Title: PSI 17/18 - Gravitational Physics - Lecture 1

Date: Jan 03, 2018 10:15 AM

URL: http://pirsa.org/18010021

Abstract:

Exterior derivative (antisymmetric differentiation)

Note - we define a derivative on functions,

$$f \to \frac{\partial f}{\partial x^\mu} dx^\mu = \underline{d}f$$

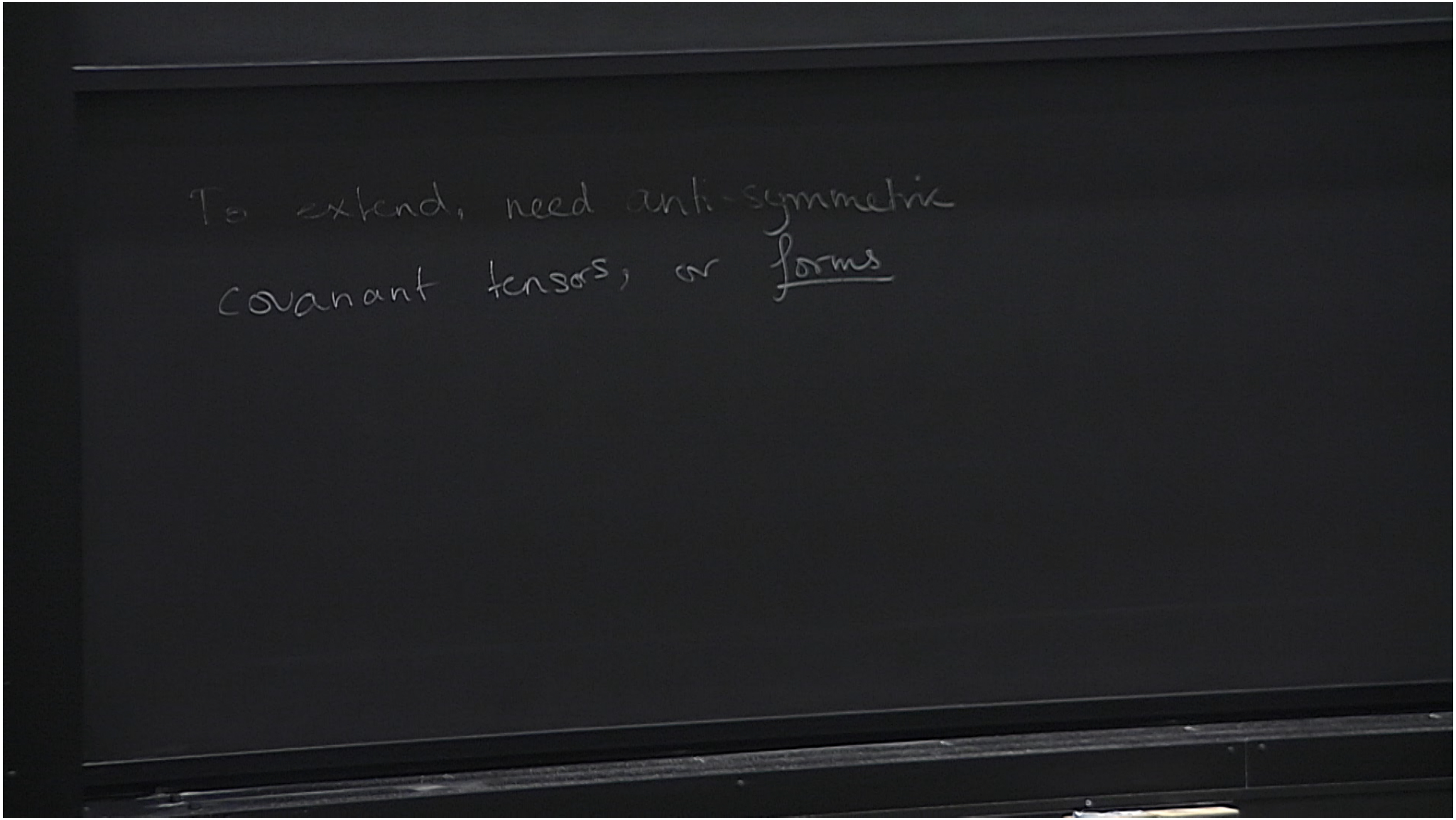
gradient of f.



covector.

$\langle \underline{d}f | T \rangle = \underline{T}f$

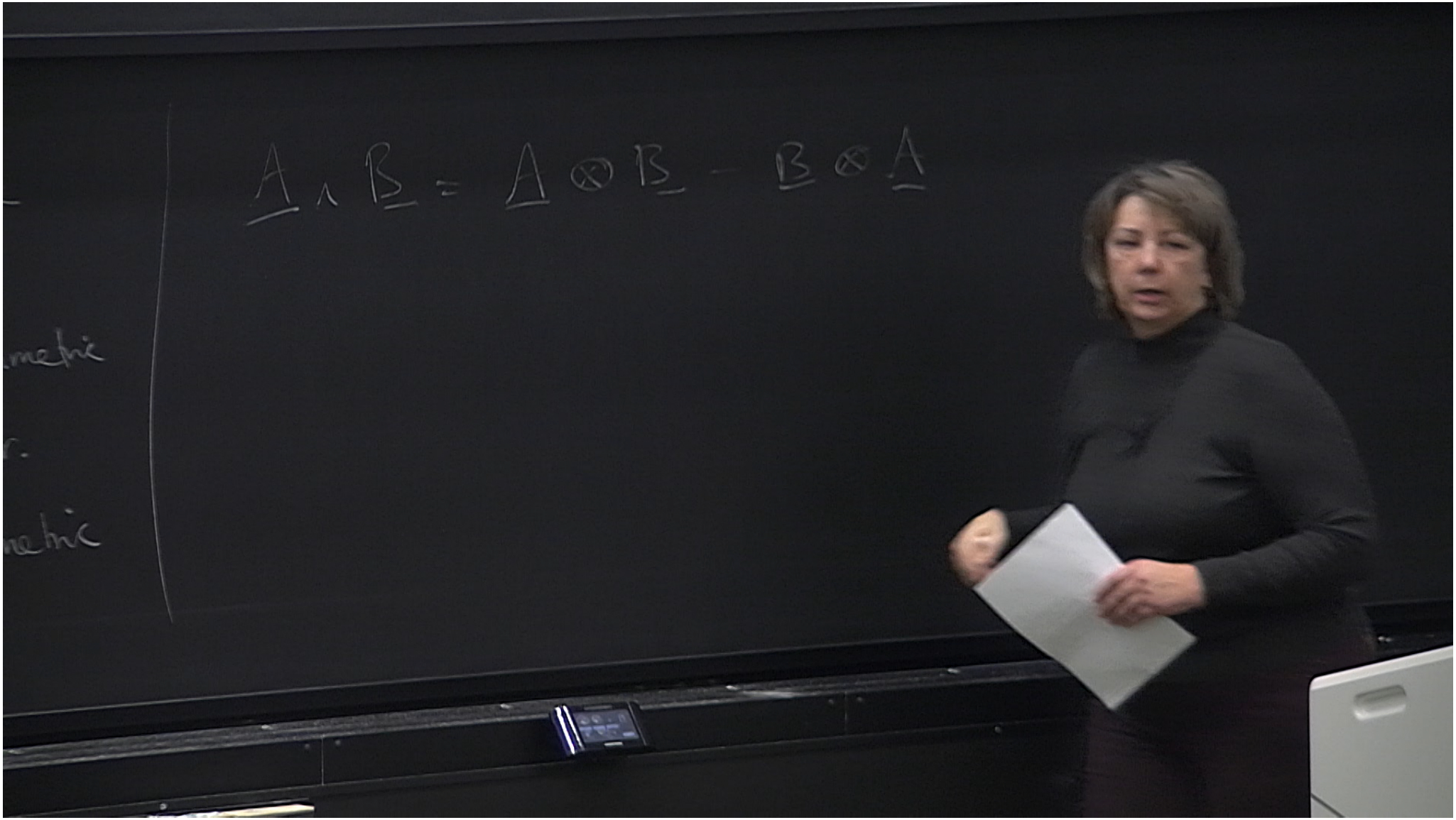
for all $\underline{T}$ at P.

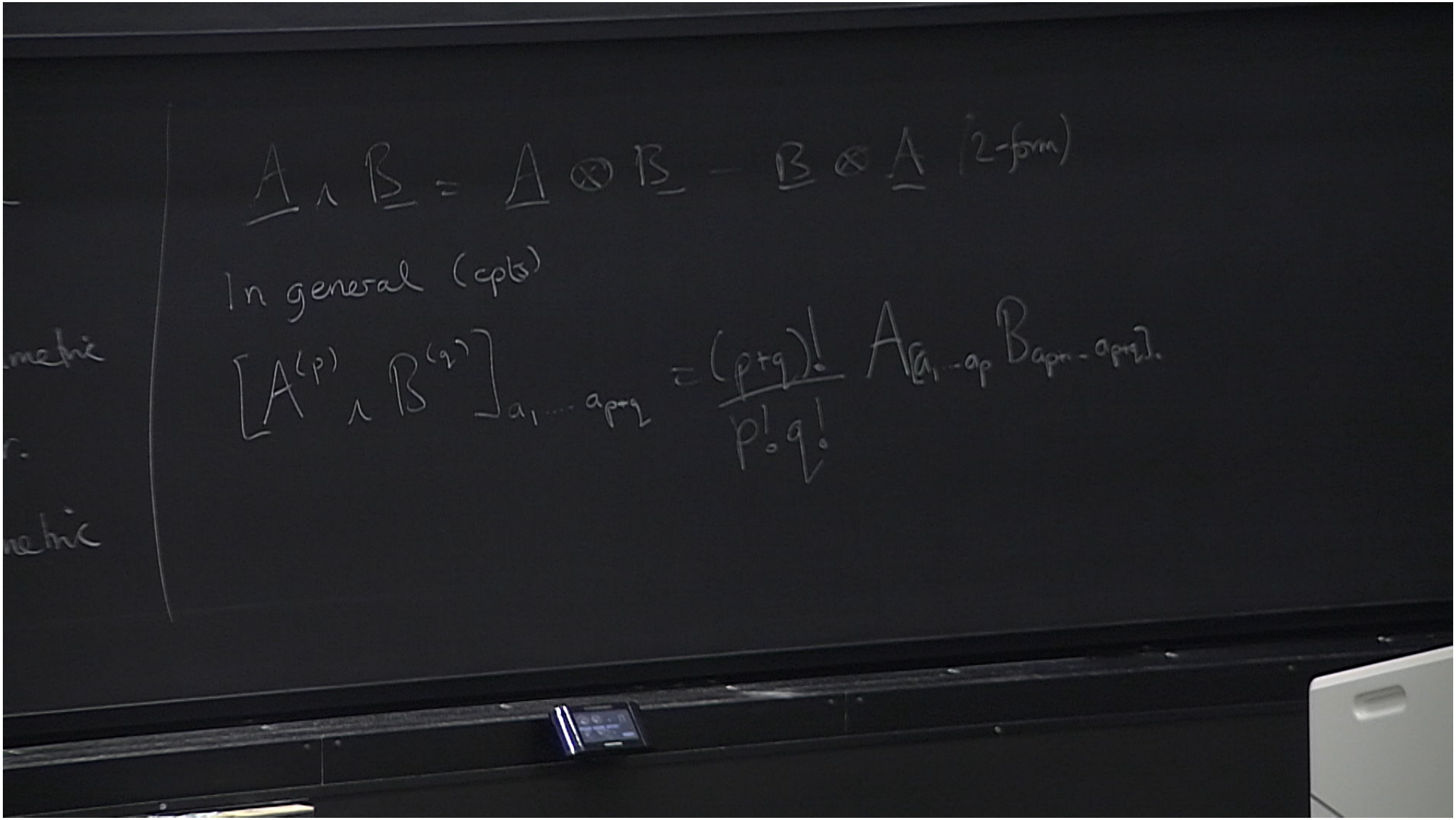

To extend, need anti-symmetric
covariant tensors; or forms

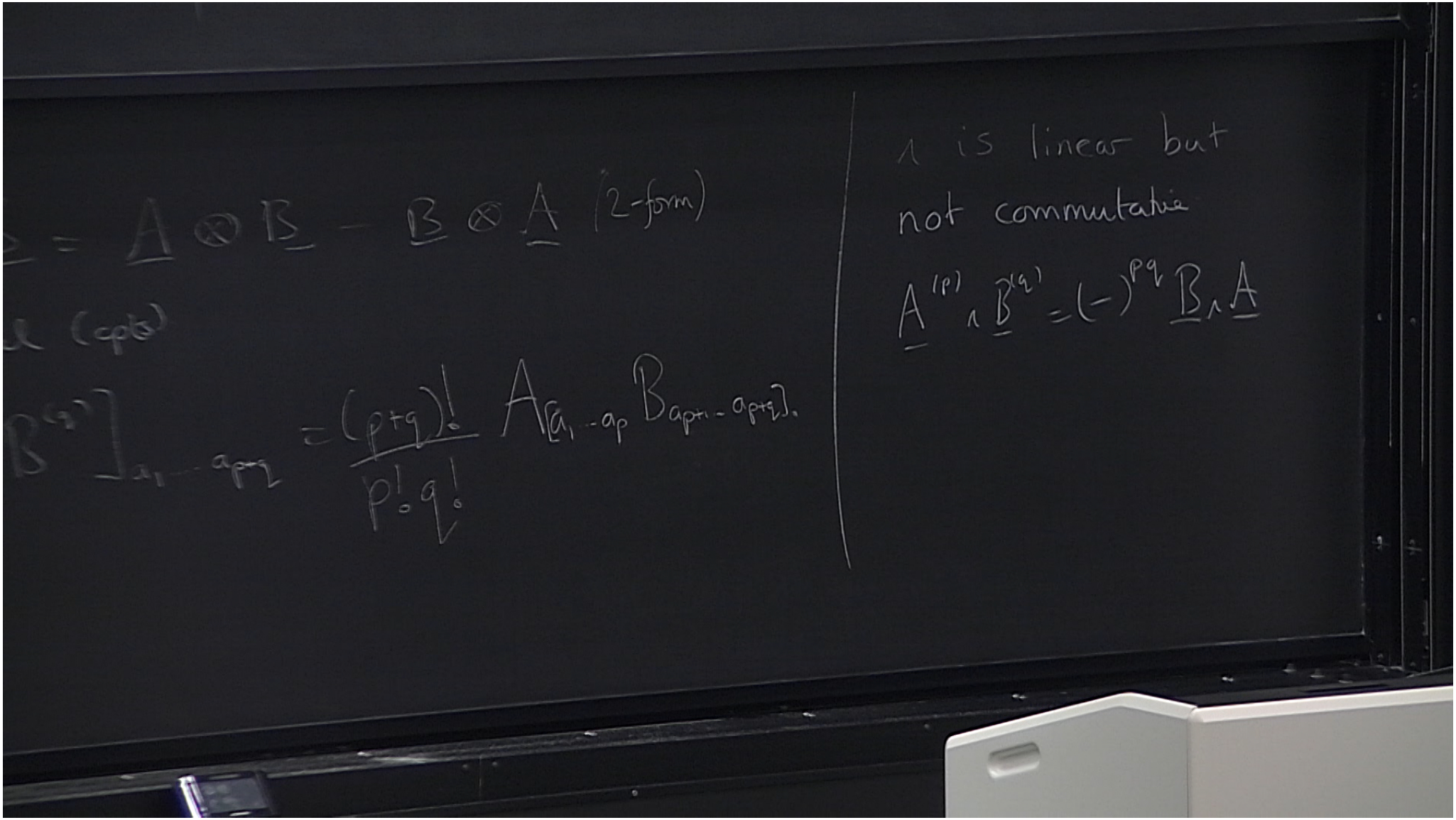
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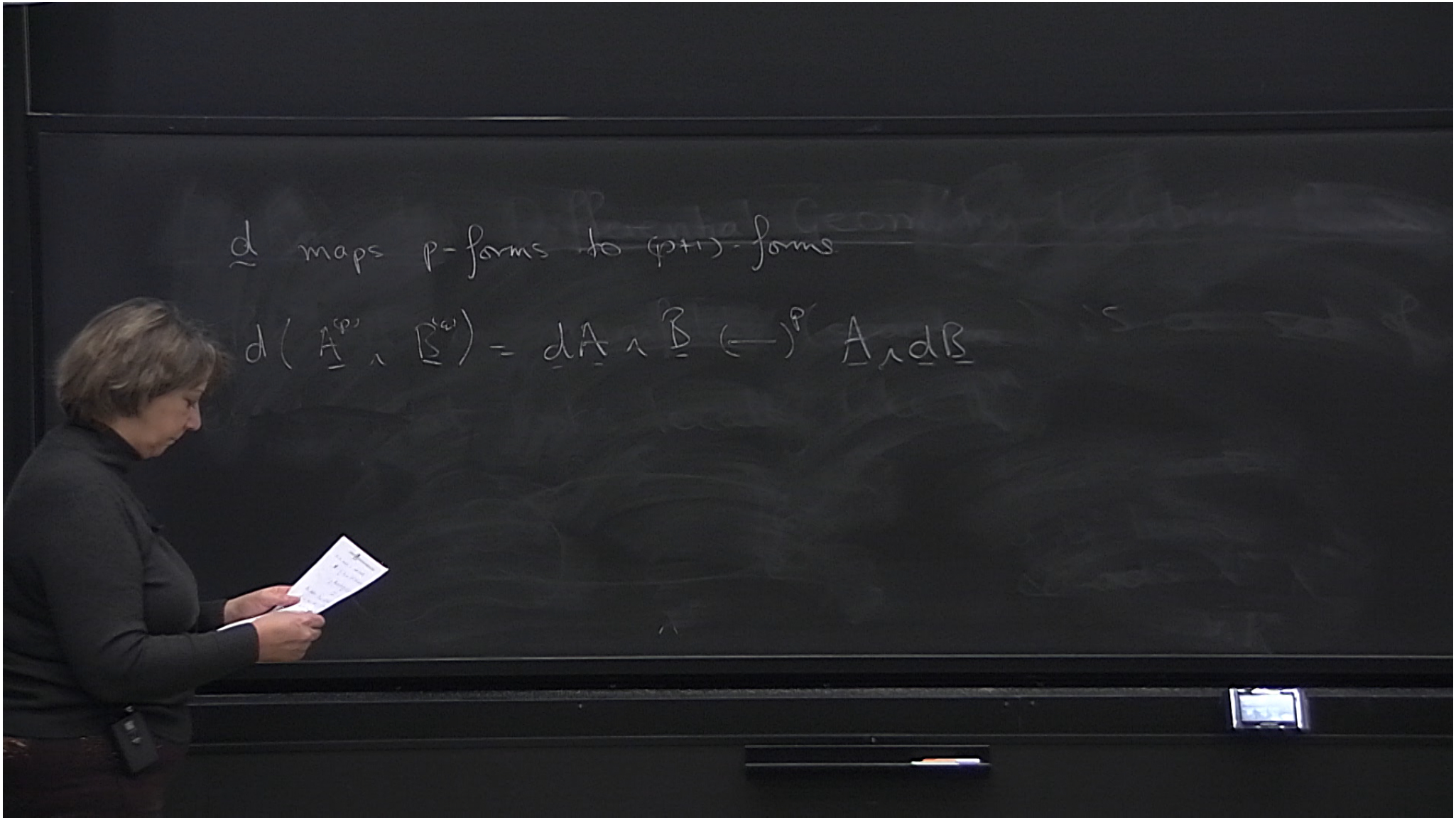
Defn a p-form is an antisymmetric rank p covariant tensor.

