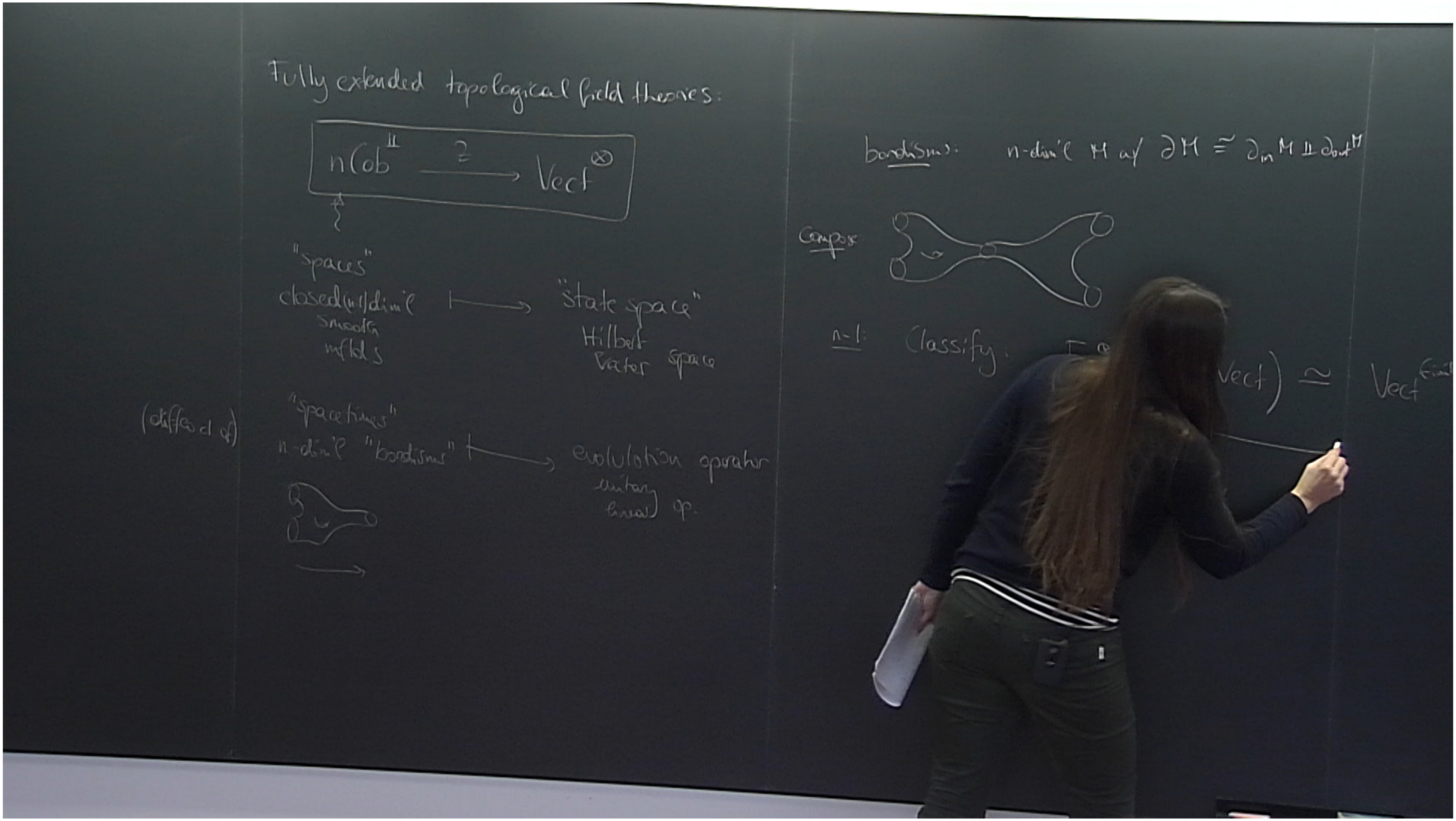
Title: Fully extended functorial field theories and dualizability in the higher Morita category

Date: Dec 11, 2017 02:00 PM

URL: http://pirsa.org/17120020

Abstract: <p>Atiyah and Segal's axiomatic approach to topological and conformal quantum field theories provided a beautiful link between the geometry of "spacetimes" (cobordisms) and algebraic structures. Combining this with the physical notion of "locality" led to the introduction of the language of higher categories into the topic.
Natural targets for extended topological field theories are higher Morita categories: generalizations of the bicategory of algebras, bimodules, and homomorphisms.

