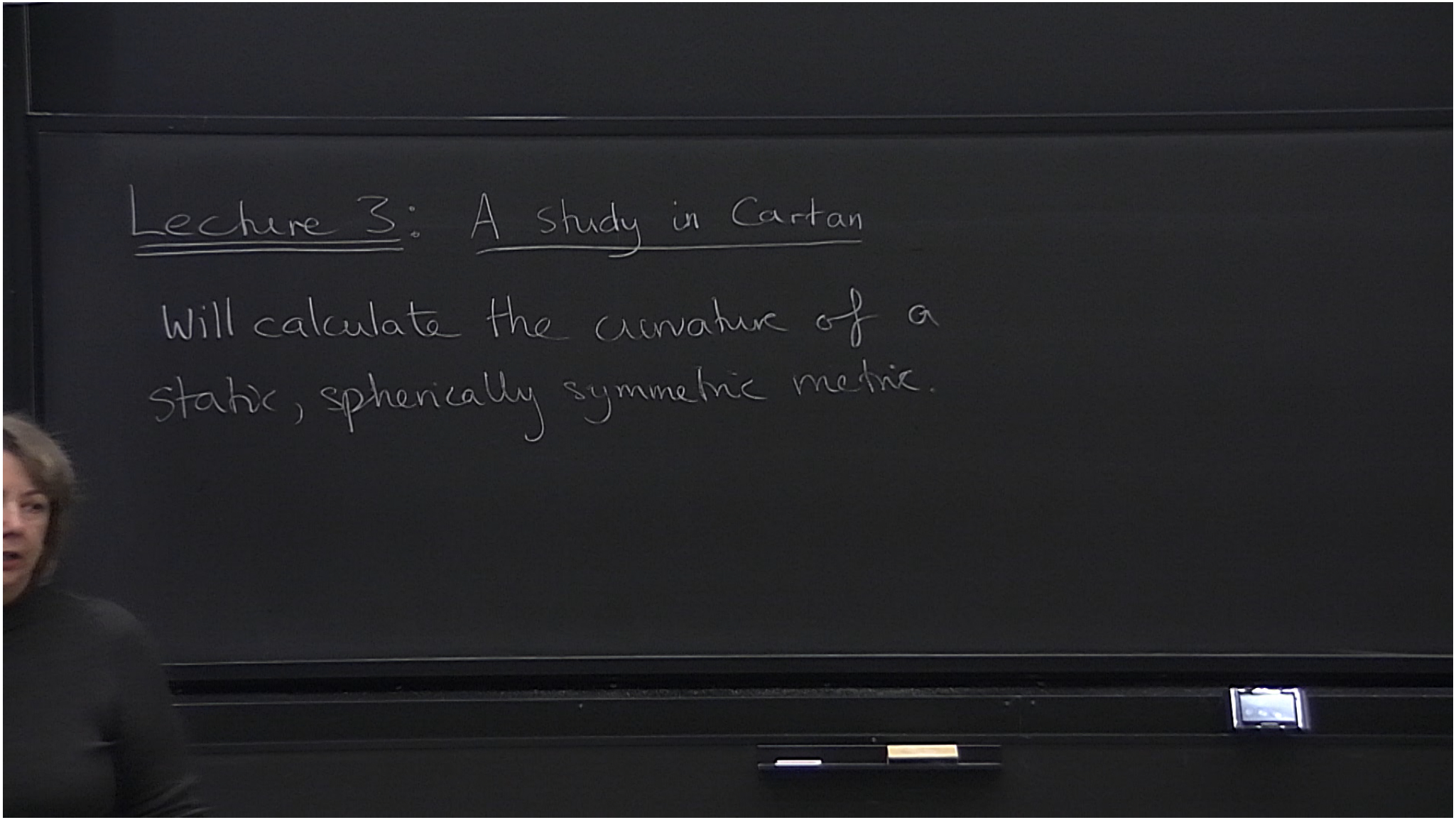
Title: PSI 17/18 - Gravitational Physics - Lecture 3

Date: Jan 05, 2018 10:15 AM

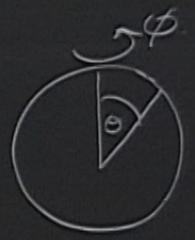
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Abstract:


Lecture 3: A study in Cartan

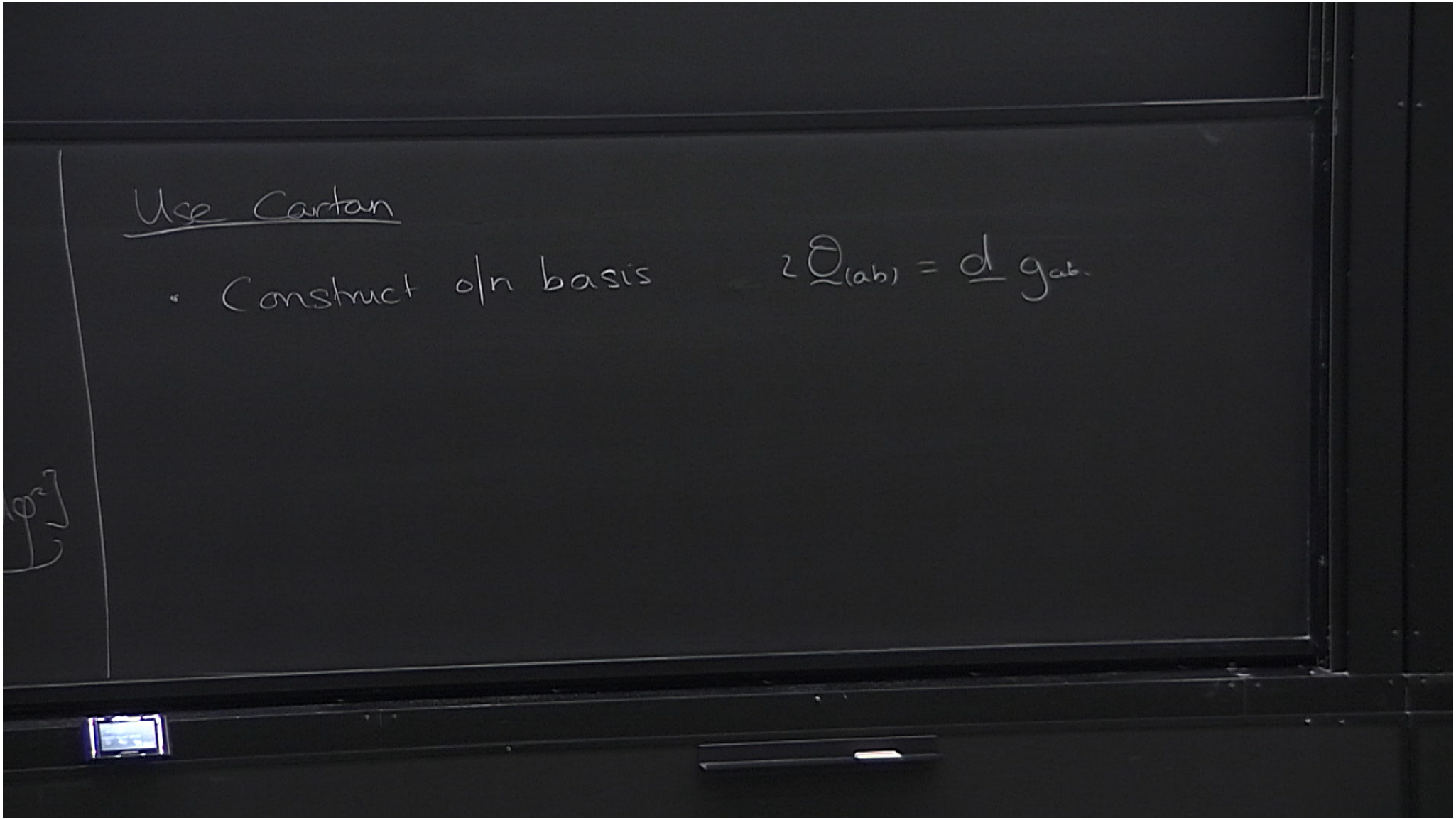

Lecture 3: A study in Cartan
Will calculate the curvature of a
static, spherically symmetric metric.

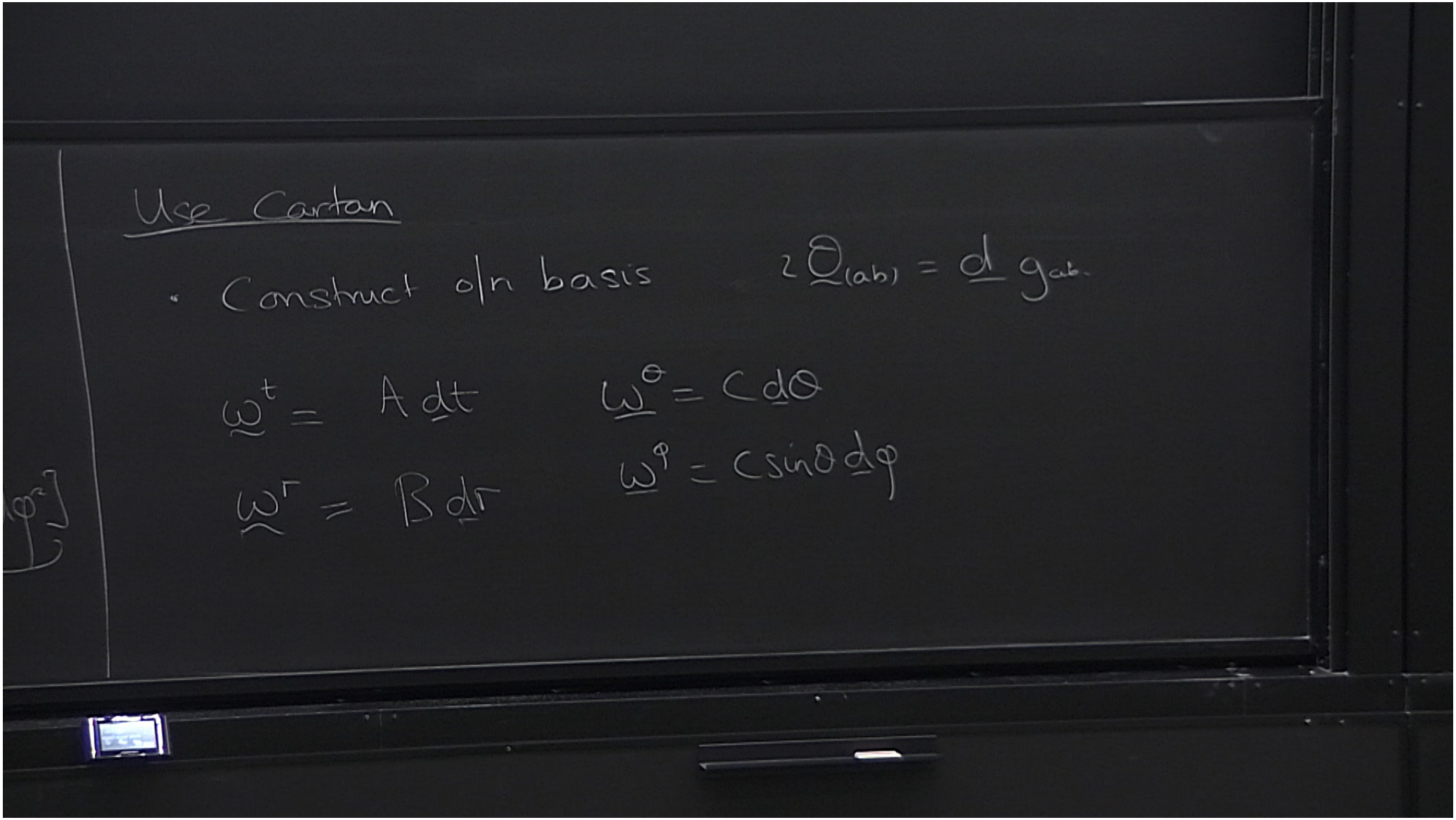
Lecture 3: A study in Cartan



Will calculate the curvature of a static, spherically symmetric metric.

$$ds^2 = A^2(r)dt^2 - \beta^2(r)dr^2 - C^2(r)\underbrace{[d\theta^2 + \sin^2\theta d\varphi^2]}_{d\Omega_{II}^2}$$


Use Cartan
• Construct o/n basis
$2\underset{\sim}{\theta}_{(ab)} = \underline{d}\, g_{ab}$.

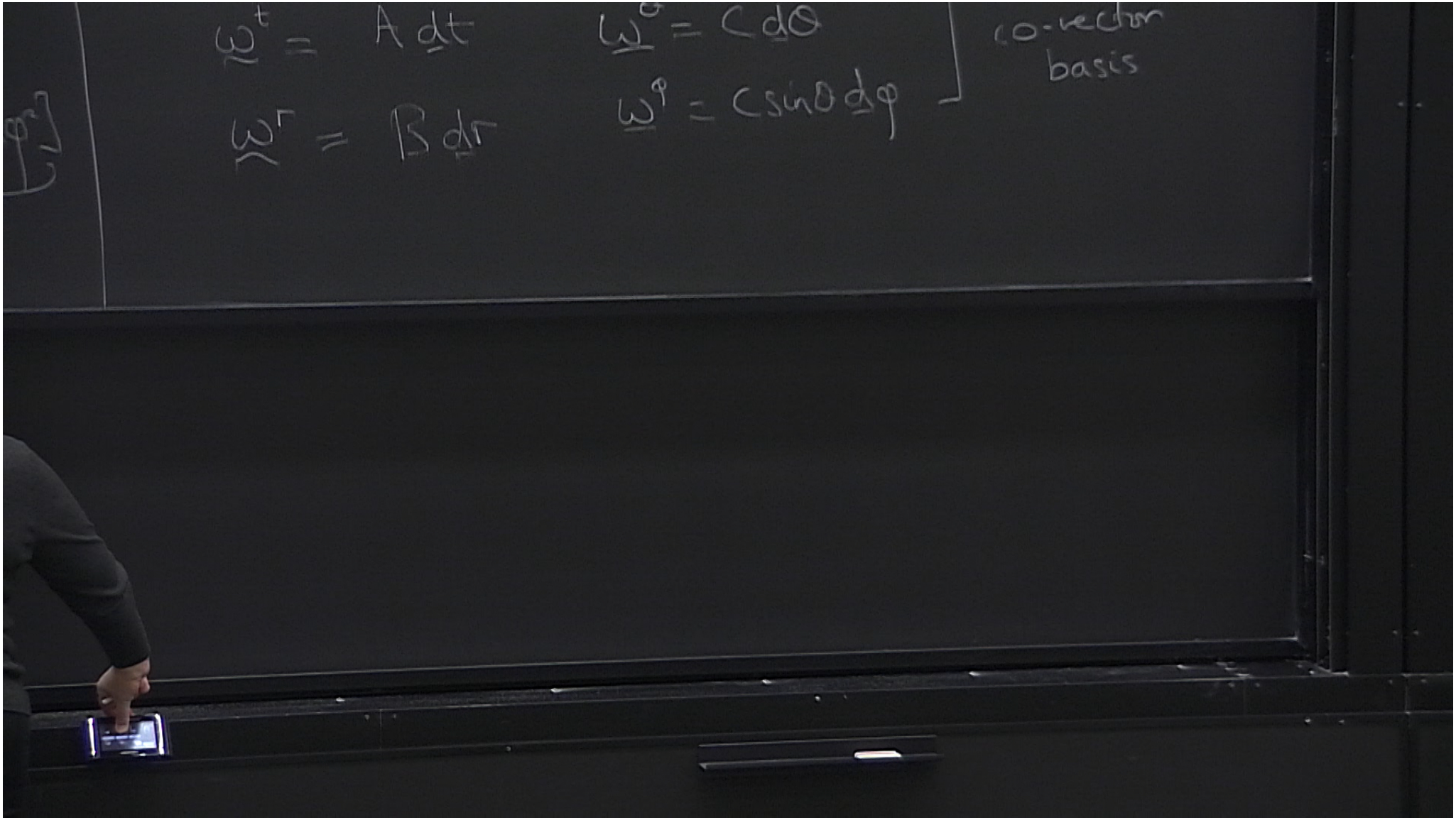

Use Cartan
• Construct o/n basis
$2\underset{\sim}{\Theta}_{(ab)} = \underline{d}\, g_{ab}$
$\underset{\sim}{\omega}^t = A\,\underline{d}t$
$\underset{\sim}{\omega}^r = B\,\underline{d}r$
$\underline{\omega}^\theta = C\,\underline{d}\theta$
$\underline{\omega}^\varphi = C\sin\theta\,\underline{d}\varphi$

Use Cartan

- Construct o/n basis $\quad 2\underset{\sim}{\theta}_{(ab)} = \underline{d}\, g_{ab}$.

$$\underset{\sim}{\omega}^t = A\,\underline{d}t \qquad \underset{\sim}{\omega}^\theta = C\,\underline{d}\theta$$

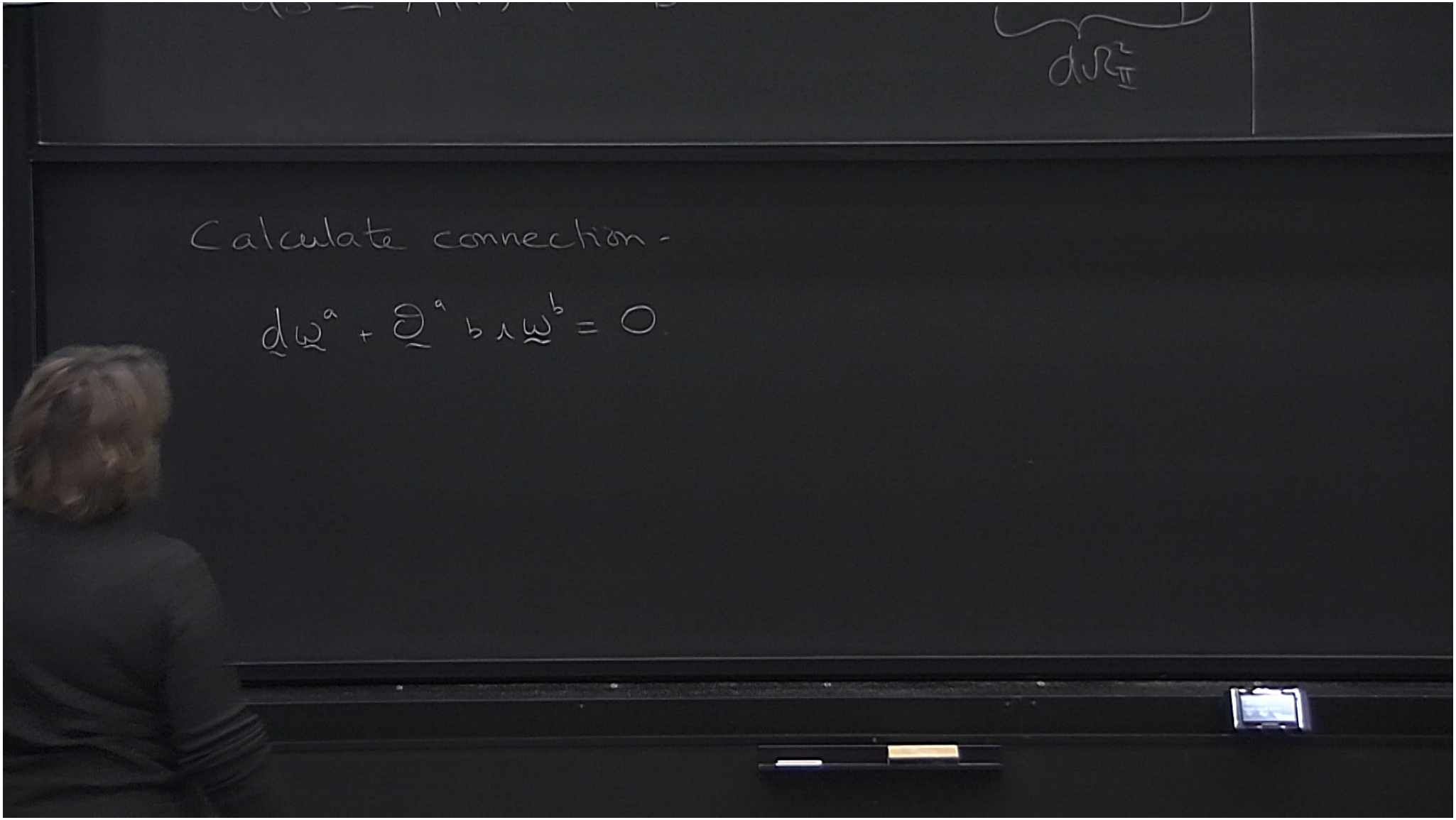
$$\underset{\sim}{\omega}^r = B\,\underline{d}r \qquad \underset{\sim}{\omega}^\varphi = C\sin\theta\,\underline{d}\varphi$$


$\omega^t = A\,dt$
$\omega^r = B\,dr$
$\omega^\theta = C\,d\theta$
$\omega^\varphi = C\sin\theta\,d\varphi$
co-vector basis

Use Cartan

- Construct o/n basis $\quad 2\underset{\sim}{\Theta}_{(ab)} = \underline{d}\, g_{ab}$.

$$\left.\begin{aligned} \underset{\sim}{\omega}^t &= A\,\underline{dt} & \underset{\sim}{\omega}^\theta &= C\,\underline{d\theta} \\ \underset{\sim}{\omega}^r &= B\,\underline{dr} & \underset{\sim}{\omega}^\varphi &= C\sin\theta\,\underline{d\varphi} \end{aligned}\right] \text{ co-vector basis}$$


$d\mathcal{R}^2_{II}$
Calculate connection.
$d\underset{\sim}{\omega}^a + \underset{\sim}{\Theta}^a{}_b \wedge \underset{\sim}{\omega}^b = 0$

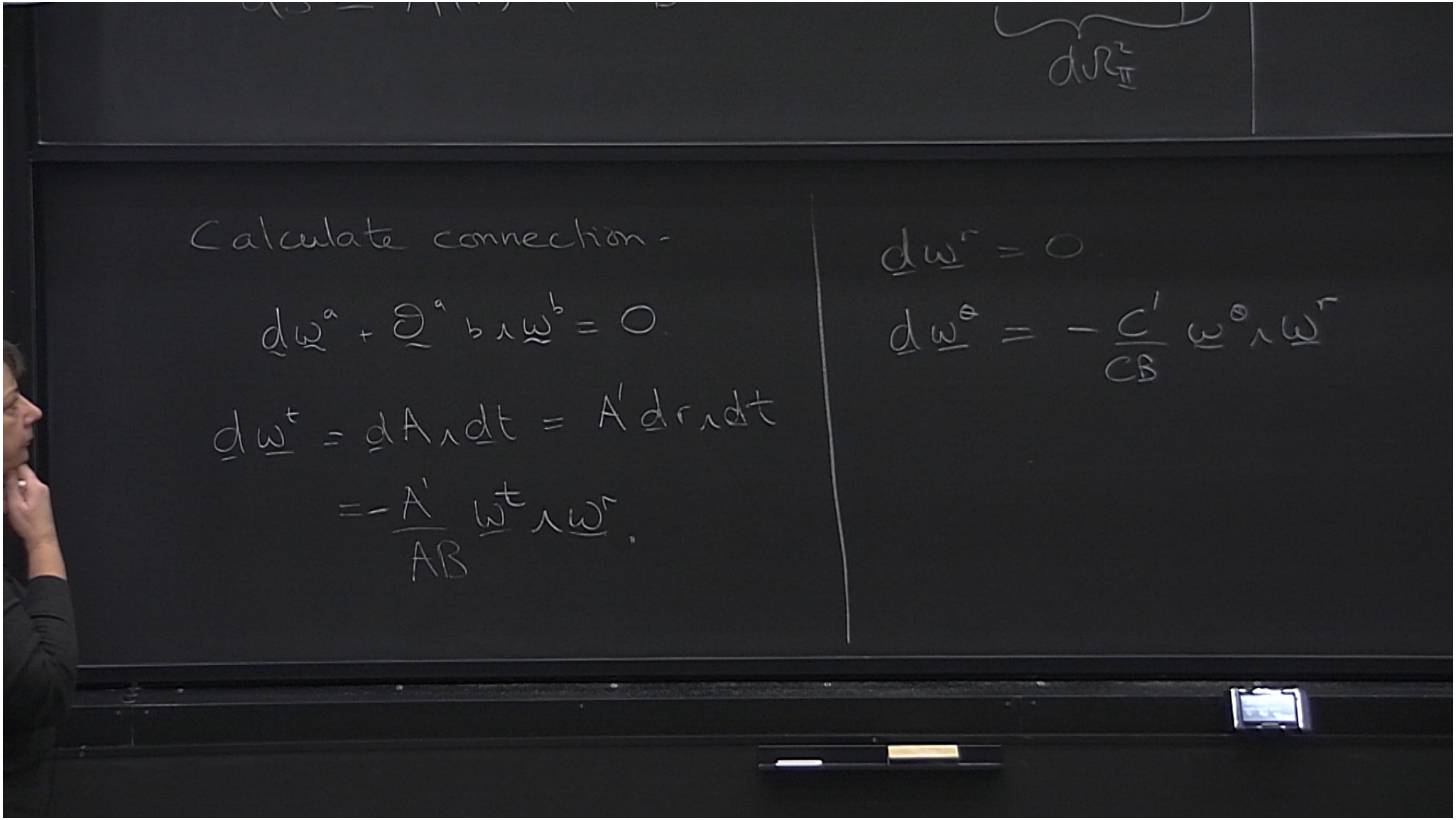
$d\Omega_{II}^2$

Calculate connection.

$$d\underline{\omega}^a + \underline{\Theta}^a{}_b \wedge \underline{\omega}^b = 0$$

$$d\underline{\omega}^t = d\underline{A} \wedge d\underline{t} = A' d\underline{r} \wedge d\underline{t}$$

$$= -\frac{A'}{AB}\, \underline{\omega}^t \wedge \underline{\omega}^r .$$


$d\omega^r = 0$
$d\omega^\theta = -\frac{C'}{CB}\,\omega^\theta \wedge \omega^r$
Calculate connection -
$d\omega^a + \Omega^a{}_b \wedge \omega^b = 0$
$d\omega^t = dA \wedge dt = A' dr \wedge dt$
$= -\frac{A'}{AB}\,\omega^t \wedge \omega^r$.

$d\Omega_{II}^2$

chion -

$= 0$

$= A' dr \wedge dt$

$\wedge \omega^r$.