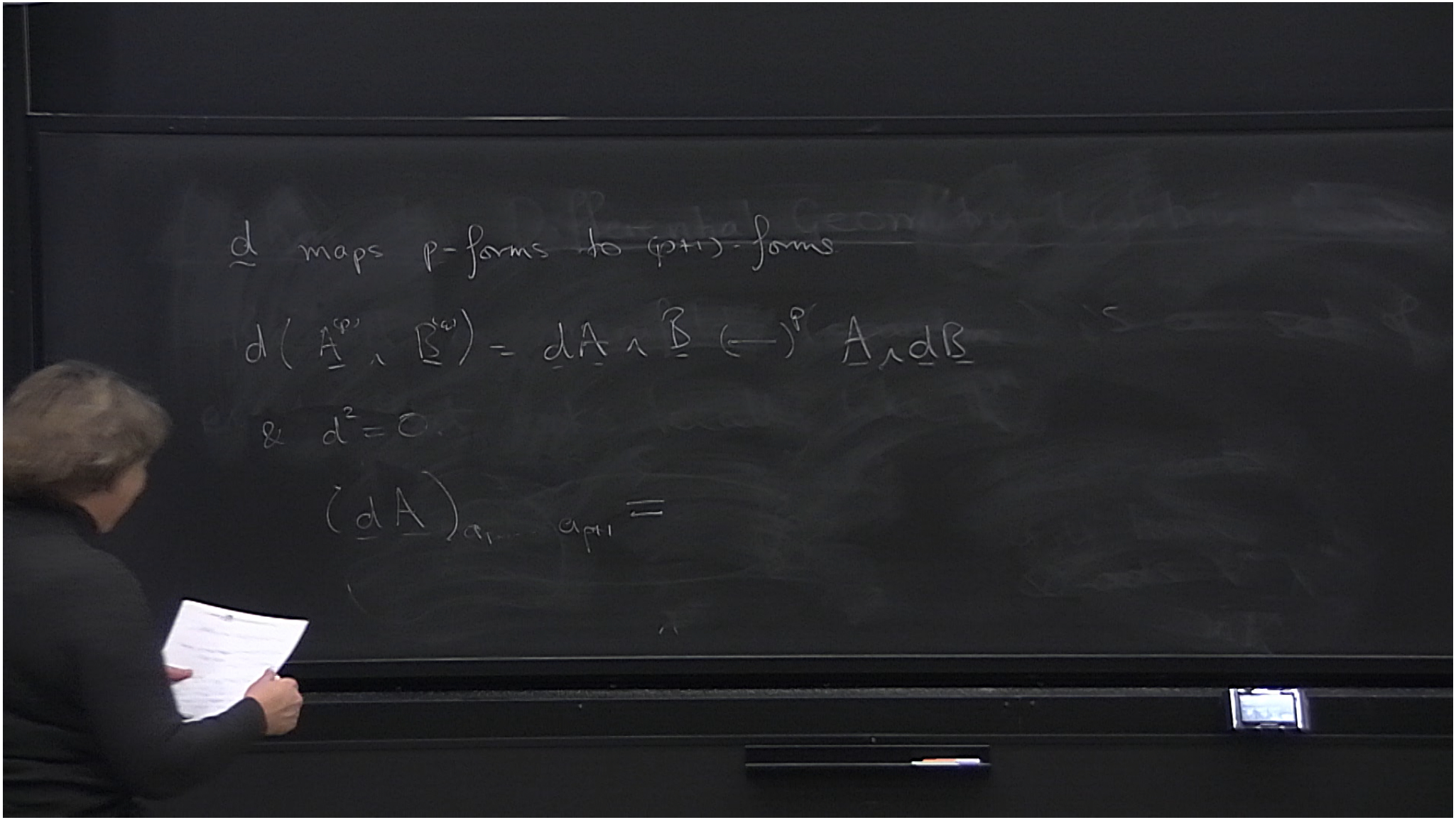
Construct using antisymmetric tensor product $\wedge$.


A ∧ B = A ⊗ B − B ⊗ A


$\underline{A} \wedge \underline{B} = \underline{A} \otimes \underline{B} - \underline{B} \otimes \underline{A}$ (2-form)
In general (cpts)
$[A^{(p)} \wedge B^{(q)}]_{a_1 \cdots a_{p+q}} = \frac{(p+q)!}{p!\,q!} A_{[a_1 \cdots a_p} B_{a_{p+1} \cdots a_{p+q}]}$.


$= \underline{A} \otimes \underline{B} - \underline{B} \otimes \underline{A}$ (2-form)
(ap⊗)
$B^{(q)}]_{a_1 \dots a_{p+q}} = \frac{(p+q)!}{p!\,q!} A_{[a_1 \dots a_p} B_{a_{p+1} \dots a_{p+q}]}$.
∧ is linear but
not commutative
$\underline{A}^{(p)} \wedge \underline{B}^{(q)} = (-)^{pq} \underline{B} \wedge \underline{A}$


$\underline{d}$ maps p-forms to (p+1)-forms
$d(\underline{A}^{(p)} \wedge \underline{B}^{(q)}) = \underline{dA} \wedge \underline{B} \; (-)^p \; \underline{A} \wedge \underline{dB}$

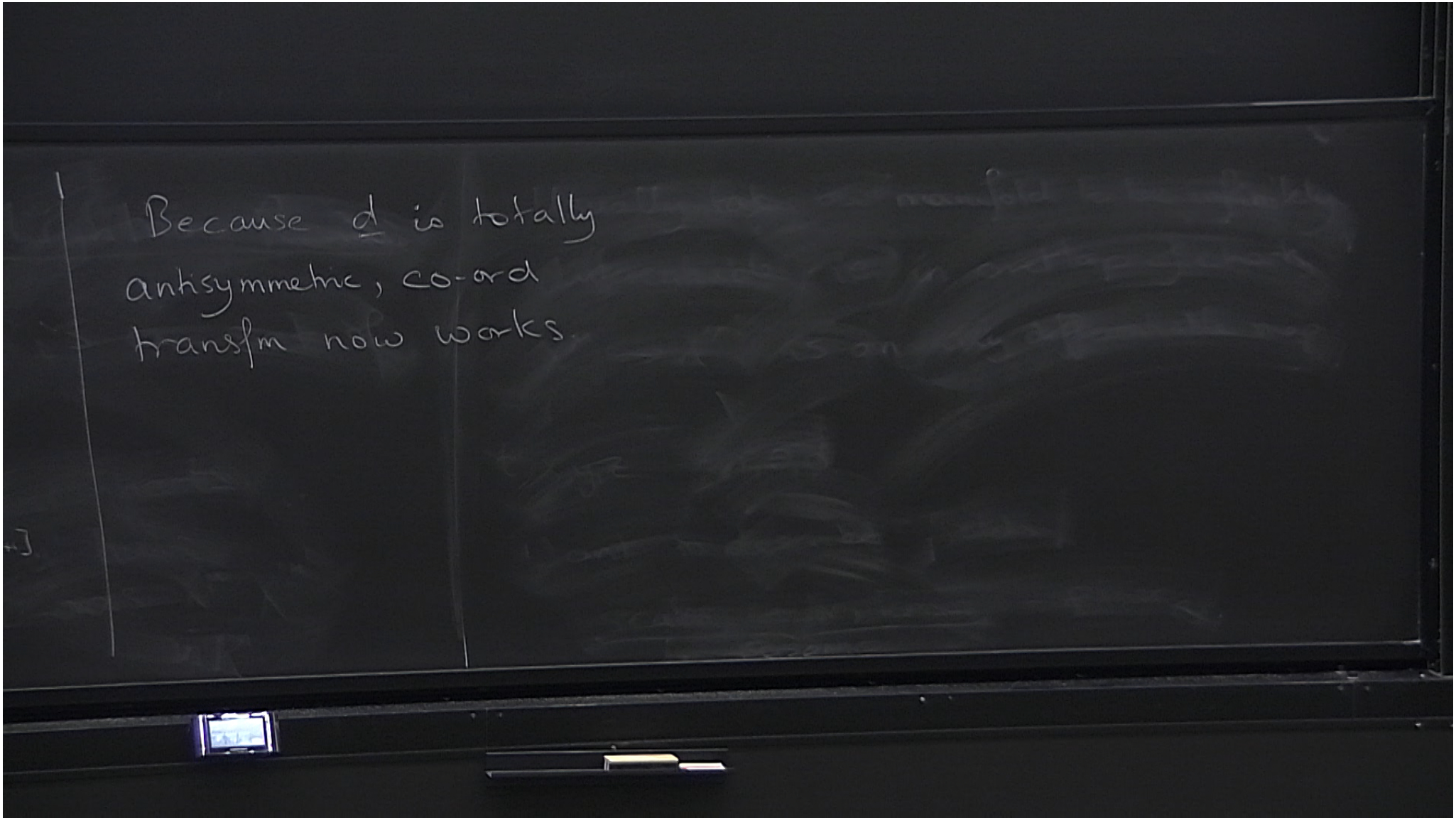

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$d(\underline{A}^{(p)} \wedge \underline{B}^{(q)}) = \underline{dA} \wedge \underline{B} (-)^p \underline{A} \wedge \underline{dB}$
& $d^2 = 0$
$(\underline{dA})_{a_1 \ldots a_{p+1}} =$

$\underline{d}$ maps p-forms to (p+1)-forms

$$d(\underline{A}^{(p)} \wedge \underline{B}^{(q)}) = d\underline{A} \wedge \underline{B} + (-)^p \underline{A} \wedge d\underline{B}$$

& $d^2 = 0$

$$(\underline{d}\underline{A})_{a_1 \ldots a_{p+1}} = \frac{(p+1)!}{p!} \partial_{[a_1} A_{a_2 \ldots a_{p+1}]}$$


Because d is totally
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Notice that we can only have up to an n-form.

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The n-form is a multiple of the alternating symbol

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The n-form is a multiple of the alternating symbol

ϵ_{abcd} = +1 if abcd even perm of 0123

−1 if odd perm

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$\epsilon_{abcd} = +1$ if abcd even perm of 0123

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$$\underline{\epsilon} = \frac{1}{n!}\epsilon_{\mu\nu\rho} dx^\mu \wedge dx^\nu \wedge dx^\lambda \wedge dx^\rho$$

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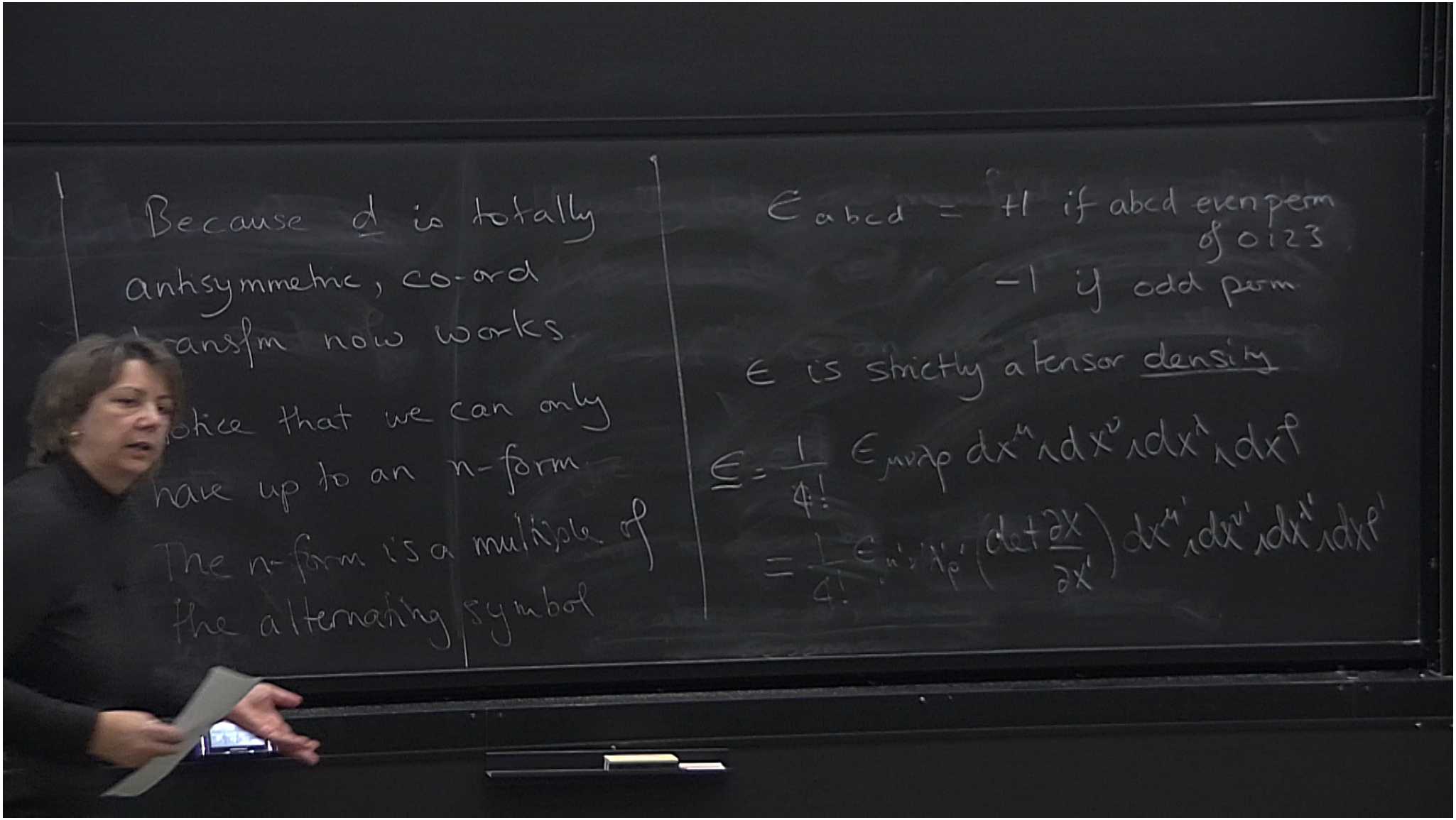
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$= \frac{1}{4!}\epsilon_{\mu'\nu'\lambda'\rho'}\left(\det \frac{\partial x}{\partial x'}\right) dx^{\mu'} \wedge dx^{\nu'} \wedge dx^{\lambda'} \wedge dx^{\rho'}$

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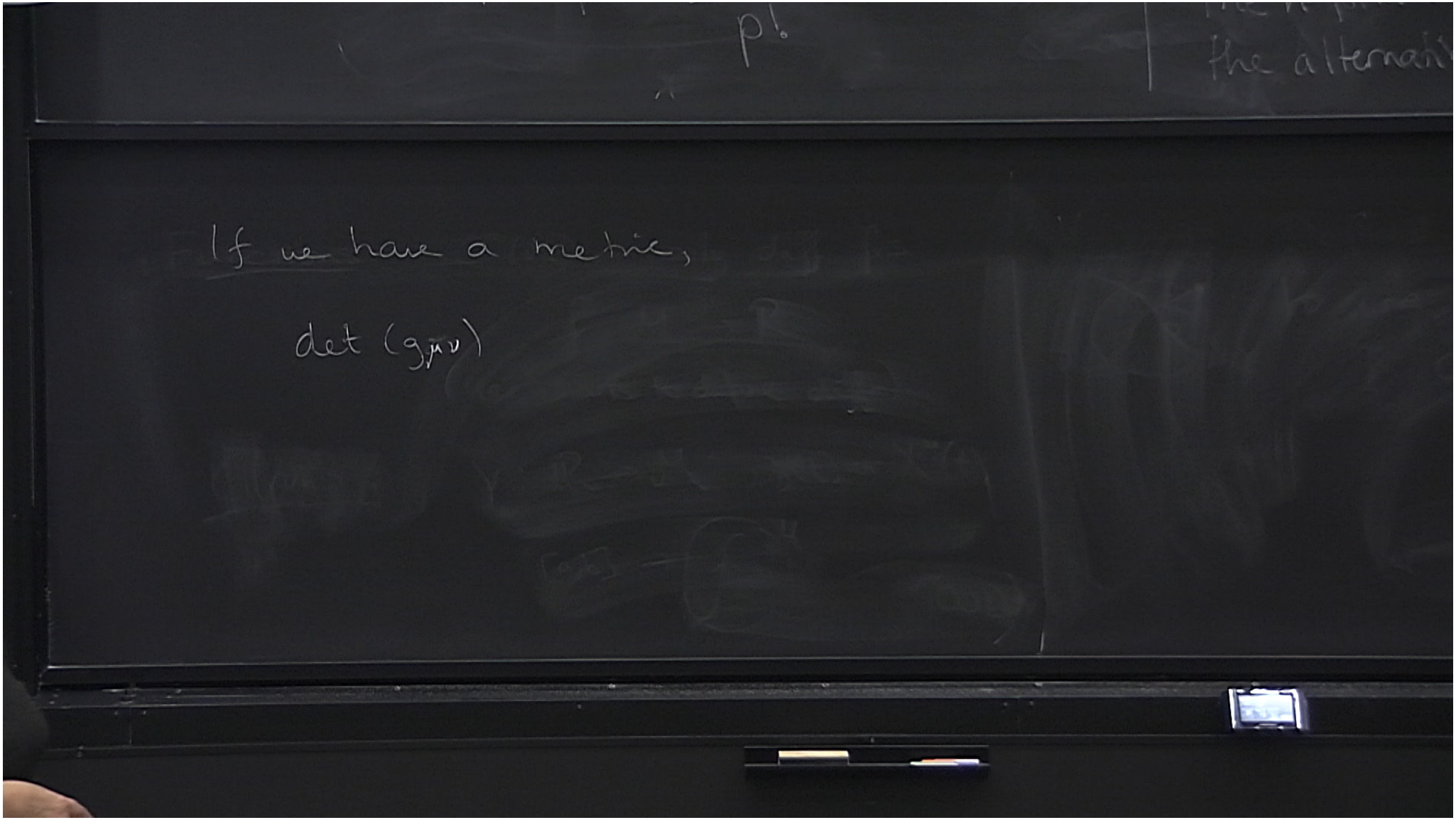
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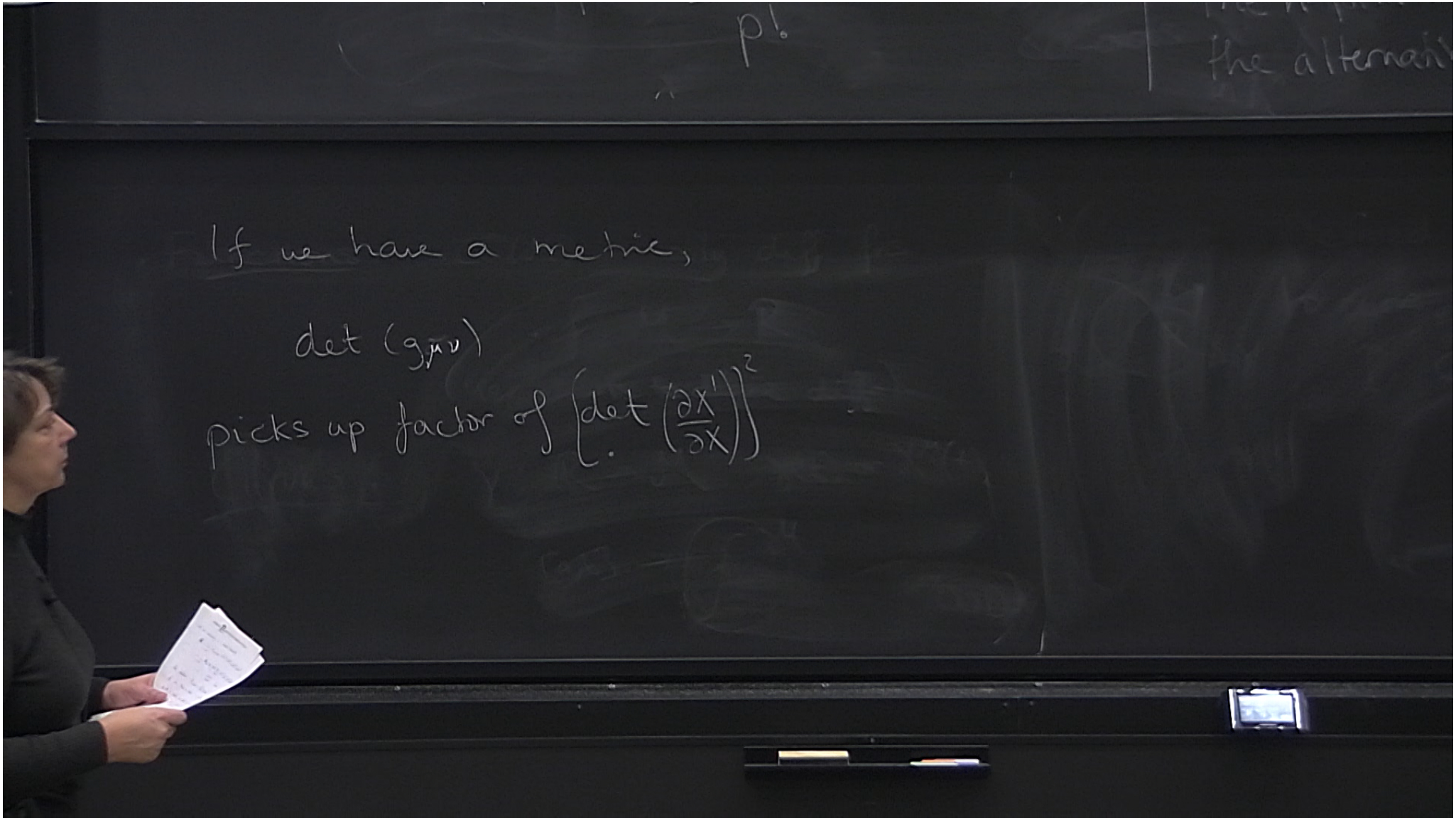
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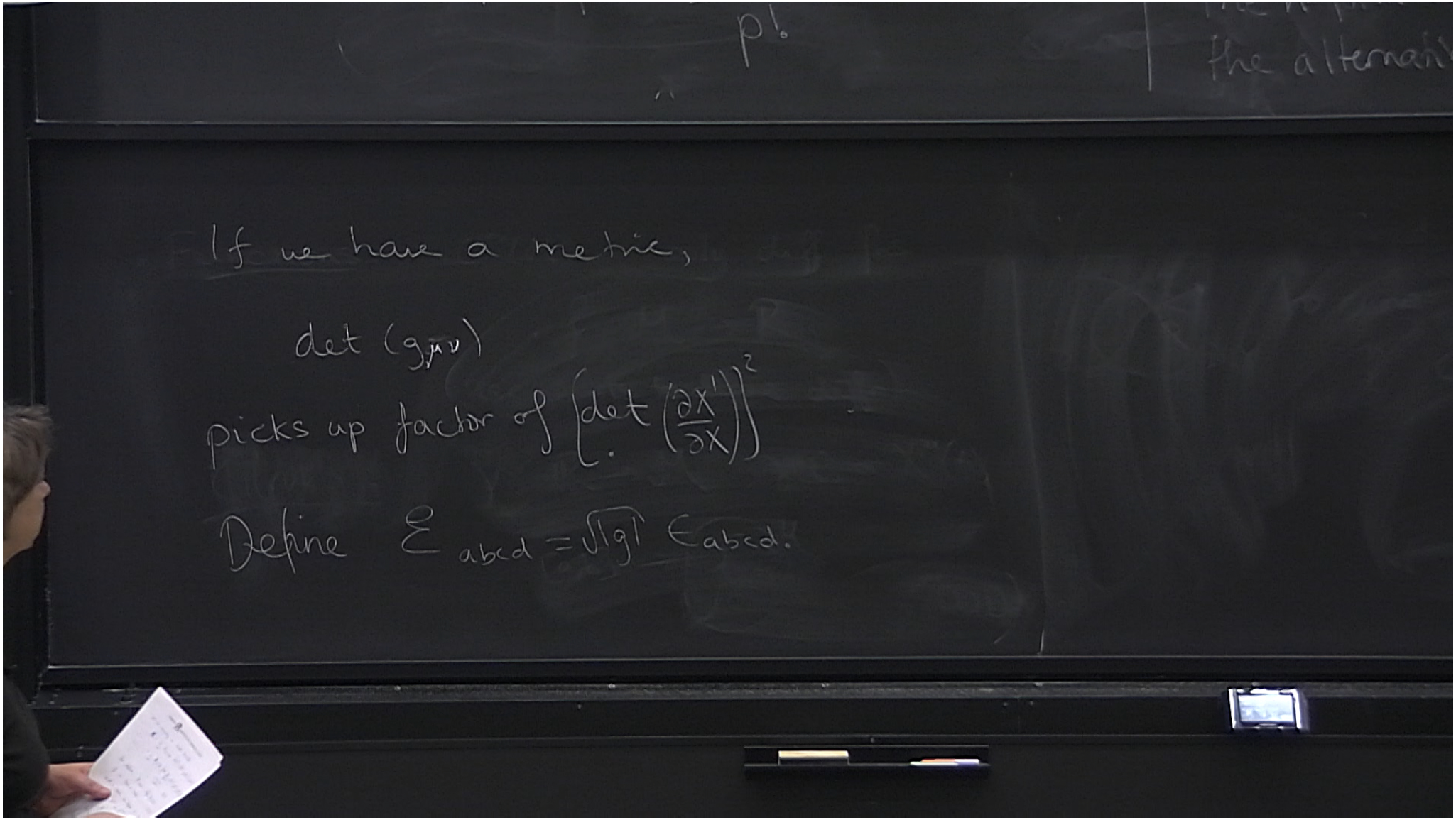
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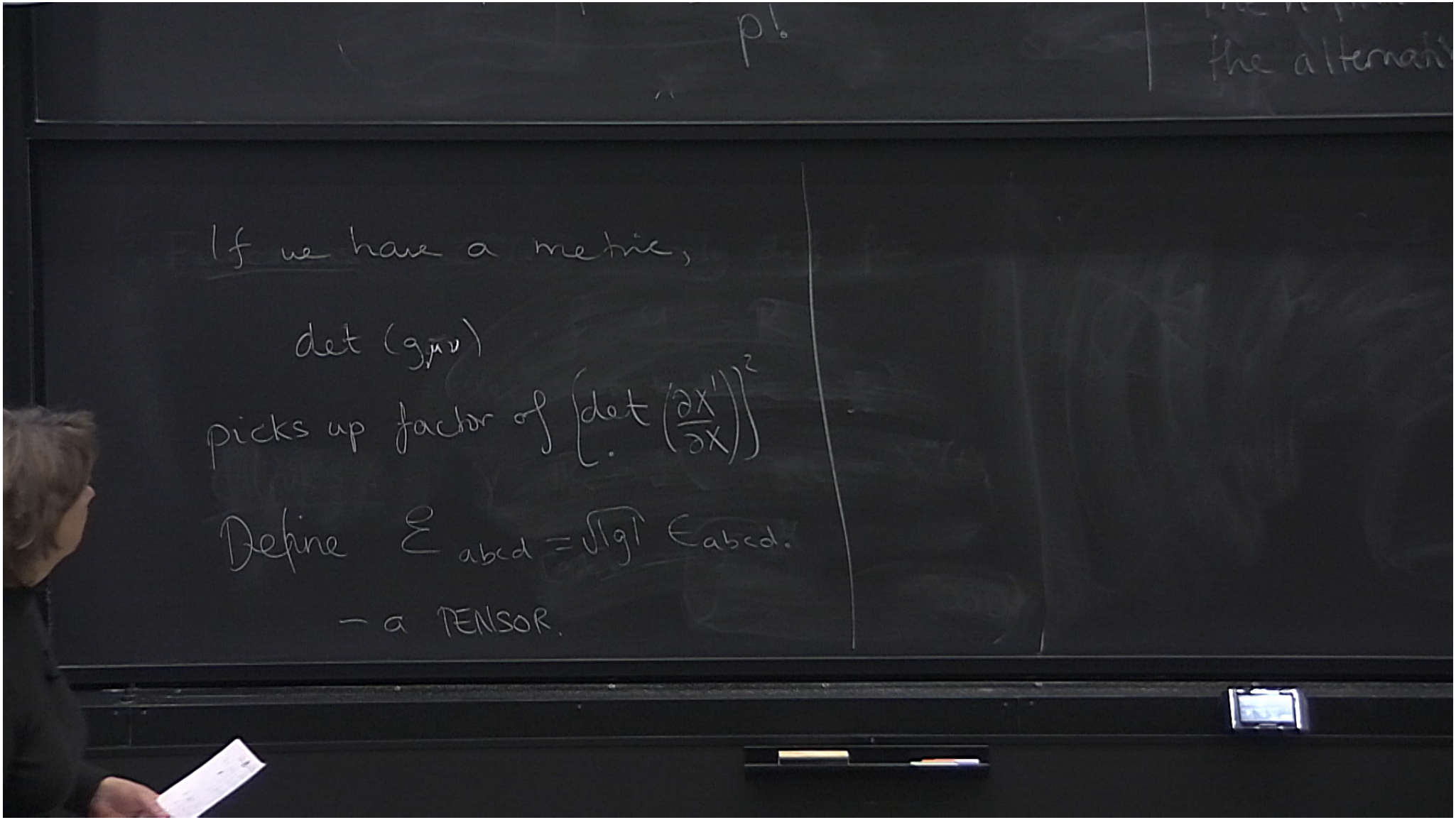
$$\underline{\epsilon} = \frac{1}{4!}\epsilon_{\mu\nu\lambda\rho}\, dx^\mu \wedge dx^\nu \wedge dx^\lambda \wedge dx^\rho$$

