Title: Space of Field Theories, UV Completeness, and Integrability

Date: Dec 05, 2017 02:30 PM

URL: http://pirsa.org/17120012

Abstract: <p>Abstract TBA</p>

# Space of Field Theories, UV Completeness, and Integrability

Perimeter Institute, December 2017

1

• UV Completeness: Generally means that QFT has physically sensible behavior at short distances. To recall the nature of the problem, let me bring up the textbook example of

$$QED \quad \text{and} \quad \text{"Landau pole"}$$

As was argued by Landau and others, the observable value of the coupling $\alpha_{\rm ph} = 1/137$ relates to its "bare" value $\alpha_0$ as

$$\alpha_{\rm ph} = \frac{\alpha_0}{1 + \beta_2\, \alpha_0 \log\frac{\Lambda}{M}}, \qquad \alpha_0 = \frac{\alpha_{\rm ph}}{1 - \beta_2\, \alpha_{\rm ph} \log\frac{\Lambda}{M}},$$

where $\Lambda$ is UV cutoff energy. When $\Lambda$ increases, the bare coupling $\alpha_0$ must be increased, too, in order to keep $\alpha_{\rm ph}$ fixed. And, as the second form of the equation makes explicit, the cutoff $\Lambda$ has an upper bound (or "Landau scale")

$$\Lambda_* = M\, e^{\frac{1}{\beta_2\, \alpha_{\rm ph}}} \simeq 10^{280}$$

at which $\alpha_0$ diverges, i.e. $1/\alpha_0$ turns to zero, and then becomes negative. Since $\alpha_0$ enters the action as

$$\mathcal{A}_{QCD} \sim \frac{1}{\alpha_0} \int F^2_{\mu\nu} + \ldots$$

Therefore, one is advised to keep $\Lambda$ below $\Lambda_*$, since otherwise the path integral behaves badly for short range field fluctuations. One says that 4D QCD is not UV complete.

It is fair to admit that this conclusion is based on a number of unsupported assumptions. The above equations relating $\alpha_0$ to $\alpha_{\rm ph}$ are derived if one neglects all terms in the beta function

$$d\alpha/d\log\mu = \beta(\alpha) = \beta_2\,\alpha^2 + ...$$

beyond the leading one. When $\alpha$ grows, higher terms become significant. Moreover, expansion in $\alpha$ is expected to have zero radius of convergence.

Nonetheless, the described behavior is believed to be qualitatively correct. In particular, it agrees well with general pattern suggested by Wilson's RG. In general setting, RG transformations act in

$$\Sigma \;=\; \text{the Space of Field Theories}$$

which may be regarded as the set of quasilocal actions

$$\mathcal{A} = \int_x \mathcal{L}\left(\phi(x), \partial_\mu\phi(x), \partial_\mu\partial_\nu\phi(x), ...\right)$$

where $\phi(x)$ stands for some number of "fundamental fields" - the integration variables in the path integral. The path integral is assumed to have UV cutoff $\Lambda$. The action may include higher derivatives, but it is assumed that the derivative expansion converges for $|k| < \Lambda$. (At this point question of unitarity is ignored) verges for $|k| < \Lambda$. (At this point question of unitarity is ignored)



RG transformations represent scale transformations in field theory. If one simply rescales $x \to x/L$ with some $L > 1$, that would also change the cutoff $\Lambda \to L\Lambda$, that is one also needs to integrate out degrees of freedom with $|k|$ between $\Lambda$ and $L\Lambda$. This leads to $L$-dependent transformation of the action $\mathcal{A} \to R_L\{\mathcal{A}\}$, or, in infinitesimal form

$$\frac{d}{dl}\mathcal{A} = B\{\mathcal{A}\}\,, \qquad l = \log L$$

Assuming that $\Sigma$ may be coordinatized with an (generally infinite) set of parameters $\{\alpha^i\}$, this translates to a system of ordinary differential equations

$$\frac{d\alpha^i}{dl} = B^i(\{\alpha\}) \qquad (B^i = -\beta^i)$$

The "RG trajectories" - the integral curves of this system of differential equations - give insight into the scale dependence of physics. Large scale properties are obtained by integrating the RG equations <u>forward</u> in "RG time" $l$ (= log of the length scale).

The problem may be mathematically difficult, but one does not expect any pathologies $\mathcal{A}_l$ with $l > 0$, as long as $\mathcal{A}_0$ is well defined. Indeed, it is assumed that path integral with $\mathcal{A}_0$ is convergent and well defined as QFT. Integrating out part of variables is not likely to change that.

However, as the system of differential equations, the RG flow equation can be integrated "backward" in $l$ as well.