$d\omega^r = 0$

$d\omega^\theta = -\frac{c'}{cB}\,\omega^\theta \wedge \omega^r$

$d\omega^\phi = c' \sin\theta\, dr \wedge d\varphi + c\cos\theta\, d\theta \wedge d\varphi$

$= -\frac{c}{cB}\omega^\varphi \wedge \omega^r - \frac{\cot\theta}{c}\,\omega^\varphi \wedge \omega^\theta$

Use Cartan

- Construct o/n basis $\quad 2\underset{\sim}{\Theta}_{(ab)} = \underset{\sim}{d}\, g_{ab}$.

$$\underset{\sim}{\omega}^t = A\,\underset{\sim}{d}t \qquad \underset{\sim}{\omega}^\theta = C\,\underset{\sim}{d}\theta$$
$$\underset{\sim}{\omega}^r = B\,\underset{\sim}{d}r \qquad \underset{\sim}{\omega}^\varphi = C\sin\theta\,\underset{\sim}{d}\varphi$$

] co-vector basis

$$\underset{\sim}{\omega}^a \otimes \underset{\sim}{\omega}^b\, g_{ab} = \underset{\sim}{g}$$

$\underset{\sim}{\omega}^\theta \wedge \underset{\sim}{\omega}^r$

Use Cartan

- Construct o/n basis $\quad 2\underline{\Omega}_{(ab)} = \underline{d}\, g_{ab}$

$$\underline{\omega}^t = A\,\underline{d}t \qquad \underline{\omega}^\theta = C\,\underline{d}\theta$$
$$\underline{\omega}^r = B\,\underline{d}r \qquad \underline{\omega}^\varphi = C\sin\theta\,\underline{d}\varphi$$

] co-vector basis

$$\underline{\omega}^a \otimes \underline{\omega}^b\, \eta_{ab} = g$$

$\underline{\omega}^\theta \wedge \underline{\omega}^r$

Use Cartan

- Construct o/n basis $\quad 2\underset{\sim}{\Theta}_{(ab)} = \underset{\sim}{d}\, g_{ab}$.

$$\underset{\sim}{\omega}^t = A\,\underset{\sim}{d}t \qquad \underset{\sim}{\omega}^\theta = C\,\underset{\sim}{d}\theta$$

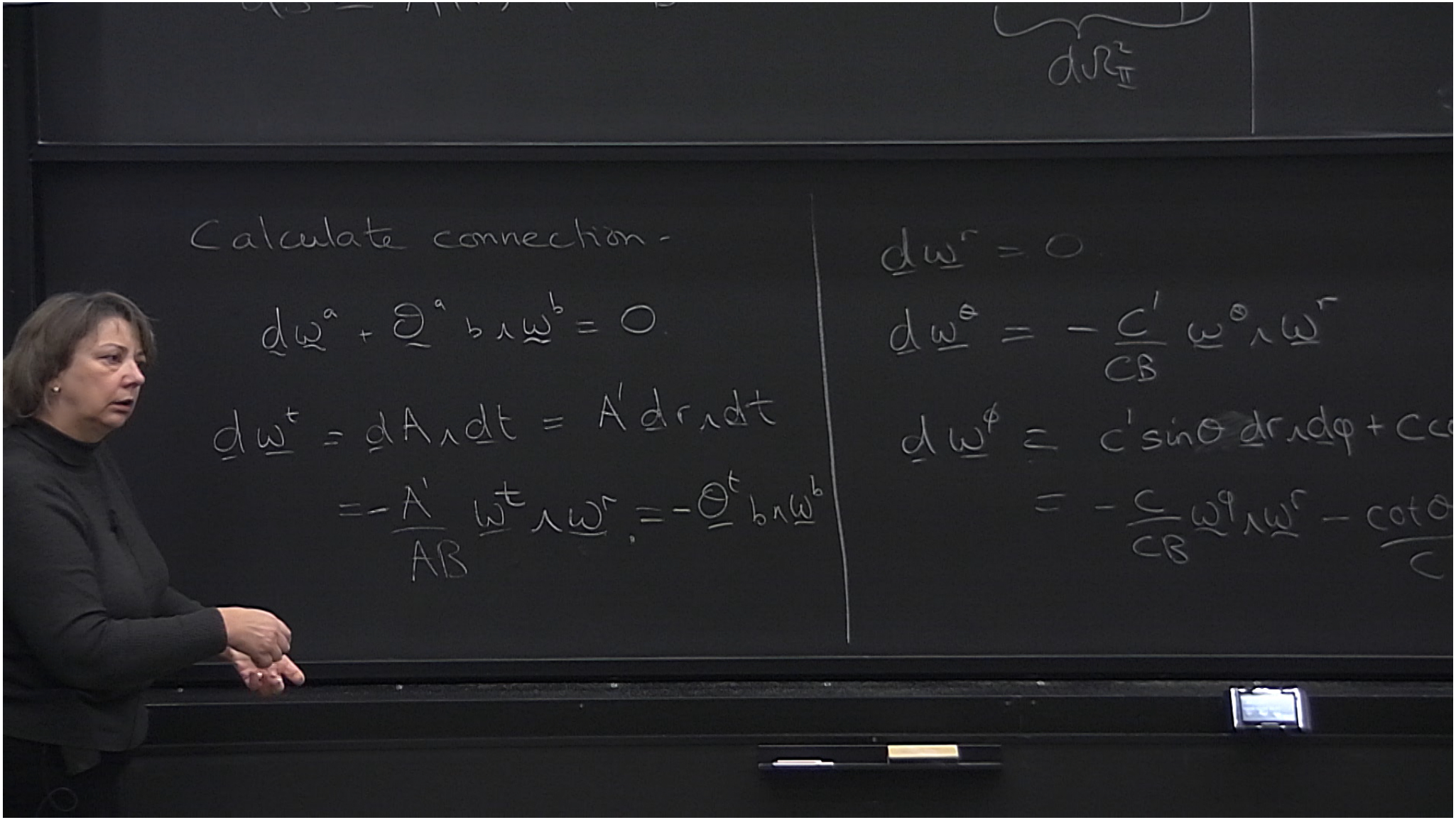
$$\underset{\sim}{\omega}^r = B\,\underset{\sim}{d}r \qquad \underset{\sim}{\omega}^\varphi = C\sin\theta\,\underset{\sim}{d}\varphi$$

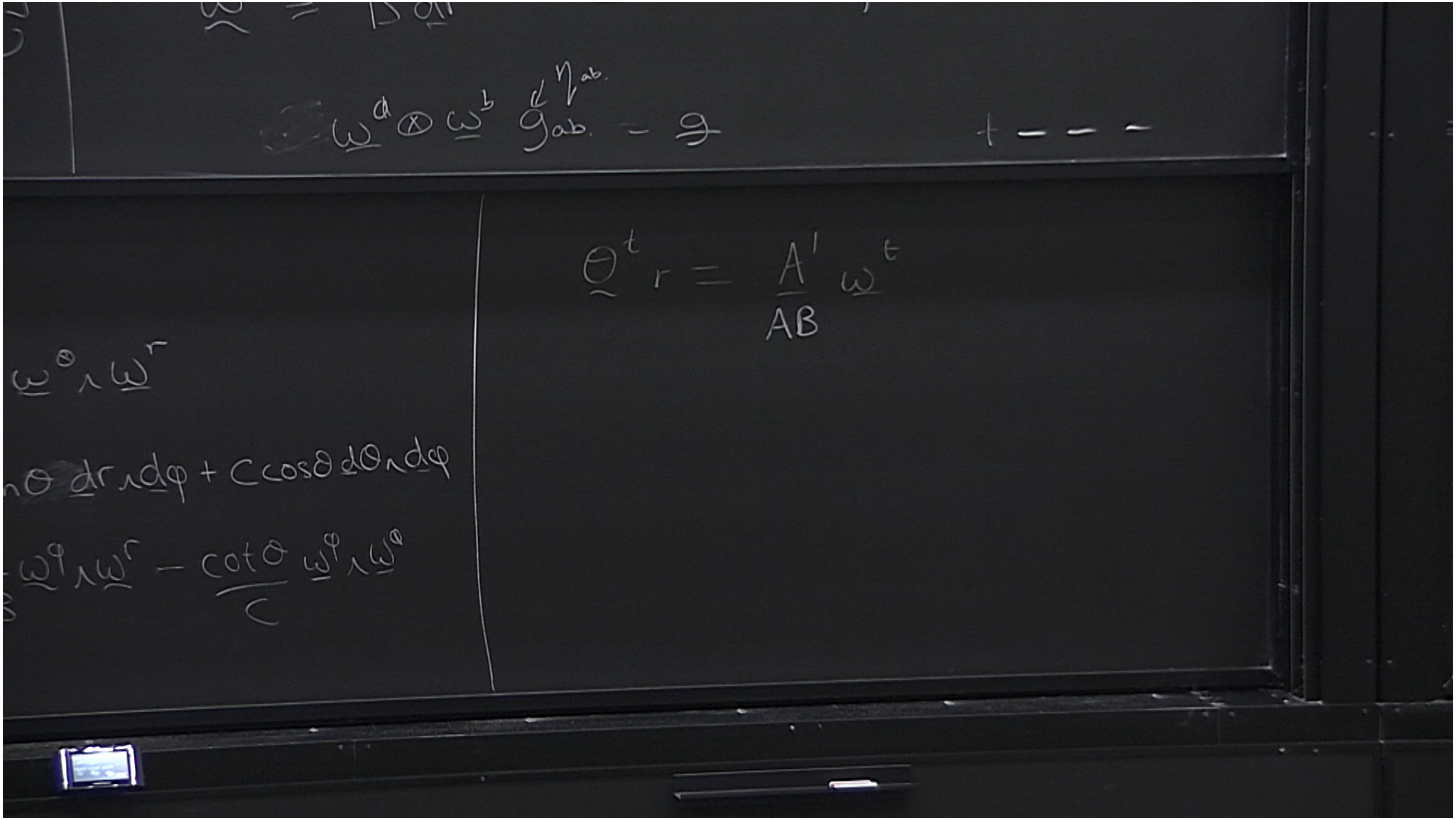
] co-vector basis

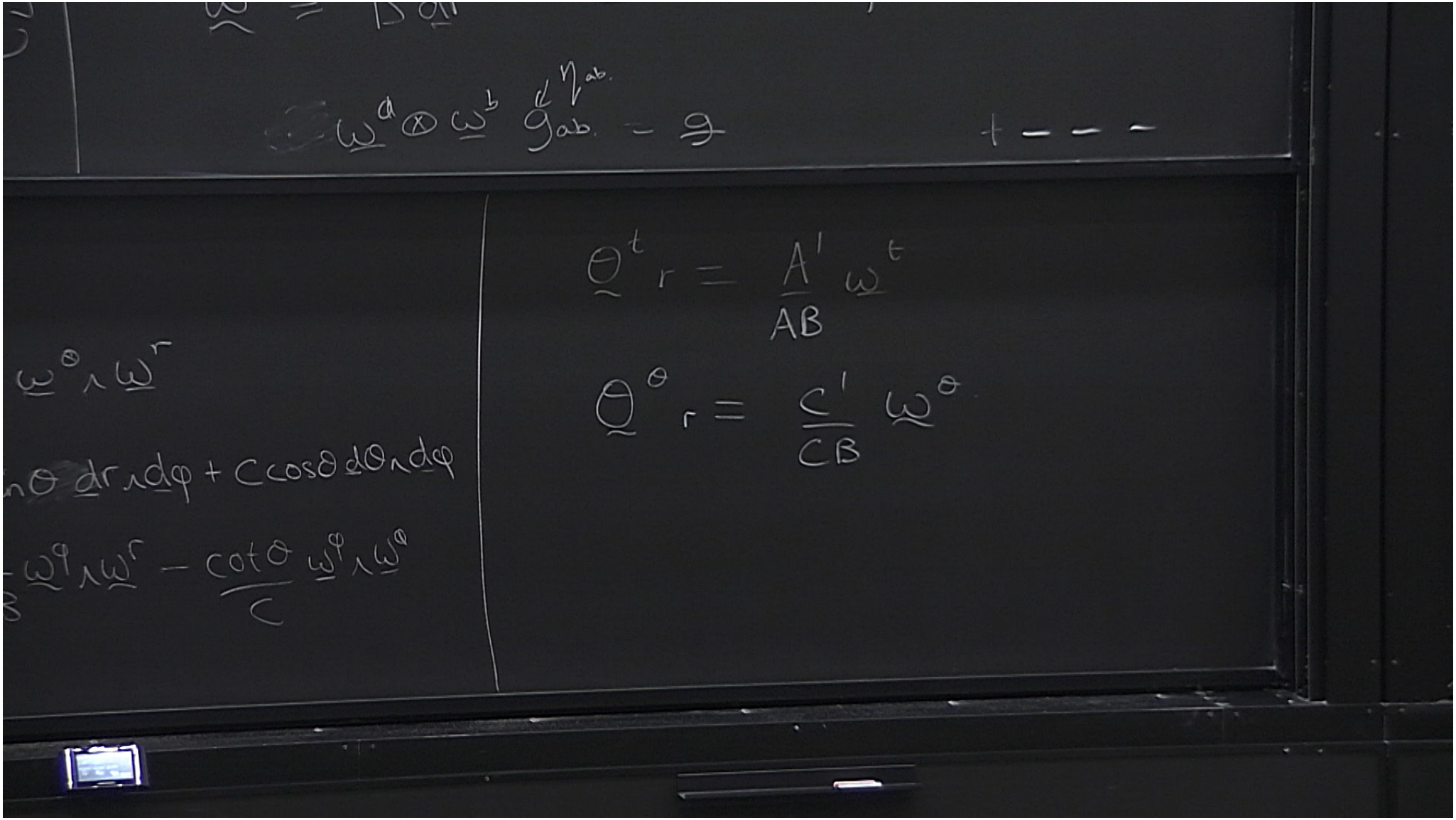
$$\underset{\sim}{\omega}^a \otimes \underset{\sim}{\omega}^b\, g_{ab} = \underset{\sim}{g} \qquad g_{ab} \propto \eta_{ab}$$

$$+ - - -$$

$$\underset{\sim}{\omega}^\theta \wedge \underset{\sim}{\omega}^r$$


$d\Omega_T^2$
Calculate connection -
$d\underline{\omega}^a + \underline{\Theta}^a{}_b \wedge \underline{\omega}^b = 0$
$d\underline{\omega}^t = d\underline{A} \wedge d\underline{t} = A' d\underline{r} \wedge d\underline{t}$
$= -\frac{A'}{AB} \underline{\omega}^t \wedge \underline{\omega}^r = -\underline{\Theta}^t{}_b \wedge \underline{\omega}^b$
$d\underline{\omega}^r = 0$
$d\underline{\omega}^\theta = -\frac{C'}{CB} \underline{\omega}^\theta \wedge \underline{\omega}^r$
$d\underline{\omega}^\phi = C' \sin\theta \, d\underline{r} \wedge d\underline{\varphi} + C$
$= -\frac{C}{CB} \underline{\omega}^\varphi \wedge \underline{\omega}^r - \cot\theta$


$\omega^a \otimes \omega^b \, \eta_{ab} = g$
$+ - - -$
$\Theta^t{}_r = \frac{A'}{AB} \omega^t$
$\omega^\theta \wedge \omega^r$
$\cos\theta \, d\theta \wedge d\varphi$
$\omega^\varphi \wedge \omega^r - \frac{\cot\theta}{C} \omega^\varphi \wedge \omega^\varphi$


$\omega^a \otimes \omega^b \eta_{ab} = g$
$+ - - -$
$\Theta^t{}_r = \frac{A'}{AB} \omega^t$
$\Theta^\theta{}_r = \frac{C'}{CB} \omega^\theta$
$\omega^\theta \wedge \omega^r$
$\sin\theta\, dr \wedge d\varphi + C\cos\theta\, d\theta \wedge d\varphi$
$\omega^\varphi \wedge \omega^r - \frac{\cot\theta}{C} \omega^\varphi \wedge \omega^\theta$

$\underset{\sim}{\omega}^a \otimes \underset{\sim}{\omega}^b \, g_{ab} = \underset{\sim}{g}$ $\quad \eta_{ab}$ $\quad + - - -$

$\underset{\sim}{\omega}^\theta \wedge \underset{\sim}{\omega}^r$

$\cdots \underset{\sim}{dr} \wedge d\varphi + C\cos\theta \, d\theta \wedge d\varphi$

$\cdots \underset{\sim}{\omega}^\varphi \wedge \underset{\sim}{\omega}^r - \dfrac{\cot\theta}{C} \underset{\sim}{\omega}^\theta \wedge \underset{\sim}{\omega}^\varphi$

$$\underset{\sim}{\Theta}^t{}_r = \frac{A'}{AB} \underset{\sim}{\omega}^t$$

$$\underset{\sim}{\Theta}^\theta{}_r = \frac{C'}{CB} \underset{\sim}{\omega}^\theta$$

$$\underset{\sim}{\Theta}^\varphi{}_r = \frac{C'}{CB} \underset{\sim}{\omega}^\varphi \qquad \underset{\sim}{\Theta}^\varphi{}_\theta = \frac{\cot\theta}{C} \underset{\sim}{\omega}^\varphi$$

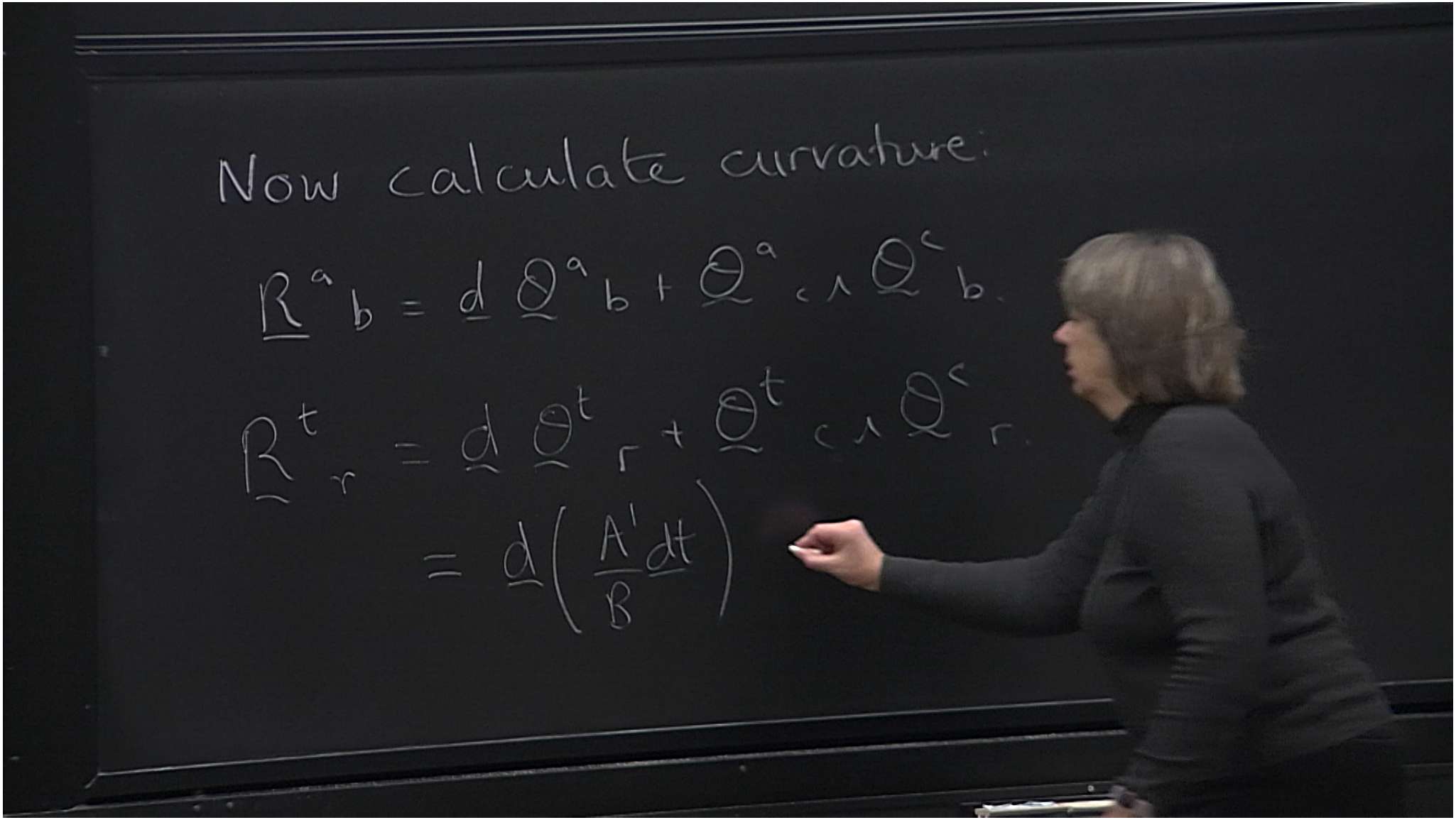
Now calculate curvature:

$$\underline{R}^{a}{}_{b} = \underline{d}\,\underline{\omega}^{a}{}_{b} + \underline{\omega}^{a}{}_{c} \wedge \underline{\omega}^{c}{}_{b}.$$

Now calculate curvature:

$$\underline{R}^{a}{}_{b} = \underline{d}\,\underline{\omega}^{a}{}_{b} + \underline{\omega}^{a}{}_{c} \wedge \underline{\omega}^{c}{}_{b}.$$

$$\underline{R}^{t}{}_{r} = \underline{d}\,\underline{\omega}^{t}{}_{r} + \underline{\omega}^{t}{}_{c} \wedge \underline{\omega}^{c}{}_{r}.$$

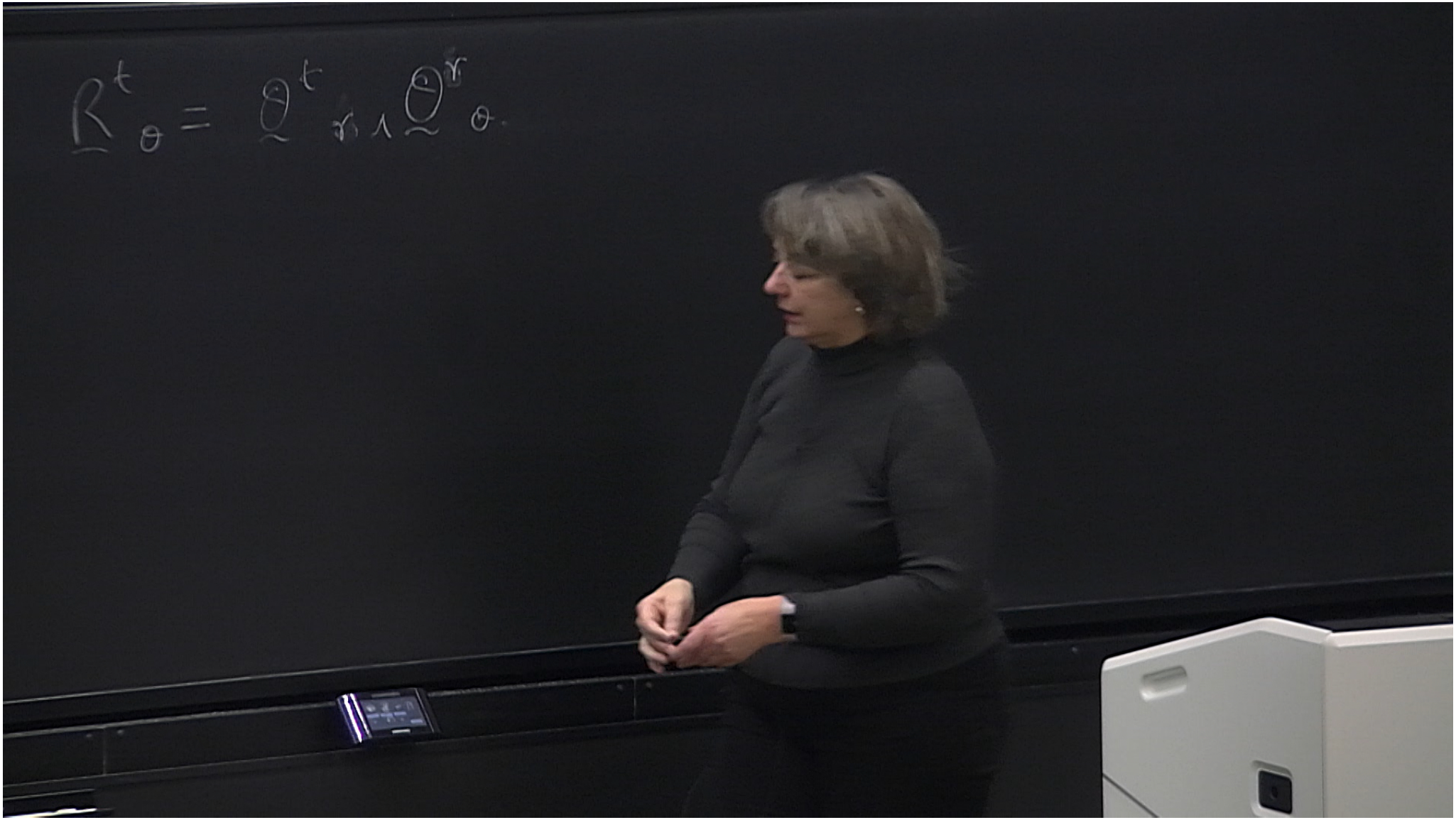

Now calculate curvature:
$\underline{R}^a{}_b = \underline{d}\,\underline{\omega}^a{}_b + \underline{\omega}^a{}_c \wedge \underline{\omega}^c{}_b$
$\underline{R}^t{}_r = \underline{d}\,\underline{\omega}^t{}_r + \underline{\omega}^t{}_c \wedge \underline{\omega}^c{}_r$
$= \underline{d}\left(\frac{A'}{B}\underline{dt}\right)$

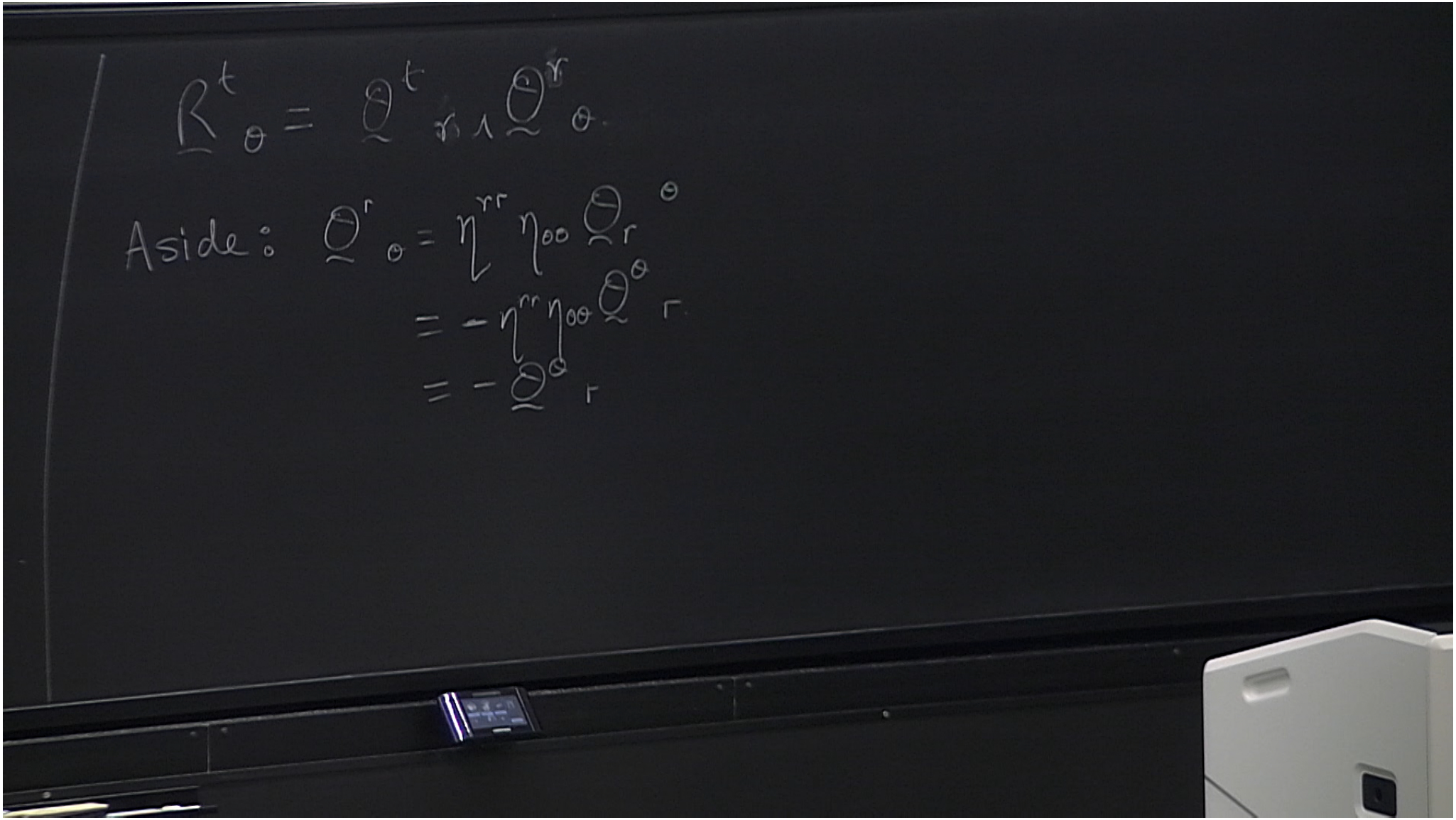
Now calculate curvature:

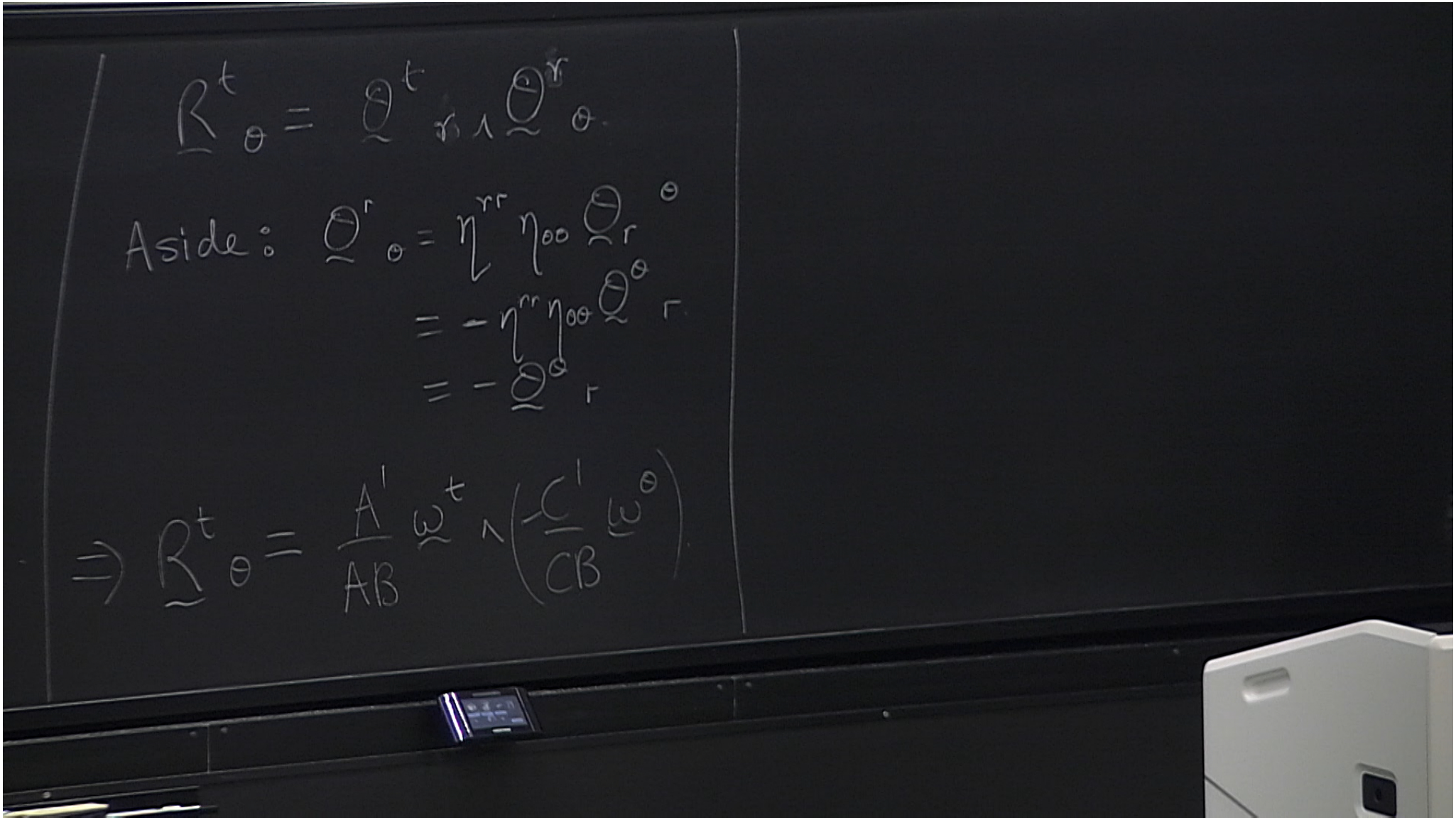
$$\underline{R}^{a}{}_{b} = \underline{d}\,\underline{\omega}^{a}{}_{b} + \underline{\omega}^{a}{}_{c} \wedge \underline{\omega}^{c}{}_{b}.$$