After giving an introduction to topological field theories, I will explain how one can use geometric arguments to obtain results on dualizablity in a ``factorization versionâ€™â€™ of the Morita category and using this, examples of low-dimensional field theories â€œrelativeâ€• to their observables. An example will be given by polynomial differential operators, i.e. the Weyl algebra, in positive characteristic and its center. This is joint work with Owen Gwilliam.</p>


Fully extended topological field theories:
nCob ⟶ Vect
"spaces"
closed (n-1) dim'l smooth mflds ⟼ "state space" Hilbert Vector space
"spacetimes"
(diffeo cl of)
n-dim'l "bordisms" ⟼ evolution operator unitary linear op.
bordisms: n-dim'l M w/ ∂M ≅ ∂in M ⊔ ∂out M
Compose
n-1: Classify:
Vect) ≃ Vect

extended topological field theories:

$$\boxed{n\mathrm{Cob}^{\amalg} \xrightarrow{Z} \mathrm{Vect}^{\otimes}}$$

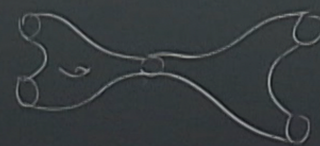
"...ces"

...(n-1)dim'l smooth mflds $\longmapsto$ "state space"
Hilbert
Vector space

"...ce times"
..."bordisms" $\longmapsto$ evolution operator
unitary
linear op.

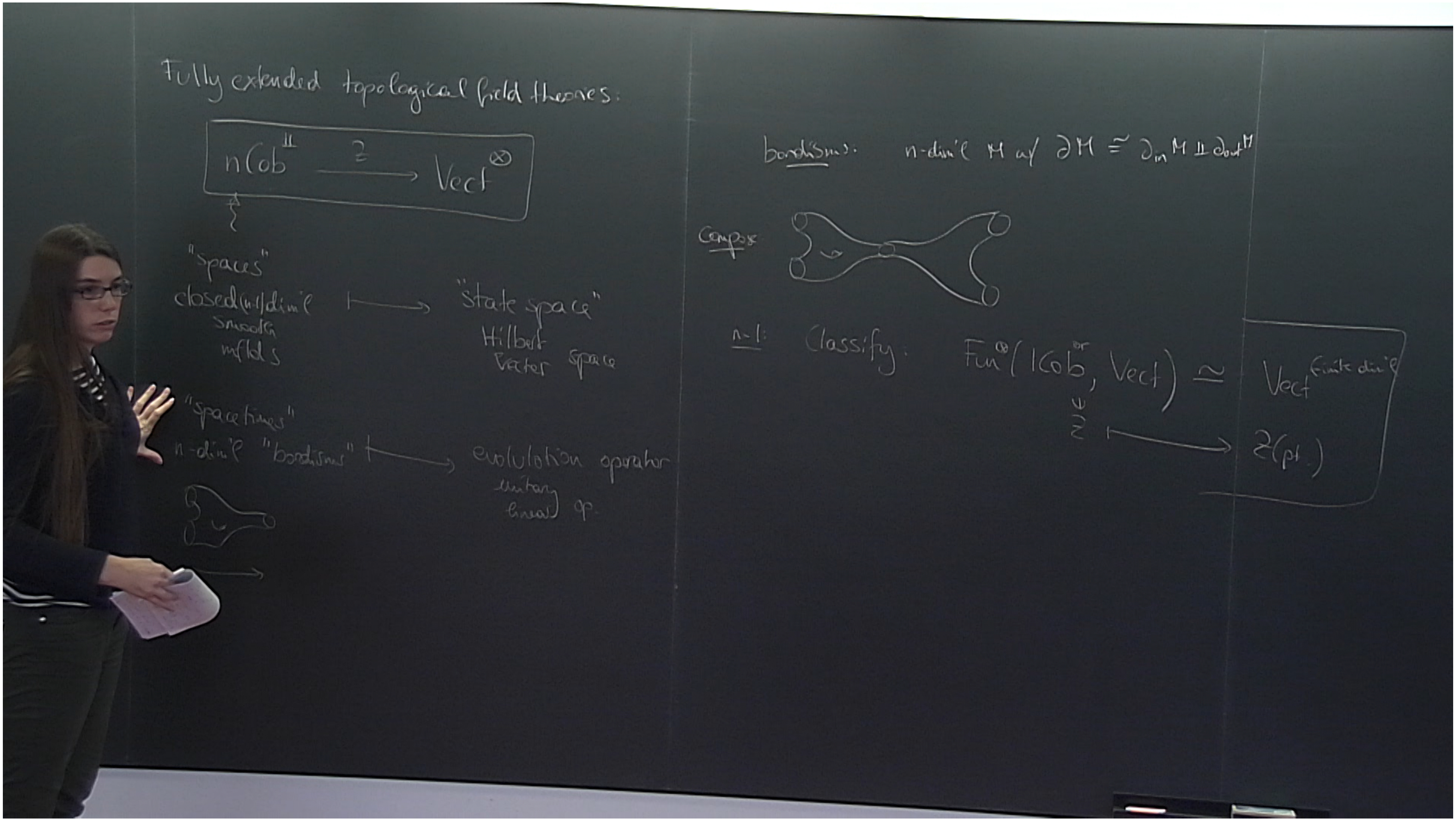
bordisms: n-dim'l M w/ $\partial M \cong \partial_{in} M \amalg \partial_{out} M$

