Title: Tensor network trial wave functions for topological phases

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Abstract: <p>The construction of trial wave functions has proven itself to be very useful for understanding strongly interacting quantum many-body systems. Two famous examples of such trial wave functions are the resonating valence bond state proposed by Anderson and the Laughlin wave function, which have provided an (intuitive) understanding of respectively spin liquids and fractional Quantum Hall states. Tensor network states are another, more recent, class of such trial wave functions which are based on entanglement properties of local, gapped systems. In this talk I will discuss the use of tensor network states for topological phases, and what we can learn from this approach. I will consider one- and two-dimensional systems, consisting of both spins and fermions. The focus will be on the different connections that can be made using tensor networks, such as connecting theory to numerics, and physical properties to ground state entanglement.</p>

Tensor network trial wave function for topological phases

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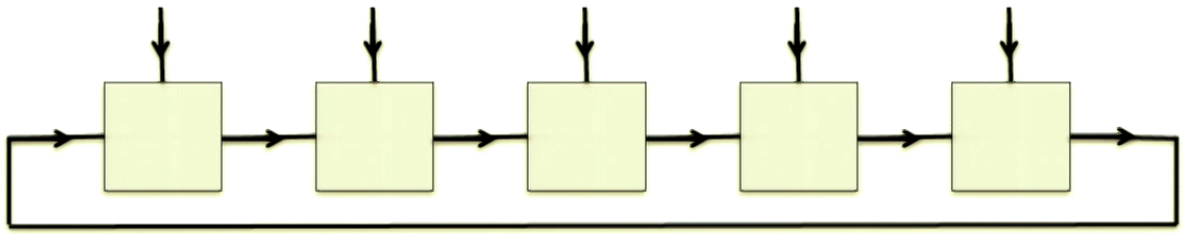
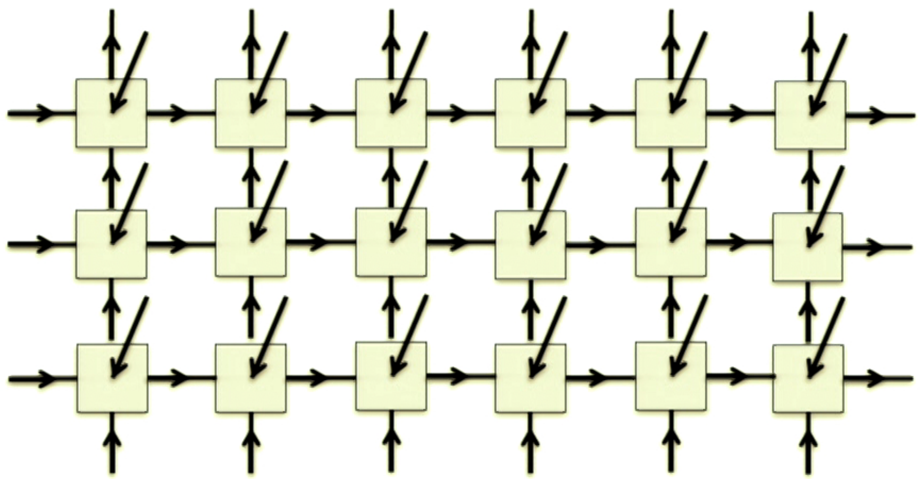
Outline

- Tensor networks as trial wave functions
- Fermionic tensor networks
- Fermionic symmetry-protected phases in one dimension
- Bosonic topological order and anyonic excitations
- Fermionic topological phases in two dimensions

Tensor network states as trial wave functions

Tensor network states: MPS, PEPS, MERA,...

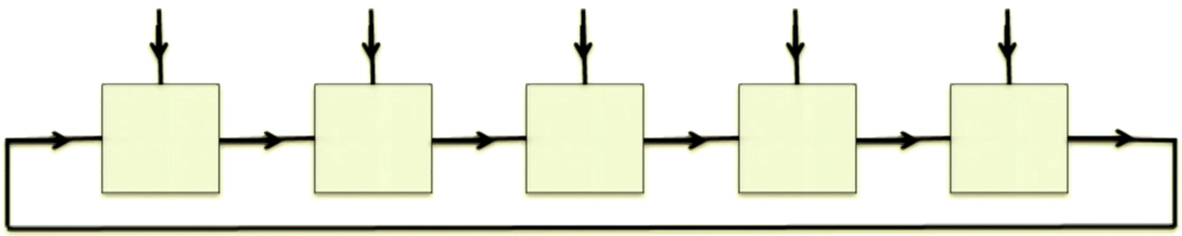




Tensor network states: MPS, PEPS, MERA,…

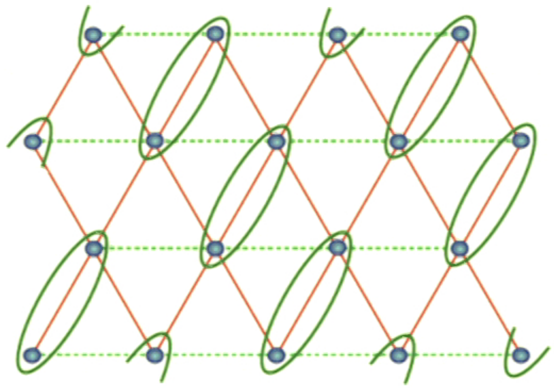
Motivating principles for the developments of tensor networks states:

- Entanglement, area law
- Computational efficiency



➡ DMRG, TEBD, iTEBD, TDVP,…

Trial wave functions



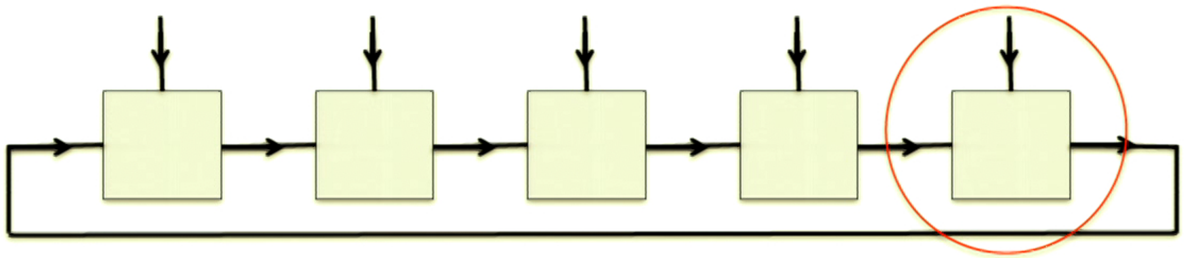
Resonating Valence Bond State (Anderson, 1973)

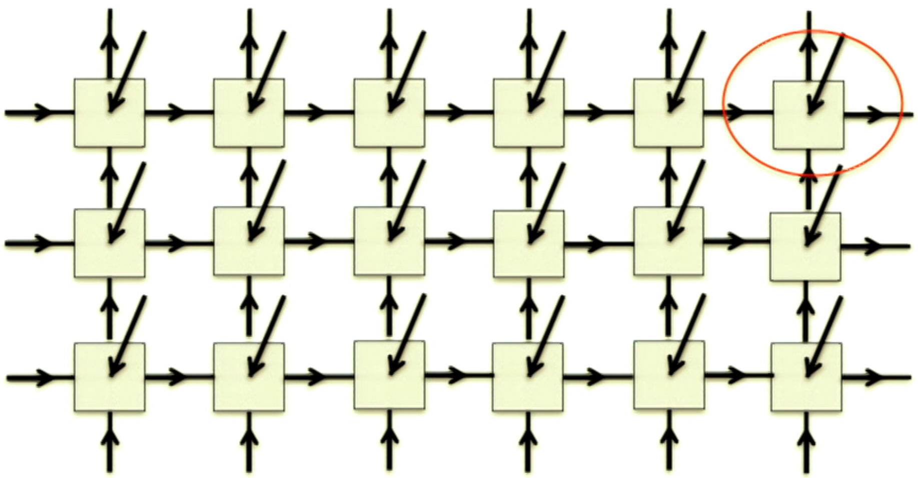
$$\prod_{i<j}(z_i - z_j)^{1/\nu}\prod_i e^{-\frac{|z_i|^2}{4l_B^2}}$$

Laughlin wavefunction (1983) and associated picture of composite fermions (Jain, 1989)

Give a physical, intuitive picture of the physics of strongly interacting quantum many-body systems

Tensor network states as trial wave functions





Use tensors, the local building blocks, to link physical properties of quantum phases to entanglement

Fermionic tensor networks

Super vector space V

- Natural direct sum structure and associated homogeneous basis vectors

$$V = V^0 \oplus V^1$$

- $\mathbb{Z}_2$ graded

$$|i\rangle \to |i| = \begin{cases} 0 & \text{if } |i\rangle \in V^0 \\ 1 & \text{if } |i\rangle \in V^1 \end{cases}$$

$$|i\rangle \otimes_{\mathfrak{g}} |j\rangle \to |i| + |j| \bmod 2$$

Super vector space V

- Dual vector space

$$\langle i| \in V^* \qquad \langle i|j\rangle = \delta_{i,j}$$

- Fermionic tensor product isomorphism

$$\mathcal{F} : V \otimes_{\mathfrak{g}} W \to W \otimes_{\mathfrak{g}} V$$

$$|i\rangle \otimes_{\mathfrak{g}} |j\rangle \to (-1)^{|i||j|} |j\rangle \otimes_{\mathfrak{g}} |i\rangle$$

$$\mathcal{F} : V^* \otimes_{\mathfrak{g}} W \to W \otimes_{\mathfrak{g}} V^*$$

$$\langle i| \otimes_{\mathfrak{g}} |j\rangle \to (-1)^{|i||j|} |j\rangle \otimes_{\mathfrak{g}} \langle i|$$