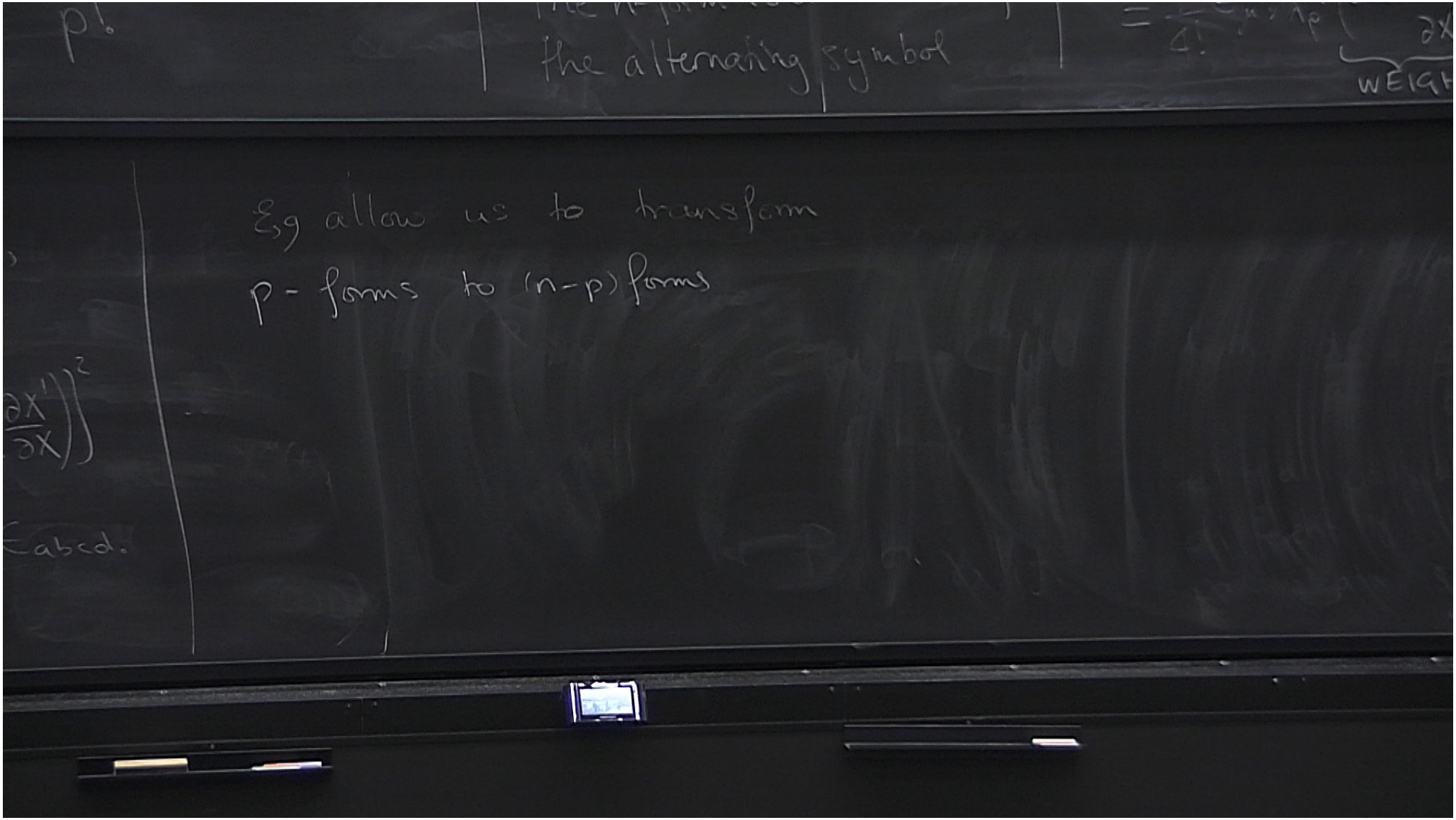
$$= \frac{1}{4!}\epsilon_{\mu'\nu'\lambda'\rho'}\underbrace{\left(\det\frac{\partial x}{\partial x'}\right)}_{\text{WEIGHT}} dx^{\mu'} \wedge dx^{\nu'} \wedge dx^{\lambda'} \wedge dx^{\rho'}$$

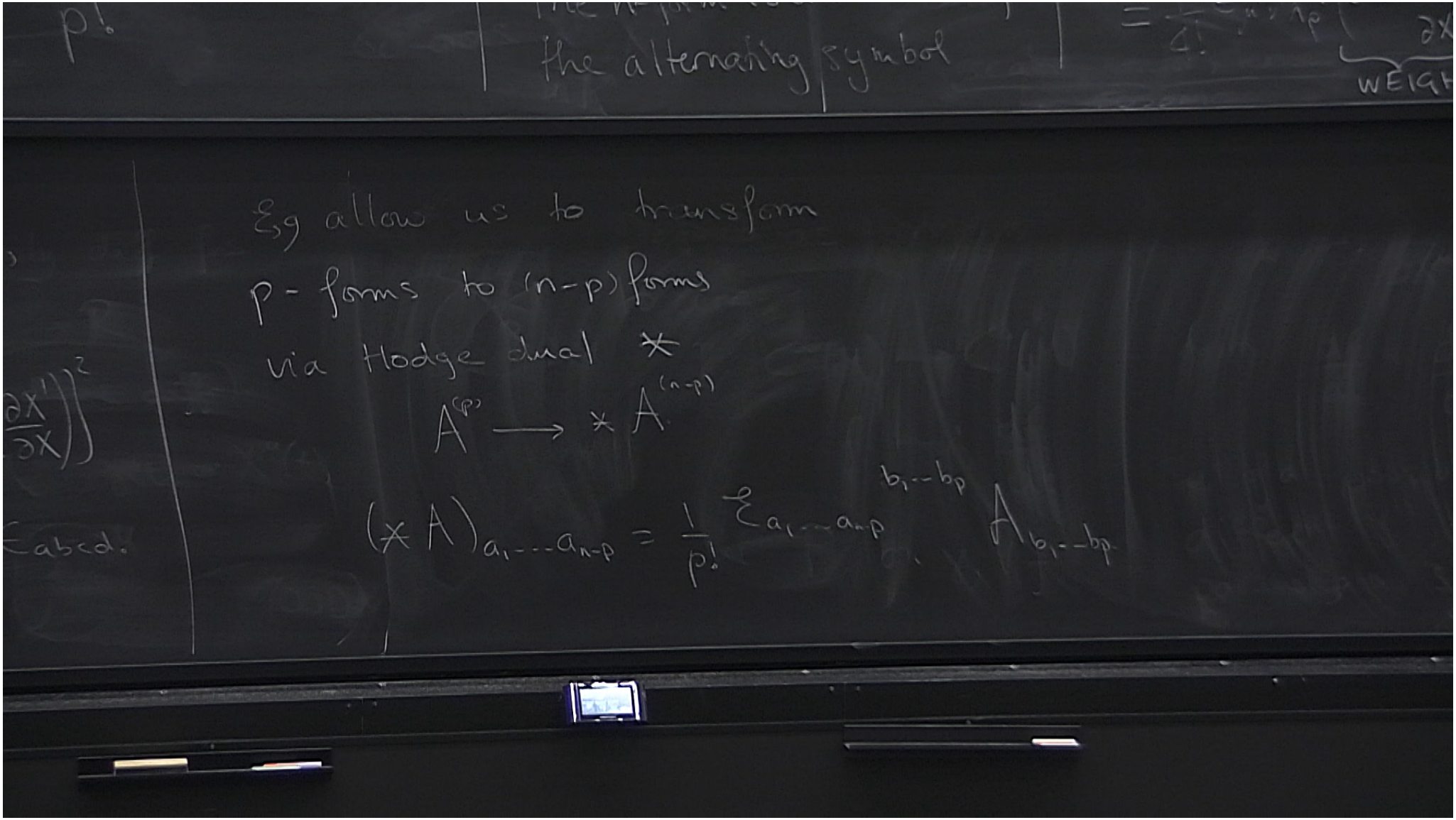

p!
the alternati
If we have a metric,
det $(g_{\mu\nu})$


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Define $\mathcal{E}_{abcd} = \sqrt{|g|}\,\epsilon_{abcd}$.


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Define $\mathcal{E}_{abcd} = \sqrt{|g|}\,\epsilon_{abcd}$.
– a TENSOR.


p!
the alternating symbol
WEIGH
Eg allow us to transform
p - forms to (n-p) forms
Eabcd.


p!
the alternating symbol
Eg allow us to transform
p - forms to (n-p) forms
via Hodge dual *
$A^{(p)} \longrightarrow *A^{(n-p)}$
$(*A)_{a_1 \cdots a_{n-p}} = \frac{1}{p!} \varepsilon_{a_1 \cdots a_{n-p}}{}^{b_1 \cdots b_p} A_{b_1 \cdots b_p}$

The n-form is a multiple of the alternating symbol

$$= \frac{1}{4!} \epsilon_{\mu'\nu'\lambda'\rho'} \underbrace{\left(\det \frac{\partial x}{\partial x'}\right)}_{\text{WEIGHT}} dx^{\mu} \wedge dx^{\nu} \wedge dx^{\lambda} \wedge dx^{\rho}$$

to transform

(n-p) forms

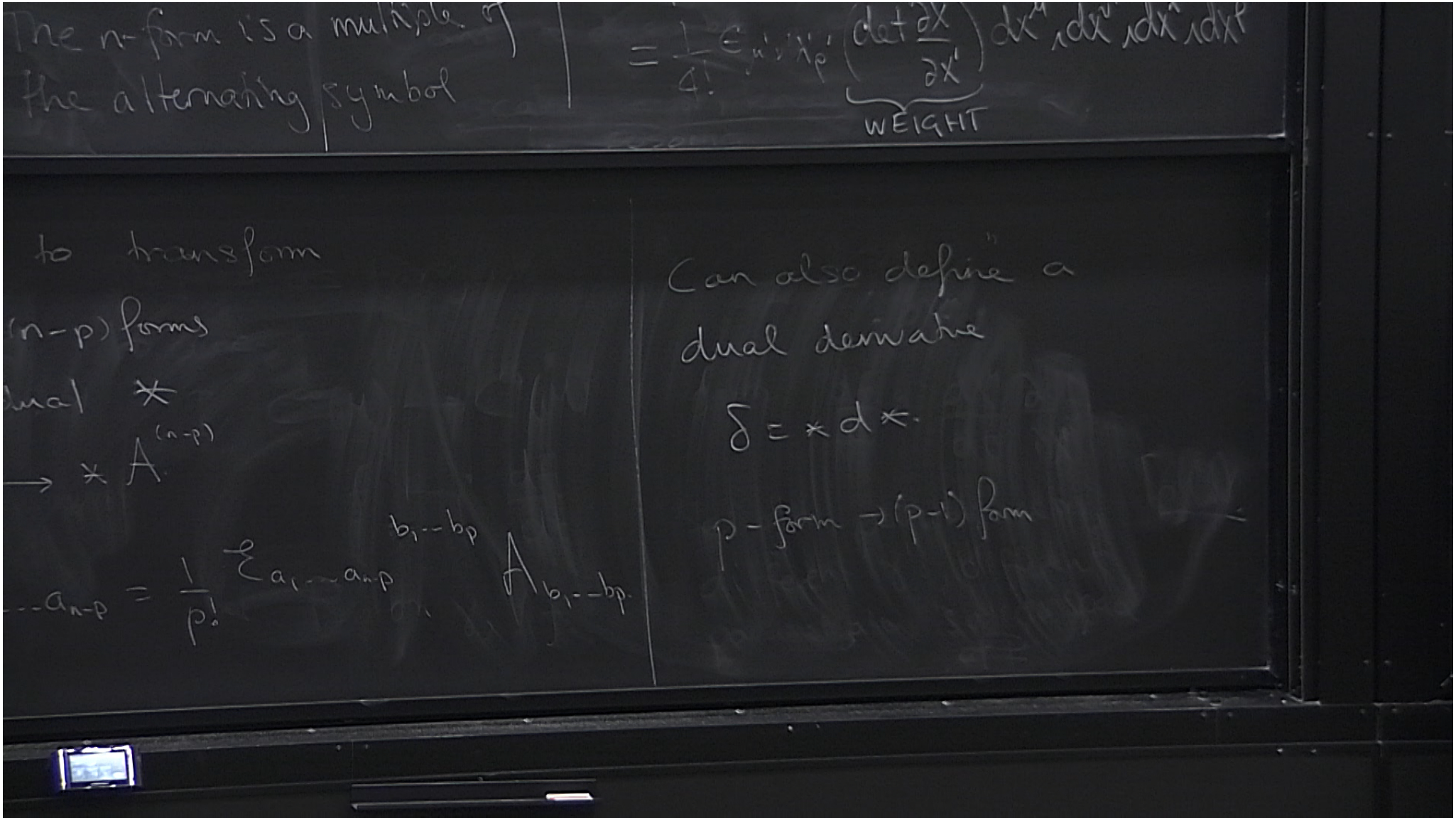
dual $*$

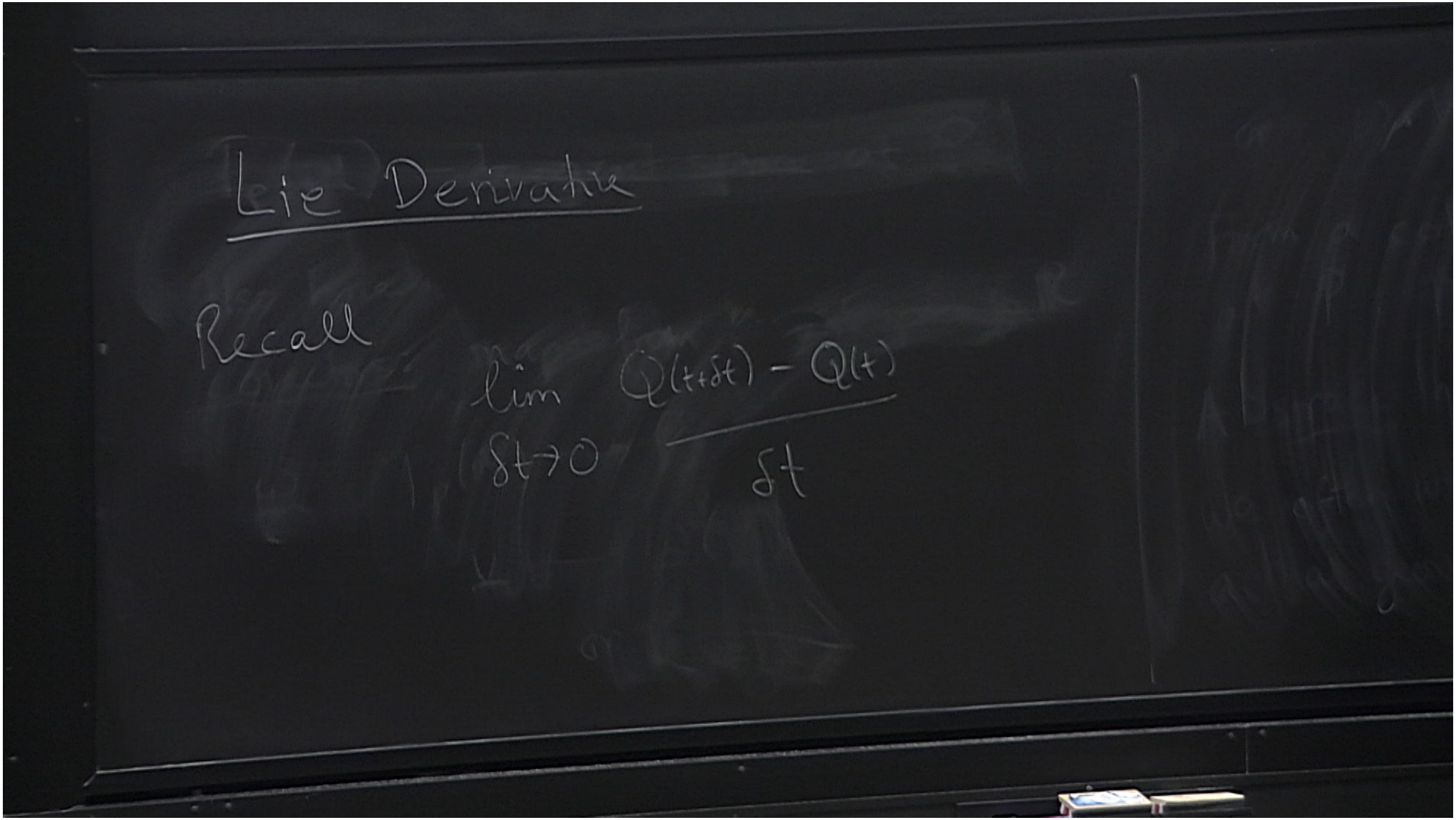
$\longrightarrow * A^{(n-p)}$

$$\ldots a_{n-p} = \frac{1}{p!} \varepsilon_{a_1 \ldots a_{n-p}}{}^{b_1 \ldots b_p} A_{b_1 \ldots b_p}$$