**Remark**: This would be true if the number of couplings $\alpha^i$ was finite. In exact RG this number - the dimensionality of $\Sigma$ - is infinite. There is an interesting question if the common properties of finite-dimensional systems (like uniqueness of solution) remain generally valid when this number becomes infinite. It is usually assumed this is the case - after all, in any practical implementation of RG transformation some finite-dimensional approximation is used. But it may be one of "dangerous" assumptions. Generally, it is not exactly clear how such "backward" integration squares with not exactly clear how such "backward" integration squares with generally "irreversible" nature of the RG transformations.



Even though it is likely possible to integrate "backward", there are no reasons to expect that this can be done indefinitely, without encountering pathologies. In fact, one expects exactly opposite. Indeed, consider $\mathcal{A}_{-l}$ with $l >> 1$. If $\mathcal{A}_{-l} \in \Sigma$, then $\mathcal{A}_0$ with cutoff $\Lambda$ is essentially equivalent to another theory with much larger cutoff $e^l\,\Lambda$, i.e. with much shorter interaction range then $\mathcal{A}_0$ itself has. This is generally unlikely, which is to say that generally $\mathcal{A}_{-l}$ leaves $\Sigma$ for sufficiently large $l$.

This picture assumes that $\Sigma$ has a boundary separating "well defined" actions from the "there be dragons" expanse; beyond the boundary all "pathological" $\mathcal{A}$ lie. $\Sigma$ includes the theories in which the action $\mathcal{A}[\phi]$ is (a) quasilocal, and (b) the path integral $\int [\mathcal{D}\phi]\, e^{-\mathcal{A}[\phi]}$ converges. Outside $\Sigma$ one or both of these properties is violated.

Thus, given $A_0$, one generally expects that at some $l = -l_*$ the

Thus, given $\mathcal{A}_0$, one generally expects that at some $l = -l_*$ the theory $\mathcal{A}_{-l}$ crosses that boundary, and then leaves $\Sigma$. If this happens at finite $l_*$, we say that the theory is "UV incomplete".

6

There is, of course, small but important subspace $\Sigma(\infty) \subset \Sigma$ for which $l_* = \infty$, i.e. the RG flow can be integrated "backwards" without limit (e.g. the flows which stem from UV fixed points, but more complicated scenarios are possible). This is the subspace of UV complete QFT, in which UV cutoff can be consistently removed.

Note that, given arbitrary $\mathcal{A}_0 \in \Sigma$, the corresponding $l_*$ is determined solely by $\mathcal{A}_0$ itself,

$$l_* = l_*(\mathcal{A})\,.$$

Then, if $l_* < \infty$, the theory $\mathcal{A}_0$ has an intrinsic UV scale

$$\epsilon_* = \Lambda_*^{-1}\,, \qquad \Lambda_* = M\, e^{l_*}$$

where $M$ is some "physical" mass scale, say the inverse correlation length $R_c^{-1}$. The "intrinsic" cutoff $\Lambda_*$ is independent of the "auxiliary" cutoff $\Lambda$ (more precisely, the ratio $\Lambda_*/M$ is RG invariant).

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In some sense, $\Lambda_*$ sets the upper limit for the cutoff $\Lambda$ - if one sets $\Lambda > \Lambda_*$, some kind of pathology is expected at the distances $< \epsilon_*$. In the example of QED $\Lambda_*$ is the "Landau scale".



Below I will describe an example (or class of examples) in $D = 2$, which admits some sort of "exact solution" - the " $(T\bar{T})$ flow". Some preparations are needed.

Consider arbitrary 2D $QFT \in \Sigma$. We will discuss in flat 2D Euclidean Space, and use notation $z$ for its points. Complex coordinates $(\mathrm{z}, \bar{\mathrm{z}})$

$$z = (\mathrm{z}, \bar{\mathrm{z}}) : \qquad \mathrm{z} = \mathrm{x} + i\mathrm{y} , \qquad \bar{\mathrm{z}} = \mathrm{x} - i\mathrm{y}$$

will be used. I will write local fields as $\mathcal{O}(z) = \mathcal{O}(\mathrm{z}, \bar{\mathrm{z}})$.

The QFT conserves energy and momentum, and the associated local densities constitute the Energy-Momentum Tensor $T_{\mu\nu}$. We assume

$$T_{\mu\nu} = T_{\nu\mu} , \qquad \partial_\mu T^{\mu\nu} = 0 .$$

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Below I use CFT-inspired notations

$$T = -(2\pi)\,T_{\mathsf{zz}}\,, \qquad \bar{T} = -(2\pi)\,\bar{T}_{\bar{\mathsf{z}}\bar{\mathsf{z}}}\,, \quad \text{and} \quad \Theta = (2\pi)\,T_{\mathsf{z}\bar{\mathsf{z}}}\,,$$

in which the continuity equation takes the form



$$\partial_{\bar{\mathsf{z}}}\,T = \partial_{\mathsf{z}}\,\Theta\,, \qquad \partial_{\mathsf{z}}\,\bar{T} = \partial_{\bar{\mathsf{z}}}\,\Theta\,.$$