Fermionic tensors

= elements in a super vector space consisting of the graded tensor product of super vector spaces and dual super vector spaces

$$\mathsf{T} = \sum_{i,j,k} T_{ijk} |i\rangle \otimes_{\mathfrak{g}} |j\rangle \otimes_{\mathfrak{g}} \langle k|$$

Homogeneous tensors:

$$\mathsf{T} = \sum_{\substack{i,j,k \\ |i|+|j|+|k|=|\mathsf{T}| \bmod 2}} T_{ijk} |i\rangle \otimes_{\mathfrak{g}} |j\rangle \otimes_{\mathfrak{g}} \langle k| \qquad |\mathsf{T}| \in \{0,1\}$$

Fermionic contraction

Define a map $\mathcal{C}$ which implements the linear action of a dual vector space on the original vector space:

$$\mathcal{C} : V^* \otimes_{\mathfrak{g}} V \to \mathbb{C} : \langle\psi| \otimes_{\mathfrak{g}} |\phi\rangle \to \langle\psi|\phi\rangle$$

Fermionic tensor contraction:

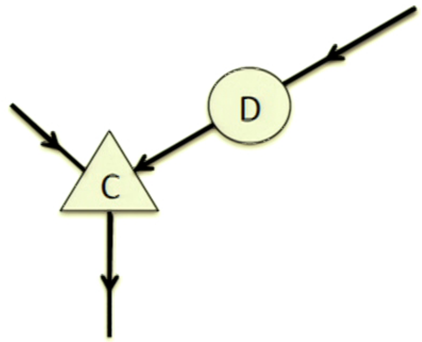
1) Take the graded tensor product of homogeneous tensors
2) Use $\mathcal{F}$ to bring the indices one wishes to contract next to each other
3) Apply $\mathcal{C}$

Fermionic contraction

- Graphical notation of bosonic tensor networks still applies

$$C = \sum_{\alpha\beta\gamma} C_{\alpha\beta\gamma} |\alpha)|\beta)(\gamma|$$

$$D = \sum_{\lambda\kappa} D_{\lambda\kappa} |\lambda)(\kappa|$$

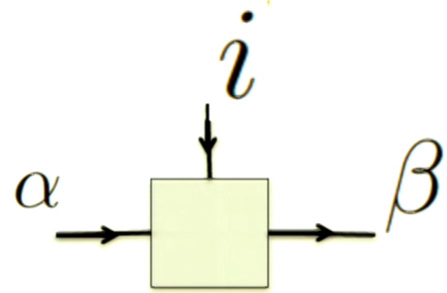


- Fermionic contract function can be easily implemented numerically
- No global ordering required
- Convenient for theoretical studies

Fermionic symmetry-protected phases in one dimension

Fermionic matrix product states (fMPS)

$$\mathsf{A} = \sum_{\alpha,i\beta} A^i_{\alpha\beta} |\alpha)|i\rangle(\beta|$$

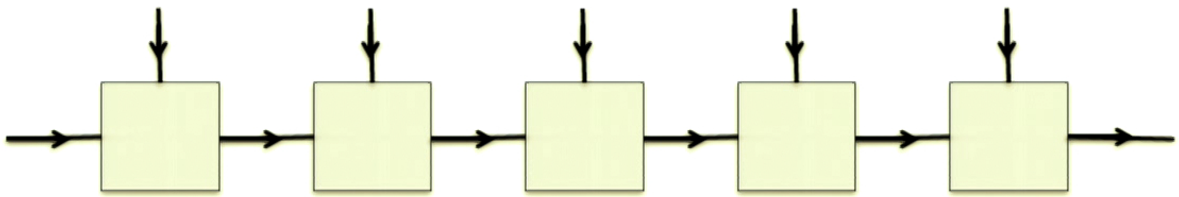


$$A^i = \begin{pmatrix} B^i & 0 \\ 0 & C^i \end{pmatrix} \quad \text{if } |i| = 0$$

$$A^i = \begin{pmatrix} 0 & D^i \\ F^i & 0 \end{pmatrix} \quad \text{if } |i| = 1$$

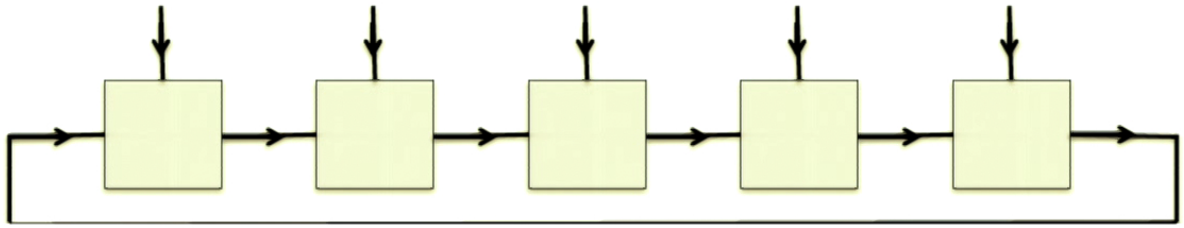
Fermionic matrix product states (fMPS)

$$\mathsf{A} = \sum_{\alpha, i\beta} A^i_{\alpha\beta} |\alpha)|i\rangle(\beta|$$



$$\begin{aligned} |\psi\rangle_{\alpha\beta} &= \mathcal{C}\left(\mathsf{A} \otimes_{\mathfrak{g}} \mathsf{A} \otimes_{\mathfrak{g}} \ldots \otimes_{\mathfrak{g}} \mathsf{A}\right) \\ &= \sum_{\{i\}} \left(A^{i_1} A^{i_2} \ldots A^{i_N}\right)_{\alpha\beta} |\alpha)|i_1\rangle|i_2\rangle \ldots |i_N\rangle(\beta| \end{aligned}$$