Can also define a dual derivative

$$\delta = * d *$$

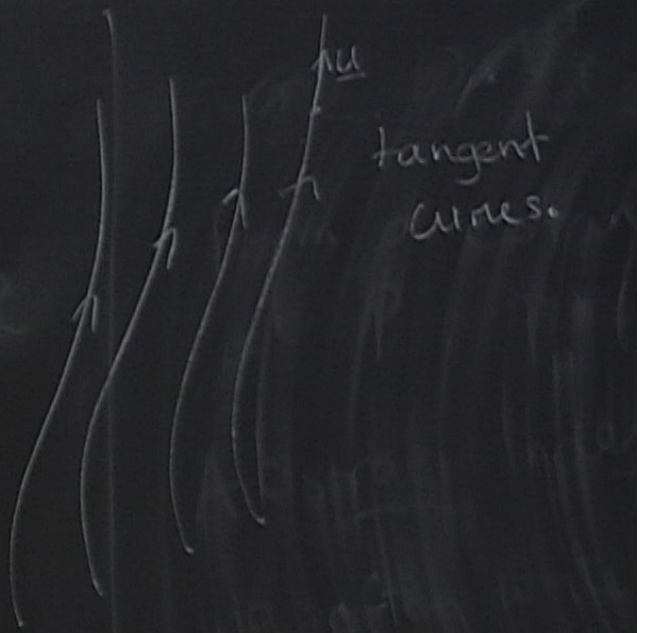

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p-form → (p-1) form

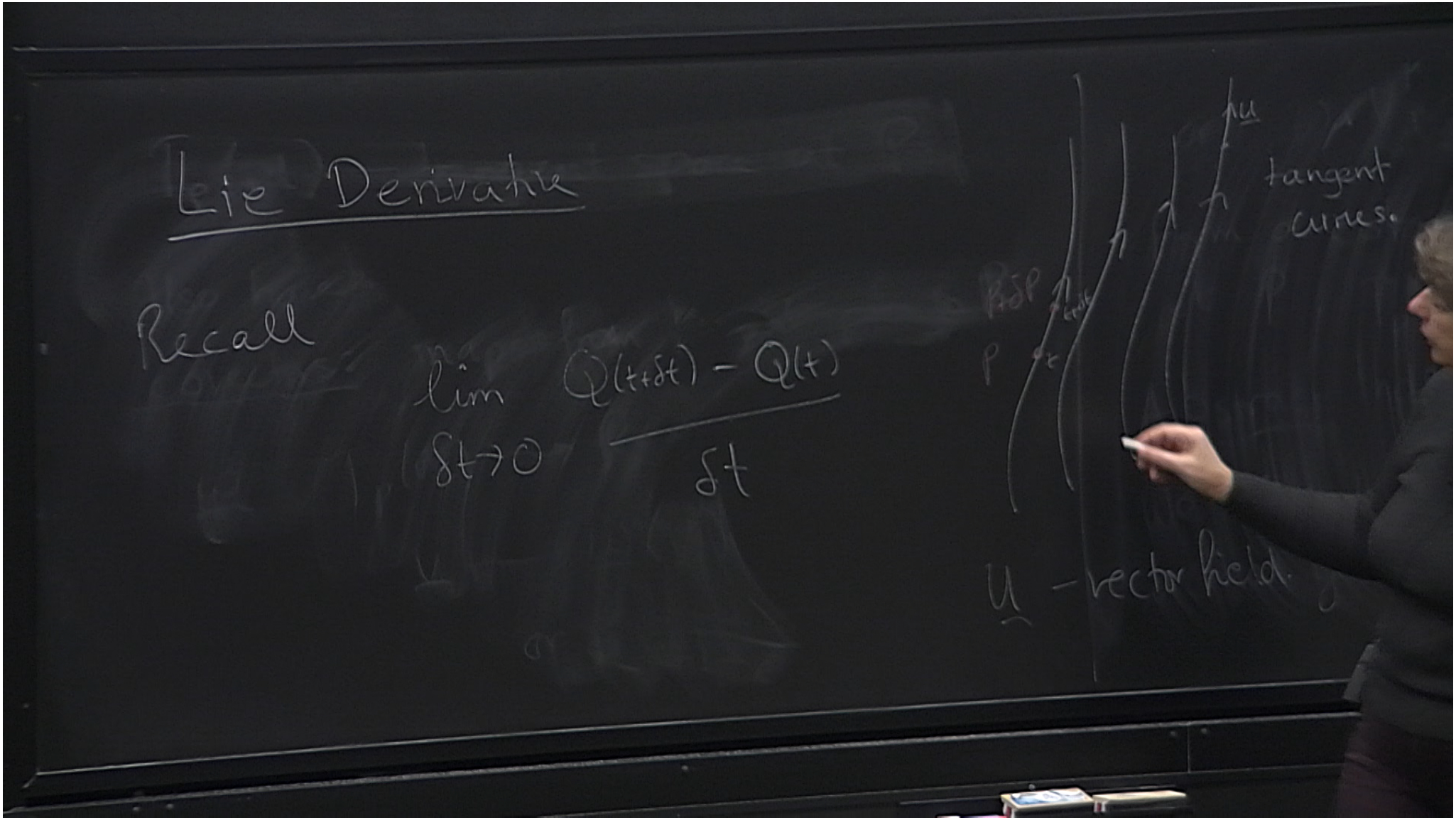

Lie Derivative
Recall
$\lim_{\delta t \to 0} \frac{Q(t+\delta t) - Q(t)}{\delta t}$

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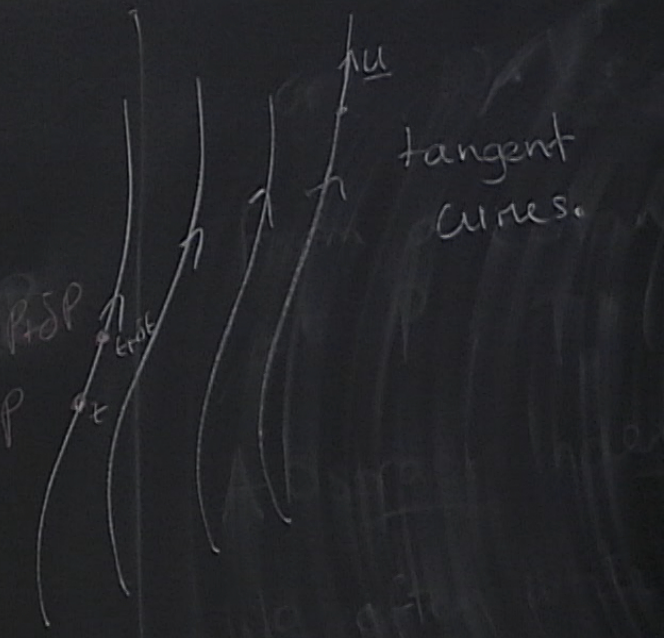


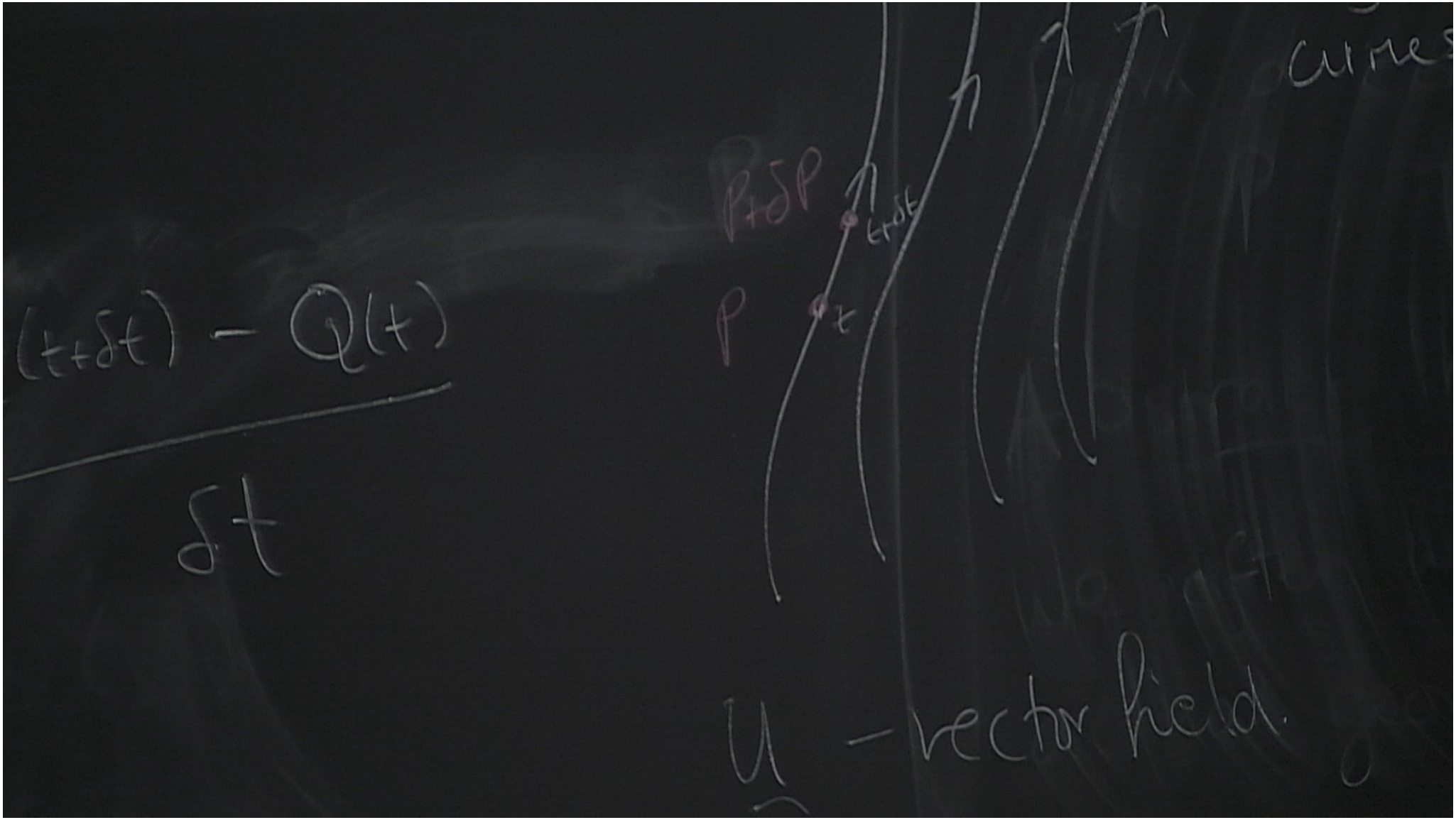

Lie Derivative
Recall
$\lim_{\delta t \to 0} \frac{Q(t+\delta t) - Q(t)}{\delta t}$
tangent curves.
$\underline{U}$ - vector field.

Lie Derivative

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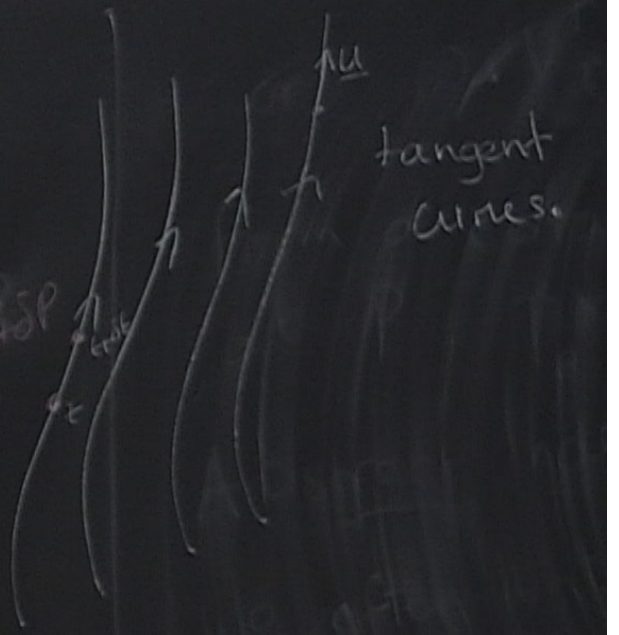

$\frac{(t+\delta t) - Q(t)}{\delta t}$
P+δP
t+δt
P
t
curves
$\underset{\sim}{u}$ - vector field.

Lie Derivative

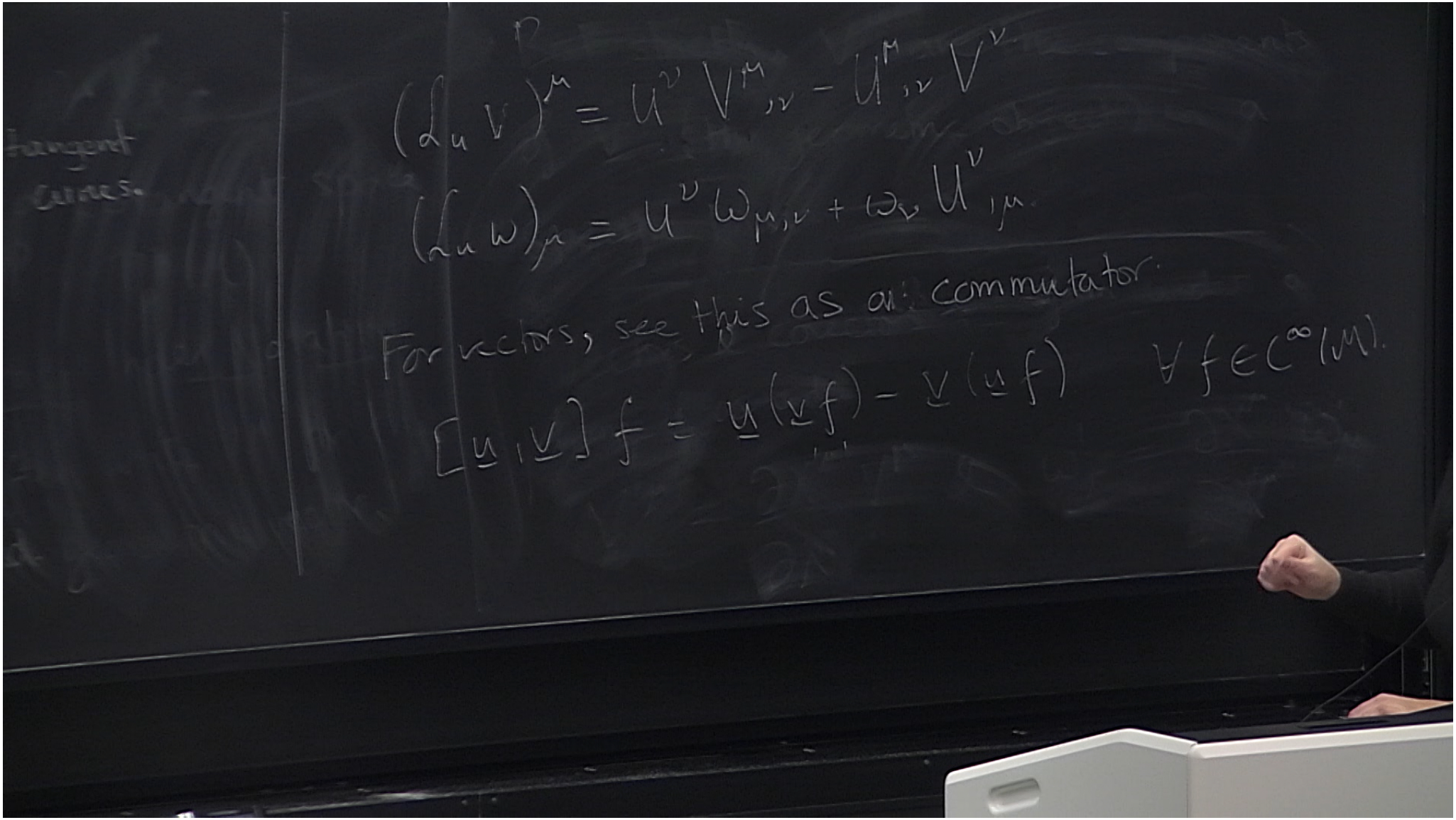
Recall

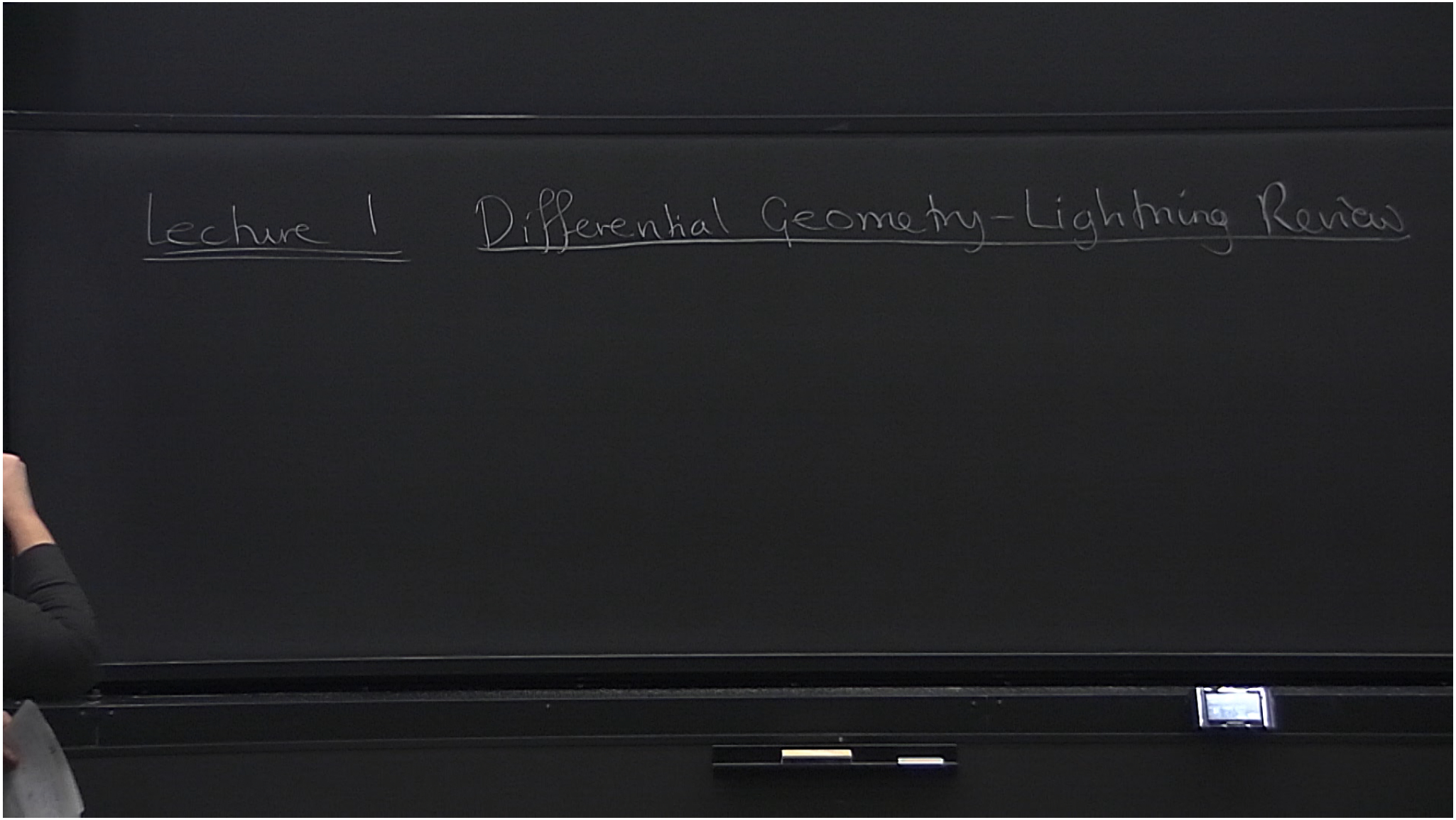
$$\mathcal{L}_u Q = \lim_{\delta t \to 0} \frac{Q(t+\delta t) - Q(t)}{\delta t}$$

↑ Lie deriv



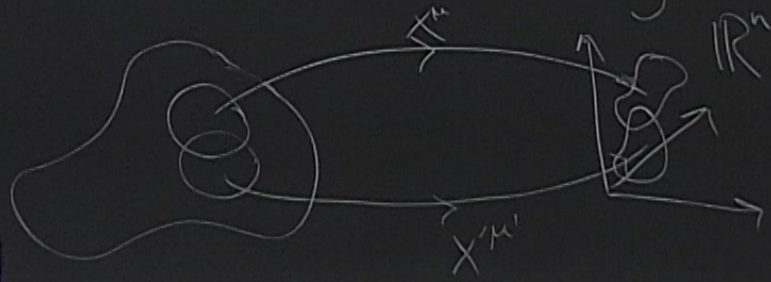
$\underset{\sim}{u}$ - vector field


$(\mathcal{L}_u v)^\mu = u^\nu v^\mu{}_{,\nu} - u^\mu{}_{,\nu} v^\nu$
$(\mathcal{L}_u w)_\mu = u^\nu w_{\mu,\nu} + w_\nu u^\nu{}_{,\mu}$
For vectors, see this as a commutator.
$[\underline{u}, \underline{v}] f = \underline{u}(\underline{v} f) - \underline{v}(\underline{u} f) \quad \forall f \in C^\infty(M)$.