Some properties of the operator products of these fields follow from the above equations alone. The one important to me now is

$$T(z)\bar{T}(z') - \Theta(z)\Theta(z') = \; X(z') + \text{derivatives}\,,$$

where

$$\text{"derivatives"} = \sum_i C^{i,\mu}(z-z')\,\partial_\mu \mathcal{O}_i(z')\,.$$

The first, non-derivative term comes with the OPE coefficient $C^X(z-z') = 1$. This follows from the identity

$$\partial_{\bar{z}}\left(T(z)\bar{T}(z') - \Theta(z)\Theta(z')\right) =$$
$$(\partial_z + \partial_{z'})\left(\Theta(z)\bar{T}(z')\right) - (\partial_{\bar{z}} + \partial_{\bar{z}'})\left(\Theta(z)\Theta(z')\right),$$

and similar identity for $\partial_z(...)$.



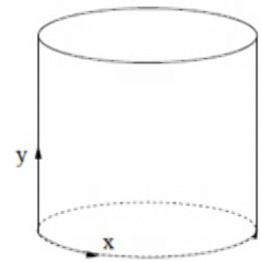
This uniquely defines a scalar local field

$$X(z) \equiv (T\bar{T})(z),$$

uniquely up to derivatives. Since scalar fields (modulo derivatives) are vectors in the tangent space $T\Sigma|_{QFT}$, it defines uniquely a tangent vector

$$X \in T\Sigma|_{QFT}.$$

A number of consequences follow. For instance, consider given QFT in the geometry of a cylinder, with the spatial coordinate compactified on a circle, $x \sim x + R$,






Then the Hamiltonian has discrete spectrum (I assume that the QFT is compact, for simplicity) and I denote $|n\rangle$ the corresponding eigenstates. Assuming that $|n\rangle$ is non-degenerate, one easily finds

$$\langle n \mid (T\bar{T})(z) \mid n\rangle = \langle n \mid T(z) \mid n\rangle\langle n \mid \bar{T}(z) \mid n\rangle - \langle n \mid \Theta(z) \mid n\rangle^2 .$$

The expectation values in the r.h.s. can be expressed in terms of the eigenvalues

$$E_n(R)\,, \qquad P_n(R) = \frac{2\pi\, k_n}{R}\,.$$

Using

$$\langle\, n \mid T_{\mathrm{yy}} \mid n\,\rangle = -\frac{1}{R}\, E_n(R)\,, \qquad \langle\, n \mid T_{\mathrm{xx}} \mid n\,\rangle = -\frac{d}{dR}\, E_n(R)\,,$$

$$\langle n \mid T_{xy} \mid n \rangle = \frac{i}{R} P_n(R)\,.$$

one derives

$$\langle n \mid (T\bar{T}) \mid n \rangle = -\frac{\pi^2}{R}\left(E_n(R)\frac{d}{dR}E_n(R) + \frac{1}{R}P_n^2(R)\right).$$

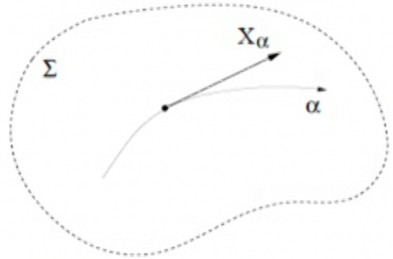
11

## What is it good for?

Define "$(T\bar{T})$ flow": Consider a curve $\mathcal{A}_\alpha$ in $\Sigma$, with $\alpha$ being the parameter along the curve, such that at each point the tangent vector to the curve is $\sim X = (T\bar{T})$,

$$\frac{d\mathcal{A}_\alpha}{d\alpha} = \frac{1}{\pi^2}\int (T\bar{T})_\alpha(z)\, d^2z\,,$$

where the subscript in $X_\alpha \equiv (T\bar{T})_\alpha$ is added to emphasize that the field belongs to the QFT $\mathcal{A}_\alpha$:




I call the curve $\mathcal{A}_\alpha$ the "$(T\bar{T})$ flow".

12

Since at any point of the curve

$$\frac{\partial E_n(R,\alpha)}{\partial \alpha} = \langle n \mid \int (T\bar{T})_\alpha(z)\, d\mathsf{x} \mid n\rangle = R\,\langle n \mid (T\bar{T})_\alpha \mid n\rangle_\alpha$$

one arrives at closed differential equation

$$\frac{\partial}{\partial\alpha} E(R,\alpha) + E(R,\alpha)\,\frac{\partial}{\partial R} E(R,\alpha) + \frac{P^2(R)}{R} = 0\,.$$

where I've dropped the index $n$ in

$$E(R,\alpha) = E_n(R,\alpha)$$

because the equation is the same for all (non-degenerate) levels.

The equation has the form of equation of motion for compressible inviscid fluid in 1D - the (inviscid) Burgers equation - with the driving force $-P^2/R = (2\pi k)^2/R^3$. Thus, given $E(R, 0)$ one can determine $E(R, \alpha)$ at all $\alpha$ along the curve.