Fermionic matrix product states (fMPS)



$$|\psi\rangle_e = \mathcal{C}_N \left(\sum_{\{i\}} \sum_{\alpha\beta} \left(A^{i_1} A^{i_2} \dots A^{i_N}\right)_{\alpha\beta} |\alpha)_N |i_1\rangle |i_2\rangle \dots |i_N\rangle (\beta|_N \right)$$

$$|\psi\rangle_e = \mathcal{C}_N \left(\sum_{\{i\}} \sum_{\alpha\beta} \left(A^{i_1} A^{i_2} \dots A^{i_N}\right)_{\alpha\beta} (-1)^{|\beta|} (\beta| \, |\alpha) |i_1\rangle |i_2\rangle \dots |i_N\rangle \right)$$

$$|\psi\rangle_e = \sum_{\{i\}} \mathrm{tr}\left(\mathcal{P} A^{i_1} A^{i_2} \dots A^{i_N}\right) |i_1\rangle |i_2\rangle \dots |i_N\rangle$$

$$\mathcal{P} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$$

Fermionic matrix product states (fMPS)

Can we build translationally invariant fMPS with odd fermion parity?

$$|\psi\rangle_o = \mathcal{C}\left(\mathsf{Y} \otimes_{\mathfrak{g}} \mathsf{A} \otimes_{\mathfrak{g}} \mathsf{A} \otimes_{\mathfrak{g}} \ldots \otimes_{\mathfrak{g}} \mathsf{A}\right)$$
$$= \sum_{\{i\}} \operatorname{tr}\left(Y A^{i_1} A^{i_2} \ldots A^{i_N}\right) |i_1\rangle |i_2\rangle \ldots |i_N\rangle$$

$$\operatorname{tr}\left(Y A^{i_1} A^{i_2} \ldots A^{i_N}\right) = \operatorname{tr}\left(Y A^{i_2} A^{i_3} \ldots A^{i_1}\right)$$

Fermionic matrix product states (fMPS)

$$\mathrm{tr}\left(Y A^{i_1} A^{i_2} \ldots A^{i_N}\right) = \mathrm{tr}\left(Y A^{i_2} A^{i_3} \ldots A^{i_1}\right)$$

$$A^i = \begin{pmatrix} B^i & 0 \\ 0 & B^i \end{pmatrix} = \mathbb{1} \otimes B^i \qquad \text{if } |i| = 0$$

$$A^i = \begin{pmatrix} 0 & B^i \\ -B^i & 0 \end{pmatrix} = y \otimes B^i \qquad \text{if } |i| = 1$$

$$y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad Y = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} = y \otimes \mathbb{1}$$

This fMPS vanishes if we try to make it even!

Fermionic matrix product states (fMPS)

Canonical form for fMPS:

Every non-zero fMPS can be written as a sum of *irreducible fMPS.* An irreducible fMPS satisfies one of following two properties:

1) The matrices A^i span a simple matrix algebra (EVEN CASE)
2) The matrices A^i are of the type

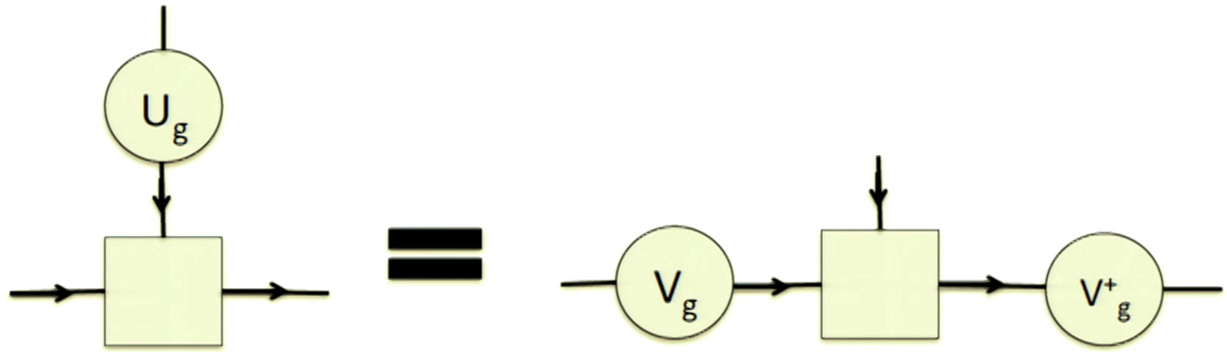
$$A^i = \begin{pmatrix} B^i & 0 \\ 0 & B^i \end{pmatrix} = \mathbb{1} \otimes B^i \qquad \text{if } |i| = 0$$

$$A^i = \begin{pmatrix} 0 & B^i \\ -B^i & 0 \end{pmatrix} = y \otimes B^i \qquad \text{if } |i| = 1$$

and the subset of even matrices generated by A^i spans a simple algebra (ODD CASE)