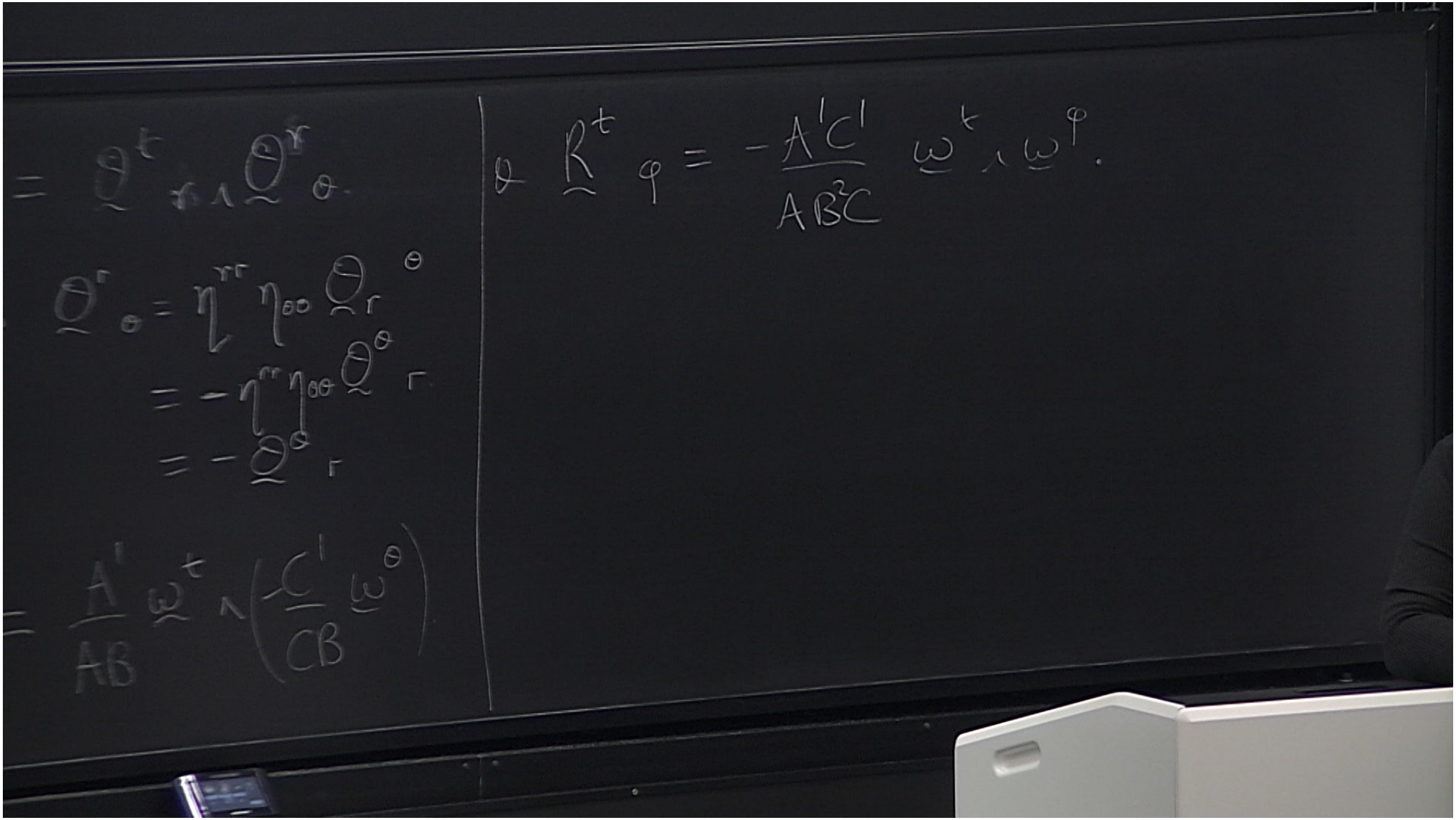
$$\underline{R}^{t}{}_{r} = \underline{d}\,\underline{\omega}^{t}{}_{r} + \underline{\omega}^{t}{}_{c} \wedge \underline{\omega}^{c}{}_{r}.$$

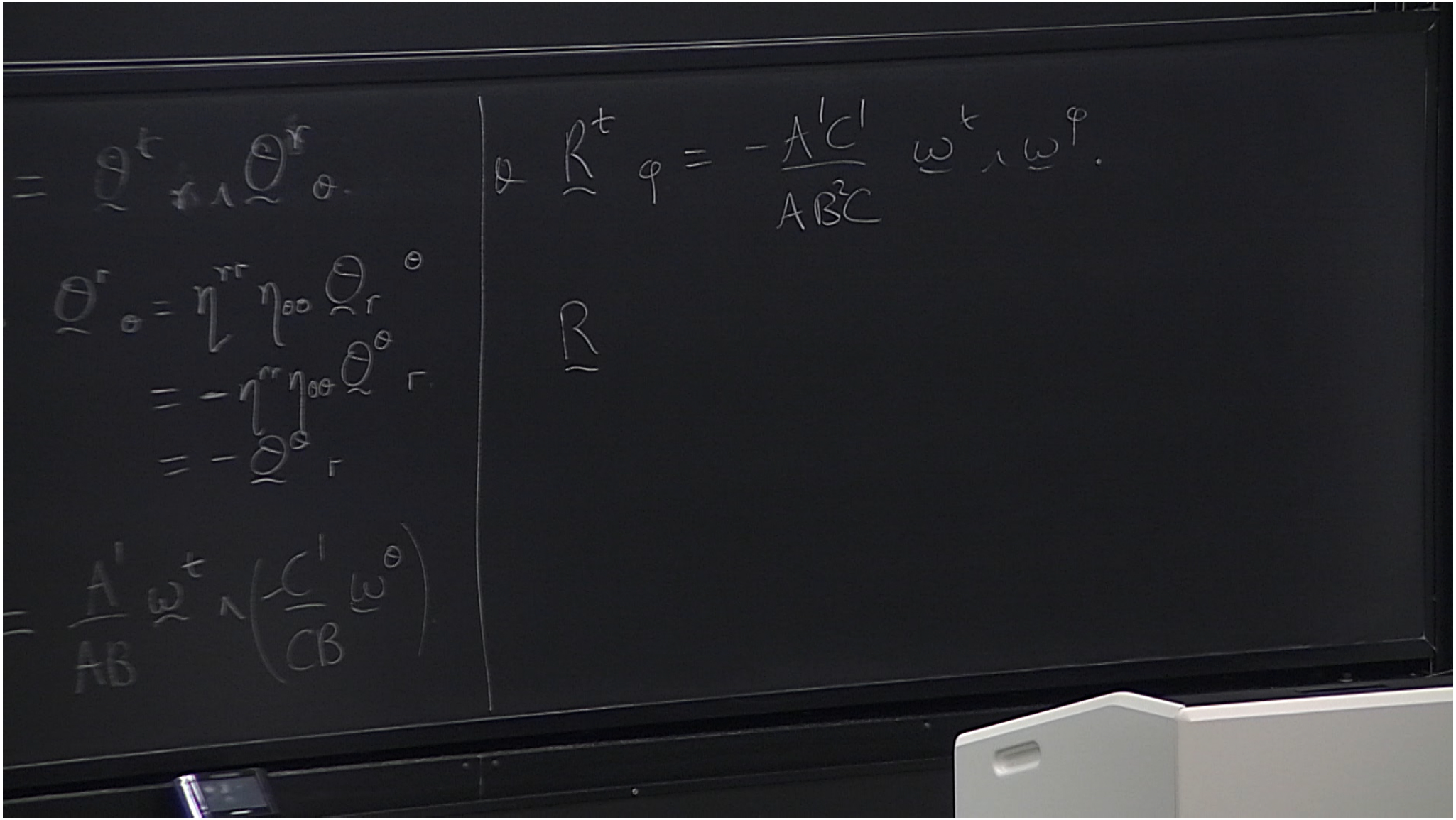
$$= \underline{d}\left(\frac{A'}{B}\underline{dt}\right) = \left(\frac{A'}{B}\right)'\underline{dr} \wedge \underline{dt}.$$

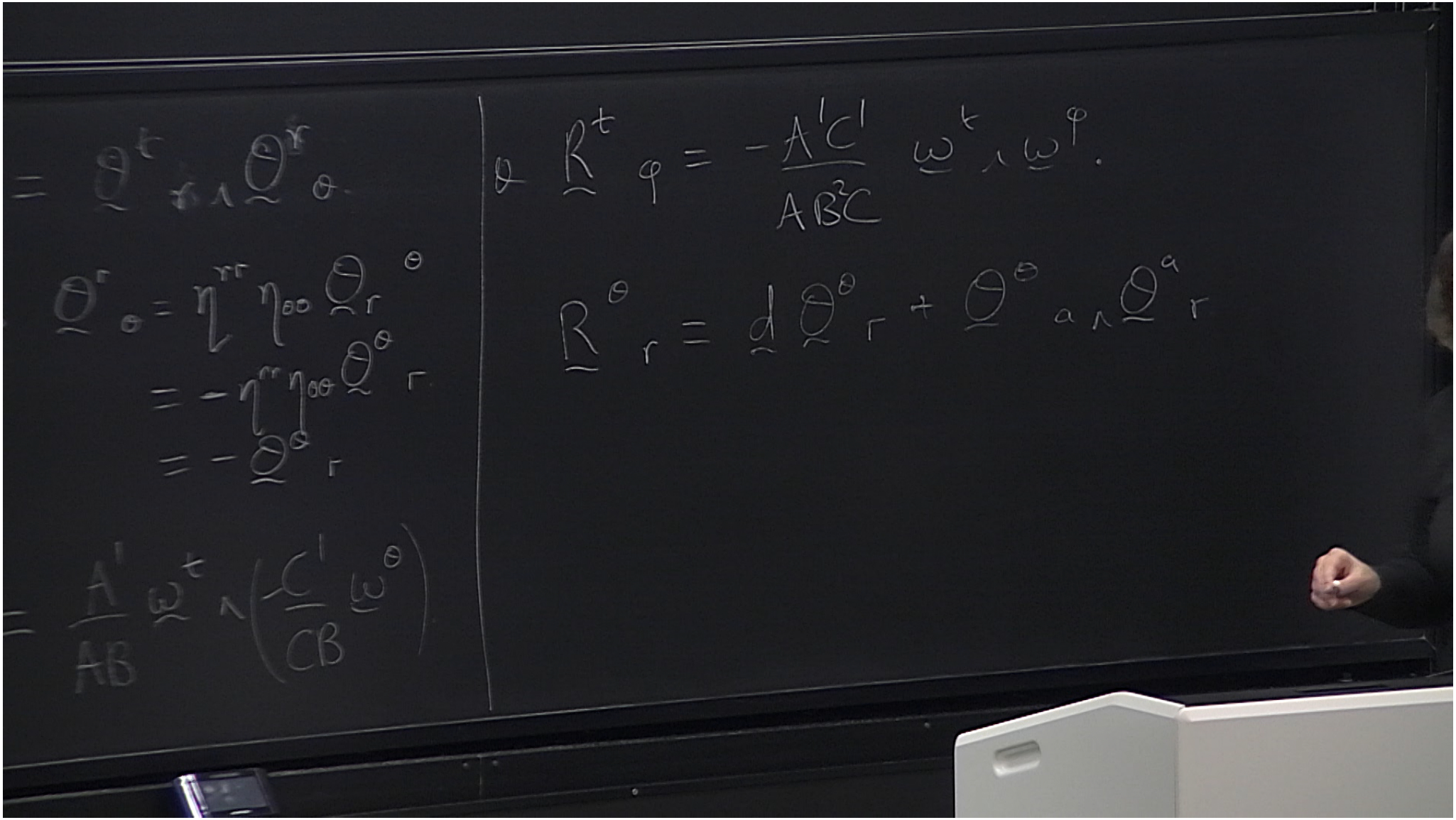

$R^t_{\theta} = \Theta^t \wedge \Theta^r_{\theta}$

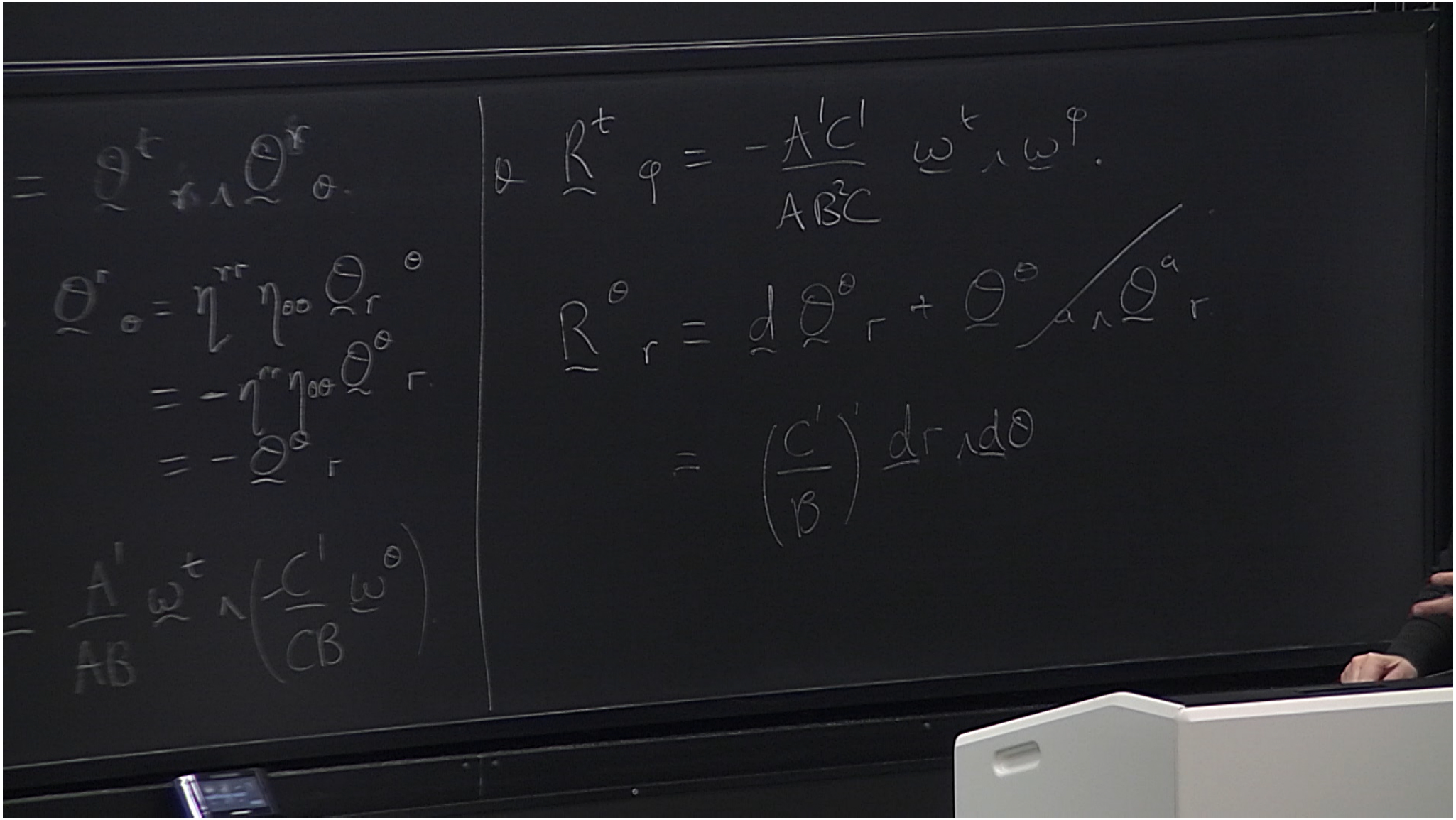

$R^t{}_\theta = \Theta^t \wedge \Theta^r{}_\theta$
Aside: $\Theta^r{}_\theta = \eta^{rr}\eta_{\theta\theta}\Theta_r{}^\theta$
$= -\eta^{rr}\eta_{\theta\theta}\Theta^\theta{}_r$
$= -\Theta^\theta{}_r$


$\underset{\sim}{R}^t{}_\theta = \underset{\sim}{\theta}^t \wedge \underset{\sim}{\theta}^r{}_\theta$
Aside: $\underset{\sim}{\theta}^r{}_\theta = \eta^{rr}\eta_{\theta\theta}\underset{\sim}{\theta}_r{}^\theta$
$= -\eta^{rr}\eta_{\theta\theta}\underset{\sim}{\theta}^\theta{}_r$
$= -\underset{\sim}{\theta}^\theta{}_r$
$\Rightarrow \underset{\sim}{R}^t{}_\theta = \frac{A'}{AB}\underset{\sim}{\omega}^t \wedge \left(\frac{-C'}{CB}\underset{\sim}{\omega}^\theta\right)$


$= \underset{\sim}{\Theta}^t{}_r \wedge \underset{\sim}{\Theta}^r{}_\theta$
$\underset{\sim}{\Theta}^r{}_\theta = \eta^{rr}\eta_{\theta\theta}\underset{\sim}{\Theta}_r{}^\theta$
$= -\eta^{rr}\eta_{\theta\theta}\underset{\sim}{\Theta}^\theta{}_r$
$= -\underset{\sim}{\Theta}^\theta{}_r$
$= \frac{A'}{AB}\underset{\sim}{\omega}^t \wedge \left(\frac{-C'}{CB}\underset{\sim}{\omega}^\theta\right)$
$\alpha\ \underset{\sim}{R}^t{}_\varphi = -\frac{A'C'}{AB^2C}\underset{\sim}{\omega}^t \wedge \underset{\sim}{\omega}^\varphi.$


$= \underset{\sim}{\Theta}^t{}_r \wedge \underset{\sim}{\Theta}^r{}_\theta$
$\underset{\sim}{\Theta}^r{}_\theta = \eta^{rr}\eta_{\theta\theta}\underset{\sim}{\Theta}_r{}^\theta$
$= -\eta^{rr}\eta_{\theta\theta}\underset{\sim}{\Theta}^\theta{}_r$
$= -\underset{\sim}{\Theta}^\theta{}_r$
$= \frac{A'}{AB}\underset{\sim}{\omega}^t \wedge \left(\frac{-C'}{CB}\underset{\sim}{\omega}^\theta\right)$
$\alpha\ \underset{\sim}{R}^t{}_\varphi = -\frac{A'C'}{AB^2C}\underset{\sim}{\omega}^t \wedge \underset{\sim}{\omega}^\varphi.$
$\underset{\sim}{R}$


$= \underset{\sim}{\omega}^t{}_r \wedge \underset{\sim}{\Theta}^r{}_\theta$
$\underset{\sim}{\Theta}^r{}_\theta = \eta^{rr}\eta_{\theta\theta}\underset{\sim}{\Theta}_r{}^\theta$
$= -\eta^{rr}\eta_{\theta\theta}\underset{\sim}{\Theta}^\theta{}_r$
$= -\underset{\sim}{\Theta}^\theta{}_r$
$= \frac{A'}{AB}\underset{\sim}{\omega}^t \wedge \left(\frac{-C'}{CB}\underset{\sim}{\omega}^\theta\right)$
$\underset{\sim}{R}^t{}_\varphi = -\frac{A'C'}{AB^2C}\underset{\sim}{\omega}^t \wedge \underset{\sim}{\omega}^\varphi$.
$\underset{\sim}{R}^\theta{}_r = \underset{\sim}{d}\underset{\sim}{\Theta}^\theta{}_r + \underset{\sim}{\Theta}^\theta{}_a \wedge \underset{\sim}{\Theta}^a{}_r$


$= \omega^t{}_r \wedge \omega^r{}_\theta$
$\omega^r{}_\theta = \eta^{rr}\eta_{\theta\theta}\omega_r{}^\theta$
$= -\eta^{rr}\eta_{\theta\theta}\omega^\theta{}_r$
$= -\omega^\theta{}_r$
$= \frac{A'}{AB}\omega^t \wedge \left(\frac{-C'}{CB}\omega^\theta\right)$
$R^t{}_\varphi = -\frac{A'C'}{AB^2C}\omega^t \wedge \omega^\varphi$
$R^\theta{}_r = d\omega^\theta{}_r + \omega^\theta{}_q \wedge \omega^q{}_r$
$= \left(\frac{C'}{B}\right)' dr \wedge d\theta$

$$= \underset{\sim}{\omega}^t{}_r \wedge \underset{\sim}{\theta}^r{}_\theta$$

$$\underset{\sim}{\theta}^r{}_\theta = \eta^{rr}\eta_{\theta\theta}\,\underset{\sim}{\theta}_r{}^\theta$$
$$= -\eta^{rr}\eta_{\theta\theta}\,\underset{\sim}{\theta}^\theta{}_r$$
$$= -\underset{\sim}{\theta}^\theta{}_r$$

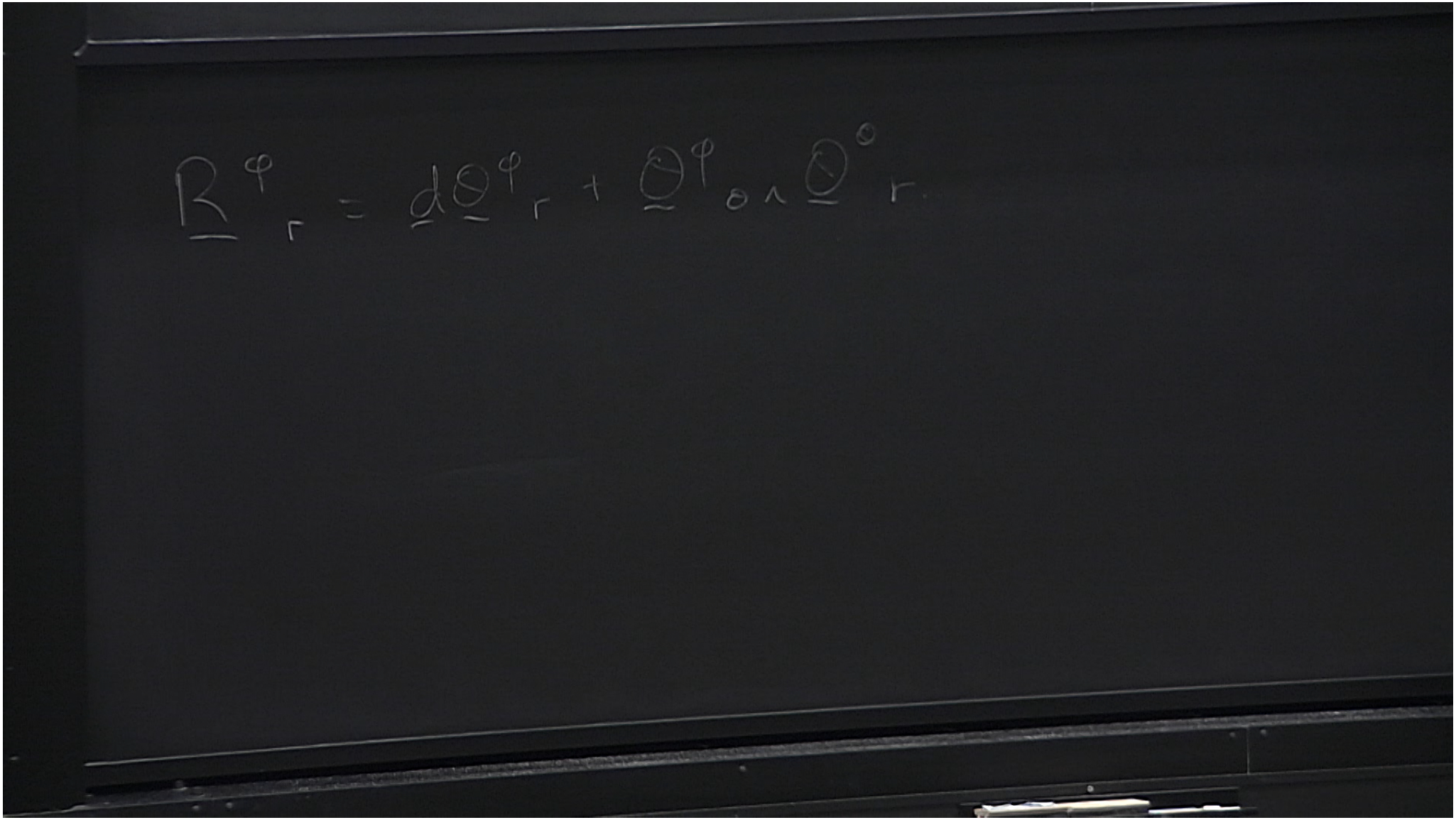
$$= \frac{A'}{AB}\,\underset{\sim}{\omega}^t \wedge \left(\frac{-C'}{CB}\,\underset{\sim}{\omega}^\theta\right)$$

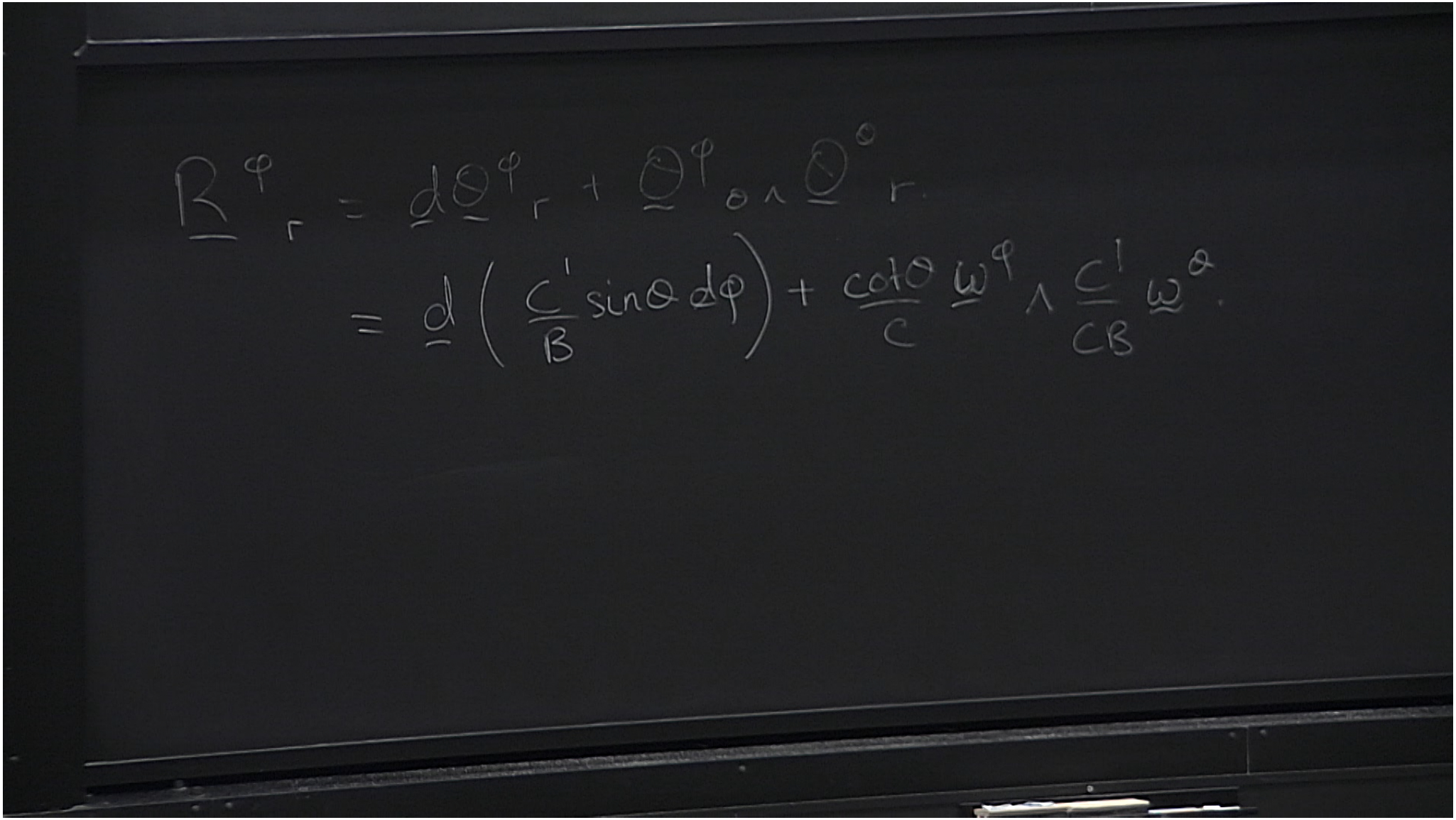
$$\theta\ \underset{\sim}{R}^t{}_\varphi = -\frac{A'C'}{AB^2C}\,\underset{\sim}{\omega}^t \wedge \underset{\sim}{\omega}^\varphi .$$

$$\underset{\sim}{R}^\theta{}_r = \underset{\sim}{d}\,\underset{\sim}{\theta}^\theta{}_r + \underset{\sim}{\theta}^\theta{}_a \wedge \underset{\sim}{\theta}^a{}_r$$

$$= \left(\frac{C'}{B}\right)' \underset{\sim}{dr} \wedge \underset{\sim}{d\theta}$$

$$= \frac{1}{CB}\left(\frac{C'}{B}\right)' \underset{\sim}{\omega}^r \wedge \underset{\sim}{\omega}^\theta$$


$\underline{R}^{\varphi}{}_{r} = d\underline{\theta}^{\varphi}{}_{r} + \underline{\theta}^{\varphi}{}_{o} \wedge \underline{\theta}^{o}{}_{r}$.