This equation can be used to derive a number of results, including exact $\alpha$-dependence of the bulk energy density and correlation length. Let me derive explicitly the $\alpha$ dependence of the $2 \to 2$ elastic scattering amplitude.

13

Assume that the theory $\mathcal{A}_0$ is massive, and $M_0$ is the mass of the lightest particle. Let $R >> M^{-1}$, and consider $|\, p, -p\rangle$ - the state of two particles, with the momenta $p$ and $-p$. The notion makes sense when the energy is below all inelastic thresholds, which we assume. Then, up to exponentially small finite size corrections

$$E(R, \alpha) = F\, R + 2\sqrt{M^2 + p^2}\,, \quad P(R) = 0\,,$$

where admissible momenta are determined by the quantization condition

condition

$$p\,R + \Delta(p) = 2\pi\,N\,.$$

This asymptotic form of $E(R,\alpha)$ satisfies the diff. equation iff

$$\frac{\partial p}{\partial \alpha} = -2\omega\,\frac{\partial p}{\partial R}\,, \qquad \omega = \sqrt{M^2 + p^2}\,.$$

Then, from the quantization condition

$$\frac{\partial \Delta(p,\alpha)}{\partial \alpha} = -2\,\omega p\,, \quad \Rightarrow \quad \Delta_\alpha(p) = \Delta_0(p) - 2\alpha\,\omega p\,.$$

14

In terms of the rapidity difference $\theta = \theta_1 - \theta_2$

$$\Delta_\alpha(\theta) = \Delta_0(\theta) - \alpha M^2 \sinh\theta\,.$$

In other words, the two-particle elastic S-matrix of $\mathcal{A}_\alpha$ differs from that of $\mathcal{A}_0$ by the factor

$$2 \to 2: \qquad S_\alpha(\theta) = S_0(\theta)\,e^{-i\alpha M^2 \sinh\theta}\,.$$

Note the abnormally fast growth of the $\alpha$-contribution to the elas-
tic scattering phase at high energy

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This analysis is valid for $2 \to 2$ elastic scattering amplitude, but in general it does not apply to multi-particle S-matrix elements because of possible inelastic channels. The exceptions are Integrable field theories, where all scattering processes are essentially elastic, and all S-matrix elements factorize in terms of the $2 \to 2$ amplitudes. If one assumes that $\mathcal{A}_0$ is integrable, it is possible to show that the whole curve consists of integrable field theories. Then the full factorizable S-matrix of $\mathcal{A}_\alpha$ is the product of the above "dressed" $2 \to 2$ S-matrices.



**Remarks**:

($M \to 0$ limit of) the factorizable S-matrix with $S_\alpha(\theta) = \exp\{-i\alpha M \sinh\theta\}$ was proposed by

S. Dubovski, R. Flauger, V. Gorbenko, (2012)

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S. Dubovski, R. Flauger, V. Gorbenko, (2012)

M. Caselle, D. Florevanti, F. Gliozi, R. Tateo (2013)

as the S-matrix of Goldstone excitations on QCD string.

Above we derived the $2 \to 2$ S-matrix in $(T\bar{T})$ flow from the Burgers equation. If the theory is integrable with the above $2 \to 2$ S-matrix, one can, conversely, derive the Burgers equation from the associated NLIE, as in

A. Cavaglia, S. Negro, I. Szecsni, R. Tateo (2016)

Another important conclusion can be made about the UV behavior of the ground state energy $E_\alpha(R) = E(R, \alpha)$. For the ground state $P = 0$, and the Burgers equation admits elementary solution

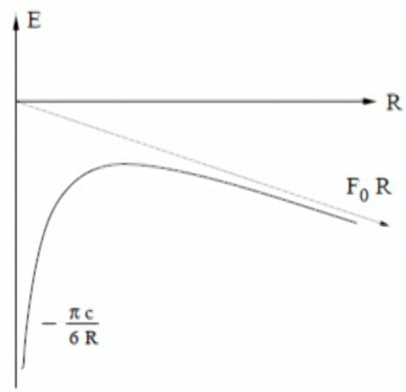
$$E_\alpha(R) = E_0\,(R - \alpha\, E_\alpha(R))\ ,$$

which takes even more transparent form in terms of the functions $R_\alpha(E)$ and $R_0(E)$, inverse to $E_\alpha(R)$ and $E_0(R)$, respectively:

$$R_\alpha(E) = R_0(E) + \alpha\, E\,.$$

The plot of $R_\alpha(E)$ is related to the plot of $R_0(E)$ by the affine transformation $E \to E,\ R \to R + \alpha E$.