Lecture 1 Differential Geometry - Lightning Review

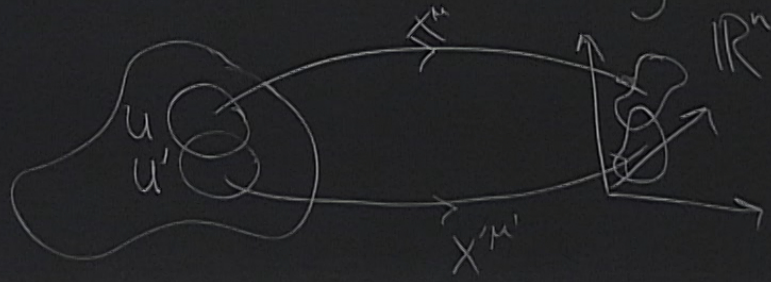
Lecture 1 Differential Geometry - Lightning Review

Recall a manifold (spacetime) is a set of events that looks locally like $\mathbb{R}^n$.



Lecture 1 Differential Geometry - Lightning Review

Recall a manifold (spacetime) is a set of events that looks locally like $\mathbb{R}^n$.



local charts U, U' have co-ords $X^\mu, X'^{\mu'}$ in $\mathbb{R}^n$.

thing Review

a set of

harts

ave

s X^μ, $X'^{\mu'}$

in $\mathbb{R}^n$.

Usually take our manifold to be infinitely differentiable, ie. in overlap of charts, $X^\mu \hookrightarrow X'^{\mu'}$ is an ooly differentiable map

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t, x, y, z $\quad$ t, r, θ, ϕ

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scalar, vector, tensor - spinor

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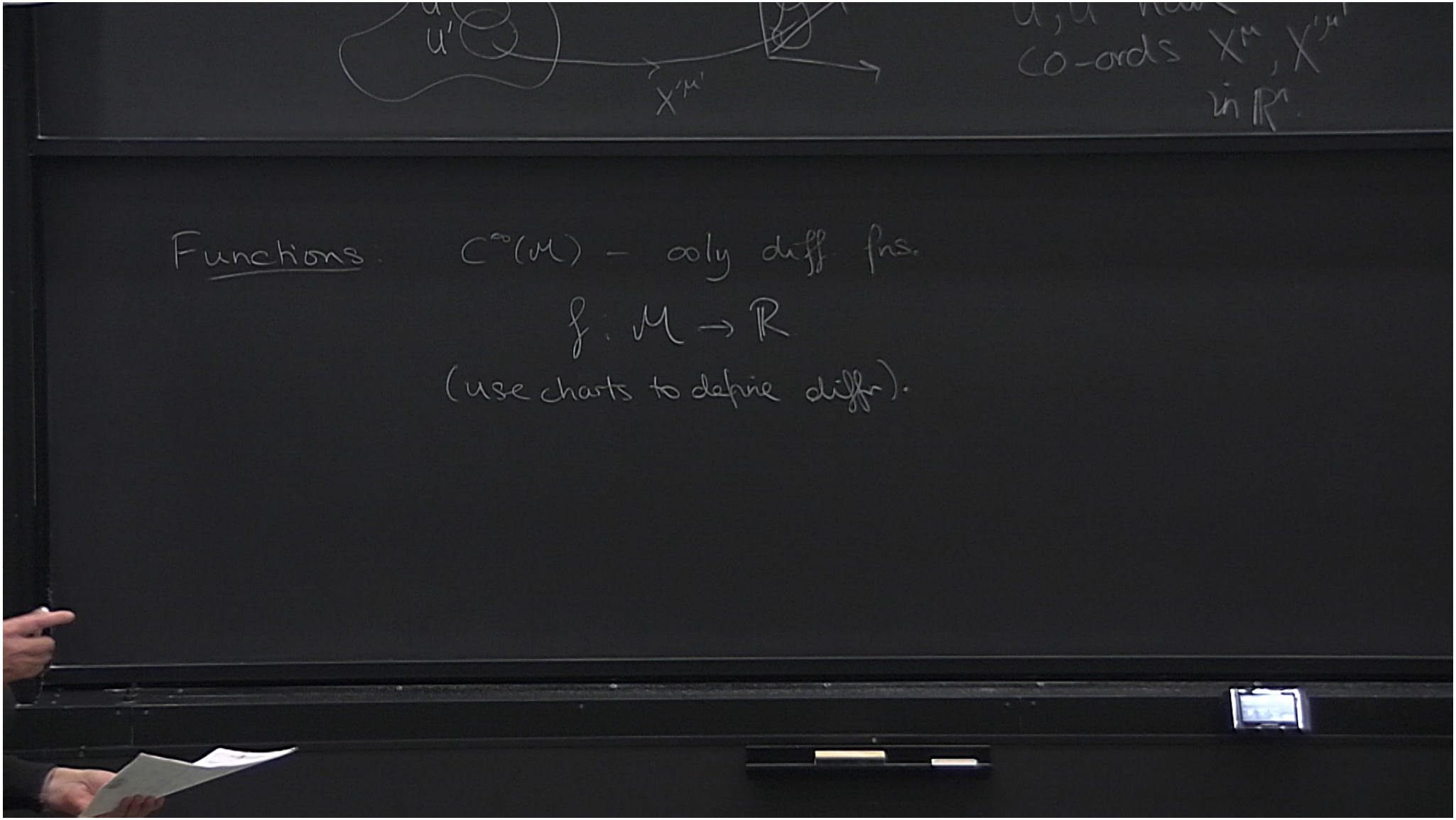
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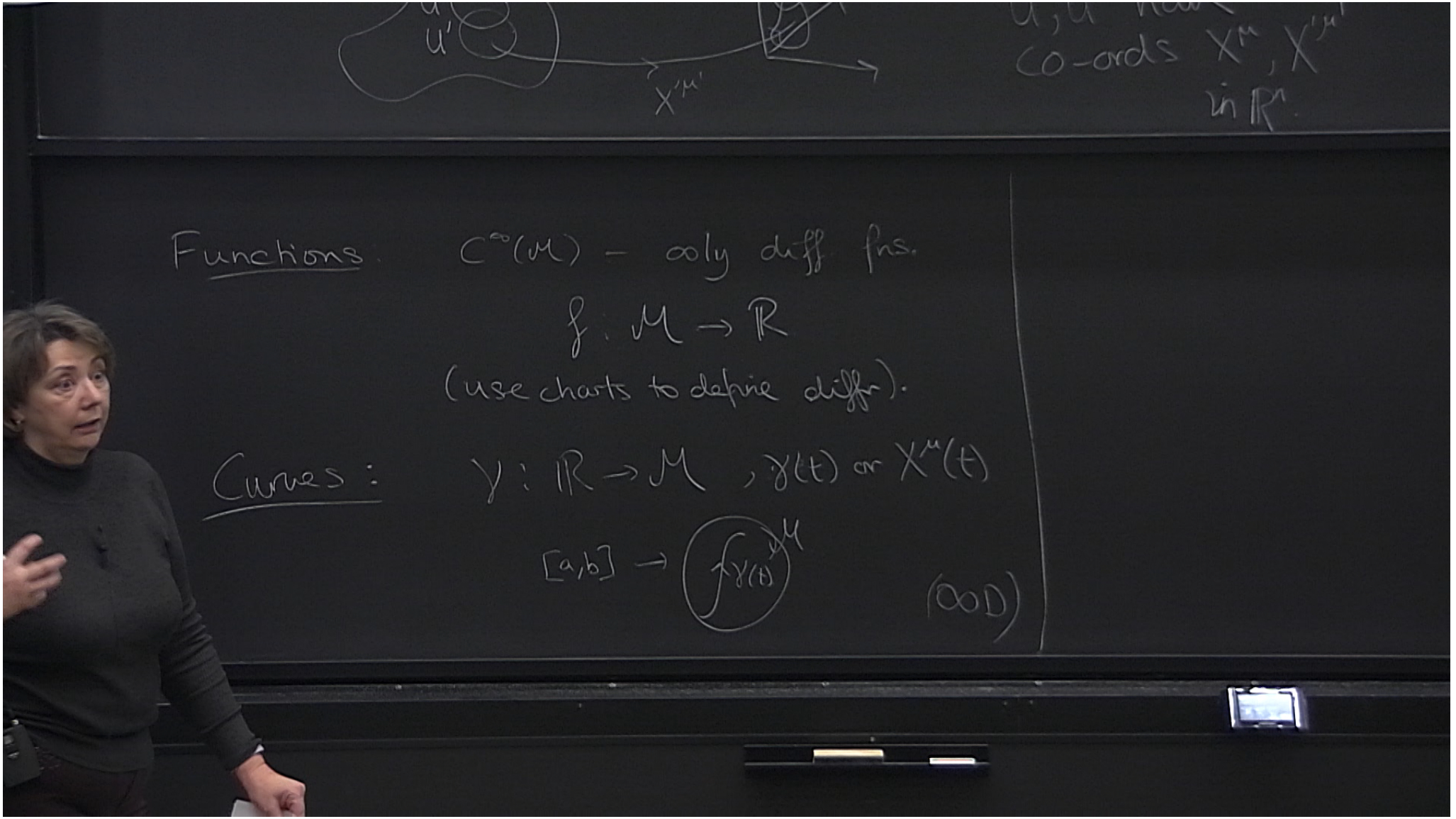
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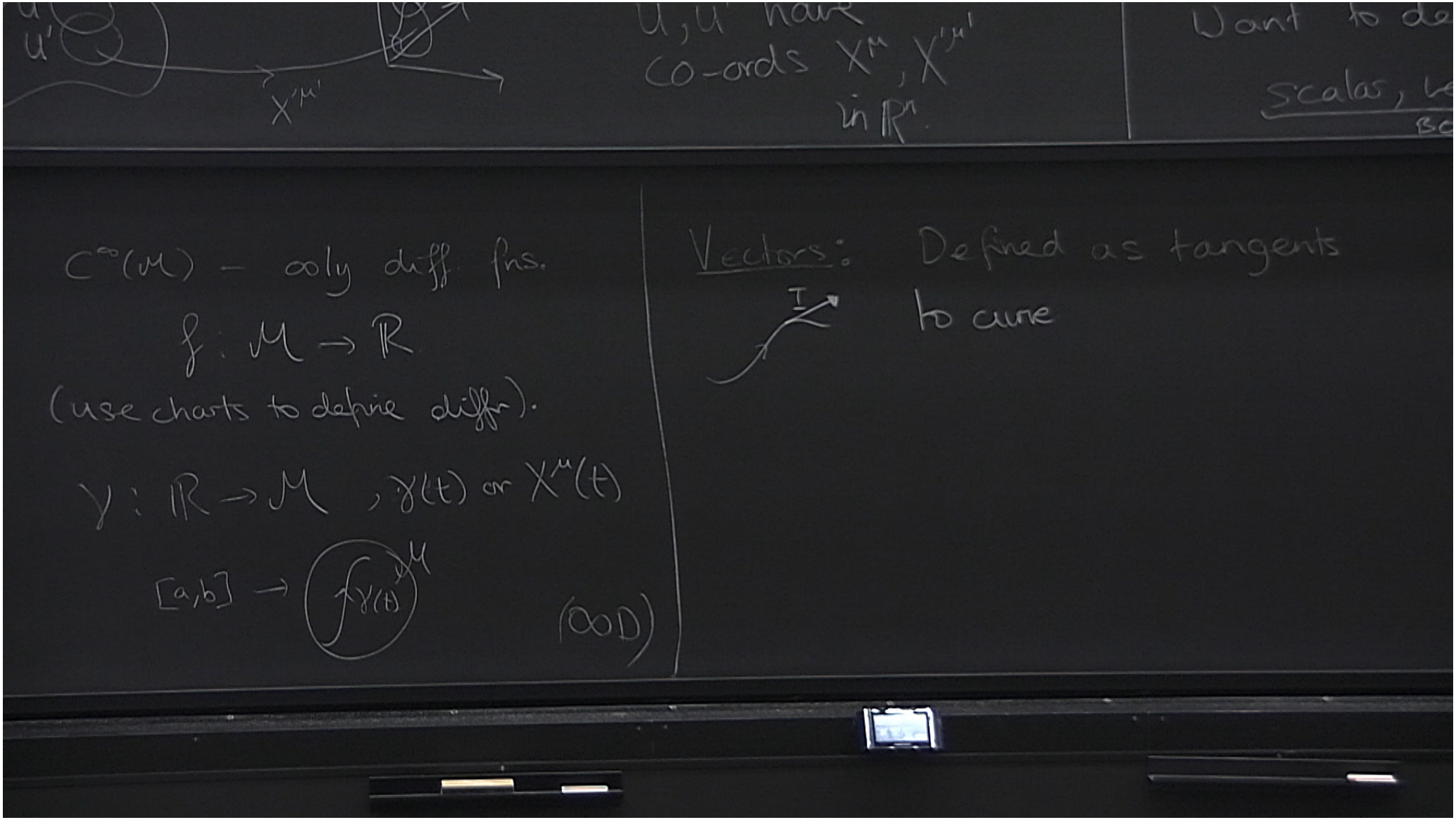
Want to describe fields.

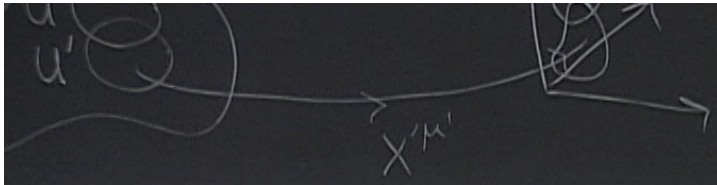
Scalar, vector, tensor - spinor

BOSONIC


U'
$x'^{\mu'}$
co-ords x^μ, $x'^{\mu'}$
in $\mathbb{R}^n$.
Functions $C^\infty(\mathcal{M})$ - ∞ly diff fns.
$f : \mathcal{M} \to \mathbb{R}$
(use charts to define diffn).


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$X'^{\mu'}$
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Functions $C^\infty(M)$ - ∞ly diff fns.
$f: M \to \mathbb{R}$
(use charts to define diffn).
Curves: $\gamma: \mathbb{R} \to M$, $\gamma(t)$ or $X^\mu(t)$
$[a,b] \to f_{\gamma(t)}$ in M
(OD)


U, U' have
co-ords X^μ, $X'^{\mu'}$
in $\mathbb{R}^n$.
Want to de
scalas, ve
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$[a,b] \to f(\gamma(t))$
(∞D)
Vectors: Defined as tangents
I
to curve



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Want to do scalas, ve

$C^\infty(M)$ – ooly diff. fns.

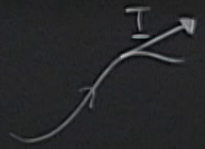
$f : M \to \mathbb{R}$

(use charts to define diffn).

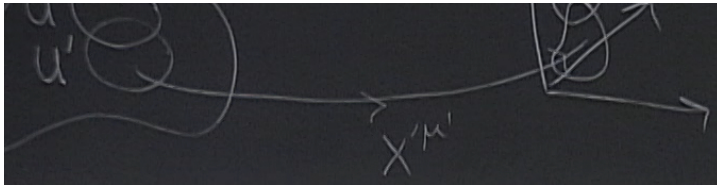
$\gamma : \mathbb{R} \to M$, $\gamma(t)$ or $x^\mu(t)$

$[a,b] \to f(\gamma(t))$ (OoD)

Vectors: Defined as tangents to curve



$\underline{I} : C^\infty(M) \to C^\infty(M)$



U, U' have co-ords $x^\mu, x'^{\mu'}$ in $\mathbb{R}^n$.

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scalas, ve

$C^\infty(M)$ – ∞ly diff. fns.

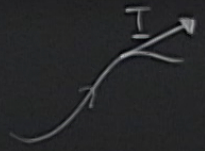
$f : M \to \mathbb{R}$

(use charts to define diffn).

$\gamma : \mathbb{R} \to M$, $\gamma(t)$ or $x^\mu(t)$

$[a,b] \to f(\gamma(t))$ (∞D)

Vectors: Defined as tangents to curve



$I : C^\infty(M) \to C^\infty(M)$

$f \mapsto \frac{df}{dt}$

U, U' have co-ords $X^\mu, X'^{\mu'}$ in $\mathbb{R}^n$.

$X'^{\mu'}$

Want to do scalas, ve

$C^\infty(M)$ – ∞ly diff. fns.

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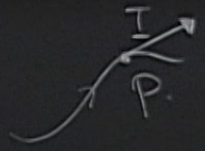
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$\gamma : \mathbb{R} \to M$, $\gamma(t)$ or $X^\mu(t)$

$[a,b] \to$ (f(γ(t)) X^μ)

(OOD)

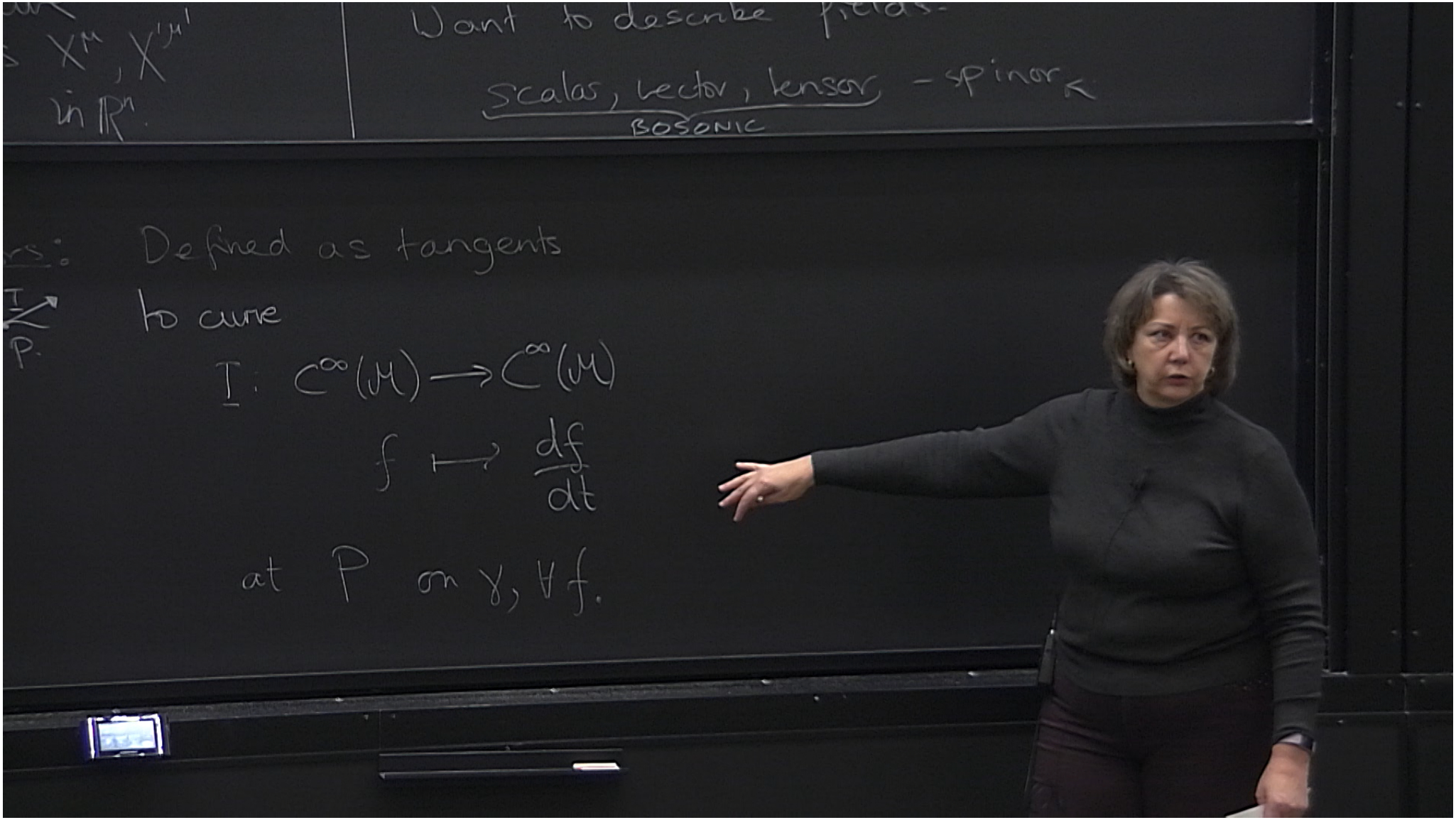
Vectors: Defined as tangents to curve



$\underline{I} : C^\infty(M) \to C^\infty(M)$

$f \longmapsto \frac{df}{dt}$

at P on γ.