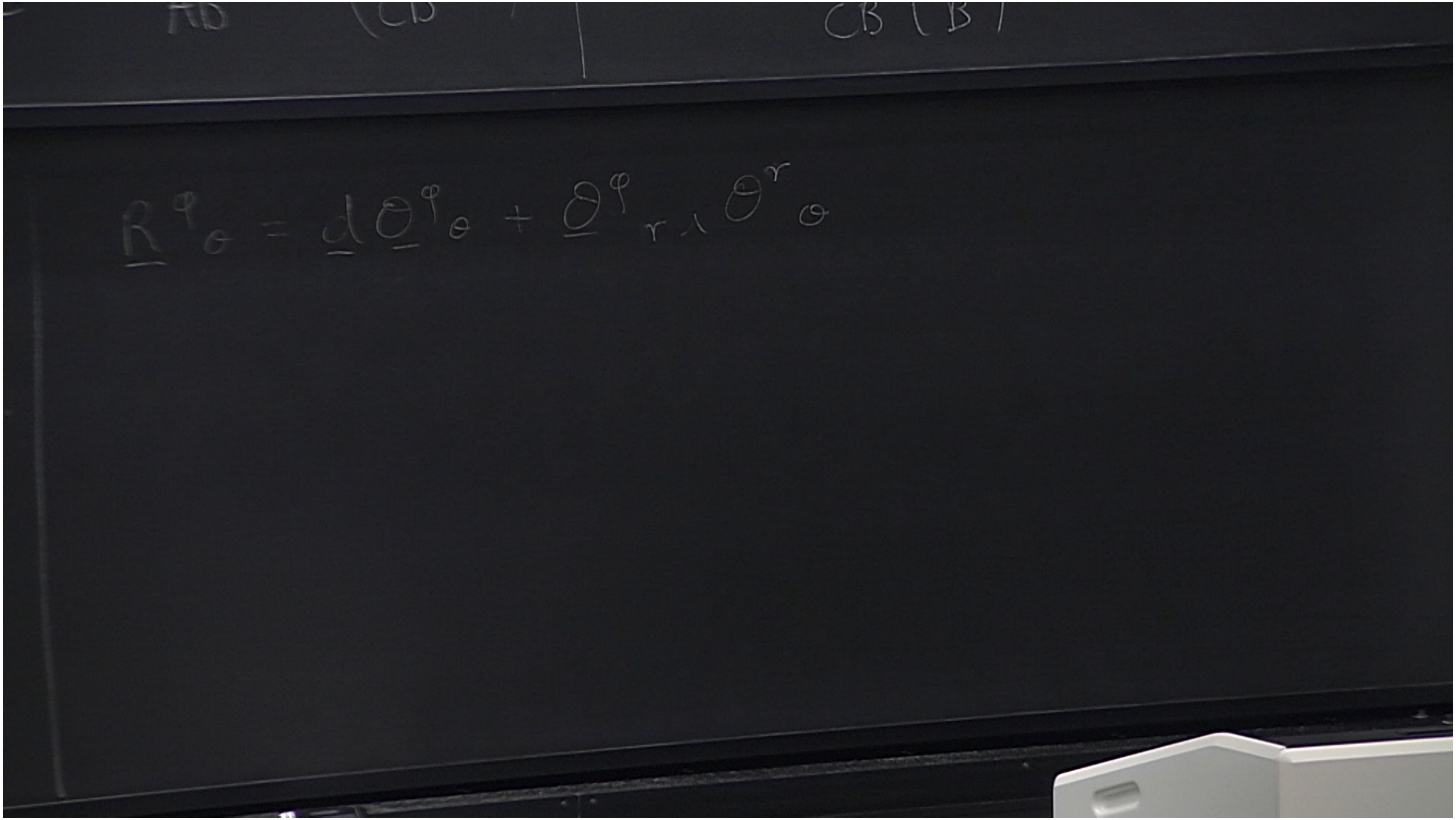

$\underline{R}^{\varphi}{}_{r} = \underline{d}\underline{\Theta}^{\varphi}{}_{r} + \underline{\Theta}^{\varphi}{}_{\theta} \wedge \underline{\Theta}^{\theta}{}_{r}$.
$= \underline{d}\left(\frac{\underline{C}^{1}}{B} \sin\theta \, d\varphi \right) + \frac{\cot\theta}{C} \underline{\omega}^{\varphi} \wedge \frac{\underline{C}^{1}}{CB} \underline{\omega}^{\theta}$.

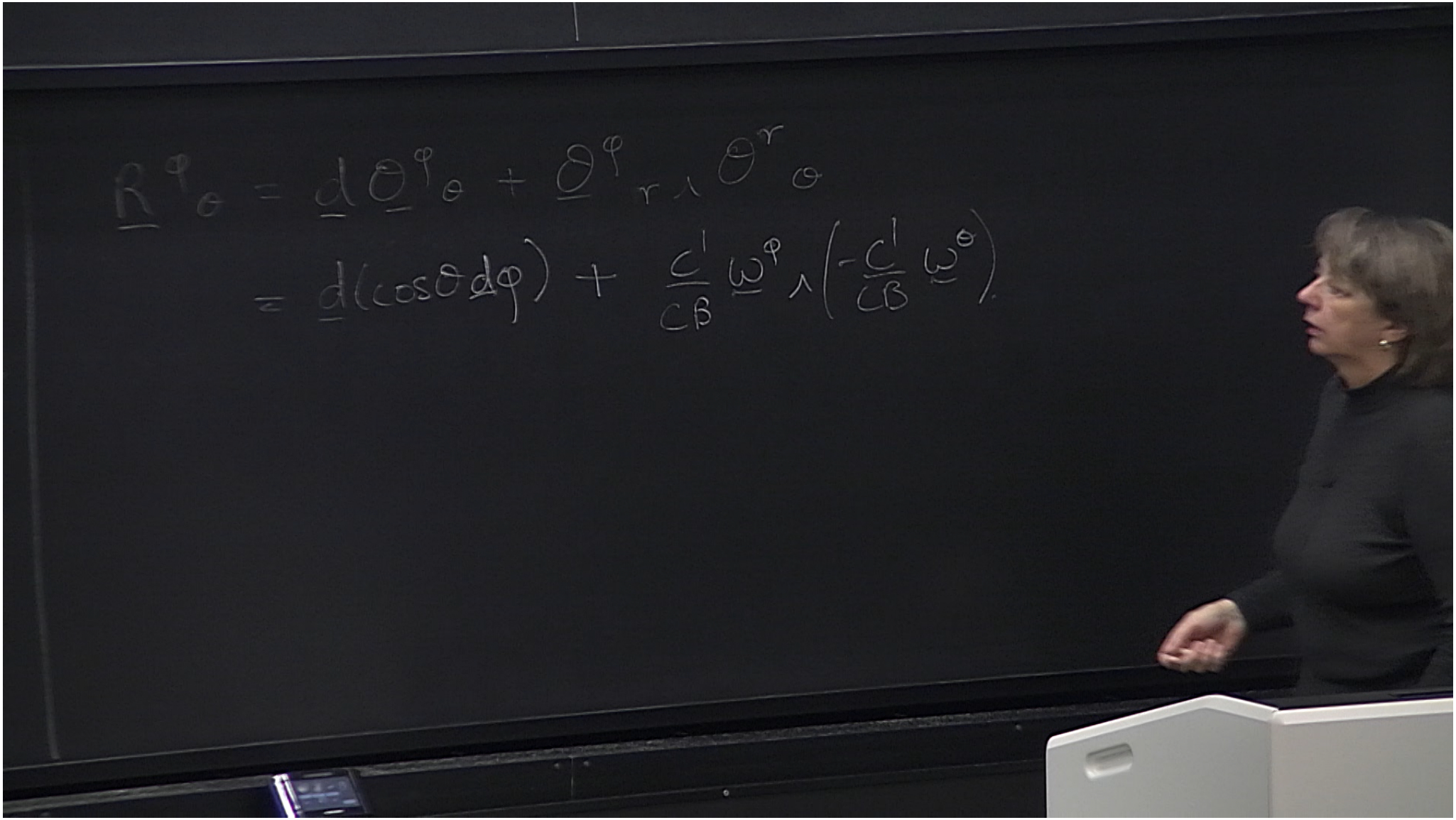
$$\underline{R}^{\varphi}{}_{r} = \underline{d}\underline{\theta}^{\varphi}{}_{r} + \underline{\theta}^{\varphi}{}_{\theta} \wedge \underline{\theta}^{\theta}{}_{r}.$$

$$= \underline{d}\left(\frac{C'}{B}\sin\theta\, d\varphi\right) + \frac{\cot\theta}{C}\,\underline{\omega}^{\varphi} \wedge \frac{\underline{C}'}{CB}\,\underline{\omega}^{\theta}$$

$$= \left(\frac{C'}{B}\right)' \sin\theta\, \underline{d}r \wedge \underline{d}\varphi + \frac{C'}{B}\cos\theta\, \underline{d}\theta \wedge d\varphi$$

$$= \frac{1}{CB}\left(\frac{C'}{B}\right)' \underline{\omega}^{r} \wedge \underline{\omega}^{\varphi} + \frac{C'\cot\theta}{C^2 B}\,\underline{\omega}^{\varphi} \wedge \underline{\omega}^{\theta}.$$


$\underline{R}^{\varphi}{}_{\theta} = \underline{d}\underline{\theta}^{\varphi}{}_{\theta} + \underline{\theta}^{\varphi}{}_{r} \wedge \underline{\theta}^{r}{}_{\theta}$

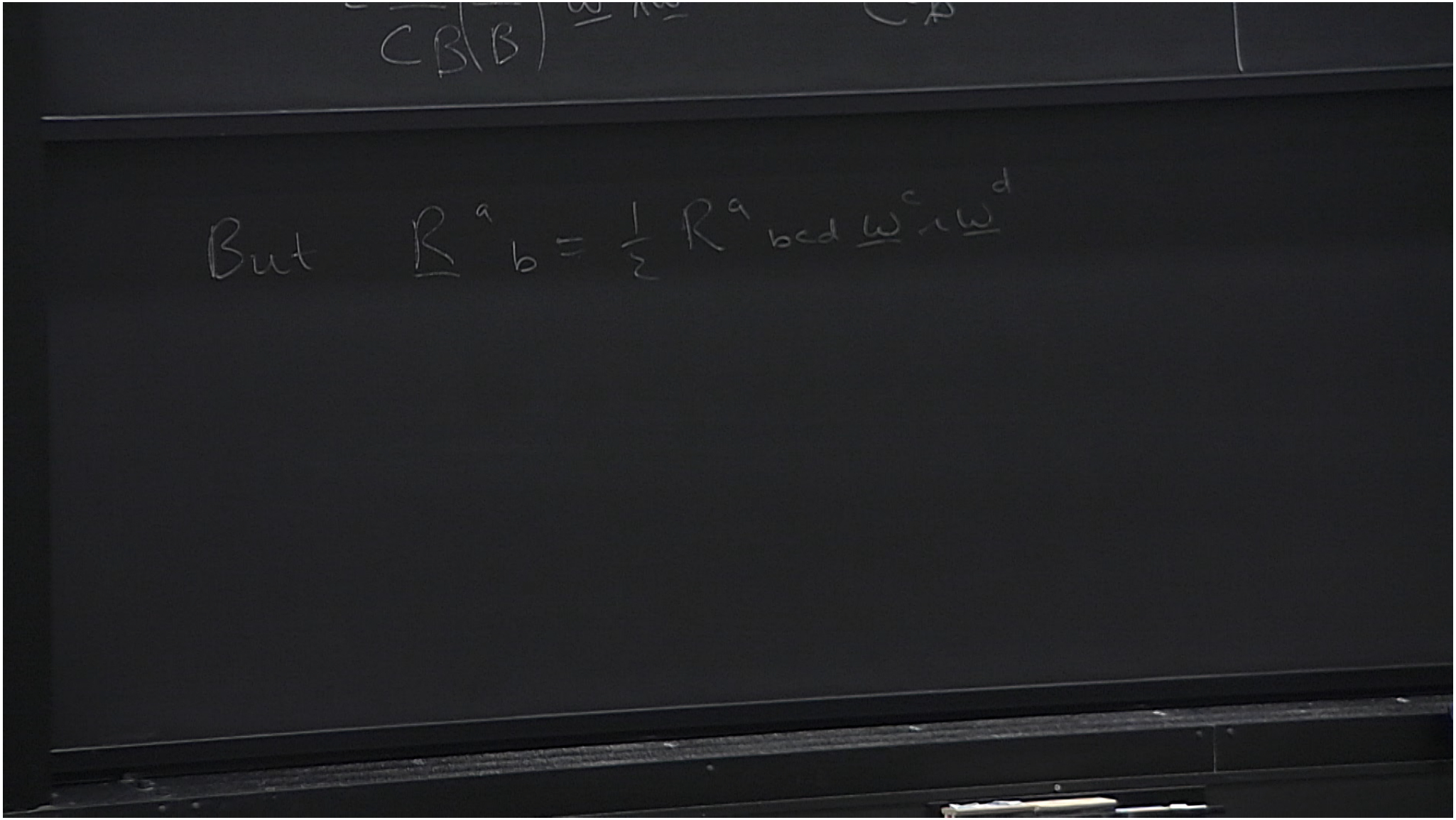

$\underline{R}^{\varphi}{}_{\theta} = \underline{d}\,\underline{\Theta}^{\varphi}{}_{\theta} + \underline{\Theta}^{\varphi}{}_{r} \wedge \Theta^{r}{}_{\theta}$
$= \underline{d}(\cos\theta\, d\varphi) + \frac{c'}{cB}\,\underline{\omega}^{\varphi} \wedge \left(-\frac{c'}{cB}\,\underline{\omega}^{\theta}\right)$

$$\underline{R}^{\varphi}{}_{\theta} = \underline{d}\,\underline{\Theta}^{\varphi}{}_{\theta} + \underline{\Theta}^{\varphi}{}_{r} \wedge \Theta^{r}{}_{\theta}$$

$$= \underline{d}(\cos\theta\, \underline{d}\varphi) + \frac{c'}{cB}\,\underline{\omega}^{\varphi} \wedge \left(-\frac{c'}{cB}\,\underline{\omega}^{\theta}\right)$$

$$= -\sin\theta\, \underline{d}\theta \wedge \underline{d}\varphi - \frac{c'^2}{c^2 B^2}\,\underline{\omega}^{\varphi} \wedge \underline{\omega}^{\theta}$$

$$= \frac{1}{c^2}\left(1 - \frac{c'^2}{B^2}\right)\underline{\omega}^{\varphi} \wedge \underline{\omega}^{\theta}$$


But $\underline{R}^a{}_b = \frac{1}{2} R^a{}_{bcd}\, \underline{\omega}^c \wedge \underline{\omega}^d$

$$\frac{1}{CB}\left(\frac{}{B}\right)\underline{\omega}\wedge\underline{\omega} \qquad C_A$$

But $\underline{R}^{a}{}_{b} = \frac{1}{2} R^{a}{}_{bcd}\, \underline{\omega}^{c}\wedge\underline{\omega}^{d}$

$$\rightarrow R^{\hat{t}}{}_{\hat{r}\hat{r}\hat{t}} = \frac{1}{AB}\left(\frac{A'}{B}\right)'$$

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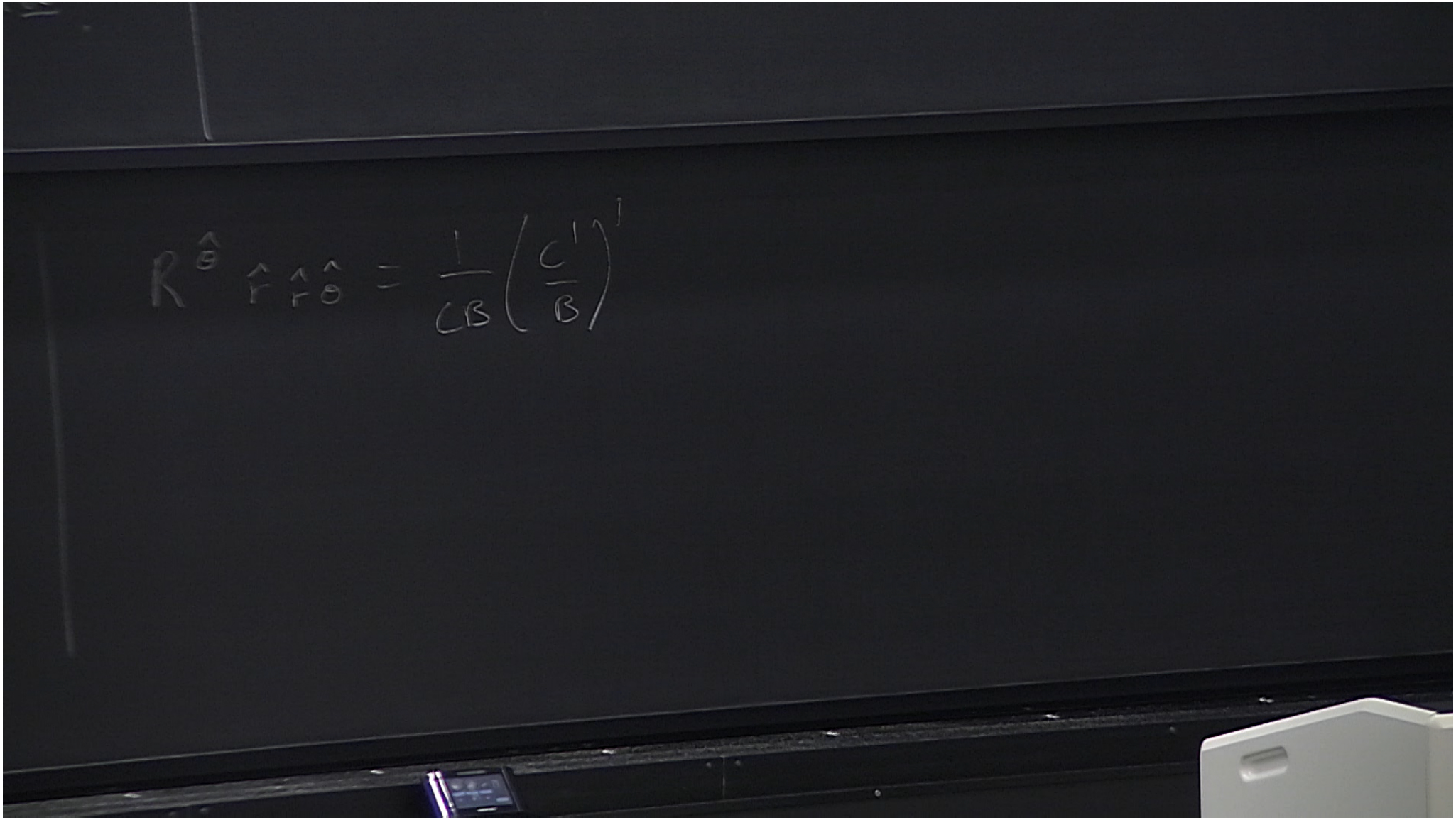
$$R^{\hat{t}}{}_{\hat{\theta}\hat{t}\hat{\theta}} = -\frac{A'C'}{AB^2C}$$

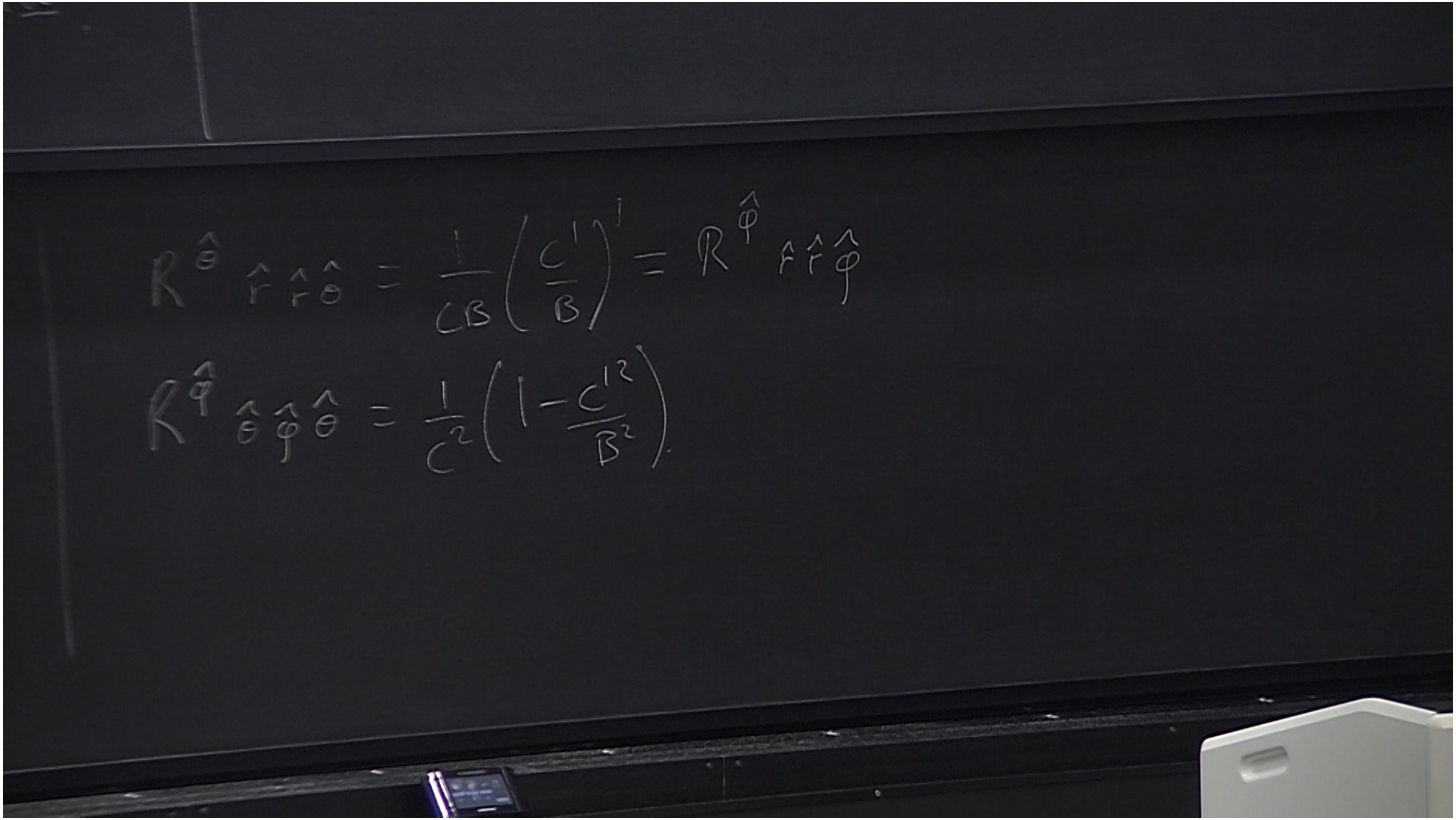
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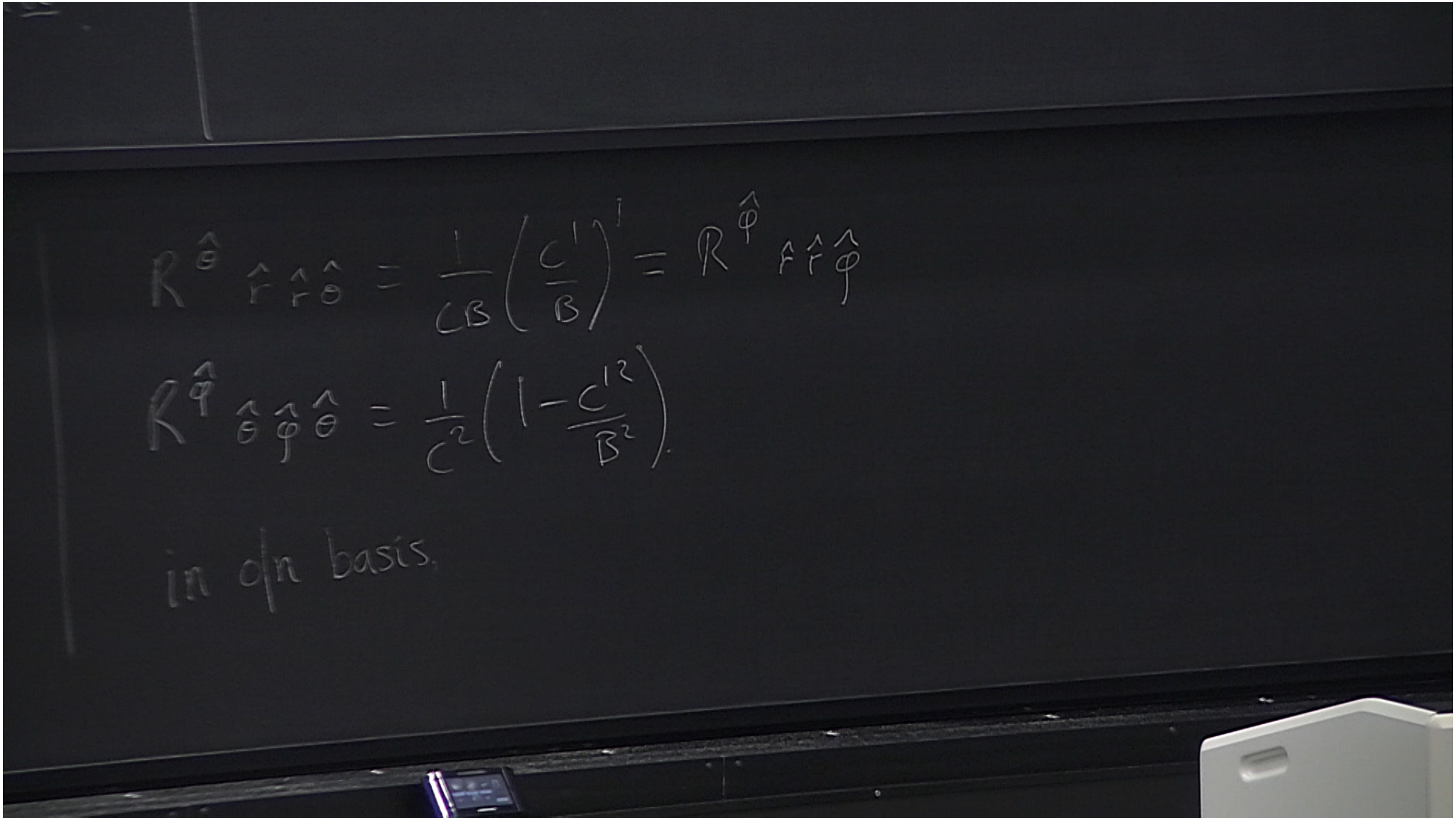
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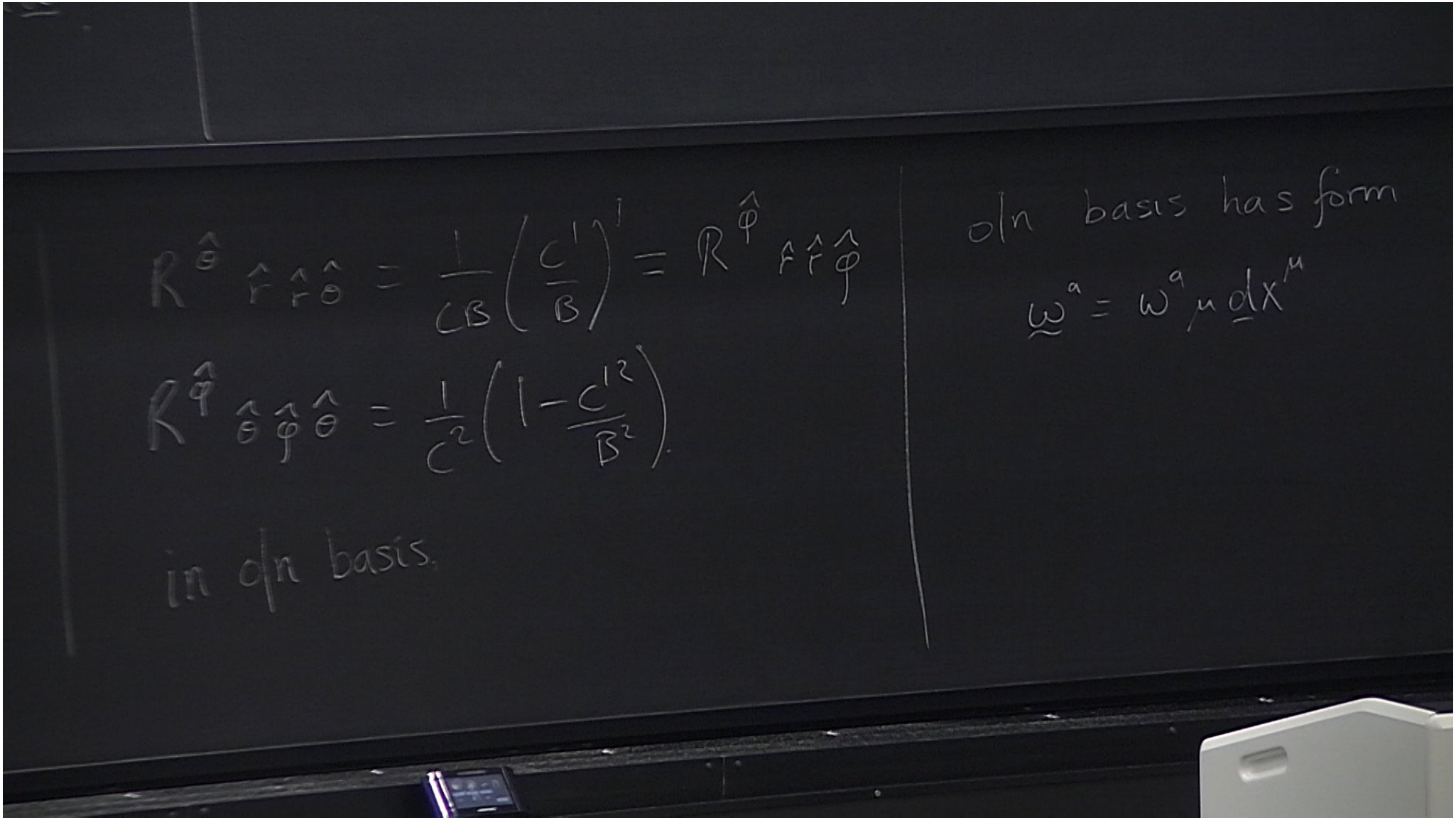
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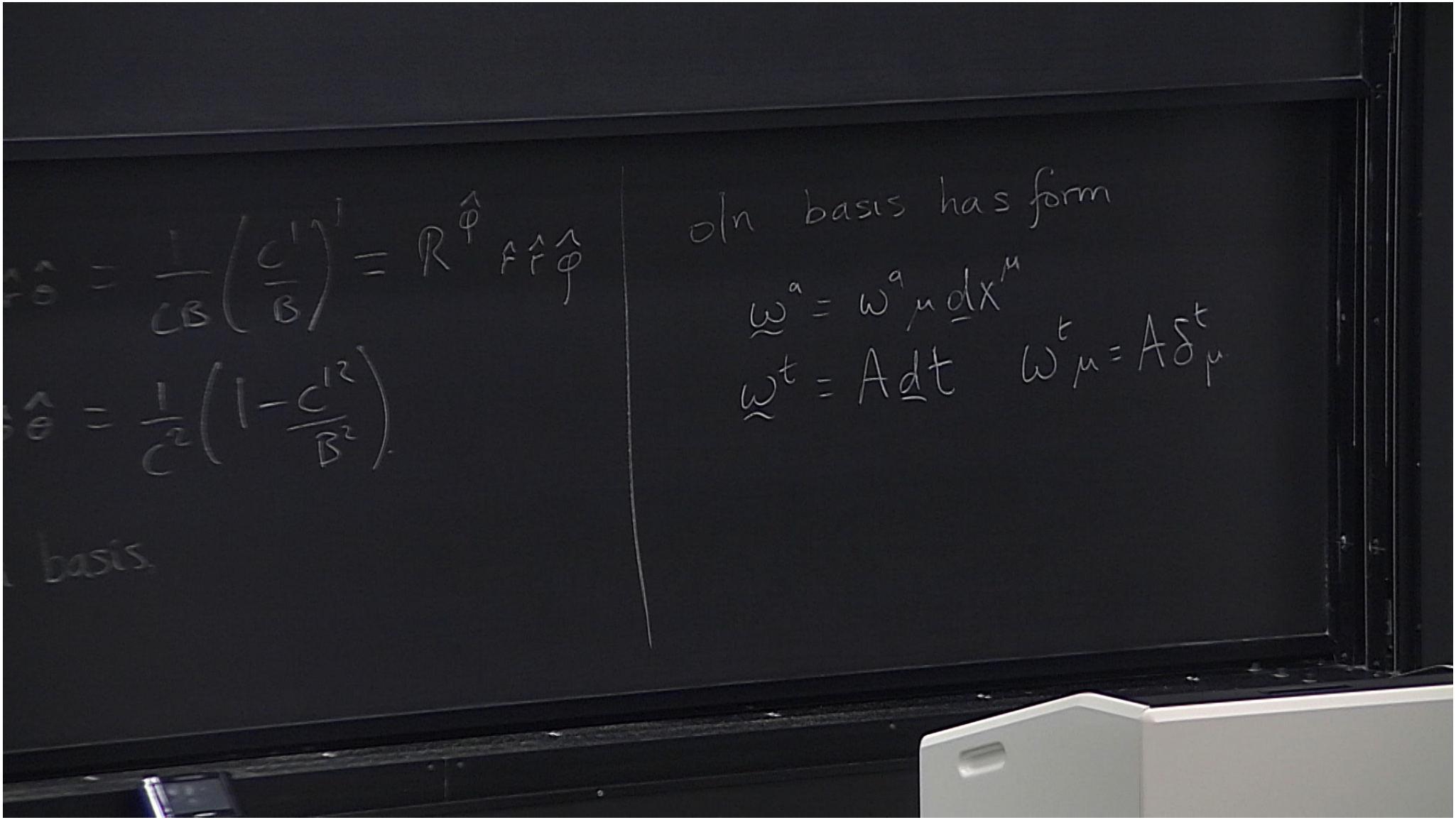
$$R^{\hat{t}}{}_{\hat{\theta}\hat{t}\hat{\theta}} = -\frac{A'C'}{AB^2C} = R^{\hat{t}}{}_{\hat{\varphi}\hat{t}\hat{\varphi}}$$