Compose

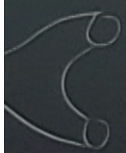


n=1: Classify: $\mathrm{Fun}^{\otimes}(1\mathrm{Cob}^{or}, \mathrm{Vect}) \simeq \boxed{\mathrm{Vect}^{\text{finite dim'l}}}$

$Z \longmapsto Z(pt.)$


Fully extended topological field theories:
nCob^⊔ → Vect^⊗
Z
"spaces"
closed (n-1) dim'l smooth mflds ↦ "state space" Hilbert Vector space
"spacetimes"
n-dim'l "bordisms" ↦ evolution operator unitary linear op.
bordisms: n-dim'l M w/ ∂M ≅ ∂_in M ⊔ ∂_out M
Compose
n=1: Classify: Fun^⊗(1Cob^or, Vect) ≃ Vect^finite dim'l
Z ↦ Z(pt.)

…n'l M w/ $\partial M \cong \partial_{in} M \amalg \partial_{out} M$



$$\mathrm{Fun}^{\otimes}(1\mathrm{Cob}^{or}, \mathrm{Vect}) \simeq \mathrm{Vect}^{\text{finite dim'l}}$$

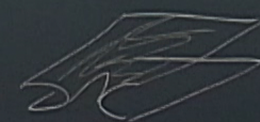
$$Z \longmapsto Z(pt)$$

$\mathrm{Bord}_{n,n-1}$

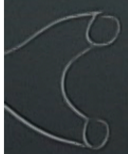
$\rightsquigarrow$

$\mathrm{Bord}_{n,n-1,\dots,n-k,\dots,0}$ add corners.

- objects. 0-dim'l mflds
- 1-mor. 1-dim'l bordisms
- 2-mor 2-dim'l "bordism of bordisms"
- ⋮
- n-mor n-dim'l …



...n'l M w/ $\partial M \cong \partial_{in} M \amalg \partial_{out} M$



$$\mathrm{Fun}^{\otimes}(1\mathrm{Cob}^{or}, \mathrm{Vect}) \simeq \mathrm{Vect}^{\text{finite dim'l}}$$

$$Z \longmapsto Z(pt)$$

$$\mathrm{Bord}_{n,n-1} \rightsquigarrow \mathrm{Bord}_{n,n-1,\dots,n-k,\dots,0}$$

add corners.

"(∞, n)-category"

objects. 0-dim'l mflds

1-mor. 1-dim'l bordisms

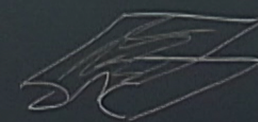
2-mor 2-dim'l "bordism of bordisms"