Fermionic matrix product states (fMPS)

Fermionic matrix product states and on-site global unitary symmetries:



$$V_g V_h = \omega(g,h) V_{gh} \qquad \mathcal{P} V_g = (-1)^{\mu(g)} V_g \mathcal{P}$$

$$(V_g \otimes_{\mathfrak{g}} W_g)\,(V_h \otimes_{\mathfrak{g}} W_h) = (-1)^{|W_g|\,|V_h|} V_g V_h \otimes_{\mathfrak{g}} W_g W_h$$

Fermionic matrix product states (fMPS)

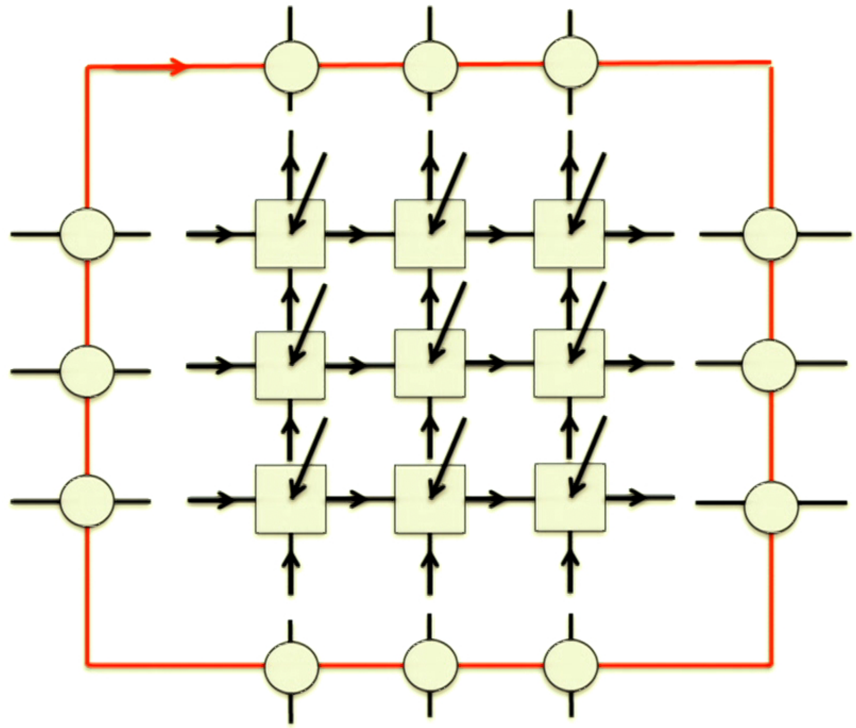
Can be extended to anti-unitary symmetries and reflection symmetries:

- $\mathbb{Z}_8$ classification of time-reversal invariant Majorana chains can be related to the eightfold periodicity in the representations of real Clifford algebras over the real numbers
- $\mathbb{Z}_8$ classification of reflection symmetric phases in one dimension crucially relies on the intrinsic fermionic formulation of the matrix product states

Bosonic topological order and anyonic excitations

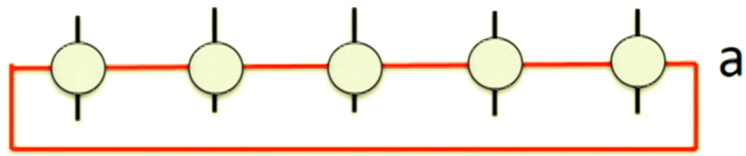
Topological information in PEPS

Similar as in the (f)MPS case: topological information is contained in the codimension-one 'symmetry' operators acting on the virtual indices



Matrix product operator algebra

$$O_a =$$



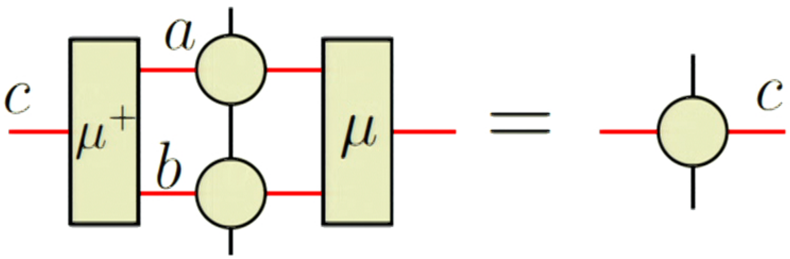
$$O_a^L O_b^L = \sum_{c=1}^{N} N_{ab}^c O_c^L$$