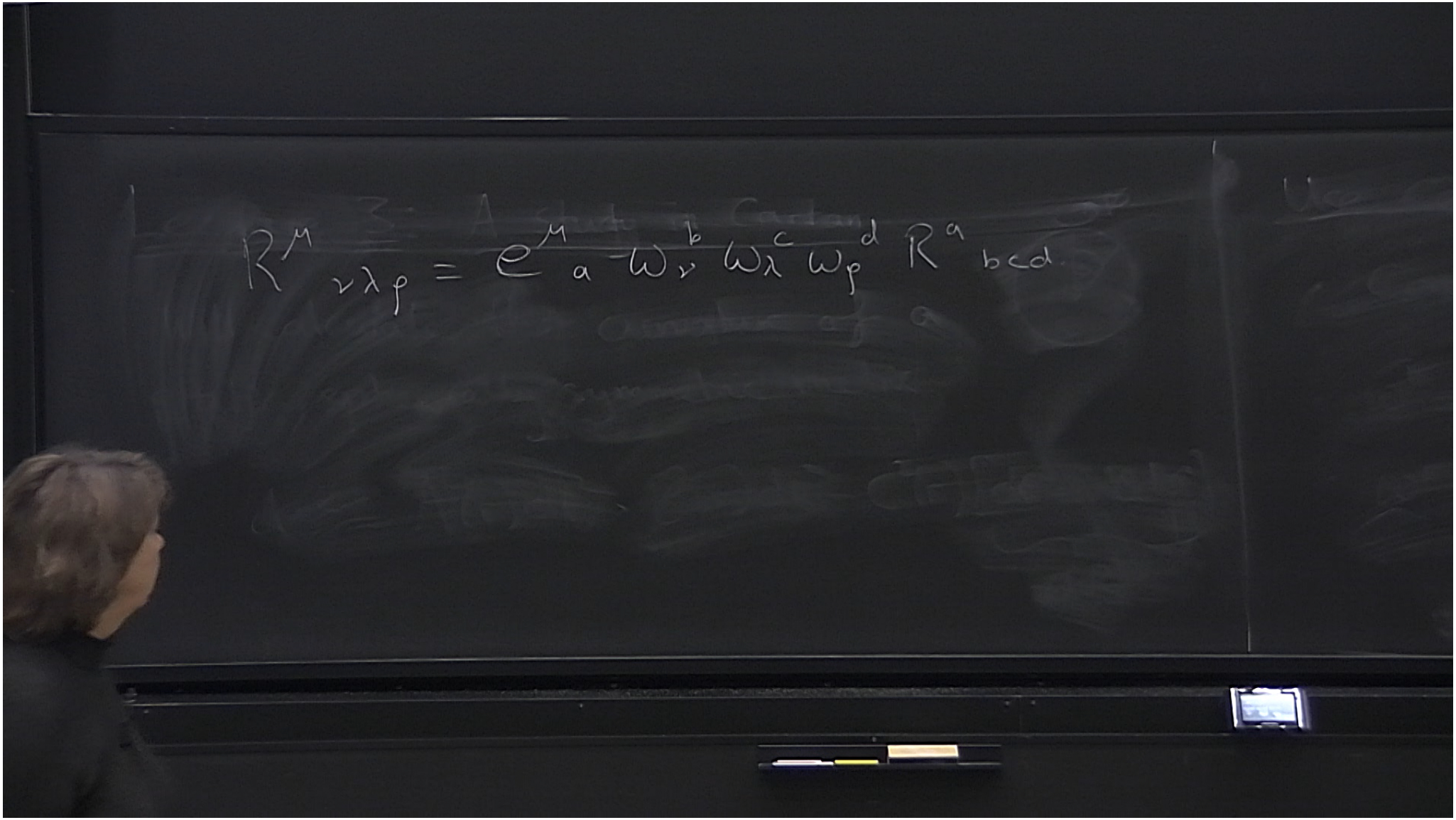

$R^{\hat{\theta}}{}_{\hat{r}\hat{r}\hat{\theta}} = \frac{1}{CB}\left(\frac{C'}{B}\right)'$


$R^{\hat{\theta}}{}_{\hat{r}\hat{r}\hat{\theta}} = \frac{1}{CB}\left(\frac{C'}{B}\right)' = R^{\hat{\varphi}}{}_{\hat{r}\hat{r}\hat{\varphi}}$
$R^{\hat{\varphi}}{}_{\hat{\theta}\hat{\varphi}\hat{\theta}} = \frac{1}{C^2}\left(1 - \frac{C'^2}{B^2}\right)$


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in o/n basis.


$R^{\hat{\theta}}{}_{\hat{r}\hat{r}\hat{\theta}} = \frac{1}{CB}\left(\frac{C'}{B}\right)' = R^{\hat{\varphi}}{}_{\hat{r}\hat{r}\hat{\varphi}}$
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o/n basis has form
$\underset{\sim}{\omega}^a = \omega^a{}_\mu \underline{d}x^\mu$


$\hat{\theta} = \frac{1}{CB}\left(\frac{C'}{B}\right)' = R^{\hat{\varphi}}{}_{\hat{r}\hat{r}\hat{\varphi}}$
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oln basis has form
$\underset{\sim}{\omega}^a = \omega^a{}_\mu \underline{d}x^\mu$
$\underset{\sim}{\omega}^t = A\underline{d}t \quad \omega^t{}_\mu = A\delta^t_\mu$


$R^{\mu}{}_{\nu\lambda\rho} = e^{\mu}_{a}\,\omega_{\nu}{}^{b}\,\omega_{\lambda}{}^{c}\,\omega_{\rho}{}^{d}\,R^{a}{}_{bcd}$

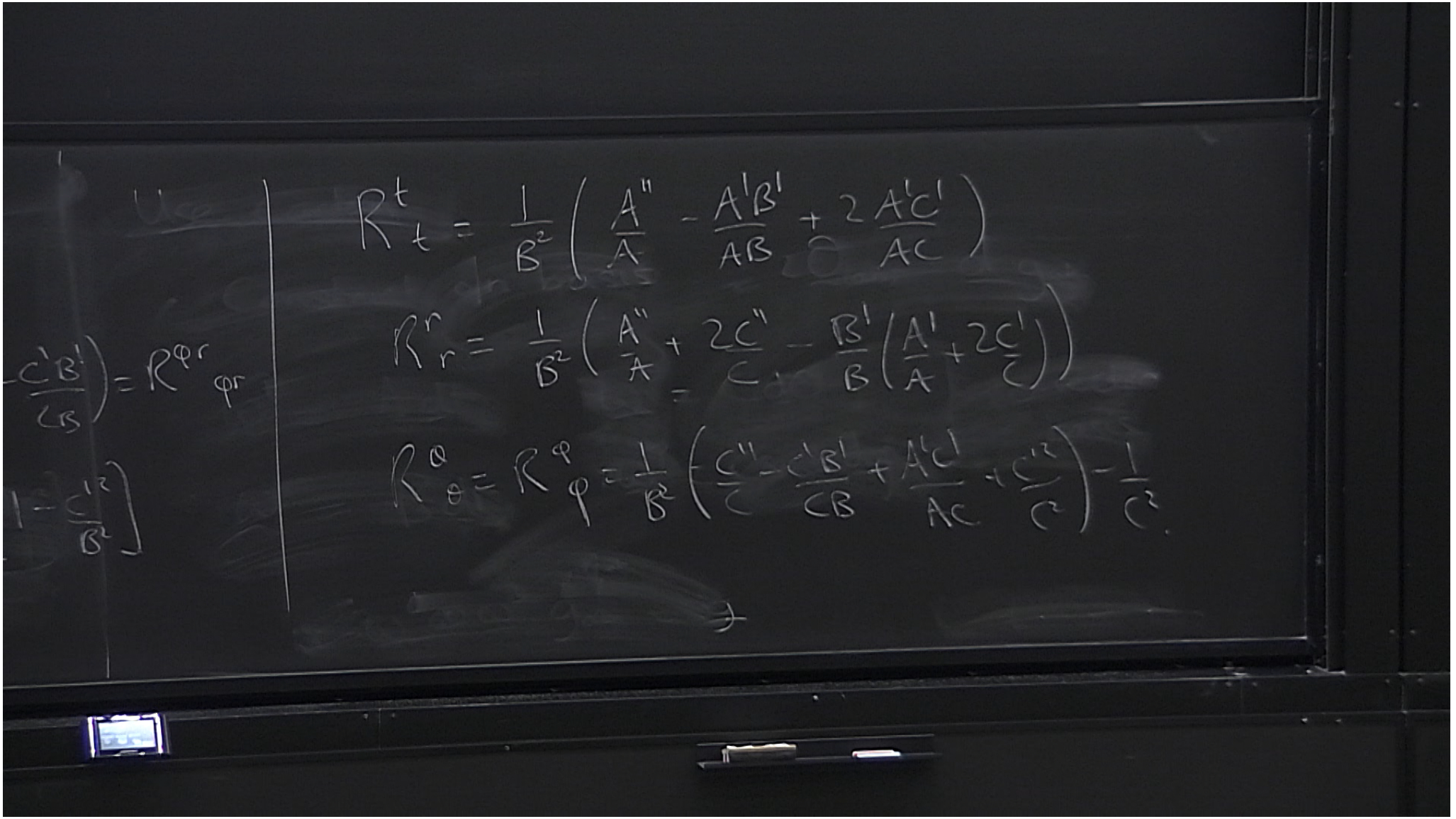
$$R^{\mu}{}_{\nu\lambda\rho} = e^{\mu}_{a}\,\omega_{\nu}^{b}\,\omega_{\lambda}^{c}\,\omega_{\rho}^{d}\,R^{a}{}_{bcd}$$

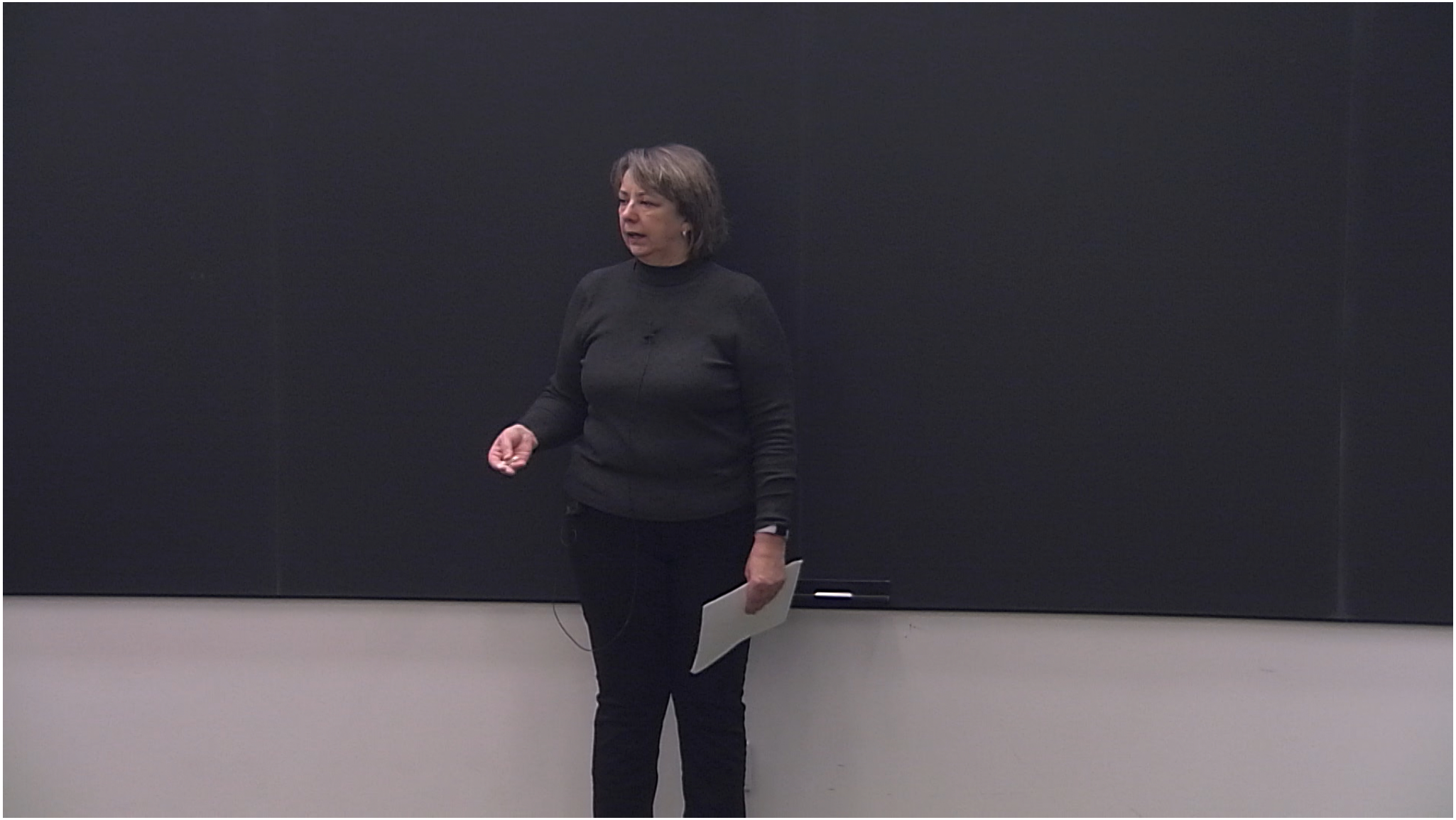
$$R^{tr}{}_{tr} = \frac{1}{B^2}\left(\frac{A''}{A} - \frac{A'B'}{AB}\right)$$

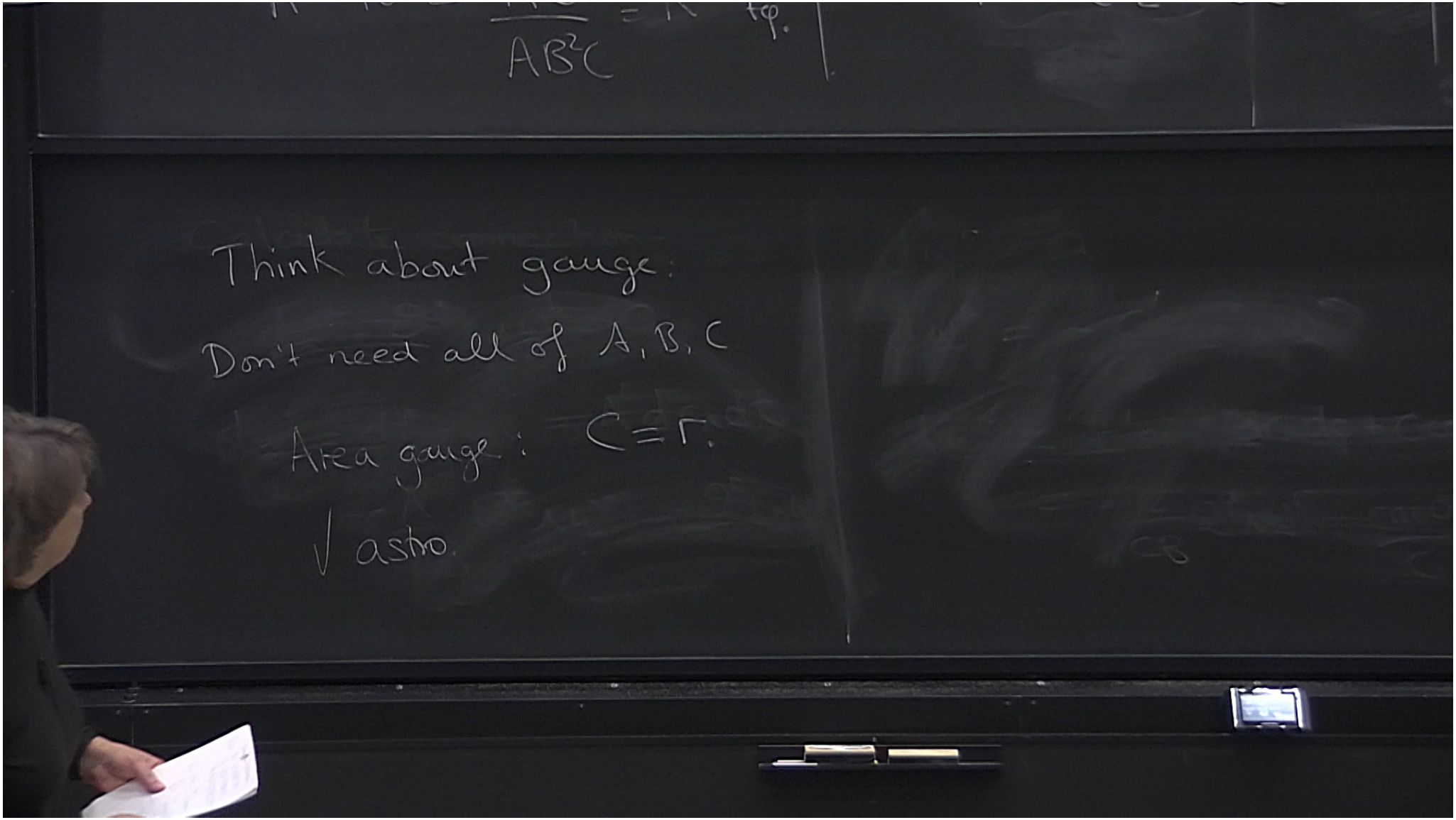
$$R^{t\alpha}{}_{t\alpha} = \frac{A'C'}{AB^2C} = R^{t\varphi}{}_{t\varphi}.$$

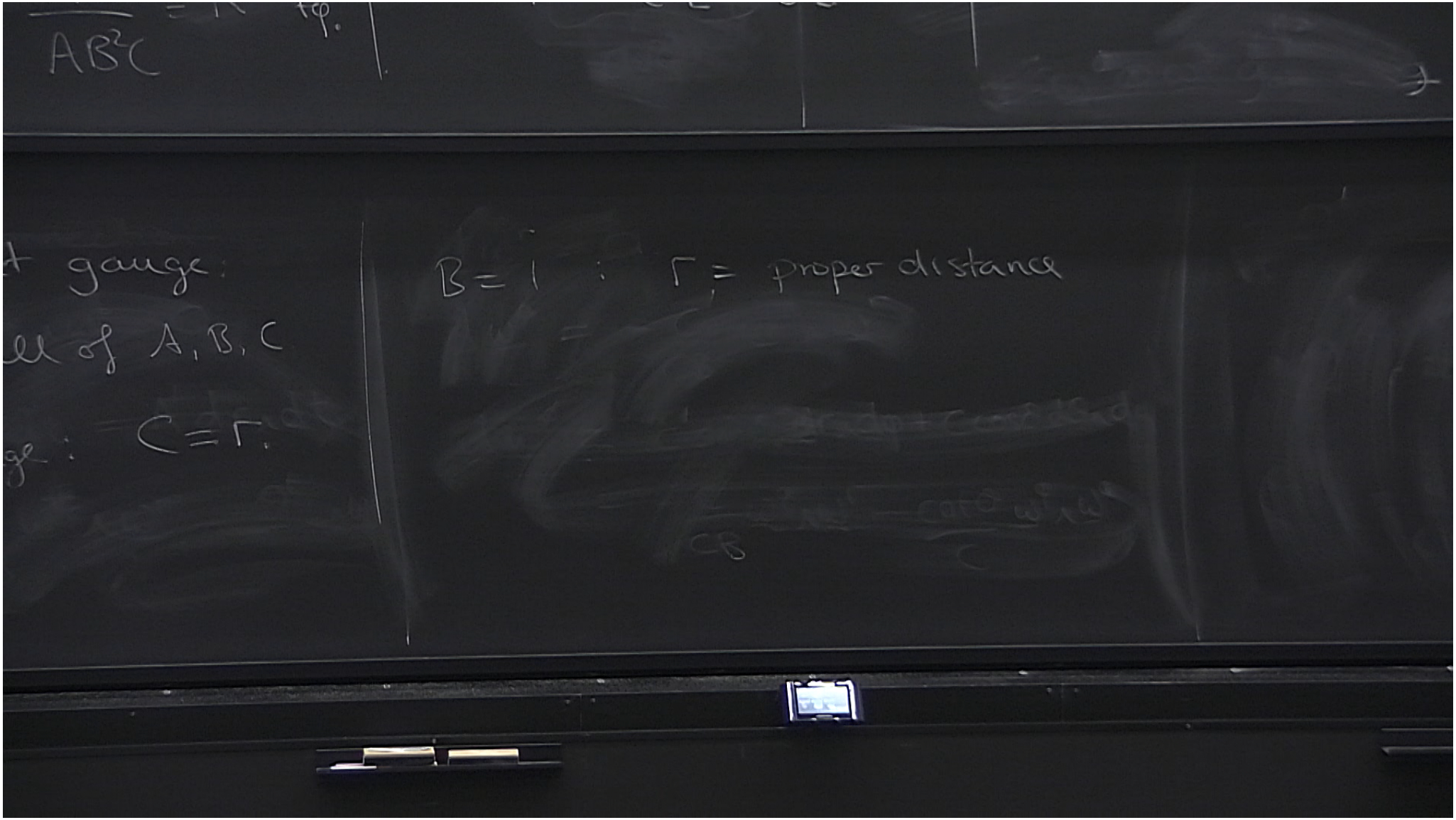
$$R^{\theta r}{}_{\theta r} = \frac{1}{B^2}\left(\frac{C''}{C} - \frac{C'B'}{CB}\right) = R^{\varphi r}{}_{\varphi r}$$