Assume that $\mathcal{A}_0$ is UV complete QFT, whose UV limit is controlled by CFT with the central charge $c > 0$. Then the plot of $E_0(R)$ looks, qualitatively, like this




At large $R$ it approaches the linear form $F_0 R$, with $F_0$ being the bulk energy density, while as $R \to 0$ it has the standard CFT behavior. The above affine transformation reveals the following form(s) of $E_\alpha(R)$

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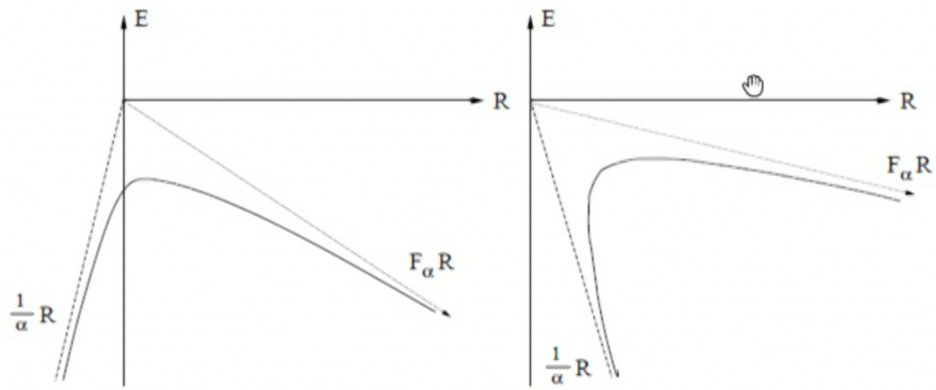


where the left plot applies to the case $\alpha > 0$, while the right plot corresponds to $\alpha < 0$.

There are reasons to believe that, when originated at UV complete $\mathcal{A}_0$, the $(T\bar{T})$ flow is ill-defined (has no ground state) at $\alpha > 0$ (I will bring up some argument later in this talk). Therefore, here I concentrate on the case of negative $\alpha$. As the plot shows, in this case $E_\alpha(R)$ develops square-root singularity at some positive $R = R_*$, so that continuation to $R < R_*$ returns complex values.

For example, if $\mathcal{A}$ is itself a CFT, i.e.

$$E_0(R) = F_0\,R - \frac{\pi c}{6R},$$

then

$$E_\alpha(R) = F_\alpha\,R + \frac{R}{2\tilde{\alpha}}\left(1 - \sqrt{1+t}\right),$$

where $F_\alpha = F_0/(1+\alpha F_0)$, $\tilde{\alpha} = \alpha\,(1+\alpha F_0)$, and

$$t = \frac{2\pi c\,\alpha}{3\,R^2}.$$

The singularity occurs at $t = -1$, i.e. at

$$R_* = \sqrt{-\frac{2\pi c\alpha}{3}}.$$

Clearly, some sort of instability develops at the scales $< R_*$, and it seems important to understand the mechanism behind this instability. Anyway, it seems that the flow $\mathcal{A}_\alpha$ originating from UV complete $\mathcal{A}_0$ is not UV complete in usual sense.

In the case when $\mathcal{A}_0$ is integrable ($\Rightarrow \mathcal{A}_\alpha$ is integrable), potential problem with locality can be inferred from the behavior of form factors. To keep things as simple as possible, let me limit attention to the two-particle form factors, the matrix elements

$$F_{\mathcal{O}}(\theta_1 - \theta_2) = \langle 0 \mid \mathcal{O}(0) \mid A(\theta_1)A(\theta_2)\rangle \,.$$

Also, I assumed for simplicity that there is only one sort of particles $A$, with the mass $M$. The form factor satisfies the well known "bootstrap" equations

$$F(\theta) = F(2\pi i - \theta)\,, \qquad F(\theta) = S(\theta)F(-\theta)\,,$$

where I disregard the suffix "$\mathcal{O}$", since the equations are the same for all $\mathcal{O}$. Let, as above

$$S_\alpha(\theta) = S_0(\theta)\, e^{-i\alpha M^2 \sinh\theta}\,.$$



The equations for $F$ have many solutions, but the simplest one is

$$F_\alpha(\theta) = F_0(\theta)\ \exp\left\{\frac{\alpha}{2\pi}\,(\theta - i\pi)\ \sinh\theta\right\}$$

Note that the $\alpha$-factor decays (at negative $\alpha$) along the real axis, and, also in the strip $0 < \Im m\theta < \pi/2$, but grows fast at $\pi/2 < \Im m\theta < \pi$. Since the intermediate-state decompositions of the products

$$\mathcal{O}(x)\mathcal{O}(y) \qquad \text{and} \qquad \mathcal{O}(y)\mathcal{O}(x)$$

are related by the shifts of the rapidity integration contours through this strips, it is not clear how local commutativity can be satisfied.

In the special case of integrable theories there is an interesting extension of the notion of the $(T\bar{T})$ flow.