Want to describe
Scalas, vector, tensor - spinor
BOSONIC
Defined as tangents
to curve
$I: C^\infty(M) \to C^\infty(M)$
$f \mapsto \frac{df}{dt}$
at P on γ, $\forall f$.

$X^\mu, X'^{\mu'}$
in $\mathbb{R}^n$.

Want to describe fields:

Scalar, vector, tensor - spinor
BOSONIC

Defined as tangents to curve

$$I: C^\infty(M) \to C^\infty(M)$$
$$f \mapsto \frac{df}{dt}$$

at P on γ, $\forall f$.

$$I = T^\mu \frac{\partial}{\partial x^\mu}$$

from $\frac{df}{dt} = \frac{dx^\mu}{dt}\frac{\partial}{\partial x^\mu} f$.

co-ords X^μ, $X'^{\mu'}$ in $\mathbb{R}^n$.

Want to describe fields:

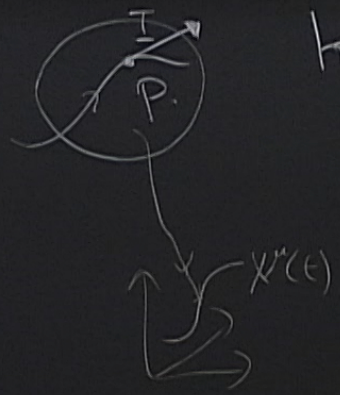
scalar, vector, tensor - spinor

BOSONIC

$X^\mu(t)$

(∞D)

Vectors: Defined as tangents to curve



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at P on γ, $\forall f$.

$$\underline{I} = I^\mu \frac{\partial}{\partial X^\mu}$$

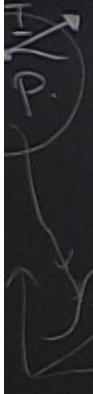
from $\frac{df}{dt} = \frac{dX^\mu}{dt}\frac{\partial}{\partial X^\mu} f$.

$X^\mu, X'^{\mu'}$ in $\mathbb{R}^n$.

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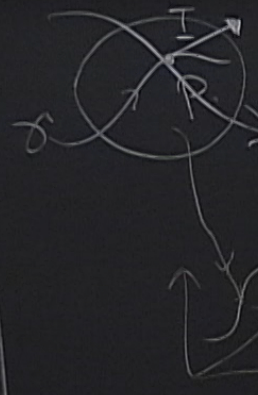
Set of all possible vectors, different γ's, different parametrizations, is a vector space at P. $T_P(M)$

co-ords X^μ, $X'^{\mu'}$ in $\mathbb{R}^n$.

Want to describe fields.

Scalar, vector, tensor - spinor
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Vectors: Defined as tangents to curve



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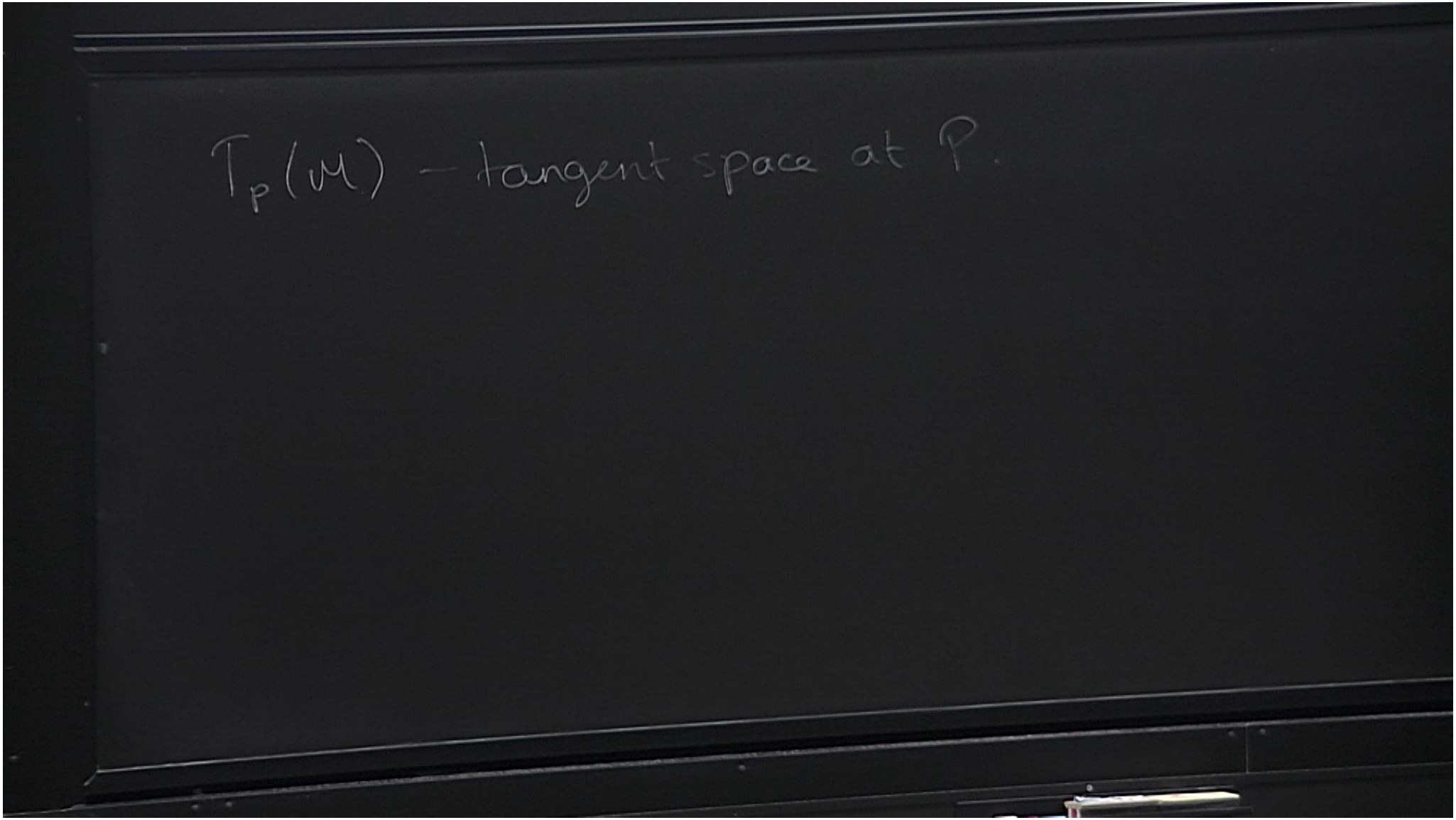
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$T_p(M)$ – tangent space at P.

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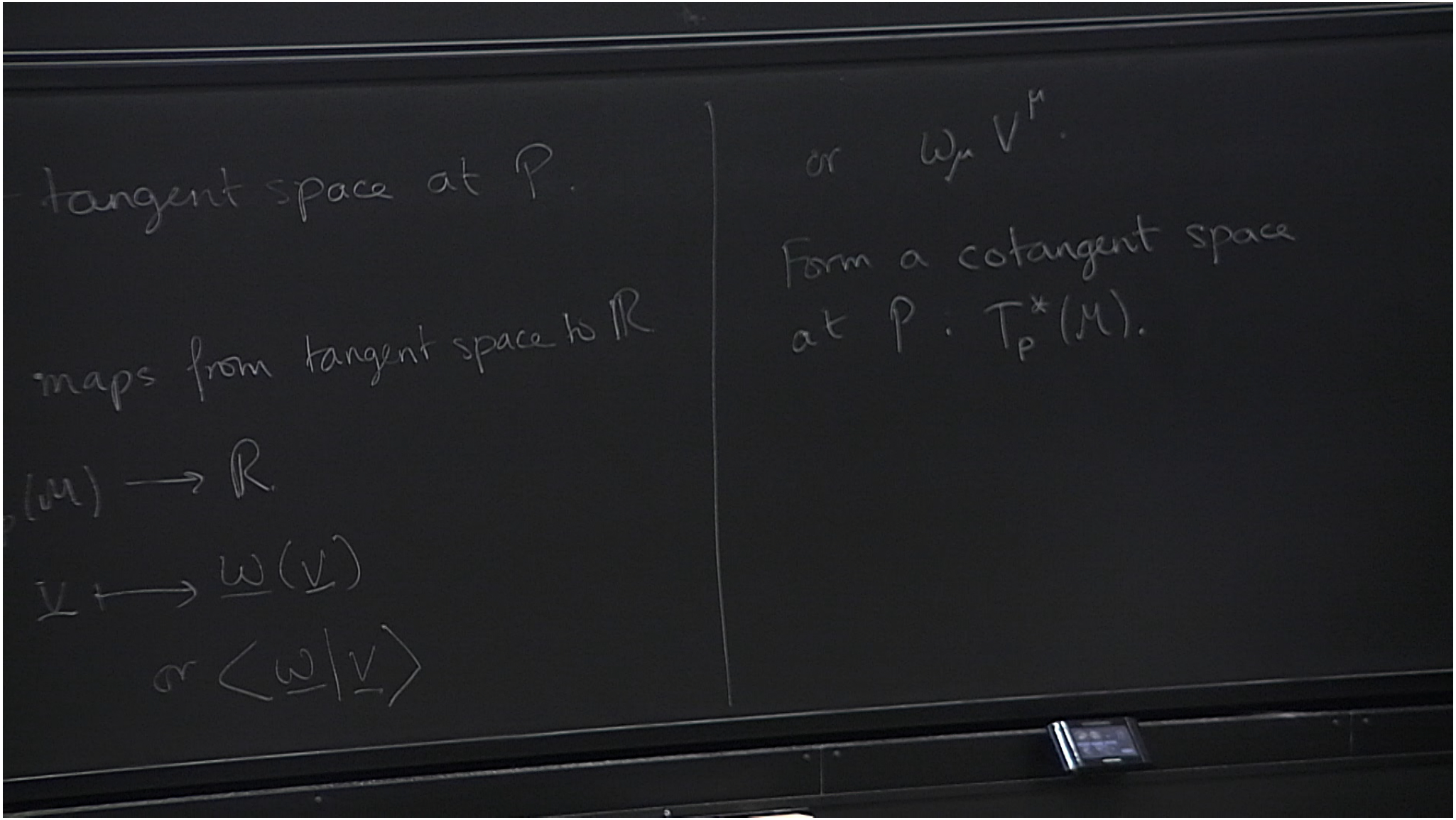
Also have

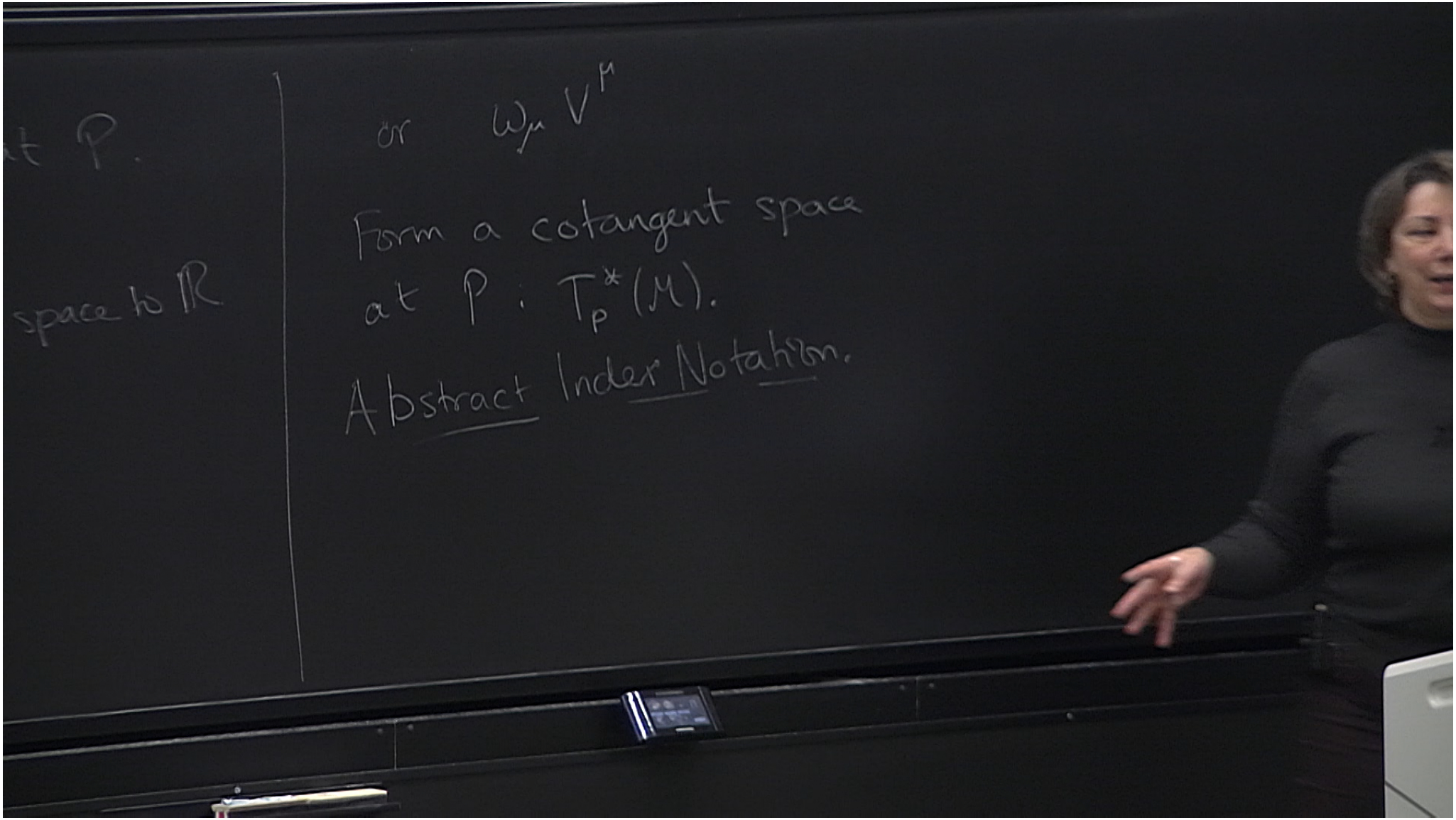
<u>covectors:</u> maps from tangent space to $\mathbb{R}$

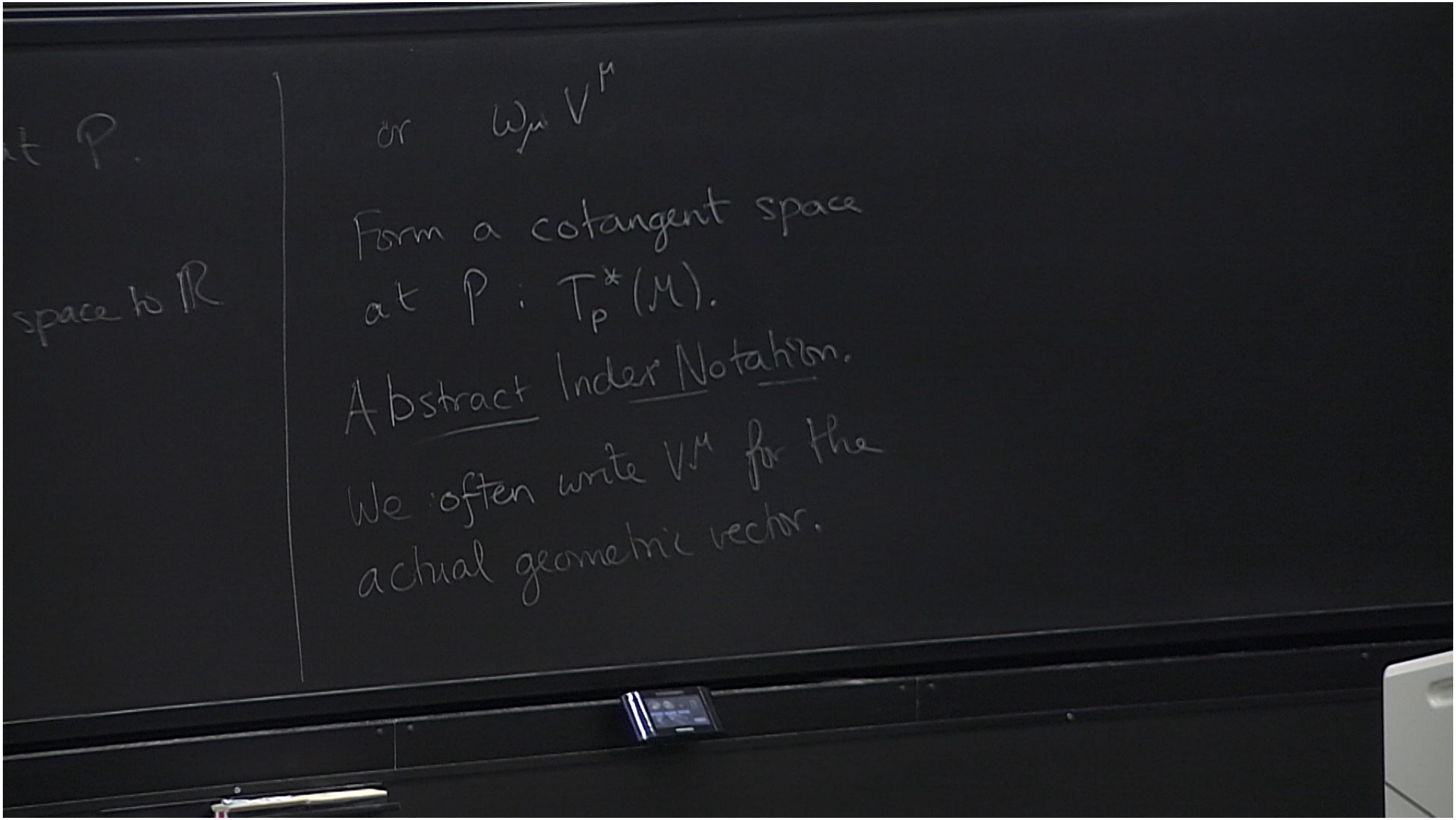
$\underline{\omega} : T_p(M) \longrightarrow \mathbb{R}$