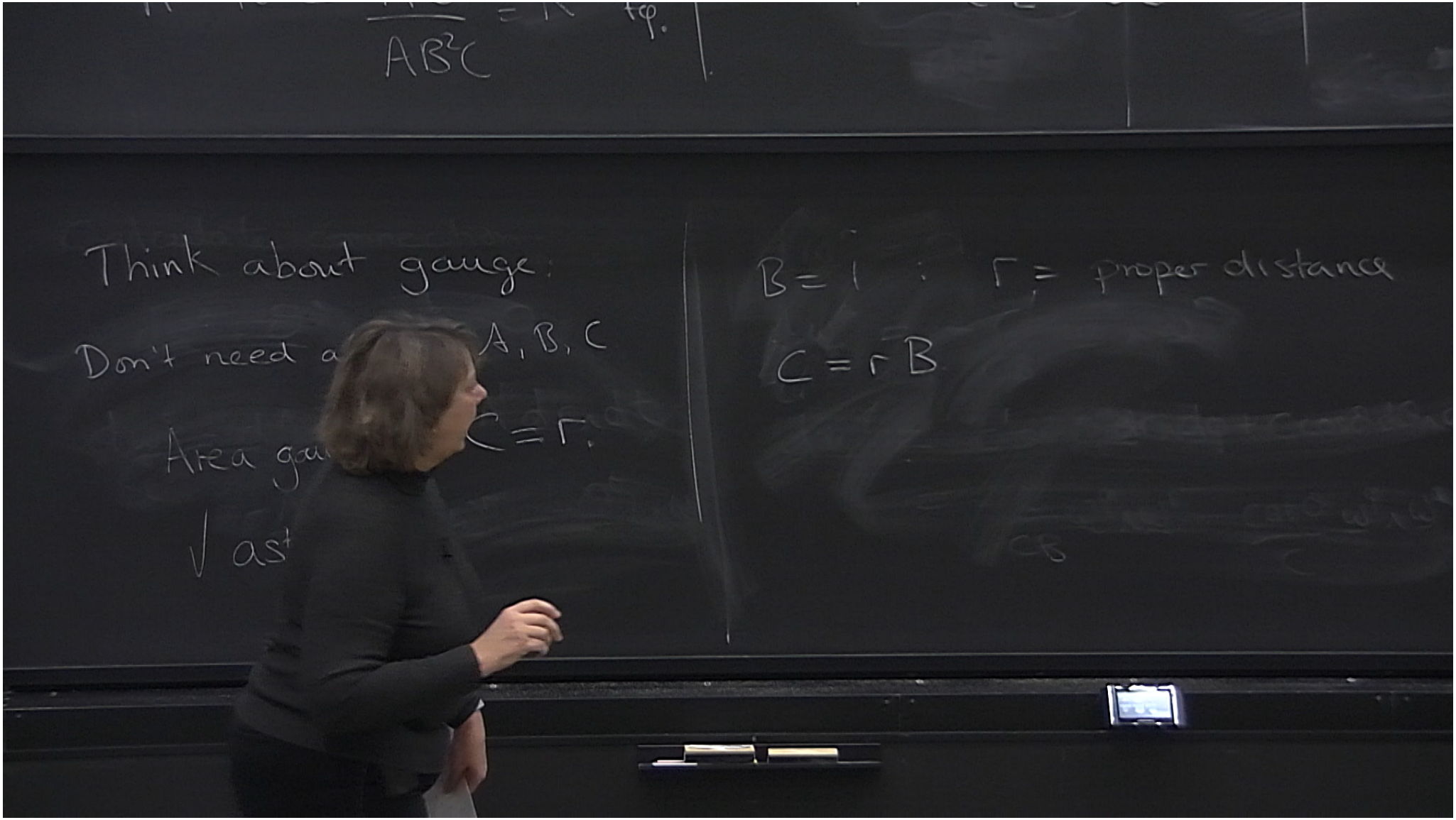
$$R^{\varphi\alpha}{}_{\varphi\theta} = \frac{-1}{C^2}\left[1 - \frac{C'^2}{B^2}\right]$$


$R^t_t = \frac{1}{B^2}\left(\frac{A''}{A} - \frac{A'B'}{AB} + 2\frac{A'C'}{AC}\right)$
$R^r_r = \frac{1}{B^2}\left(\frac{A''}{A} + 2\frac{C''}{C} - \frac{B'}{B}\left(\frac{A'}{A} + 2\frac{C'}{C}\right)\right)$
$R^\theta_\theta = R^\varphi_\varphi = \frac{1}{B^2}\left(\frac{C''}{C} - \frac{C'B'}{CB} + \frac{A'C'}{AC} + \frac{C'^2}{C^2}\right) - \frac{1}{C^2}$
$-\frac{C'B'}{CB}\Big) = R^{\varphi r}{}_{\varphi r}$
$1 - \frac{C'^2}{B^2}\Big]$




$\frac{AC}{AB^2C} = K$ fφ.
Think about gauge:
Don't need all of A, B, C
Area gauge: C=r.
√ astro.


gauge:
B=1 : r = proper distance


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√ as
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Don't need all of A, B, C

Area gauge: $C = r$.

√ astro.

$B = 1$: r = proper distance

$C = rB$ – homogeneous gauge

$B^2[dr^2 + r^2 d\Omega_{II}^2]$

= proper distance

homogeneous gauge

$d\Omega_{II}^2$]

Area $ds^2 = A^2 dt^2 - B^2 dr^2 - r^2 d\Omega_{II}^2$

$$G^t_t = \frac{1}{C^2} - \frac{1}{B^2}\left(\frac{2C''}{C} - \frac{2C'B'}{CB} + \frac{C'^2}{C^2}\right)$$

= proper distance

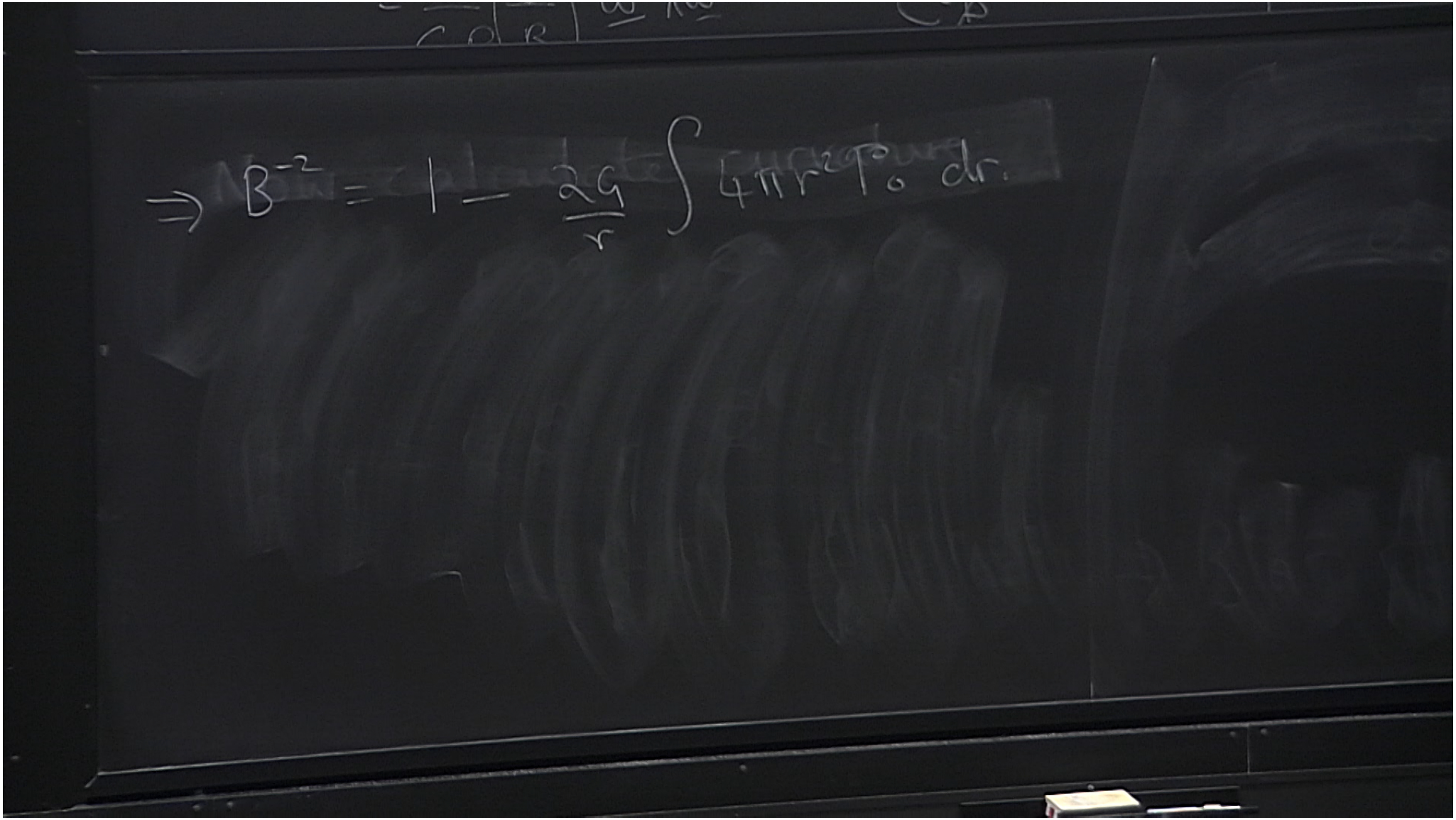
homogeneous gauge

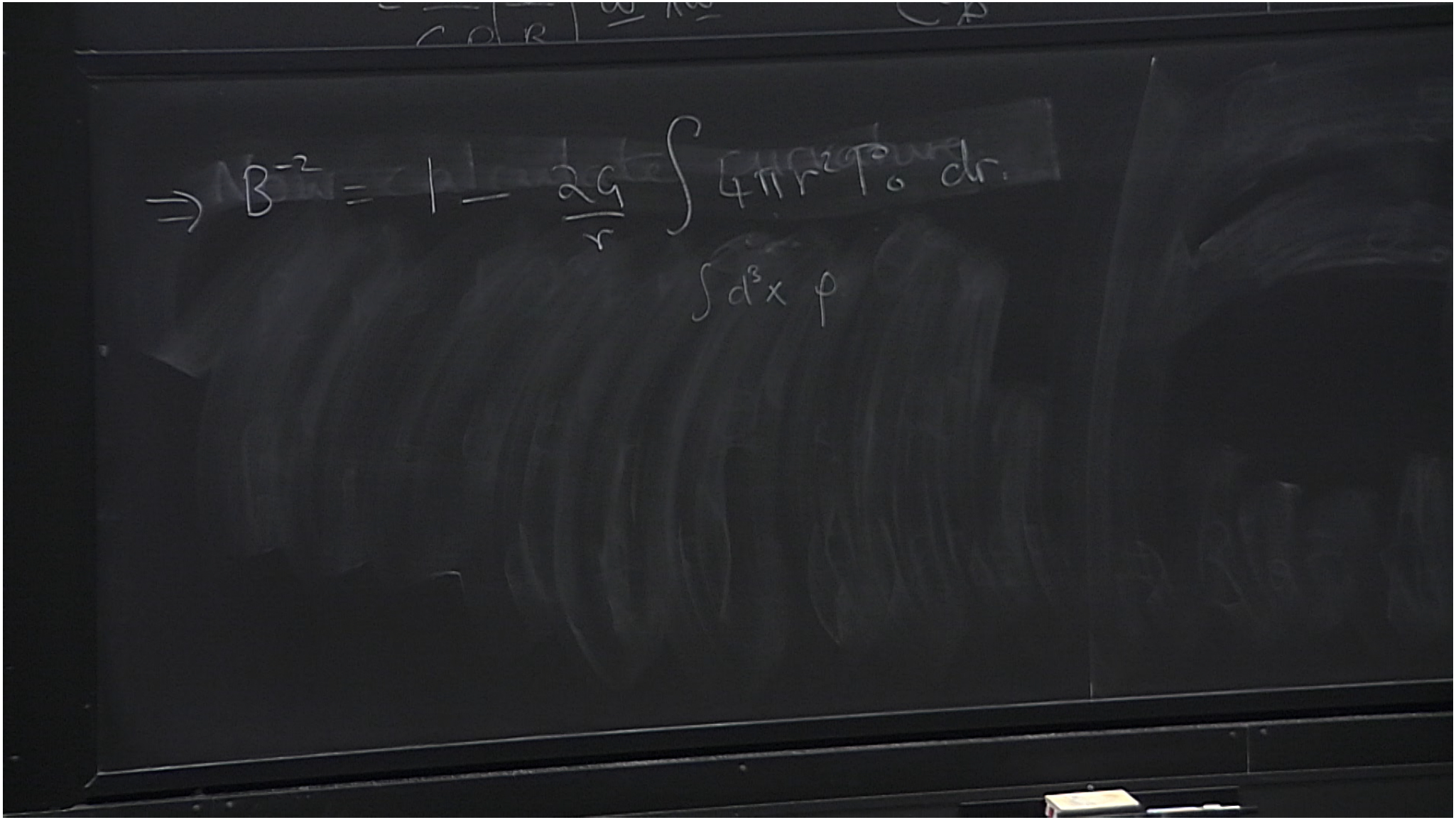
$d\Omega_{II}^2]$

Area $\quad ds^2 = A^2 dt^2 - B^2 dr^2 - r^2 d\Omega_{II}^2$

$$G^t_t = \frac{1}{C^2} - \frac{1}{B^2}\left(\frac{2C''}{C} - \frac{2C'B'}{CB} + \frac{C'^2}{C^2}\right)$$

$$\rightarrow \quad \frac{1}{r^2} - \frac{1}{r^2}\left(\frac{r}{B^2}\right)' = 8\pi G T^0_0$$

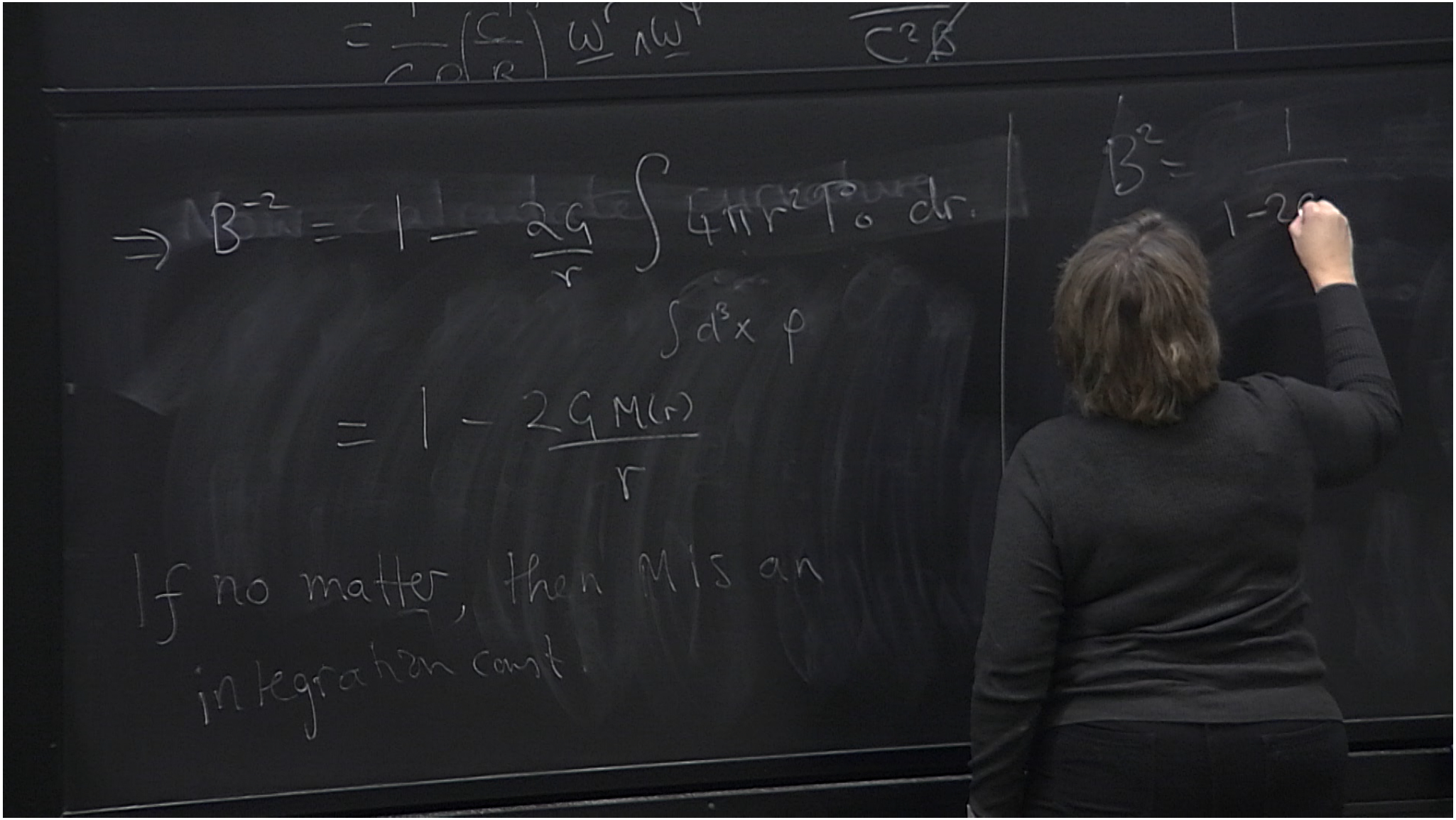

$\Rightarrow B^{-2} = 1 - \frac{2G}{r} \int 4\pi r^2 T_0 \, dr.$

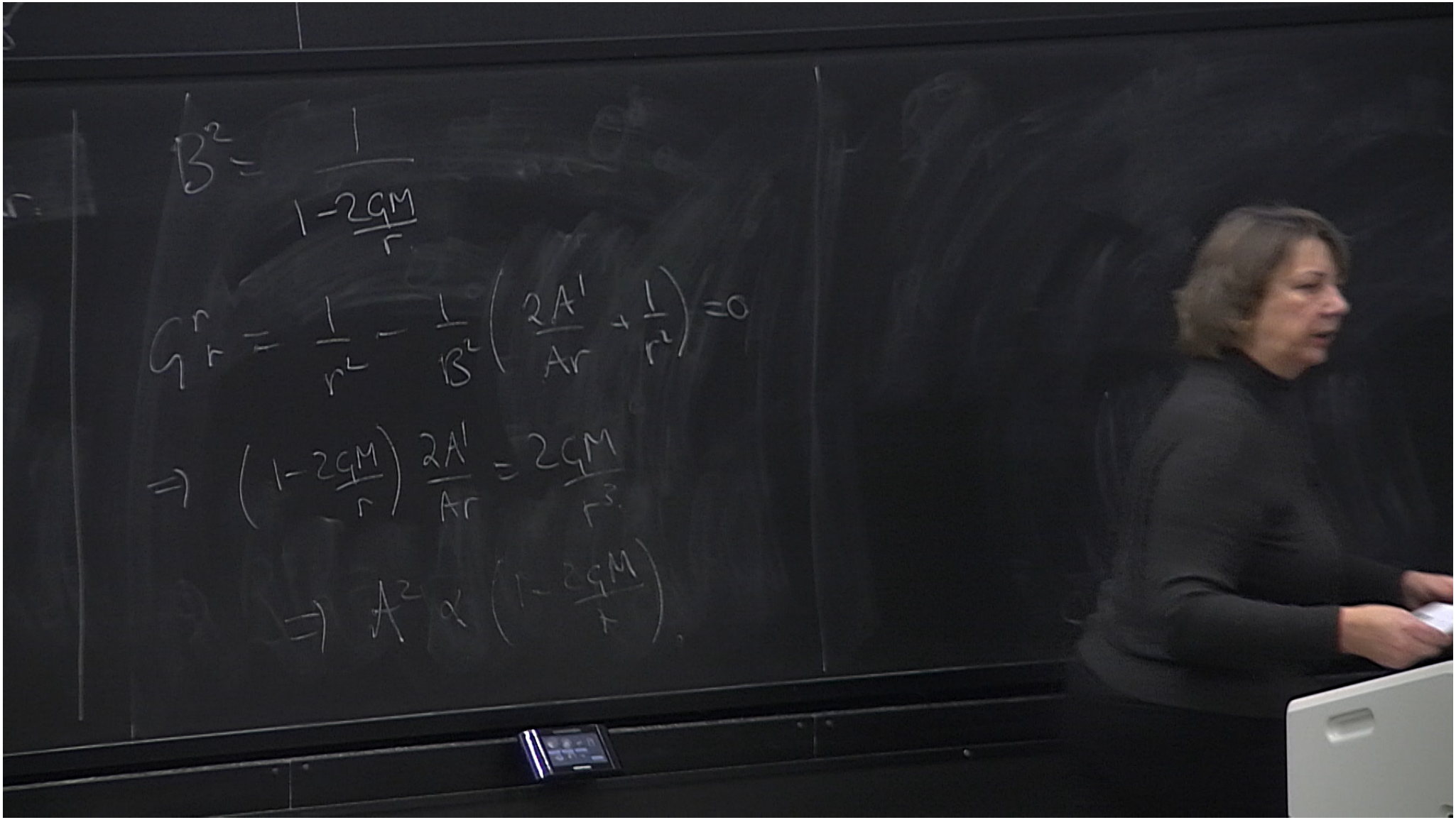

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$\int d^3x \, \rho$

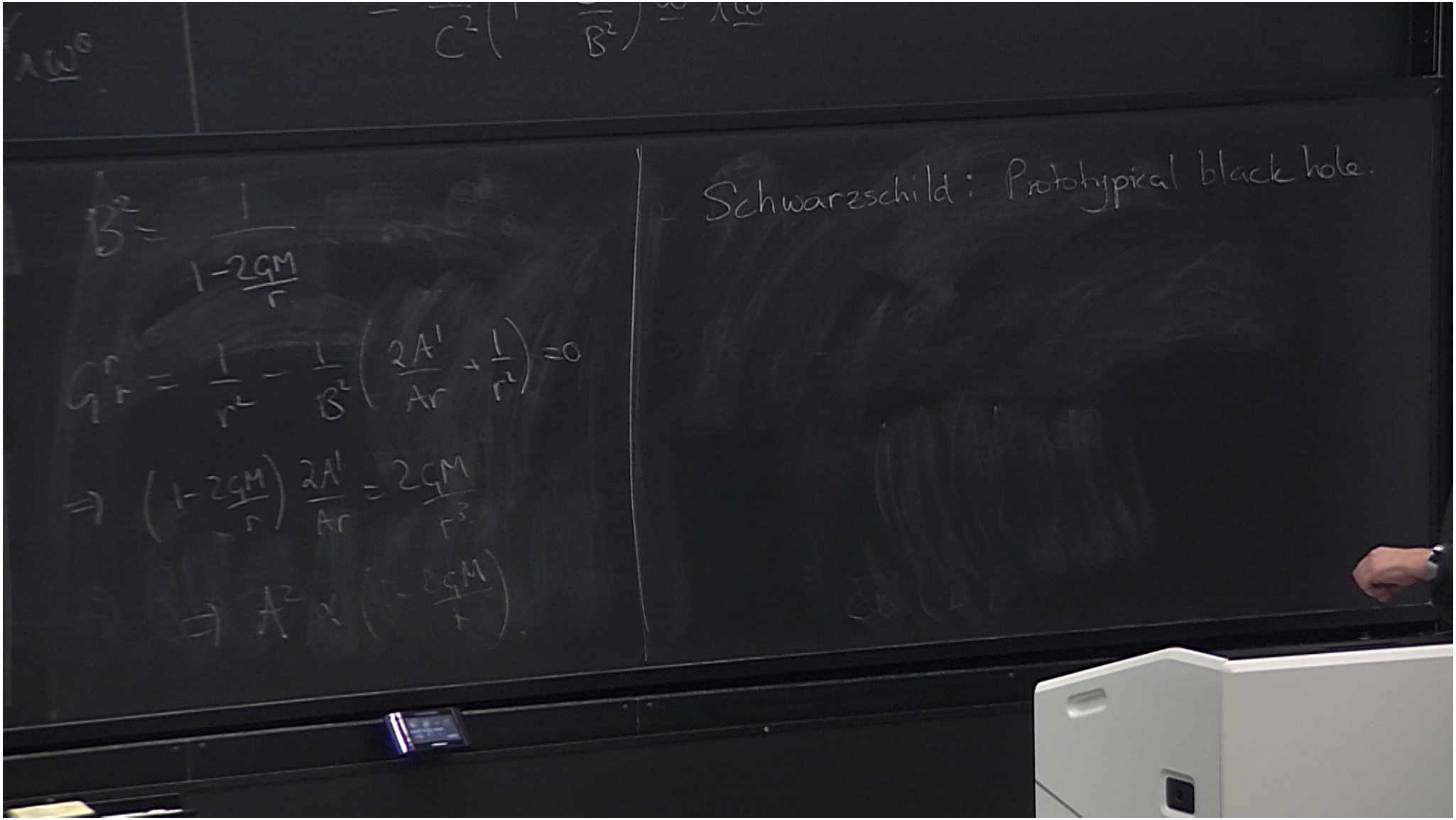
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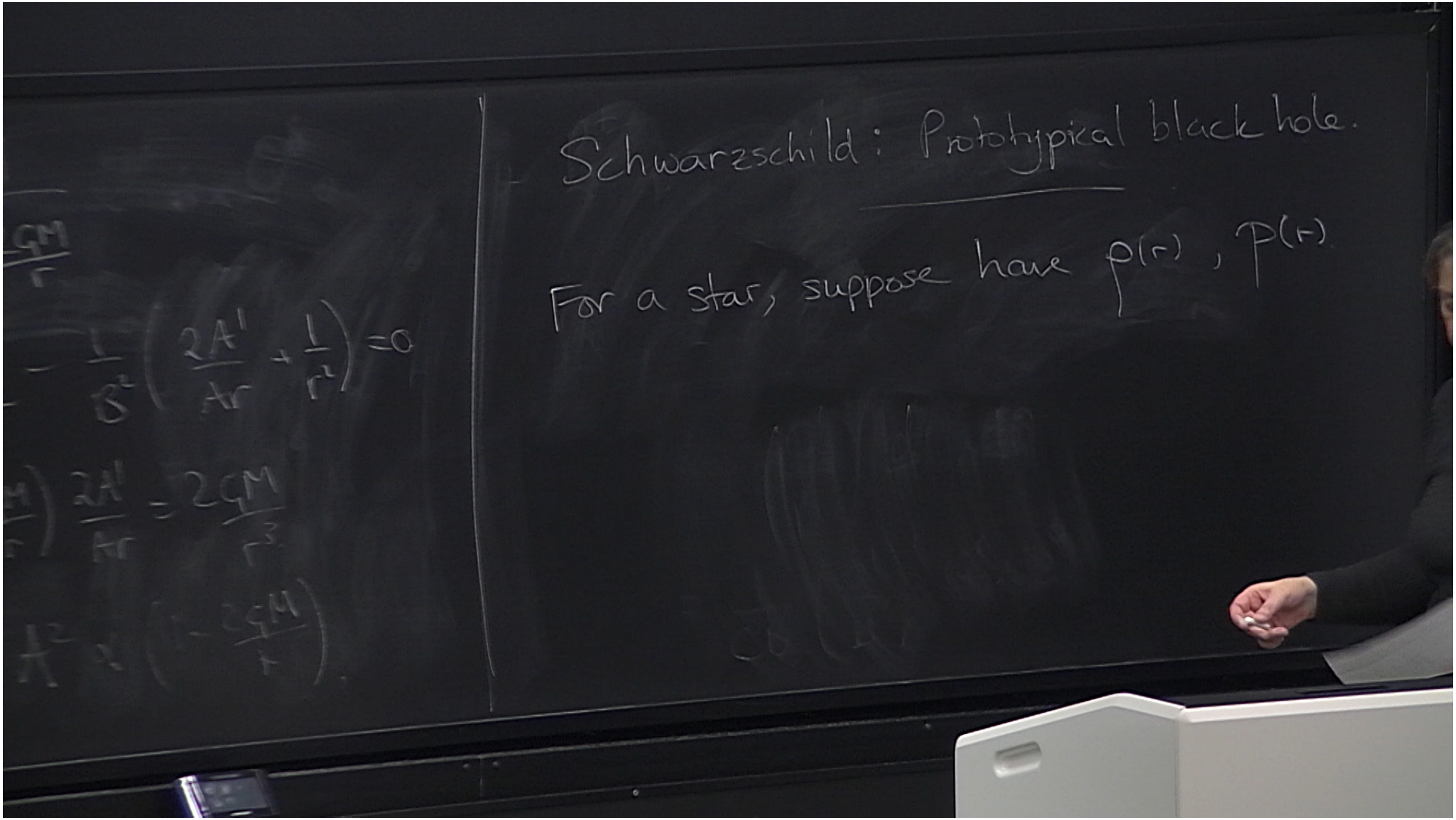
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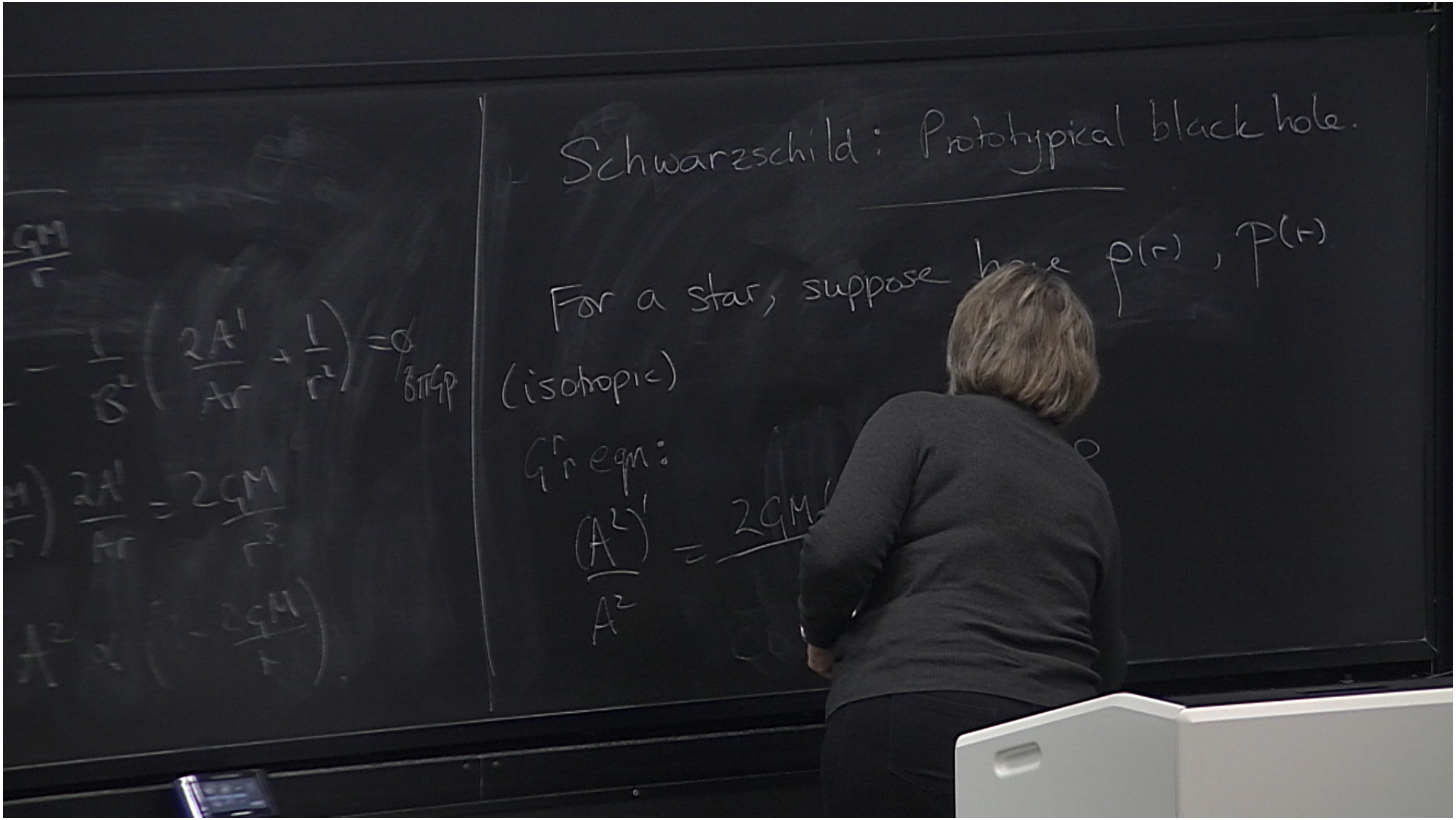
$$= 1 - \frac{2GM(r)}{r}$$