$$\left(O_a^L\right)^\dagger \equiv O_{a^*}^L$$

Algebra of one-dimensional operators

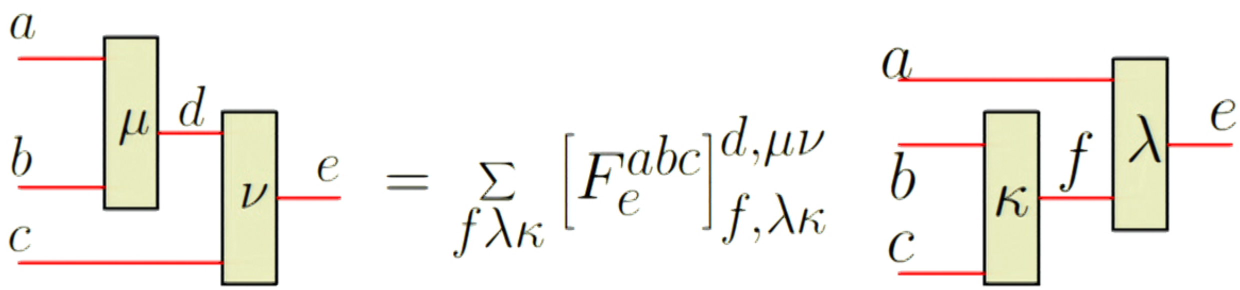
Matrix product operator algebra

Closed under multiplication:

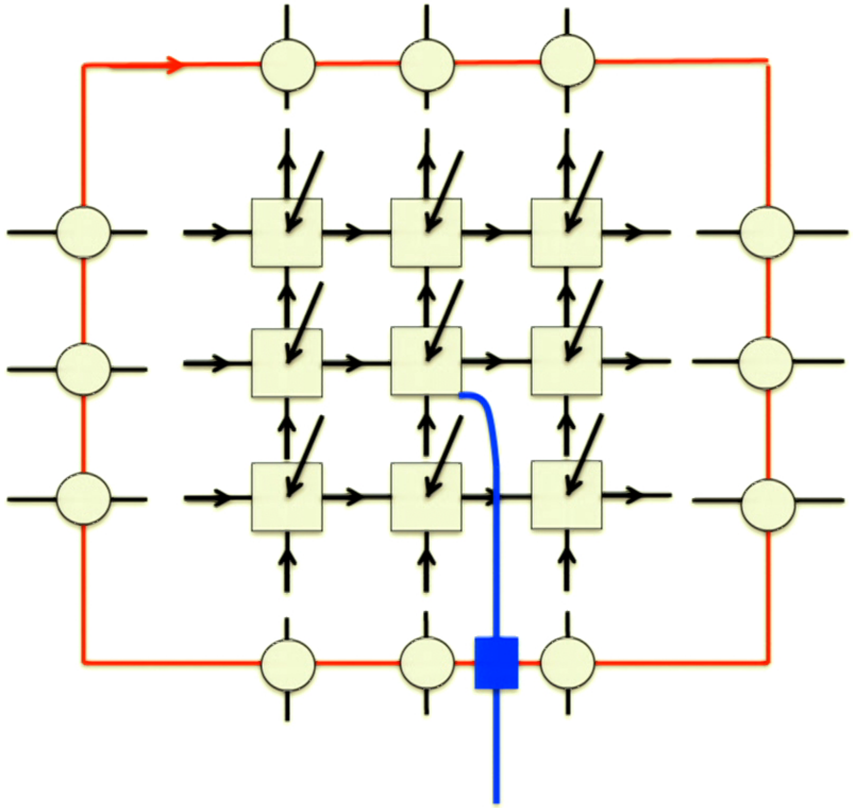


Zero-dimensional tensor identities

Associativity:



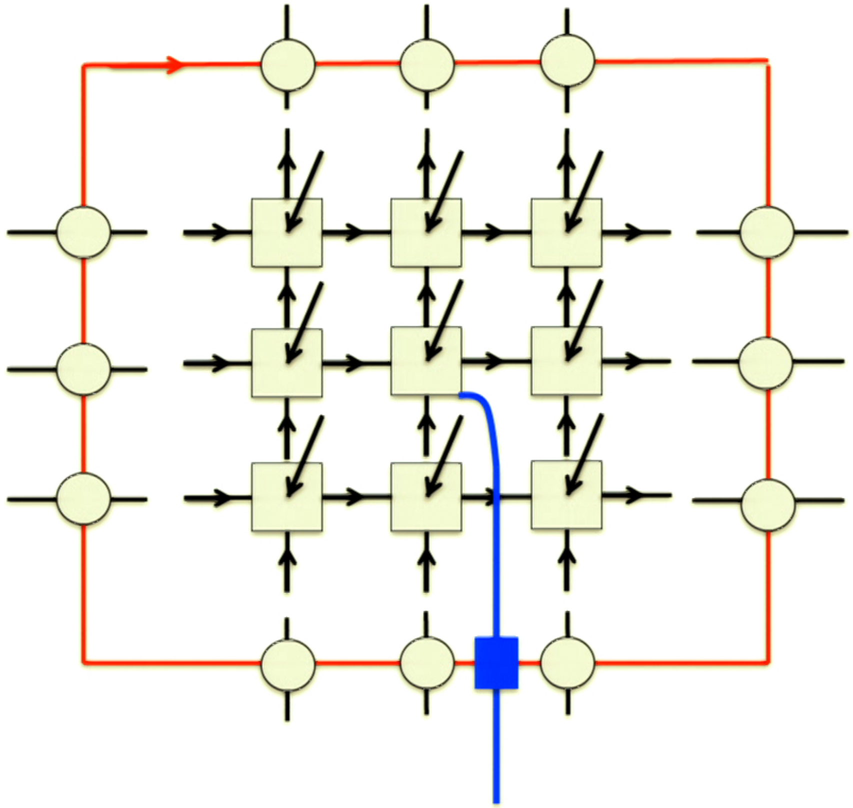
Anyonic excitations



Anyonic excitations

- Simple algorithm for the calculation of anyonic excitations
- Anyons as entanglement superselection sectors
- Gives direct access to topological spins and S matrix
- Lagrangian subgroup is easy to recognize; the topological phases always have zero thermal Hall conductivity

Anyonic excitations



Anyonic excitations

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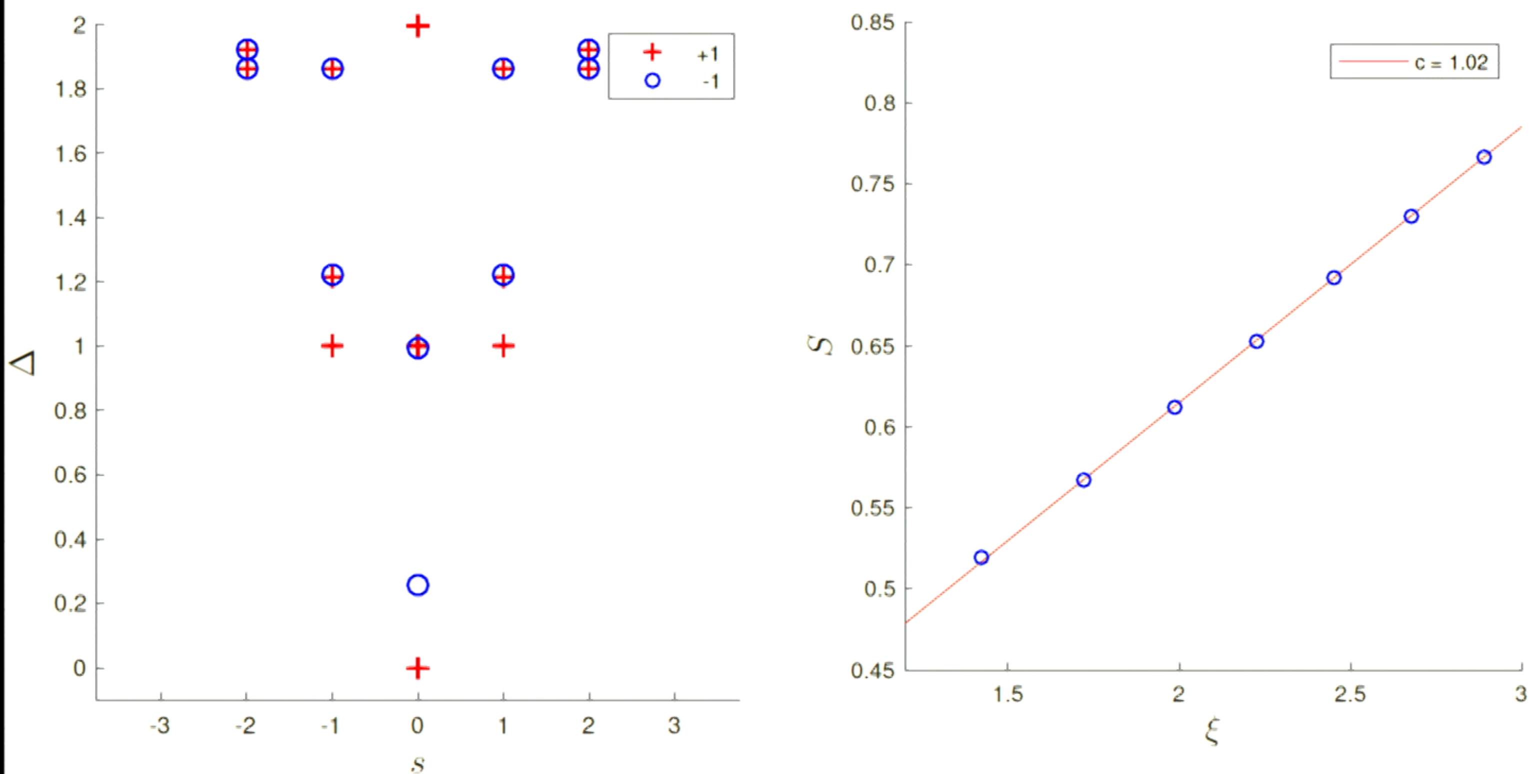
- Anyons as entanglement superselection sectors
- Gives direct access to topological spins and S matrix
- Lagrangian subgroup is easy to recognize; the topological phases always have zero thermal Hall conductivity