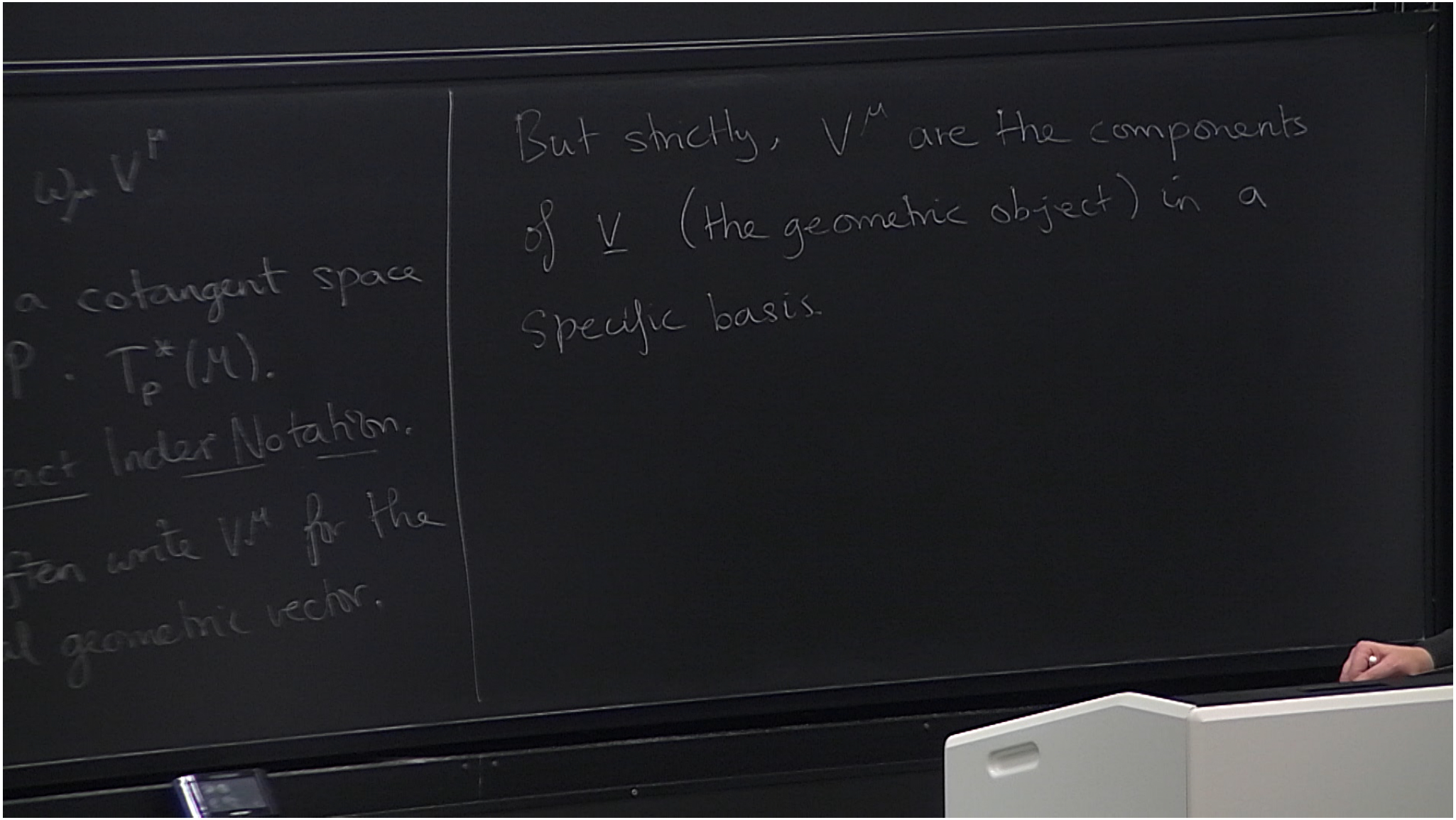
$\underline{V} \longmapsto \underline{\omega}(\underline{V})$

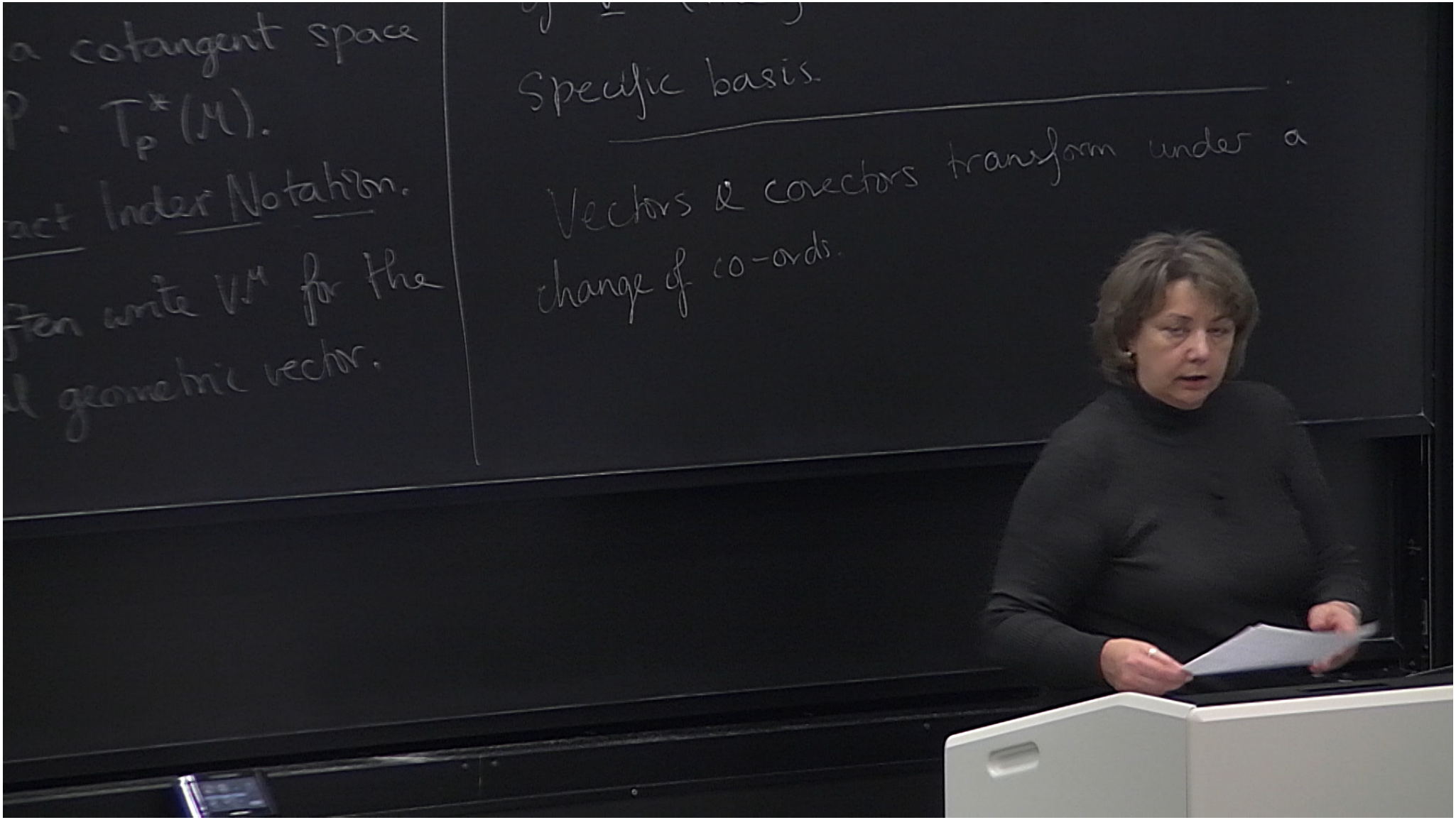
or $\langle \underline{\omega} | \underline{V} \rangle$

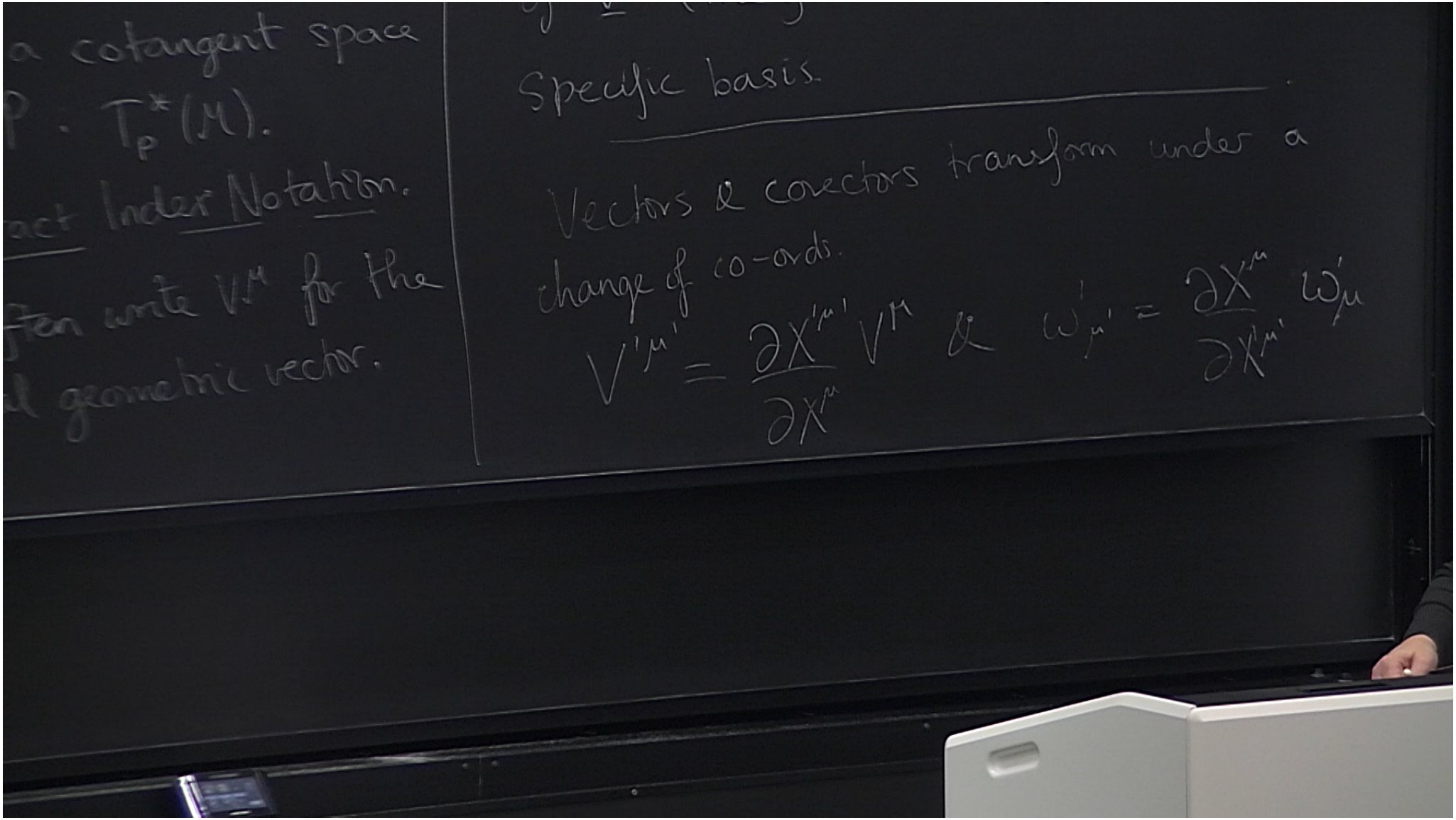

tangent space at P.
maps from tangent space to ℝ
(M) ⟶ R
V ⟶ ω(V)
or ⟨ω|V⟩
or $\omega_\mu V^\mu$.
Form a cotangent space
at P. $T_p^*(M)$.

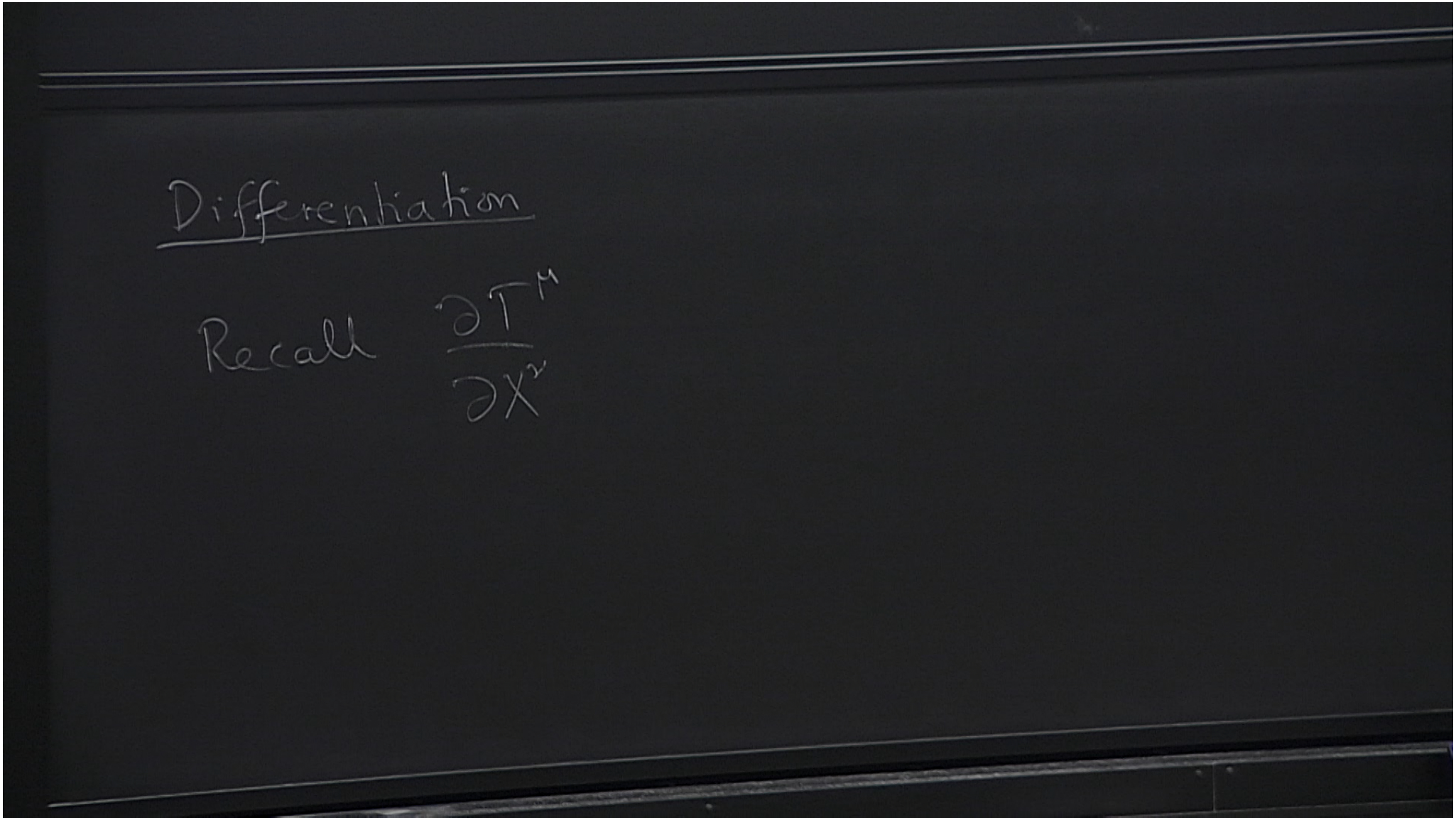

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Abstract Index Notation.

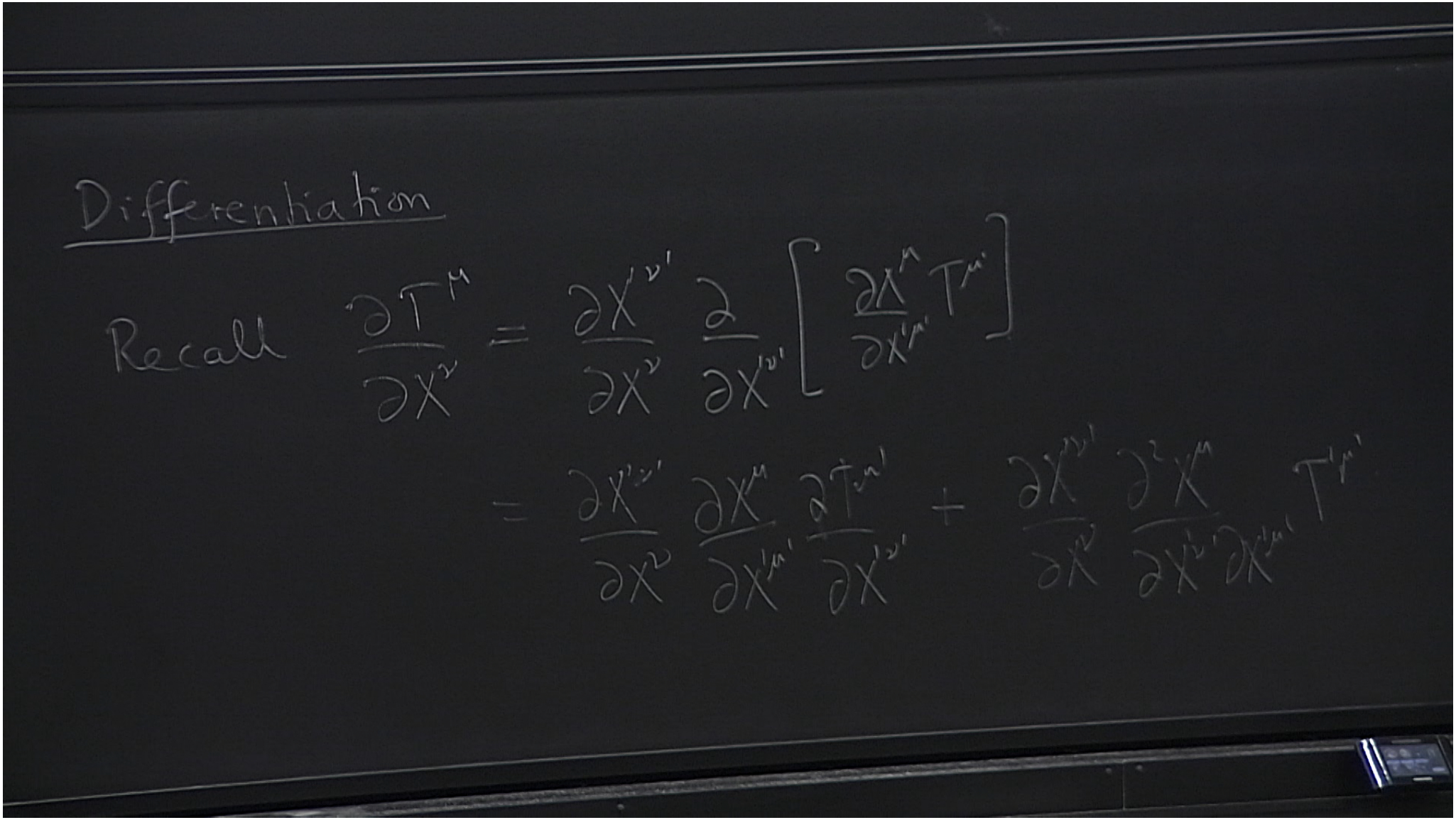

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We often write V^μ for the
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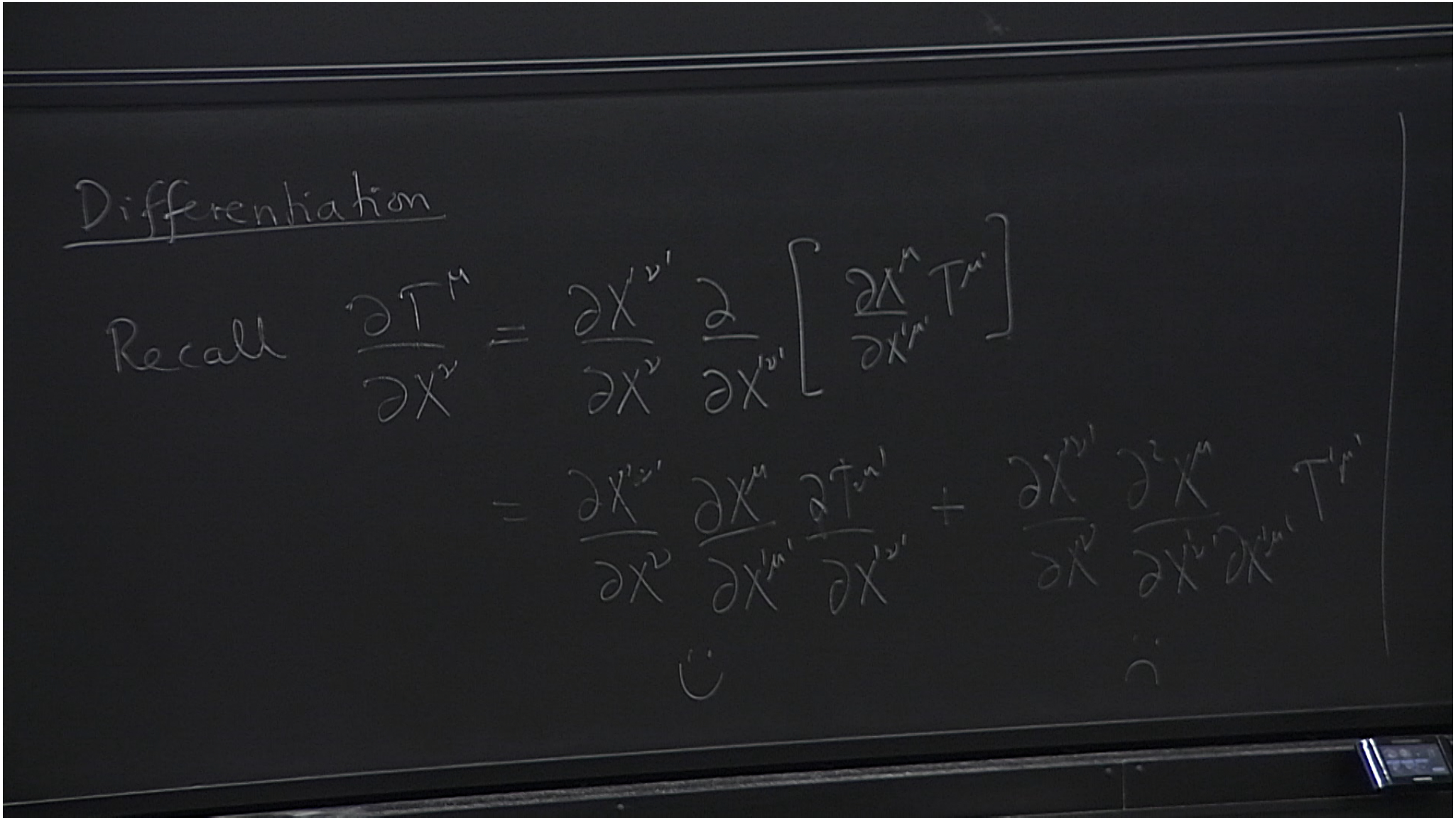

V^μ
a cotangent space
$T_p^*(M)$.
Index Notation.
Often write V^μ for the
geometric vector.
But strictly, V^μ are the components
of $\underline{V}$ (the geometric object) in a
specific basis.