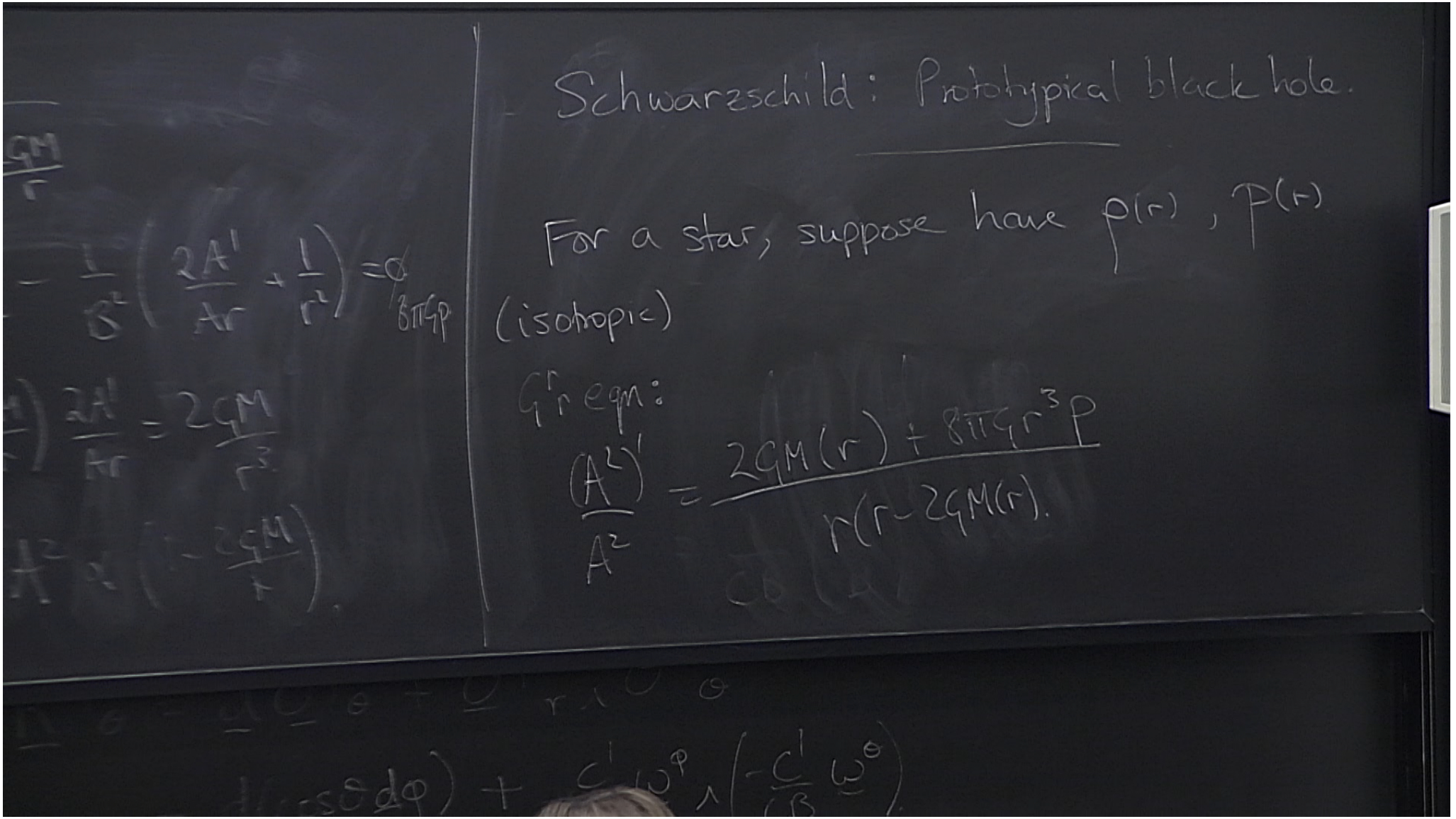

$\Rightarrow B^{-2} = 1 - \frac{2G}{r}\int 4\pi r^2 T_0 \, dr.$
$\int d^3x \, \rho$
$= 1 - \frac{2GM(r)}{r}$
If no matter, then M is an integration const.
$B^2 = \frac{1}{1-2}$

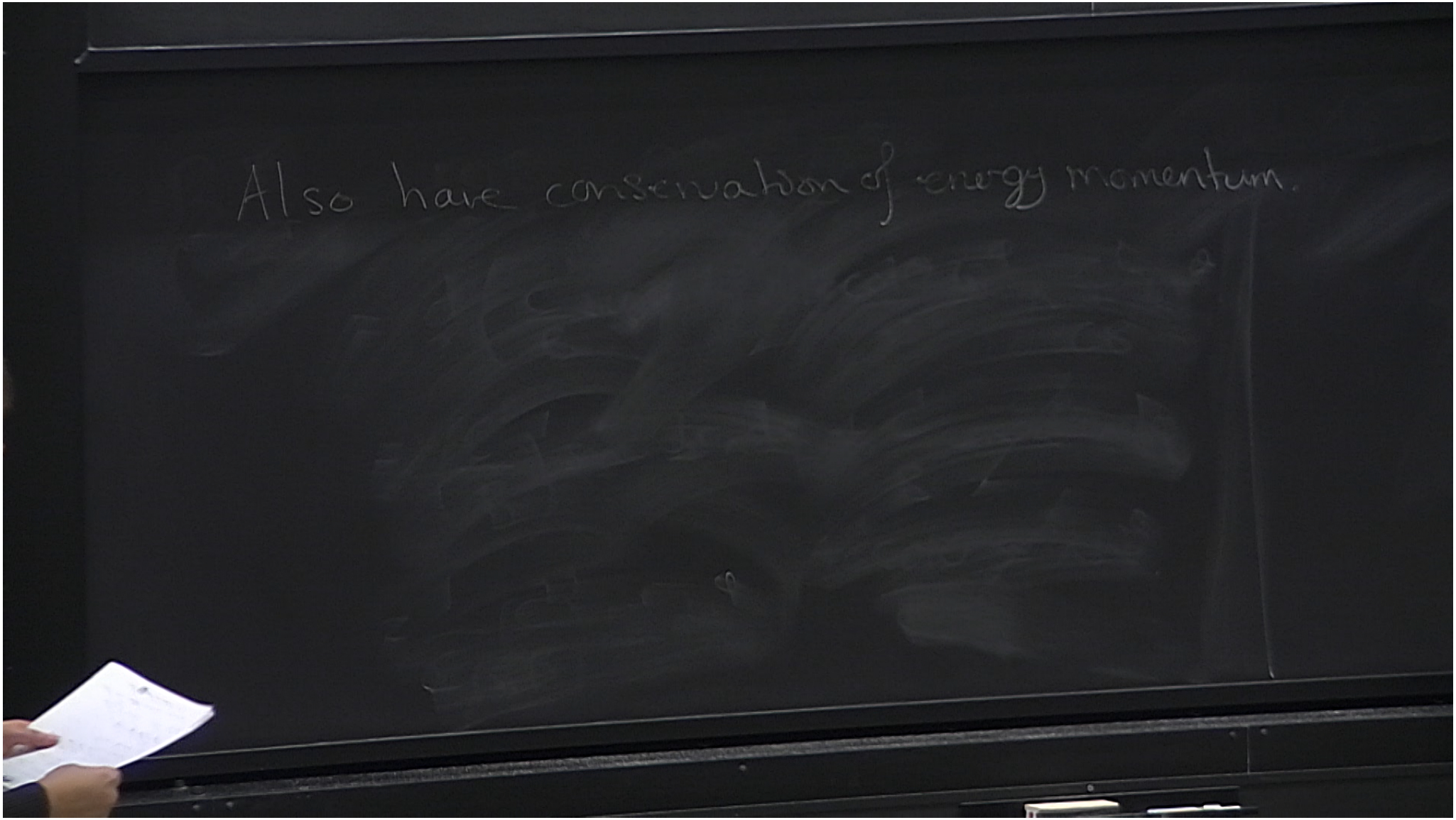

$B^2 = \frac{1}{1-\frac{2GM}{r}}$
$G^r_r = \frac{1}{r^2} - \frac{1}{B^2}\left(\frac{2A'}{Ar} + \frac{1}{r^2}\right) = 0$
$\Rightarrow \left(1-\frac{2GM}{r}\right)\frac{2A'}{Ar} = \frac{2GM}{r^3}$
$\Rightarrow A^2 \propto \left(1-\frac{2GM}{r}\right)$

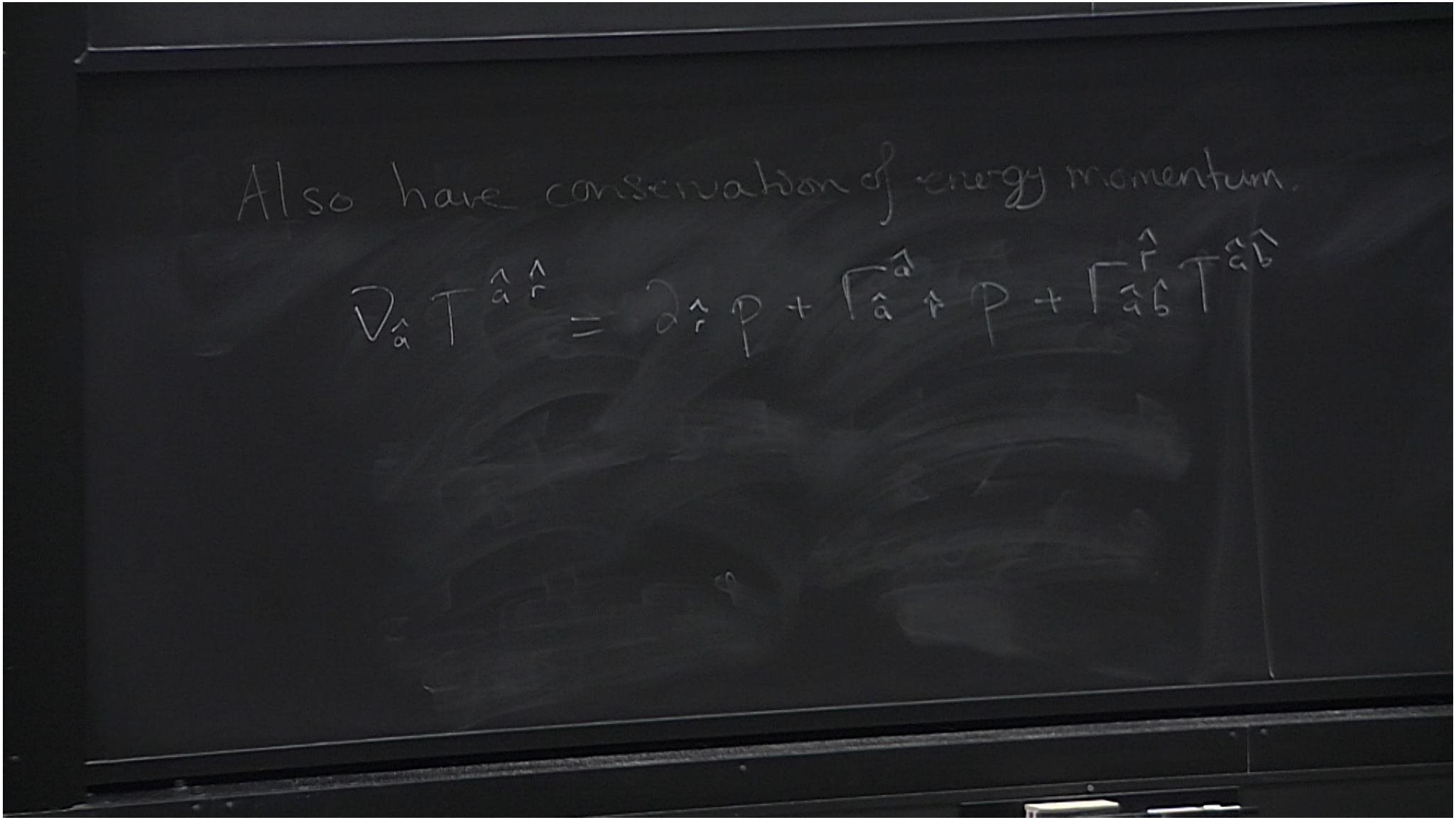

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Schwarzschild: Prototypical black hole.

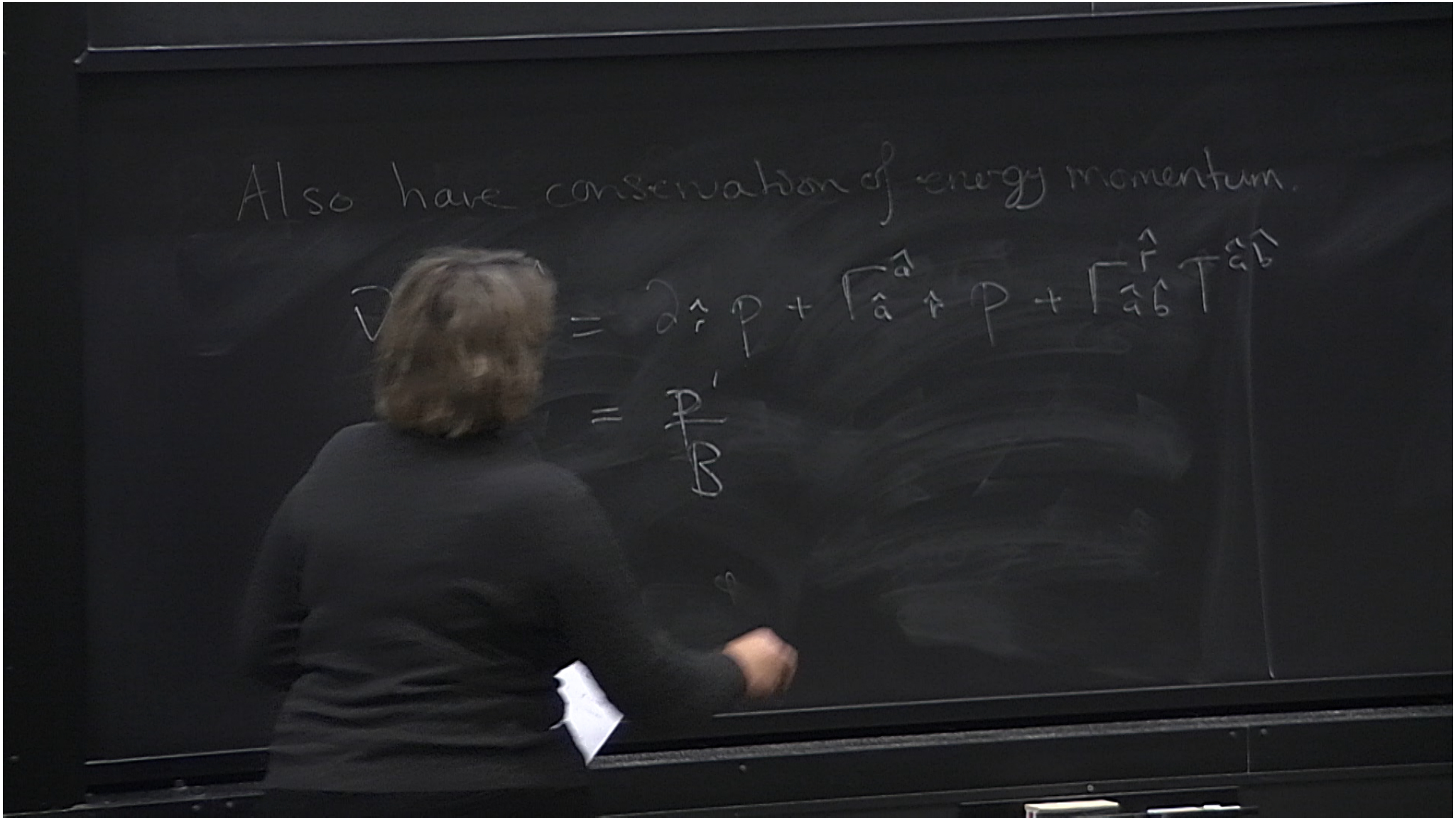

Schwarzschild: Prototypical black hole.
For a star, suppose have ρ(r), P(r)

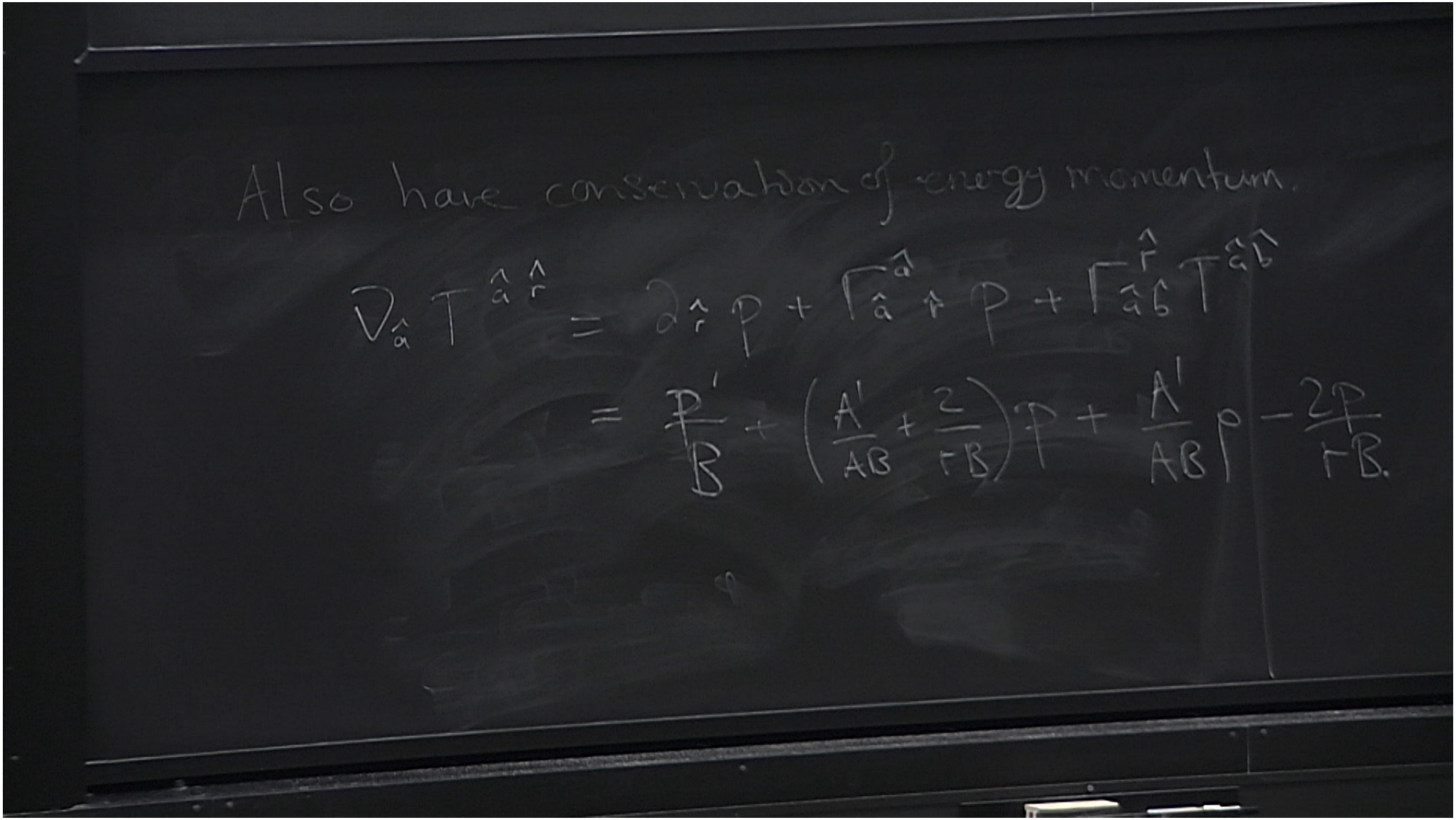

Schwarzschild: Prototypical black hole.
For a star, suppose
(isotropic)
G_rr eqn:

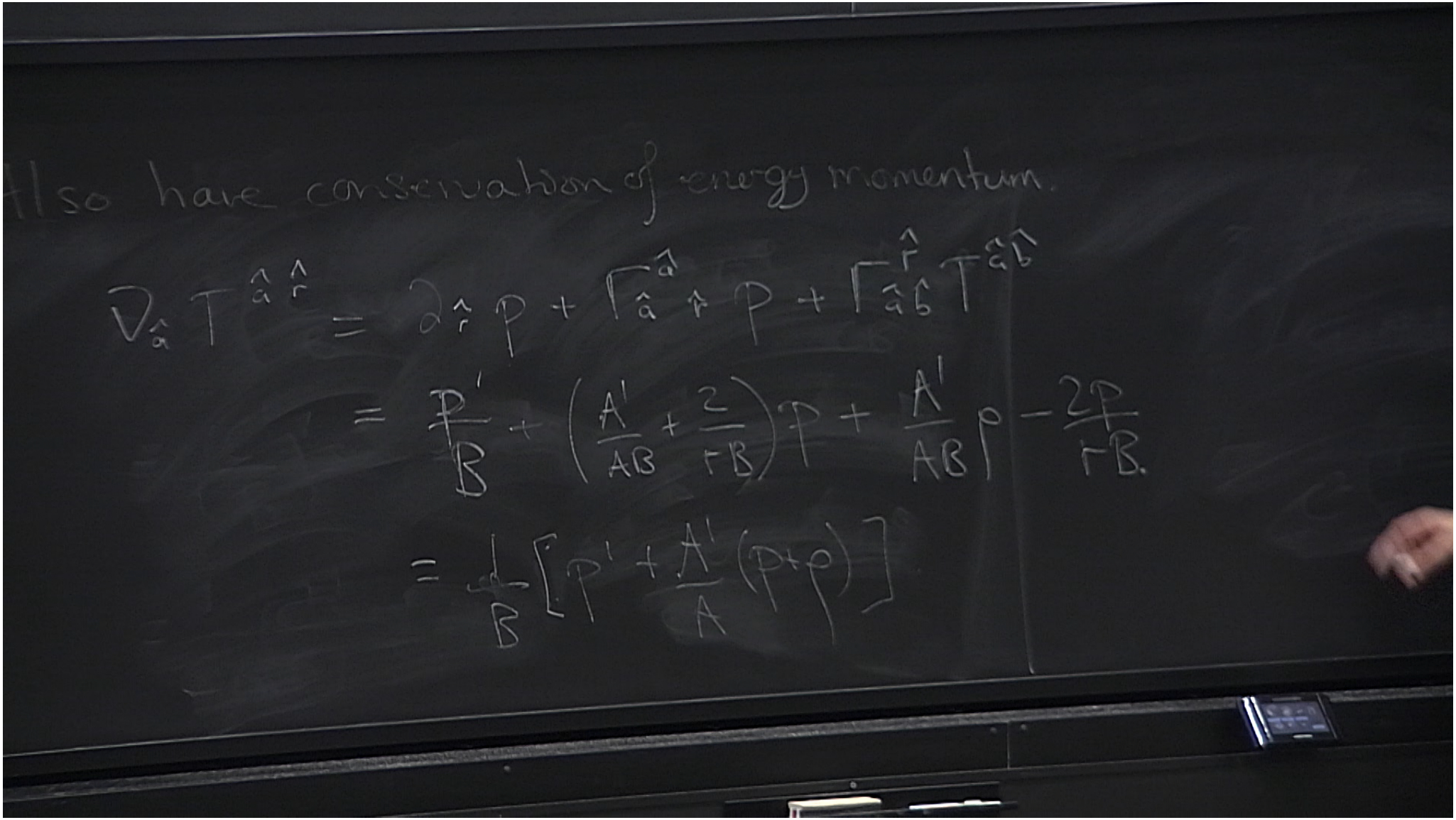

Schwarzschild: Prototypical black hole.
For a star, suppose have ρ(r), P(r)
(isotropic)
G^r_r eqn:
(A²)'/A² = (2GM(r) + 8πGr³P) / (r(r − 2GM(r)))

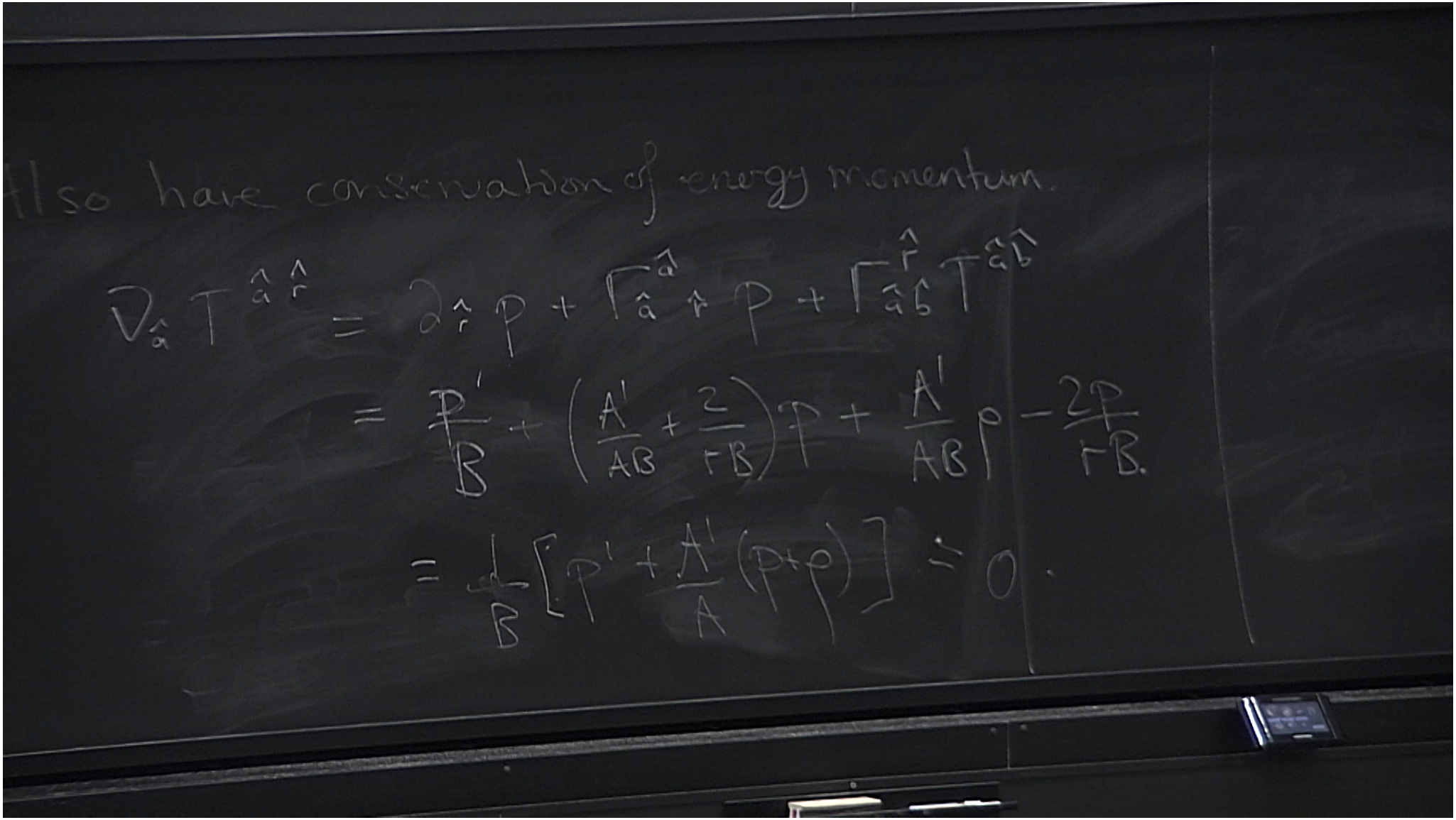

Also have conservation of energy momentum.