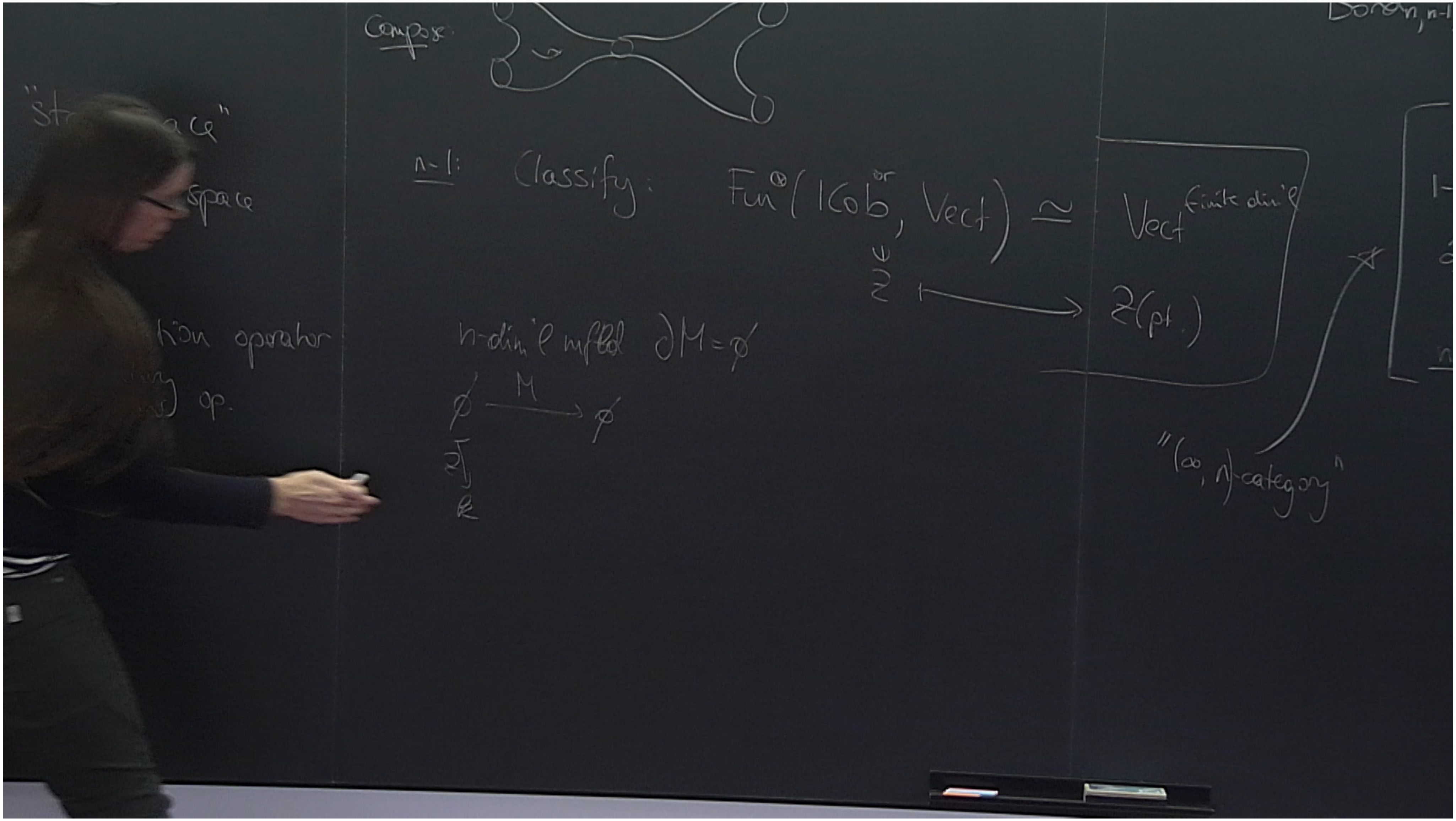
⋮

n-mor. n-dim'l ...

⋮

diffeomorphisms

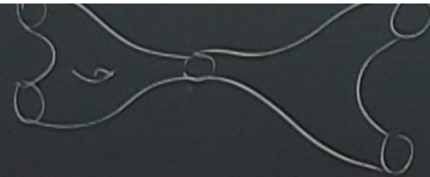



Compose:
n=1: Classify: $\mathrm{Fun}^{\otimes}(1\mathrm{Cob}^{or}, \mathrm{Vect}) \simeq \mathrm{Vect}^{\text{finite dim'l}}$
$Z \longmapsto Z(pt.)$
n-dim'l mfld $\partial M = \emptyset$
$\emptyset \xrightarrow{M} \emptyset$
k
"(∞, n)-category"
space
operator
op.

"state space"
Hilbert
Vector space

→ evolution operator
unitary
linear op.

Compose:



n-1: Classify: $\mathrm{Fun}^{\otimes}(1\mathrm{Cob}^{or}, \mathrm{Vect}) \simeq \mathrm{Vect}^{\text{finite dim'l}}$

$Z \longmapsto Z(pt.)$

"(∞, n)-category"

n-dim'l mfld $\partial M = \emptyset$

$$\begin{array}{ccc} \emptyset & \xrightarrow{M} & \emptyset \\ \downarrow Z & & \downarrow \\ \mathbb{C} & \xrightarrow{Z(M)} & \mathbb{C} \end{array}$$

Bord$_{n,n-1}$

"Fully ext. nTFT"

$\mathrm{Fun}^{\otimes}(\mathrm{Bord}_n, \mathcal{C})$

?

Examples of possible $\mathcal{C}$'s

$\underline{n=2}$: "obj" $\mathbb{C}$-algebras

"1-mor" bimodules

"2-mor"

$$\begin{array}{c} {}_A\circlearrowright M \circlearrowleft_B \\ \downarrow f \\ {}_A\circlearrowright N \circlearrowleft_B \end{array}$$

$\underline{n=3}$.

$\underline{\text{obj}}$: tensor categories $\supseteq$ fusion categories

$\underline{\text{1-mor}}$: bimodule categories

$\underline{\text{2-mor}}$: functors

$\underline{\text{3-mor}}$: nat. transf.