## Integrable QFT (IQFT)

One of the common properties of all known IQFT is the presence of an infinite set of higher-spin local Integrals of Motion (IM)

$$P_s = \frac{1}{2\pi}\int_C T_{s+1}(z)\,d\mathrm{z} + \Theta_{s-1}(z)\,d\bar{\mathrm{z}}$$
$$\bar{P}_s = \frac{1}{2\pi}\int_C \bar{T}_{s+1}(z)\,d\bar{\mathrm{z}} + \bar{\Theta}_{s-1}(z)\,d\mathrm{z}$$

where $(T_{s+1}, \Theta_{s-1})$ and $(\bar{T}_{s+1}, \bar{\Theta}_{s-1})$ are components of local currents which satisfy the continuity equations

$$\partial_{\bar{\mathrm{z}}} T_{s+1}(z) = \partial_{\mathrm{z}}\Theta_{s-1}(z)\,, \qquad \partial_{\mathrm{z}}\bar{T}_{s+1}(z) = \partial_{\bar{\mathrm{z}}}\bar{\Theta}_{s-1}(z)\,.$$

The index $s$ (associated with the Lorentz spin of the IM) runs certain subset $\{s\} \subset \mathbb{Z}_+$, characteristic of the IQFT. The IM all commute

commute

$$[P_s, P_{s'}] = [P_s, \bar{P}_{s'}] = [\bar{P}_s, \bar{P}_{s'}] = 0\,.$$

23

From these continuity equations, exactly as in the case of the Energy-Momentum tensor, one can derive the relations

$$T_{s+1}(z)\bar{T}_{s-1}(z') - \Theta_{s-1}(z)\bar{\Theta}_{s-1}(z') = X_s(z') + \text{derivatives}\,,$$

which define, up to derivatives, the scalar fields $X_s(z)$. Thus, it define $X_s$ as vectors in $T\Sigma|_{IQFT}$. Moreover, it can be shown that the infinitesimal deformations

$$\mathcal{A} \;\rightarrow\; \mathcal{A} + \sum_{s\in\{s\}} \delta\alpha_s \int X_s(z)\, d^2z$$

preserves all the IM $P_s$, $\bar{P}_s$. Starting from some IQFT $\mathcal{A}_0$, one can integrate the infinitesimal deformations into infinite-dimensional subspace $\Sigma^{\text{Int}} \subset \Sigma$, with local coordinates $\alpha_s$, so that

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$$X_s \in T\Sigma^{\text{Int}}|_{\text{IQFT}}\,.$$

Thus, in the integrable case, the notion of $(T\bar{T})$ flow can be extended to an infinite-dimensional space.

If IQFT is massive, it can be uniquely associated with a factorizable S-matrix. The full S-matrix is expressed in terms of the

$$2 \to 2 \quad \text{amplitude} \quad \widehat{S}(\theta)\,, \qquad \theta = \theta_1 - \theta_2$$

(can be an operator in the particle's "flavor" spaces). It satisfies

- Yang-Baxter equation
- Unitarity + Crossing
- "Bootstrap equations".

These conditions fix $\widehat{S}(\theta)$ up to the "CDD factor", i.e. leaves the ambiguity

$$\widehat{S}(\theta) \to \widehat{S}(\theta)\,\Phi(\theta)\,,$$

where the factor $\Phi$ is to satisfy

$$\left\{ \begin{array}{c} \Phi(\theta) = \Phi(i\pi - \theta) \\ \Phi(\theta)\Phi(-\theta) \;=\; 1 \end{array} \right\} \quad \Rightarrow \quad \Phi(2\pi i + \theta) = \Phi(\theta)\,,$$

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plus possibly additional constraints from the bound-state structure (the bootstrap conditions).

Formal but general representation

$$\Phi(\theta) = \exp\Big\{ -i \sum_{s\in\{s\}} \alpha_s \sinh(s\theta)\Big\}$$

(converges for sufficiently small $\theta$, and defined by analytic continuation beyond this domain). Here $s$ runs positive integers, but the bootstrap conditions limit the admitted values to some infinite subset $\{s\} \subset \mathbb{Z}_+$, which always coincides with the set of spins of the local IM $P_s$. The curve $\{\alpha_s = 0, s > 1; \alpha_1 = \alpha\}$ reduces to the $(T\bar{T})$ flow. Generally

| Infinitesimal CDD deformations of $\hat{S}$ | $\leftrightarrow$ | $\{X_s\}$ deformations of $\mathcal{A}_{\text{IQFT}}$ |
|---|---|---|

Parameters $\{\alpha_s\}$ can be regarded as local coordinates in the space of factorizable S-matrices ($= \Sigma^{\text{Int}}$). Alternative coordinates $\{B\}$ are given by the conventional representation

$$\Phi(\theta) = \prod_{p=1}^{N} \frac{B_p - i\,\sinh\theta}{B_p + i\,\sinh\theta}, \qquad \{B\} = \cup_N \{B_p\}\,.$$

Note that for finite $N$ such CDD factors have normal high energy behavior.