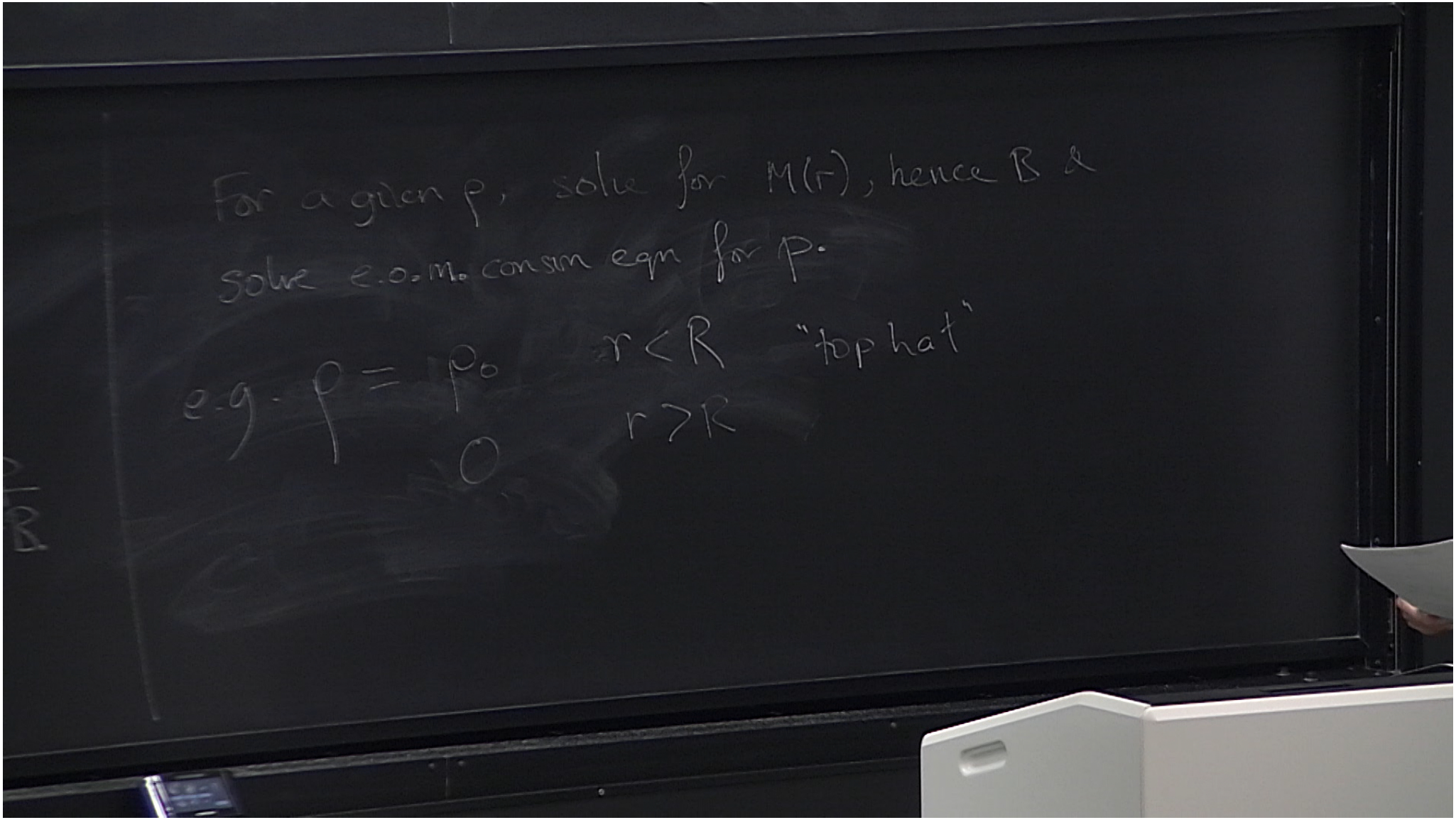

Also have conservation of energy momentum.
$\nabla_{\hat{a}} T^{\hat{a}\hat{r}} = \partial_{\hat{r}} p + \Gamma^{\hat{a}}_{\hat{a}\hat{r}} p + \Gamma^{\hat{r}}_{\hat{a}\hat{b}} T^{\hat{a}\hat{b}}$

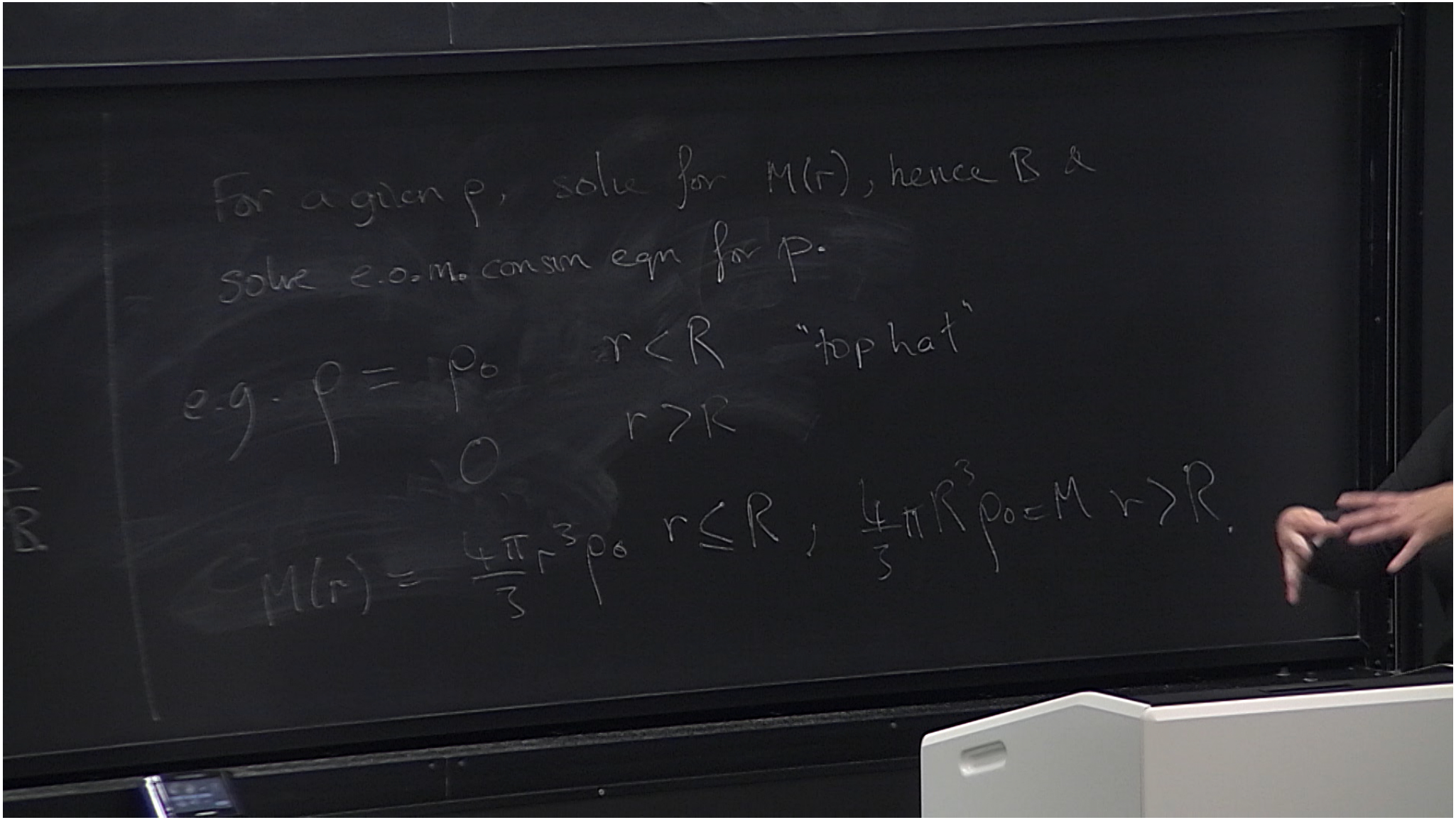

Also have conservation of energy momentum.


Also have conservation of energy momentum.
$\nabla_{\hat{a}} T^{\hat{a}\hat{r}} = \partial_{\hat{r}} \rho + \Gamma^{\hat{a}}_{\hat{a}\hat{r}} p + \Gamma^{\hat{r}}_{\hat{a}\hat{b}} T^{\hat{a}\hat{b}}$
$= \frac{p'}{B} + \left(\frac{A'}{AB} + \frac{2}{rB}\right)(p + \rho) - \frac{2p}{rB}.$


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$\nabla_{\hat{a}} T^{\hat{a}\hat{r}} = \partial_{\hat{r}} \rho + \Gamma^{\hat{a}}_{\hat{a}\hat{r}} p + \Gamma^{\hat{r}}_{\hat{a}\hat{b}} T^{\hat{a}\hat{b}}$
$= \frac{p'}{B} + \left(\frac{A'}{AB} + \frac{2}{rB}\right) p + \frac{A'}{AB} \rho - \frac{2p}{rB}$
$= \frac{1}{B}\left[p' + \frac{A'}{A}(p+\rho)\right]$


Also have conservation of energy momentum.
$\nabla_{\hat{a}} T^{\hat{a}\hat{r}} = \partial_{\hat{r}} \rho + \Gamma^{\hat{a}}_{\hat{a}\hat{r}} (\rho + P) + \Gamma^{\hat{r}}_{\hat{a}\hat{b}} T^{\hat{a}\hat{b}}$
$= \frac{P'}{B} + \left(\frac{A'}{AB} + \frac{2}{rB}\right)(P + \rho) + \frac{A'}{AB}\rho - \frac{2P}{rB}$
$= \frac{1}{B}\left[P' + \frac{A'}{A}(P+\rho)\right] = 0.$


For a given ρ, solve for M(r), hence B &
solve E.O.M. consm eqn for ρ.
e.g. ρ = ρ0 r<R "top hat"
0 r>R


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solve e.o.m. consrn eqn for ρ.
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0 r>R
M(r) = 4π/3 r^3 ρ0 r≤R ; 4/3 π R^3 ρ0 = M r>R.

Inside star:

$$p' = -(p+\rho_0)\frac{[GM(r)+4\pi G p r^3]}{r(r-2GM(r))}$$

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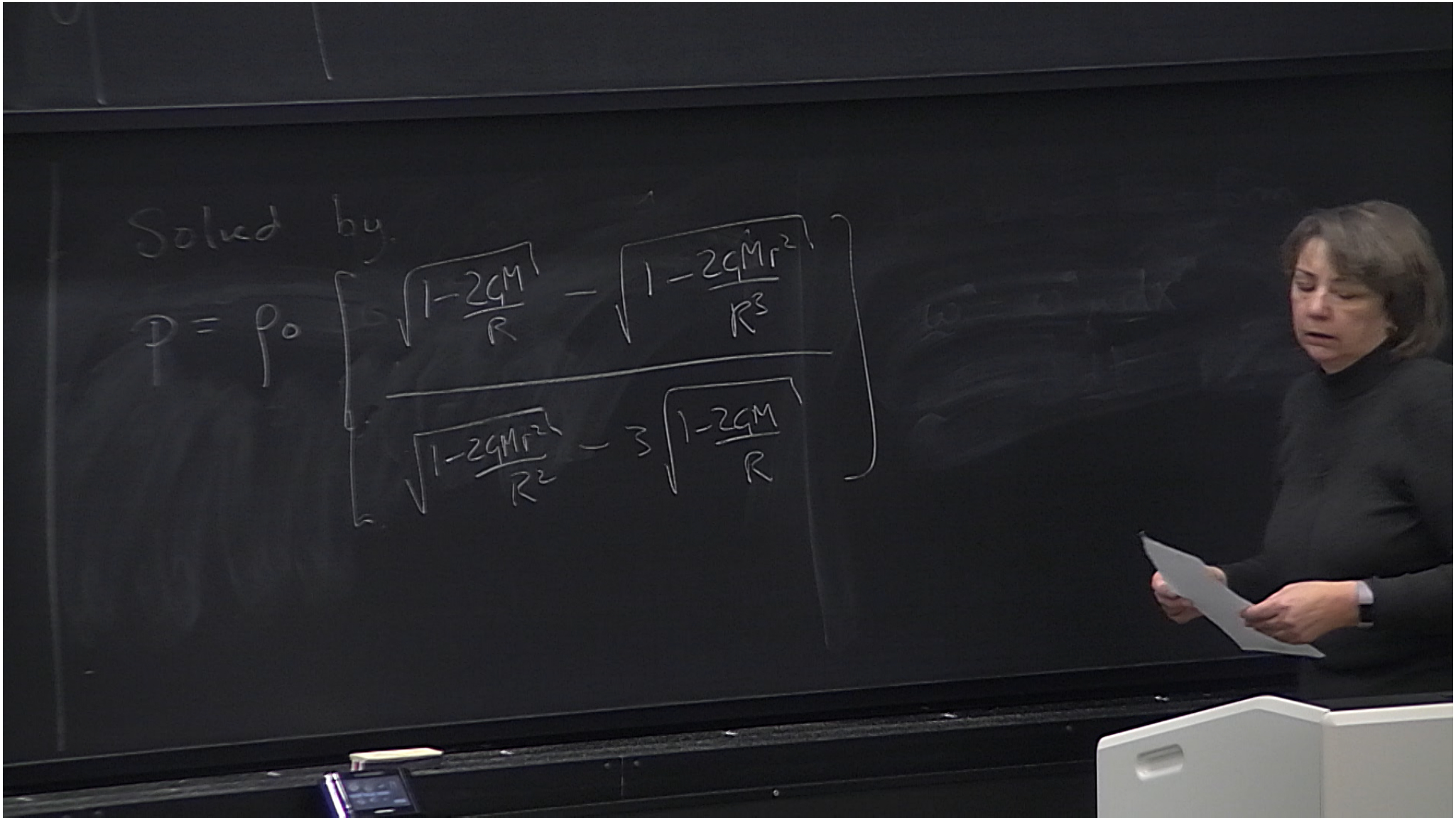
$$= -(p+\rho_0)\cdot\frac{4\pi G r(3p+\rho_0)}{(3-8\pi G r^2\rho_0)}$$

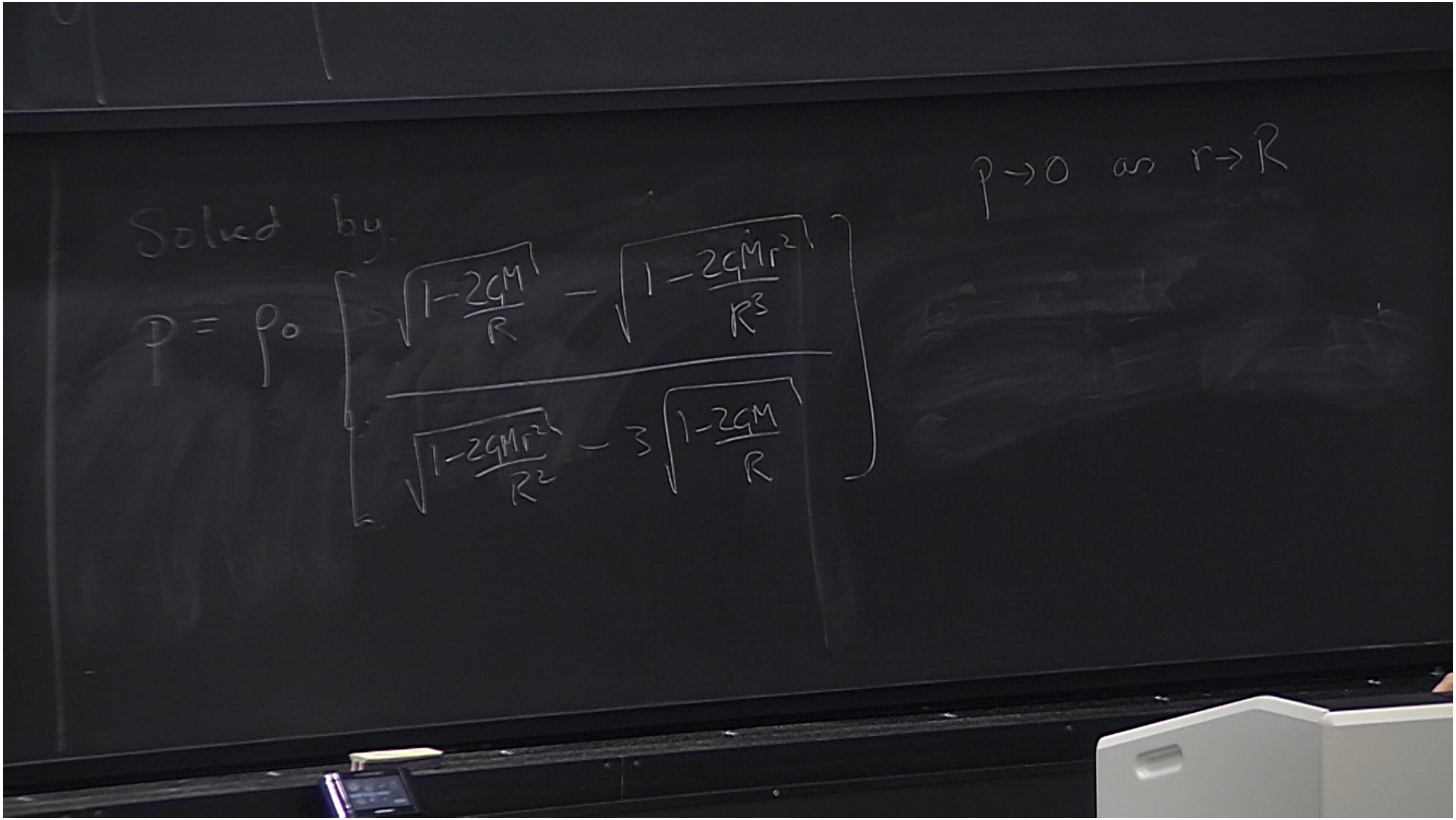
Inside star:

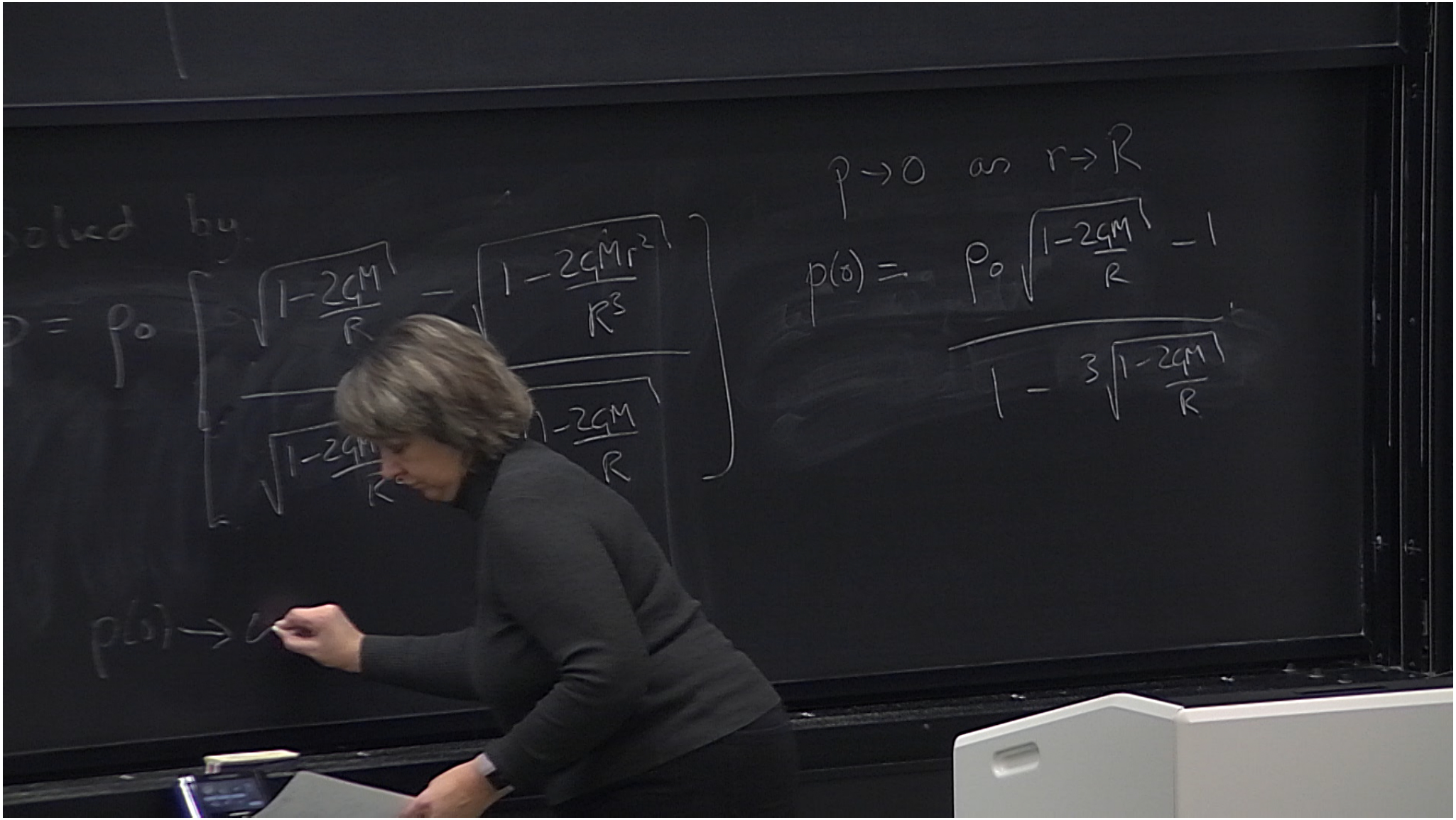
$$p' = -(p+\rho_0)\frac{[GM(r)+4\pi G p r^3]}{r(r-2GM(r))}$$

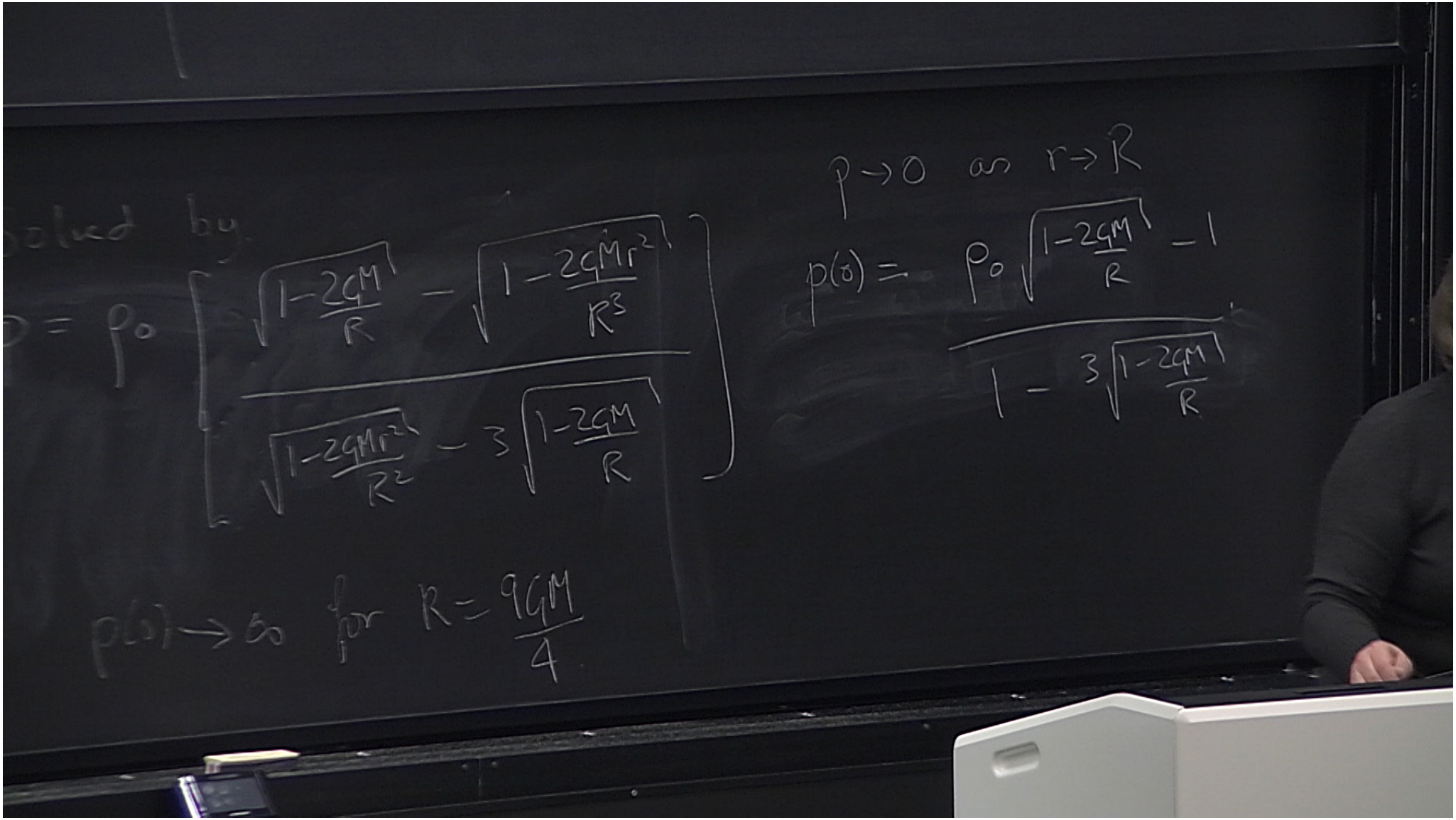
$$= -(p+\rho_0)\frac{4\pi G r(3p+\rho_0)}{(3-8\pi G r^2\rho_0)}$$

TOLMAN-OPPENHEIMER-VOLKOFF


Solved by.
$p = \rho_0 \left[\frac{\sqrt{1-\frac{2GM}{R}} - \sqrt{1-\frac{2GMr^2}{R^3}}}{\sqrt{1-\frac{2GMr^2}{R^2}} - 3\sqrt{1-\frac{2GM}{R}}} \right]$


Solved by.
$p = p_0 \left[\frac{\sqrt{1-\frac{2GM}{R}} - \sqrt{1-\frac{2GMr^2}{R^3}}}{\sqrt{1-\frac{2GMr^2}{R^2}} - 3\sqrt{1-\frac{2GM}{R}}} \right]$
$p \to 0$ as $r \to R$


p → 0 as r → R
p(0) = p₀ (1 − 3√(1 − 2GM/R))


$p \to 0$ as $r \to R$
$p(0) = p_0 \dfrac{\sqrt{1-\frac{2GM}{R}} - 1}{1 - 3\sqrt{1-\frac{2GM}{R}}}$
$p = p_0 \left[\dfrac{\sqrt{1-\frac{2GM}{R}} - \sqrt{1-\frac{2GMr^2}{R^3}}}{\sqrt{1-\frac{2GMr^2}{R^2}} - 3\sqrt{1-\frac{2GM}{R}}} \right]$
$p(0) \to \infty$ for $R = \frac{9GM}{4}$