"Fully ext. nTFT"

$$\begin{tikzcd} \mathrm{Fun}^{\otimes}(\mathrm{Bord}_n^{fr}, \mathcal{C}) \arrow[d, "\simeq \text{ of ``}\infty\text{-gpds''}"', "ev_{pt}"] \\ \mathcal{C} \arrow[u, squiggly, bend right, "?"'] \end{tikzcd}$$

$\mathcal{C}$ "n-dualizable"

Examples of possible $\mathcal{C}$'s

n=2: "obj" $\mathbb{C}$-algebras

"1-mor" bimodules

"2-mor"

$$\begin{tikzcd} {}_{A}\circlearrowright M \circlearrowleft_{B} \arrow[d, "f"] \\ {}_{A}\circlearrowright N \circlearrowleft_{B} \end{tikzcd}$$

n=3:

obj: tensor categories $\ni$ fusion categories

1-mor: bimodule categories

2-mor: functors

3-mor: nat. transf.

"Fully ext. nTFT"

$$\begin{array}{c} \mathrm{Fun}^{\otimes}(\mathrm{Bord}_n^{fr}, \mathcal{C}) \\ \text{of } \overset{\simeq}{\text{"}\infty\text{-grpds"}} \Big\downarrow ev_{pt} \quad ? \\ \mathcal{C} \end{array}$$

"n-dualizable"

Examples of possible $\mathcal{C}$'s

$n=2$:

"obj"	$\mathbb{C}$-algebras	"2-dualizable"
"1-mor"	bimodules	$=$ semi-simple/$\mathbb{C}$
"2-mor"	${}_A M_B \xrightarrow{f} {}_A N_B$	+ finite dim'l/$\mathbb{C}$

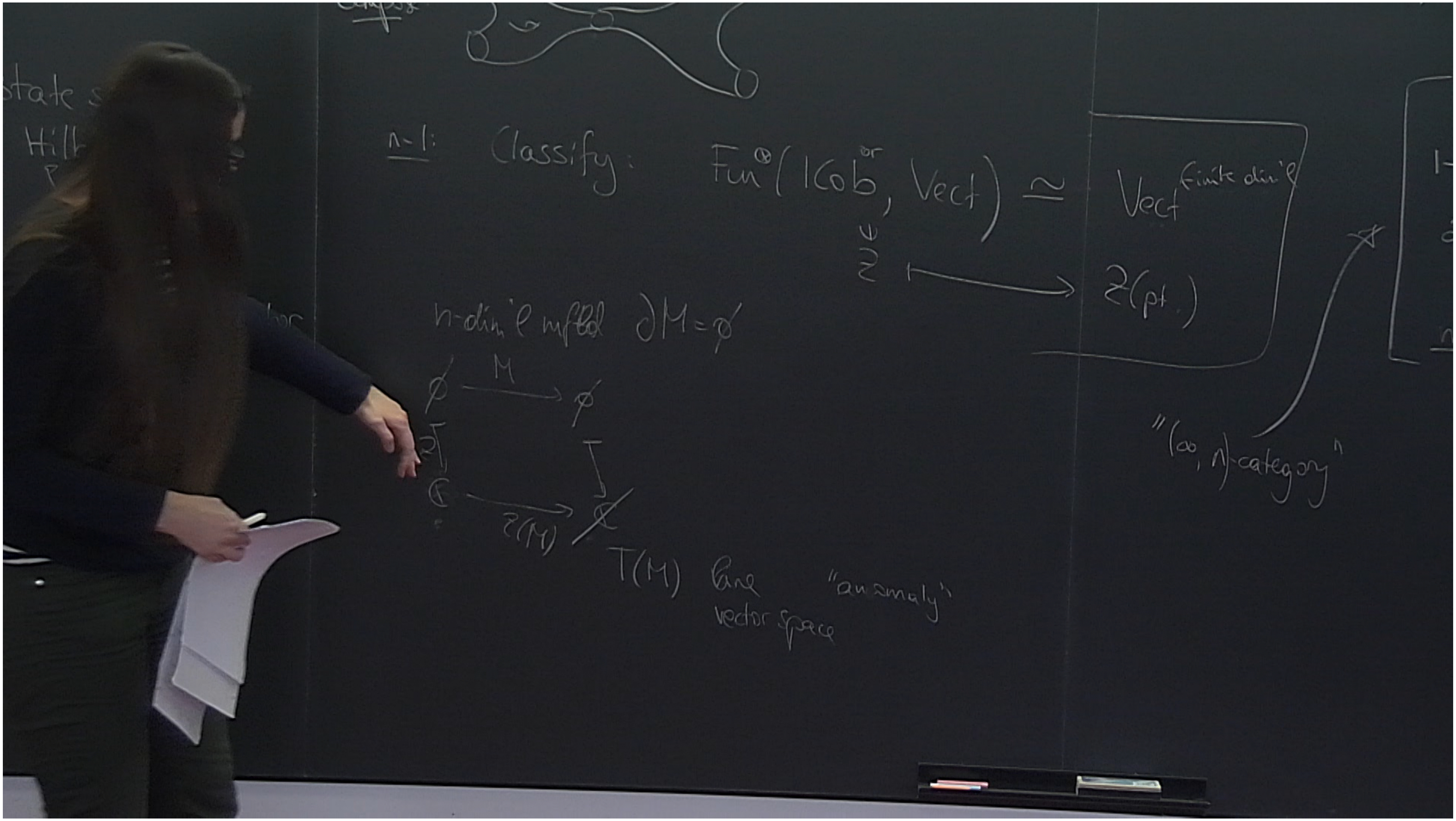
$n=3$:

obj: tensor categories $\ni$ fusion categories

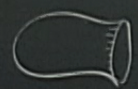
1-mor: bimodule categories

2-mor: functors

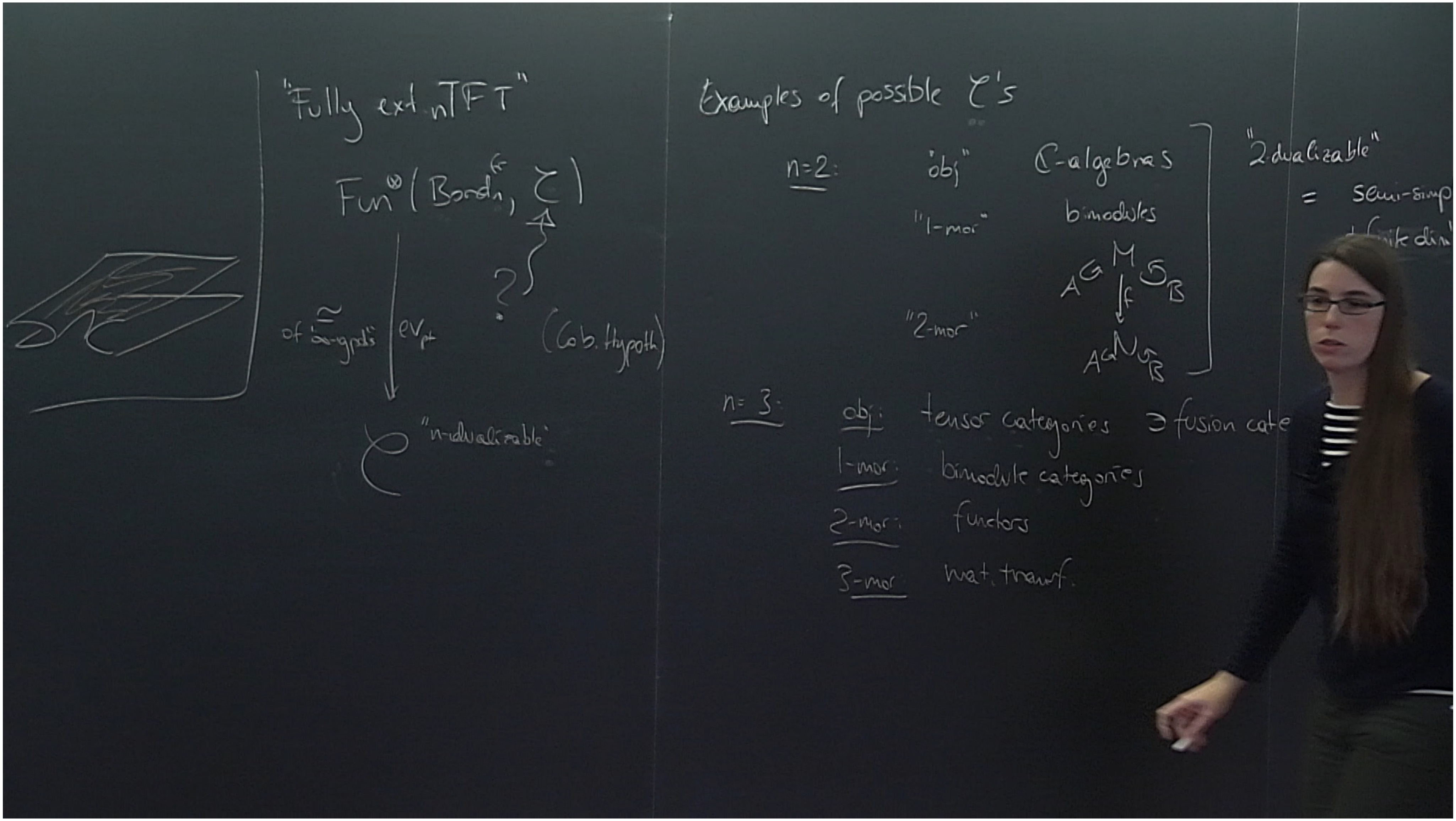
3-mor: nat. transf.