It is expected that, just as in $\Sigma$ itself, majority of QFT in $\Sigma^{\text{Int}}$ are not UV complete. On the other hand, as was observed, all elements of $\Sigma^{\text{Int}}$ are in correspondence with factorizable S-matrices. Generally, S-matrix determines, in principle, all physical content of the theory, and no UV cutoff needs to be introduced. Then, how the "UV incompleteness" shows up in the S-matrix approach?

Consider again the ground state in a finite size geometry of a cylinder of circumference $R$. In IQFT, given $\hat{S}(\theta)$, one can systematically compute the ground state energy $E_{\text{vac}}(R)$ using the "Thermodynamic Bethe Ansatz" (TBA) equation.

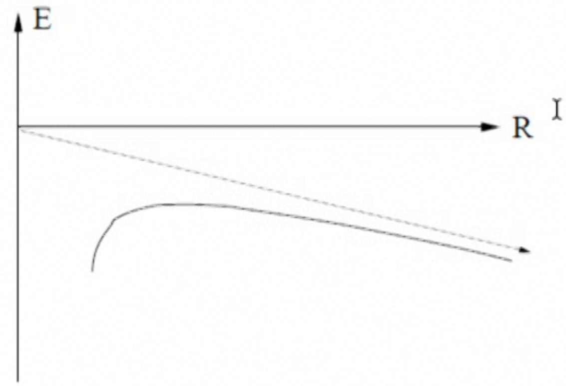
TBA is a device which, with the input of $\widehat{S}(\theta)$, computes (in most cases numerically) the vacuum energy $E_{\text{vac}}(R)$,

$$\text{TBA} : \qquad \widehat{S}(\theta) \quad \rightarrow \quad E_{\text{vac}}(R) \, .$$

Then one can taste the behavior of $E_{\text{vac}}(R)$ for arbitrary CDD factor $\Phi(\theta)$. Technically, one can start with the product forms of the CDD factors, with small number $N$ of the CDD factors, and then go up in $N$.

No systematic analysis was ever done, to the best of my knowledge. However, preliminary calculations of this sort was done in early 90th by Al. Zamolodchikov and myself, inspired by the "staircase model". We studied several sample CDD deformations with small number of factors (up to three). The results could be summarizes as follows:

Unless the parameters $B_p$ are fine-tuned to special values where UV complete behavior is observed (e.g. the "staircase model"), the function $E_{\text{vac}}(R)$ develops a singularity at some finite $R_*$ ,




Moreover, high precision calculation at $R$ close to $R_*$ reveals that in all cases the singularity is a square root branching point.

• These universal character of the singularity suggests the common mechanism of developing of instability. Note that solution for the $(T\bar{T})$ flow exhibits the same singularity.

• Since the finite $N$ CDD do not have abnormal high-energy behavior, it is unlikely that the UV problem is related to the too fast growth of the scattering phase.

Semiclassical analysis:

Some insight can be gained in the case when

- $\mathcal{A}_0 = CFT$, has no extra symmetry beyond Virasoro.
- $c \to \infty$, i.e. $T, \bar{T}$ are classical fields.

I assume that (i) The ground state is determined by some classical configuration $T_{cl}(z), \bar{T}_{cl}(z)$, and (ii) $T_{cl}(z)$ and $\bar{T}_{cl}(z)$ are constants, independent of $z$. In the following discussion I drop the subscript $cl$ under $T, \bar{T}$.

Generally, the currents $T_{s+1}$, $s = 2n-1$ are polynomials in $T$ and its derivatives,

$$T_{s+1} = T^n + a_1\, T^{n-3}(T')^2 + \ldots ,$$

and when the classical configuration is constant, the derivatives can be dropped, so that

$$T_{2n} = T^n\,, \qquad X_{2n-1} = (T\bar{T})^n\,.$$



Therefore, for sufficiently small $\alpha_{2n-1}$, the general $X_s$-deformed action (for the purpose of the ground state calculation) can be replaced by

$$\mathcal{A} = \mathcal{A}_{CFT} + \sum_{n=1}^{\infty} \alpha_{2n-1} \int (T\bar{T})^n \, d^2z = \mathcal{A}_{\mathsf{CFT}} + \frac{1}{\alpha} \int U(\alpha^2 T\bar{T})\, d^2z$$

where I wrote $\alpha_{2n-1} = \alpha^{2n-1}\, C_n$ with dimensionless $C_n$, the Taylor coefficients of $U$. When $\alpha_s$ are finite, the currents $T_{s+1}, \Theta_{s-1}$, etc, receive $\alpha$-dependent corrections, but clearly for constant $T, \bar{T}$ this structure is preserved.

Now, introducing the auxiliary fields $\mu, \bar{\mu}$, one can further replace the above action with

$$\mathcal{A}_\mu - \frac{4}{\alpha} \int W(\mu\bar{\mu})\, d^2z\,, \qquad \mathcal{A}_\mu = \mathcal{A}_{\mathsf{CFT}} + \frac{1}{\pi} \int (T\mu + \bar{T}\bar{\mu})\, d^2z\,,$$

where $W(\mu\bar{\mu})$ is the double (in $T$ and $\bar{T}$) Legendre transform of $U$.