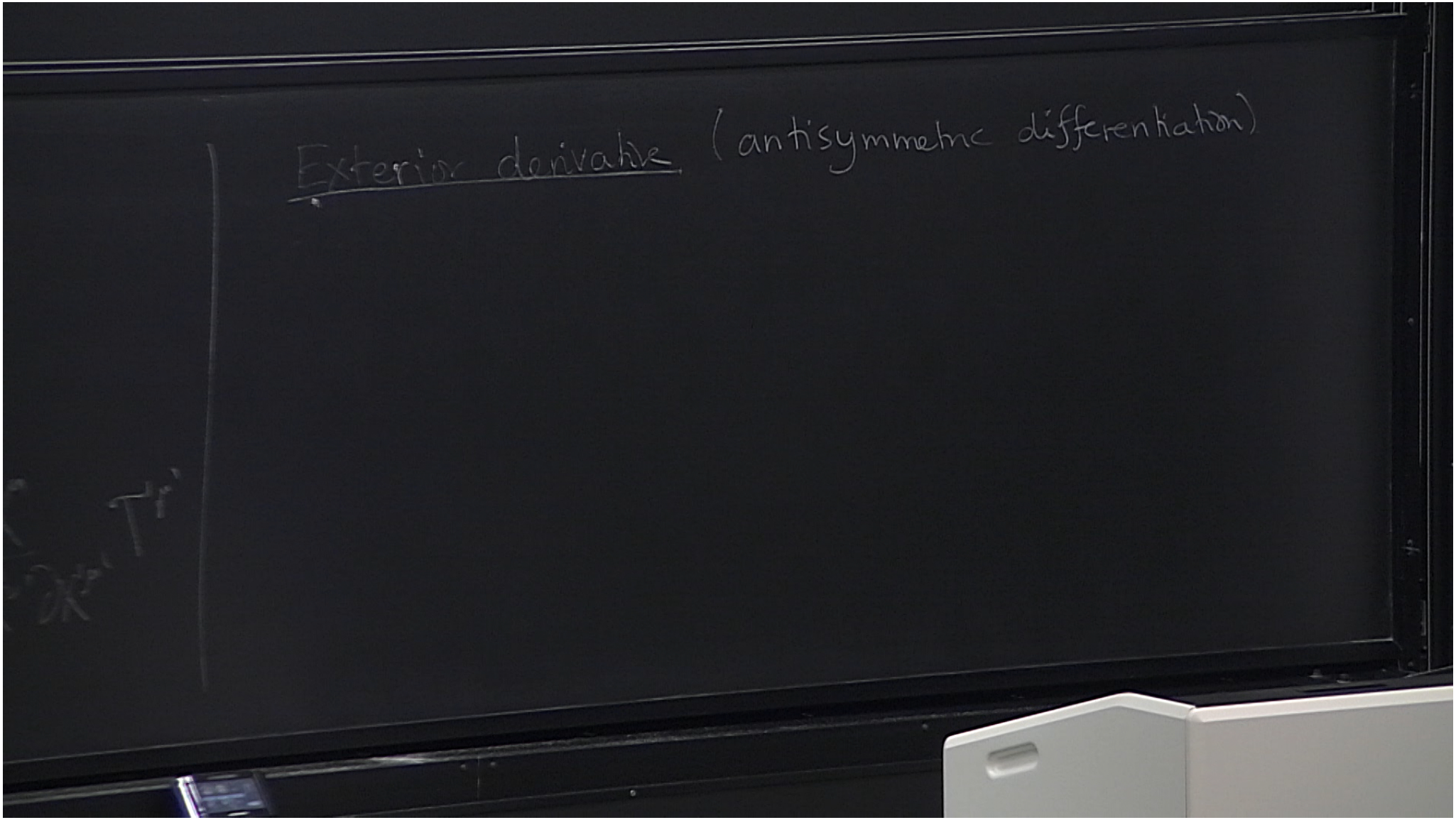

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Specific basis.
Vectors & covectors transform under a
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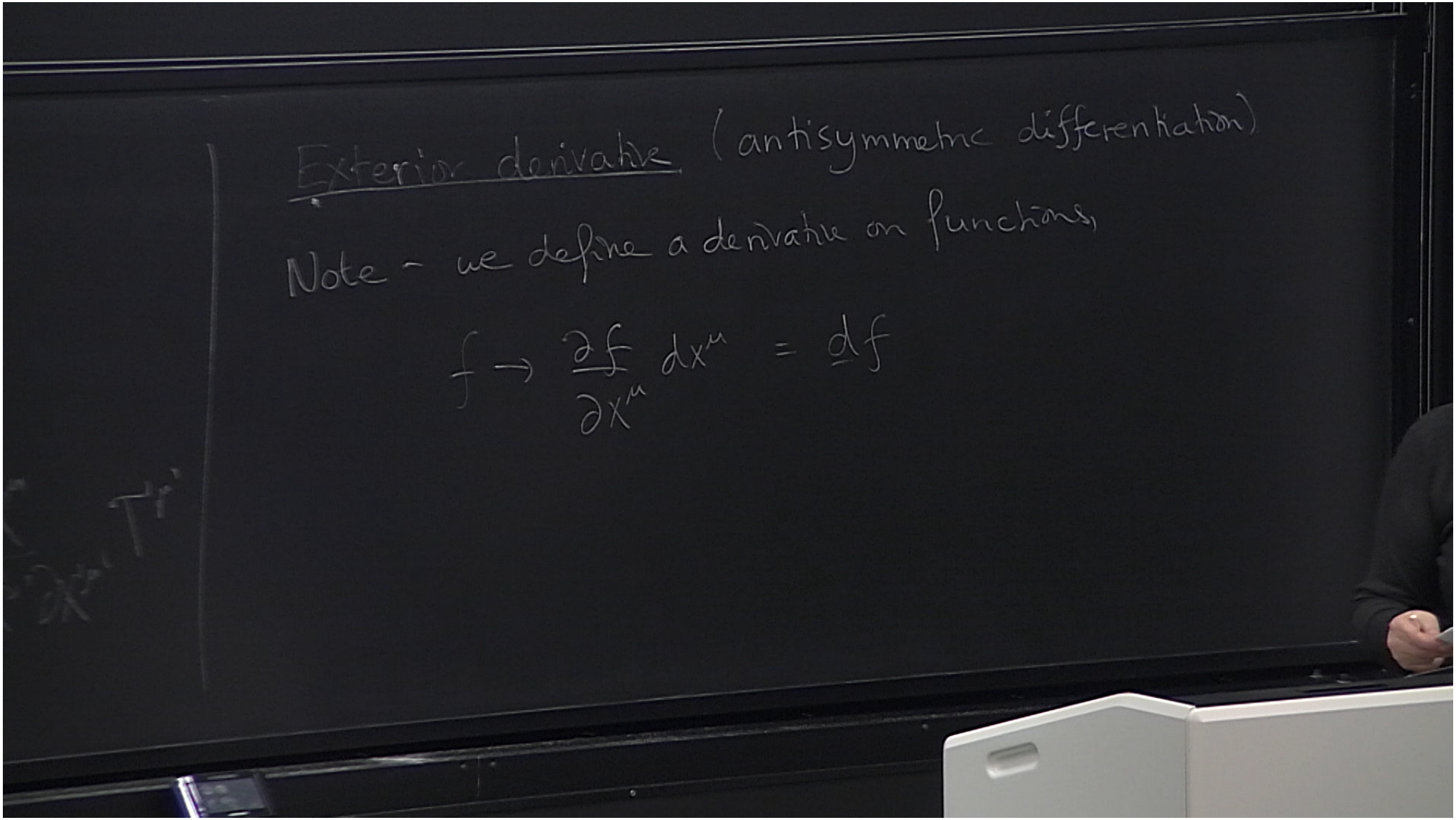

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$V'^{\mu'} = \frac{\partial X'^{\mu'}}{\partial X^\mu} V^\mu$ & $\omega'_{\mu'} = \frac{\partial X^\mu}{\partial X'^{\mu'}} \omega_\mu$

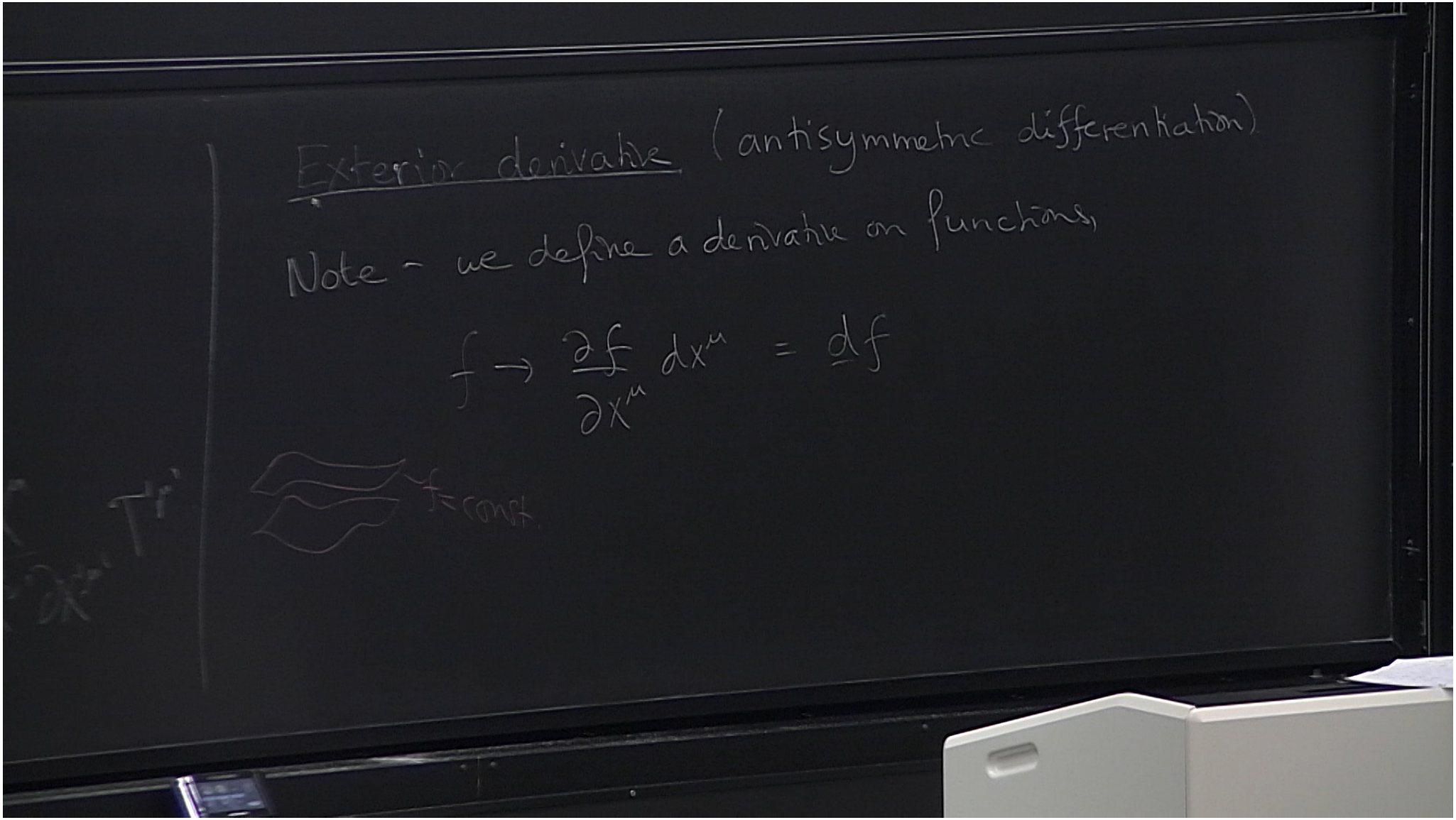

Differentiation
Recall $\frac{\partial T^{\mu}}{\partial X^{\nu}}$

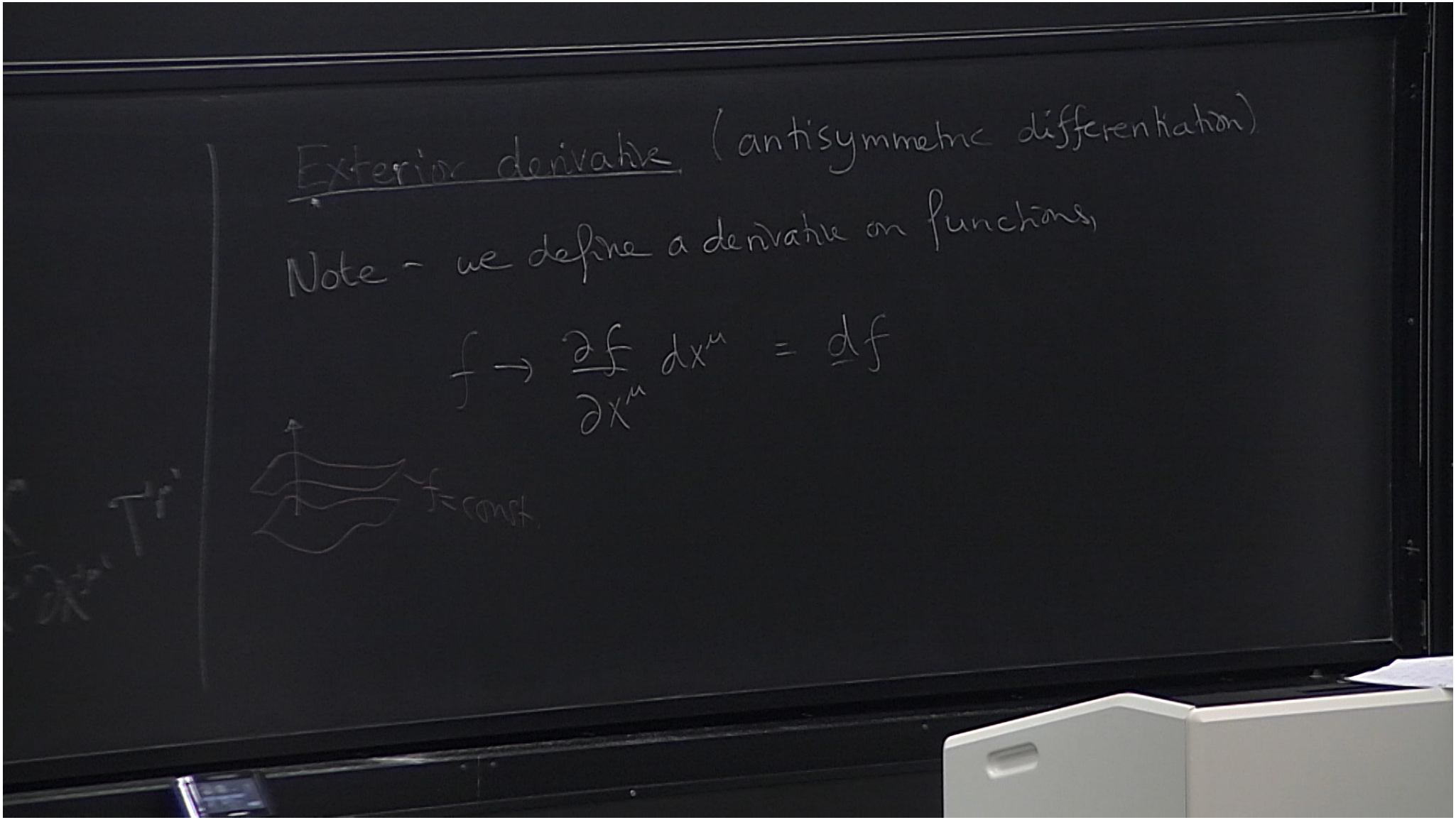

Differentiation
Recall $\frac{\partial T^{\mu}}{\partial x^{\nu}} = \frac{\partial x'^{\nu'}}{\partial x^{\nu}} \frac{\partial}{\partial x'^{\nu'}} \left[\frac{\partial x'^{\mu}}{\partial x^{\mu'}} T'^{\mu'} \right]$
$= \frac{\partial x'^{\nu'}}{\partial x^{\nu}} \frac{\partial x^{\mu}}{\partial x'^{\mu'}} \frac{\partial T'^{\mu'}}{\partial x'^{\nu'}} + \frac{\partial x'^{\nu'}}{\partial x^{\nu}} \frac{\partial^2 x'^{\mu}}{\partial x'^{\nu'} \partial x'^{\mu'}} T'^{\mu'}$

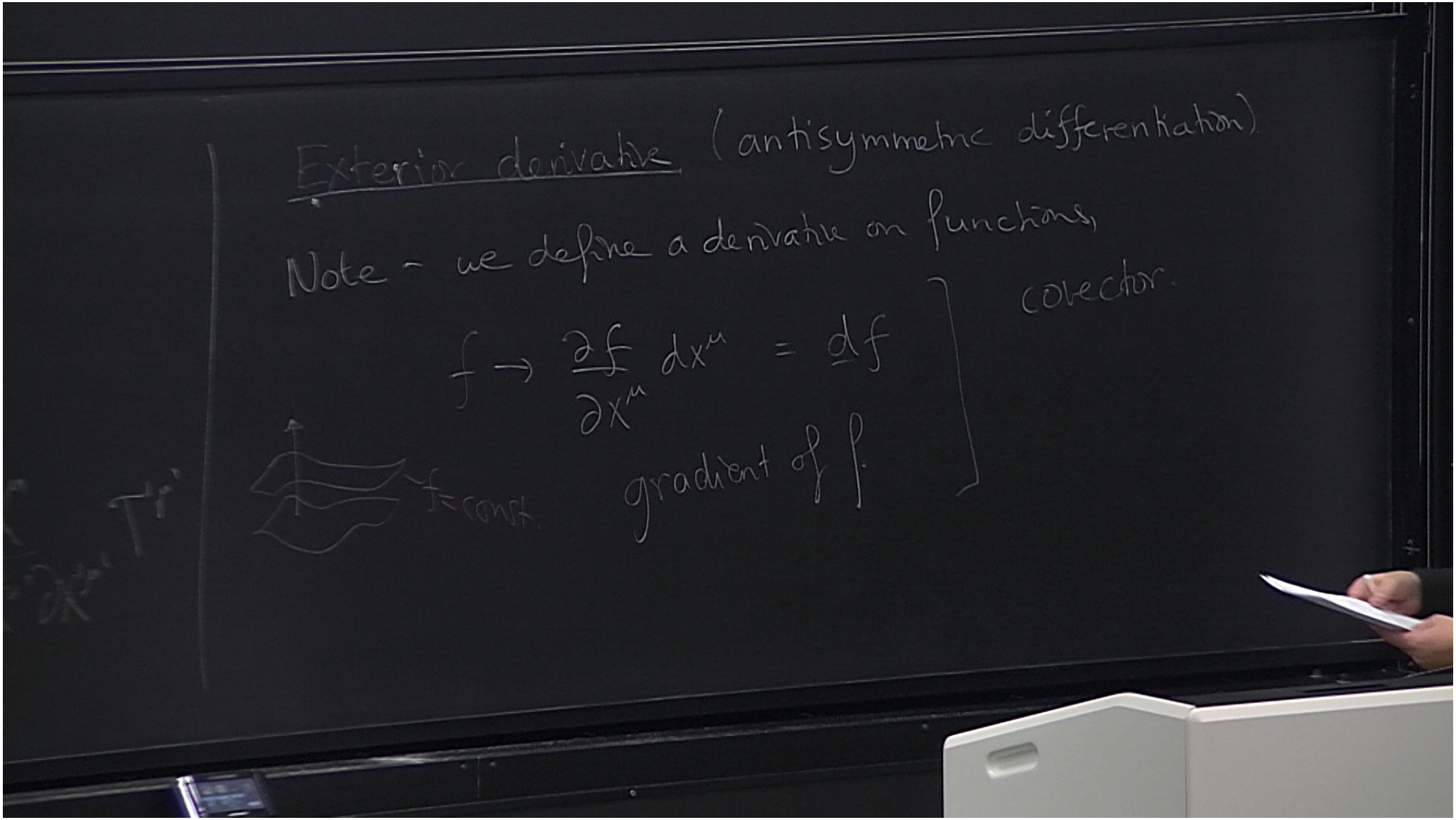

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Exterior derivative (antisymmetric differentiation)


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Note - we define a derivative on functions,
$f \to \frac{\partial f}{\partial x^\mu} dx^\mu = \underline{d} f$


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covector.
gradient of f.
f=const.