n-1:
Classify:
Fun⊗(1Cob^or, Vect) ≃ Vect^finite dim'l
Z
Z(pt.)
n-dim'l mfld ∂M=∅
M
Z(M)
T(M)
line
vector space
"anomaly"
"(∞, n)-category"

Want: "relative notion of (T) field theories"

 $\rightsquigarrow$ Stolz-Teichner "twisted field theory"

Bord_n


"Fully ext. nTFT"
Fun⊗(Bord, C)
of ∞-grpd's
≃
ev
?
(Cob. Hypoth)
"n-dualizable"
Examples of possible C's
n=2:
"obj"
C-algebras
"1-mor"
bimodules
"2-mor"
"2 dualizable"
= semi-simp
n=3:
obj:
tensor categories
∋ fusion cate
1-mor:
bimodule categories
2-mor:
functors
3-mor
nat. transf.

Want: "relative notion of (T) field theories"

$\rightsquigarrow$ Stolz - Teichner "twisted field theory"

$$\begin{tikzcd} \mathrm{Bord}_n \arrow[r, bend left, "\mathcal{C}"] \arrow[r, bend right, "T"'] \arrow[r, phantom, "\Downarrow"] & \tilde{\mathcal{C}} = (n+1)\text{-Vect} \end{tikzcd}$$

(op)lax nat transf

higher cat: Theo-$\mathcal{C}$

Characterization (uses Cob hyp)

Equiv. are

$$\left\{ \begin{tikzcd} \mathrm{Bord}_n^{fr} \arrow[r, bend left, "\mathcal{C}"] \arrow[r, bend right, "T"'] \arrow[r, phantom, "\Downarrow"] & \tilde{\mathcal{C}} \end{tikzcd} \right\} \xrightarrow{\text{Ext}} \left\{ \mathcal{C} \longrightarrow T(pt) \text{ in } \tilde{\mathcal{C}} \right\}$$

w/ certain properties

$\boxed{\text{Ex:}}$ $\underline{n=1}$:

$$\begin{tikzcd} 1\mathrm{Cob}^{or} \arrow[r, bend left, "\mathcal{C}"] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \mathrm{Alg} \end{tikzcd}$$

$\boxed{\text{Ex:}}$ $\underline{n=1}$:

$$\begin{tikzcd} 1\mathrm{Cob}^{or} \arrow[r, bend left, "\mathbb{C}"] \arrow[r, bend right, "A"'] \arrow[r, phantom, "\Downarrow"] & \mathrm{Alg} \end{tikzcd}$$

$\mathbb{C} \longrightarrow A$ morphism in Alg $\longleftrightarrow$ ${}_{\mathbb{C}}M_A$

the property is: M is finitely pres + proj / $\mathbb{C}$ ("lax")

A ("oplax")

$\to T(pt)$ in $\mathcal{C}$

$\boxed{\text{Ex:}}$ $\underline{n=1}$:

$$1\mathrm{Cob}^{or} \begin{tikzcd} {} \arrow[r, bend left, "\mathbb{C}"] \arrow[r, bend right, "A"'] \arrow[r, phantom, "\Downarrow"] & {} \end{tikzcd} \mathrm{Alg}$$

$\mathbb{C} \longrightarrow A$ morphism in Alg $\longleftrightarrow$ ${}_{\mathbb{C}}M_{A}$

the property is: M is finitely pres + proj $/\,\mathbb{C}$ ("lax")

A ("oplax")

$\underline{\text{Ex}}$: ① ${}_{\mathbb{C}}A_{A}$ $\rightsquigarrow$ A determines oplax thy!

$\boxed{\text{Ex:}}$ $\underline{n=1}$:

$$\begin{tikzcd} 1\mathrm{Cob}^{or} \arrow[r, bend left=50, "\mathbb{C}"] \arrow[r, "A" description] \arrow[r, bend right=70, "\mathbb{C}"'] \arrow[r, phantom, shift left=2.5ex, "\Downarrow"] \arrow[r, phantom, shift right=3ex, "\Downarrow"] & \mathrm{Alg} \end{tikzcd}$$

$\mathbb{C} \longrightarrow A$ morphism in Alg $\longleftrightarrow$ ${}_{\mathbb{C}}M_{A}$

the property is: M is finitely pres + proj $/ \mathbb{C}$ ("lax")

A ("oplax")

$\underline{\text{Ex}}$: ① ${}_{\mathbb{C}}A_{A}$ $\rightsquigarrow$ A determines oplax thy!