For constant $\mu, \bar{\mu}$

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For constant $\mu, \bar{\mu}$

$$\langle e^{-\mathcal{A}_\mu} \rangle_{\mathsf{CFT}} = e^{-E_\mu L}\,,$$

31

$$E_\mu = -\frac{\pi c}{6R}\left[\frac{1}{1+\mu} + \frac{1}{1+\bar{\mu}} - 1\right],$$

where I assumed the geometry of a cylinder of the circumference $R$ and the length $L$. Finally, the ground state energy is obtained by minimizing the function

$$E(\mu, \bar{\mu}) = -\frac{R}{\alpha}\left[\frac{t}{1+\mu} + \frac{t}{1+\bar{\mu}} - t + W(\mu\bar{\mu})\right]$$

with respect to $\mu$ and $\bar{\mu}$. Here again

$$t = \frac{2\pi c\,\alpha}{3R^2}\,.$$

The problem deserves systematic analysis with different forms of $W$, which is still under way. However, it is possible to identify $W$ associated with the $(T\bar{T})$ flow. It turns out

$$W_{T\bar{T}\ \mathsf{flow}}(\mu\bar{\mu}) = \frac{\mu\bar{\mu}}{1-\mu\bar{\mu}}\,.$$

With this form the above minimization problem yields the vacuum

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$$E_{\text{vac}} = -\frac{R}{2\alpha}\left(\sqrt{1+t} - 1\right),$$

which is exactly our result for the $(T\bar{T})$ flow from the Burgers equation.

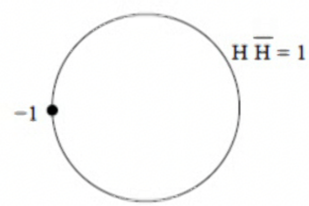
It is interesting to observe the mechanism behind the singularity formation in this model. Looking again at the energy function

$$E(\mu,\bar{\mu}) = -\frac{R}{\alpha}\left[\frac{t}{1+\mu} + \frac{t}{1+\bar{\mu}} - t + \frac{\mu\bar{\mu}}{1-\mu\bar{\mu}}\right]$$

we observe that the configuration space is divided into two distinct domains,

$$M_{\text{in}}: \quad \mu\bar{\mu} < 1 \qquad \text{and} \qquad M_{\text{out}}: \quad \mu\bar{\mu} > 1\,,$$

which are separated by an infinite "barrier" at $\mu\bar{\mu} = 1$.




33

It is natural to assume that the relevant domain is $M_{\text{in}}$, since it is the one that contains the point $(\mu, \bar{\mu}) = 0$, the saddle point at $\alpha = 0$. Then, obviously, with positive $\alpha$ the energy function is unbounded from below in the $M_{\text{in}}$ at any $R$, suggesting that at $\alpha > 0$ the theory is ill defined.

At $\alpha < 0$, instead, the $W$-term is bounded from below in $M_{\text{in}}$, and it grows indefinitely as one approaches the "boundary". However, the other terms are singular at special point $(\mu, \bar{\mu}) = (-1, -1)$ on the boundary, and diverge to $-\infty$ as one approaches this point from $M_{\text{in}}$. Competition of these two term makes $E(\mu, \bar{\mu})$ bounded from below at $R > R_*$, but it looses the lower bound at $R < R_*$. Thus, at $R < R_*$ the theory is unlikely to have a ground state.

In general semiclassical models, it is tempting to explore various types of UV behavior which emerges under different choices of $W$. However, it is important to understand how reliable the leading classical analysis is. Also, it is not yet clear how the classical description in terms of the "potential" $W$ relates to the realization via the CDD factor.

It is also not clear if the short-scale singularity in the finite-size energy really signals violation of locality. For that one would like to get hold of the correlation functions. The latter generally admit the intermediate-state decomposition

$$\langle 0 \mid \mathcal{O}(R)\mathcal{O}(0) \mid 0\rangle =$$
$$\sum_{N=0}^{\infty} \frac{1}{N!} \int |\langle 0 \mid \mathcal{O}(0) \mid A(\theta_1)...A(\theta_N)\rangle|^2 \, e^{-MR\sum_{i=1}^{N} \cosh\theta_i} \prod_{i=1}^{N} \frac{d\theta_i}{2\pi},$$

where I assumed that the separation is purely in space. Note that in IQFT with finite-product CDD deformations the high-energy behavior of the form factors is quite normal, so that the integrals in every individual $N$-particle term should converge just fine. Potential source of trouble could be the convergence of the sum over $N$.

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It is possible that, while convergent at sufficiently large $R$, the sum starts to diverge at short $R$, yielding a singularity at some $R_*$. The associated discontinuity at $R < R_*$ would then manifest non-zero commutator at the space-like separations. This important problem remains open...

To be continued...