$\boxed{\text{Ex:}}$ $\underline{n=1:}$

$$\begin{tikzcd} 1\mathrm{Cob}^{or} \arrow[r, bend left=60, "\mathbb{C}"{name=U}] \arrow[r, "A"{name=M, description}] \arrow[r, bend right=80, "\mathbb{C}"'{name=D}] & \mathrm{Alg} \arrow[from=U, to=M, phantom, "\Downarrow"] \arrow[from=M, to=D, phantom, "\Downarrow"] \end{tikzcd}$$

$\mathbb{C} \longrightarrow A$ morphism in Alg $\longleftrightarrow$ ${}_{\mathbb{C}}M_A$

the property is: M is finitely pres + proj$/\mathbb{C}$ ("lax")

A ("oplax")

$\underline{\text{Ex:}}$ ① ${}_{\mathbb{C}}A_A$ $\rightsquigarrow$ A determines oplax thy!

② V v.sp. ${}_{\mathrm{End}\,V}V_{\mathbb{C}}$ $\rightsquigarrow$ $\underline{\text{always}}$ gives an oplax 1TFT

$\boxed{\text{Ex:}}$ $\underline{n=1}$:

$$\begin{tikzcd} 1\mathrm{Cob}^{or} \arrow[r, bend left, "\mathbb{C}"] \arrow[r, phantom, "\Downarrow"] \arrow[r, bend right, "A"'] \arrow[r, bend right=80, "\mathbb{C}"'] & \mathrm{Alg} \end{tikzcd}$$

$\mathbb{C} \longrightarrow A$ morphism in Alg $\longleftrightarrow$ ${}_{\mathbb{C}}M_A$

the property is: M is finitely pres + proj$/\mathbb{C}$ ("lax")

A ("oplax")

$\underline{\text{Ex:}}$ (1) ${}_{\mathbb{C}}A_A$ $\rightsquigarrow$ A determines oplax thy!

(2) V v.sp. ${}_{\mathrm{End}\,V}V_{\mathbb{C}}$ $\rightsquigarrow$ $\underline{\text{always}}$ gives an lax 1TFT

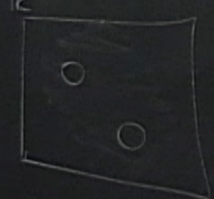
orator

$n=2$ Need 3-cat target!

$$\begin{tikzcd} \mathrm{Bord}_2^{fr} \arrow[r, bend left, "Z"] \arrow[r, bend right, "T"'] \arrow[r, phantom, "\Downarrow"] & \mathrm{Alg}_2 \end{tikzcd}$$

objects: E_2-algebra

$\mathbb{R}^2$



Ex: ·) braided monoidal category

·) commutative alg (Vect)

·) P_2-algebra (Ch_k)

· comm. mult.

· Poisson bracket of degree -1

1-mor: bimodules

2-mor: bimodules of bimodules

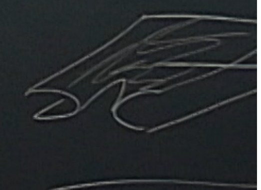
3-mor: homomorphisms

Cor $_A M_B$ determines an (op)lax thy if

① M is finitely pres + proj $/A$

② M is separable $/A$

(f p + p over $M \otimes_A M^{op}$)



-cat target!

$\to$ Alg_e

alge

d monoidal category

ommutative alg (Vect)

b (Ch_k)

mult.

oisson bracket of degree -1

es

Cor ${}_A M_B$ determines an (op)lax thy if

① M is finitely pres + p.g. $/A$

② M is separable $/A$

(f.p + p. over $M \otimes_A M^{op}$)

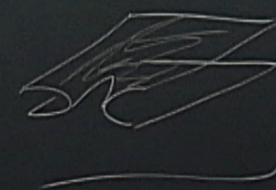
Ex: ② vector space $\rightsquigarrow$ algebra R

$\mathrm{End}(V)$ $\rightsquigarrow$ center $Z(R)$

$HH^*(R)$

$\mathrm{End}_V\, \mathbb{C}$

$Z(R)$ R $\mathbb{C}$ Satisfies ① & ② $\iff$ R is Azumaya $/Z(R)$



"Fully ext. nTFT"

$\mathrm{Fun}^{\otimes}(\mathrm{Bord}_n^{fr}, \mathcal{C})$

$\simeq$ of ∞-grpds $\downarrow \mathrm{ev}_{pt}$?

$\mathcal{C}^{\text{"n-dualizable"}}$

(Cob