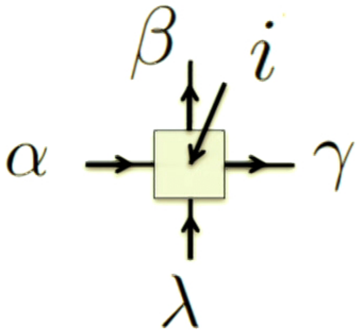
Symmetry-protected phases



Entanglement spectrum is a free boson CFT. The anomalous symmetry action can be used to numerically diagnose the SPT order.

Fermionic topological phases in two dimensions

Fermionic PEPS and spin structures

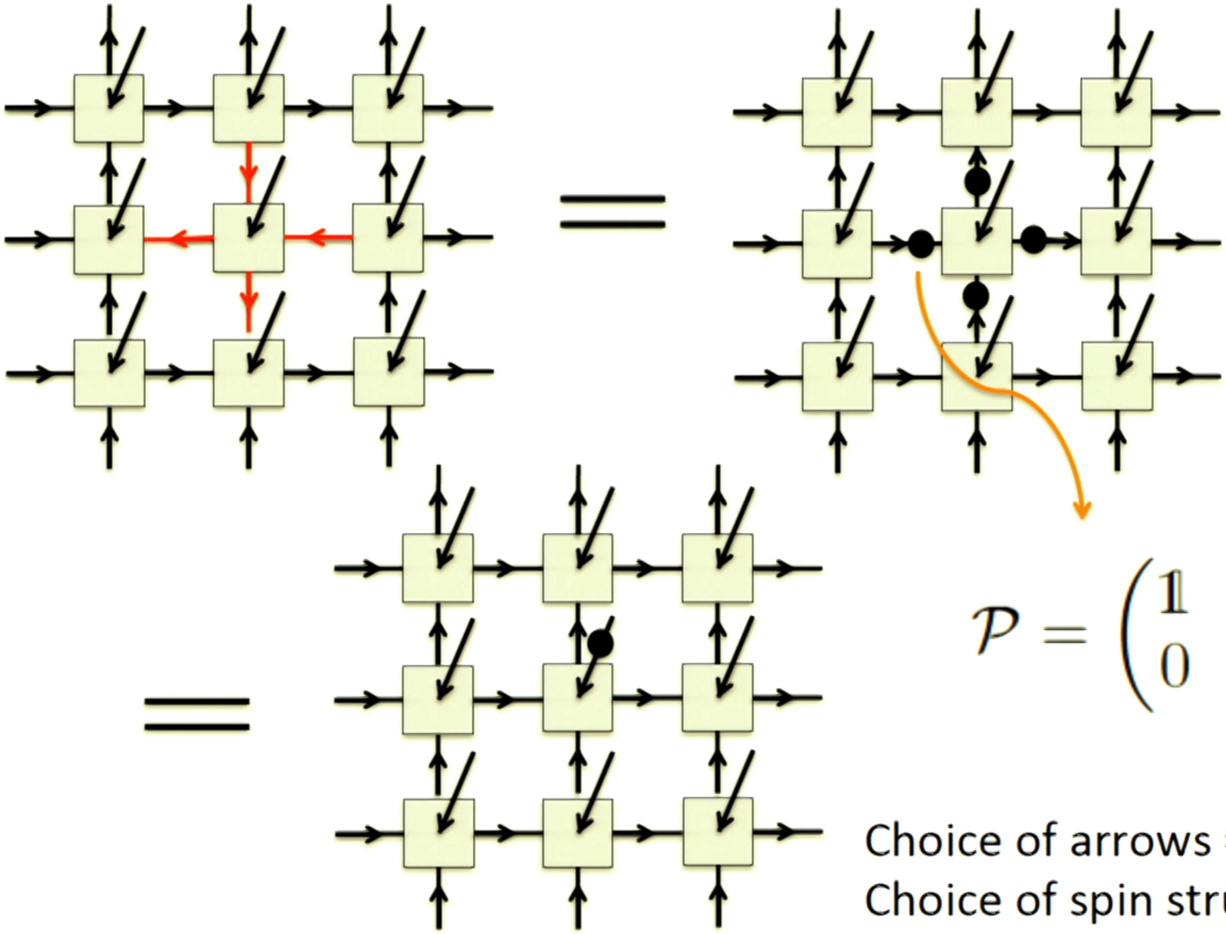


$$\mathsf{A} = \sum_{i,\alpha,\beta,\gamma,\lambda} A^{i}_{\alpha\beta\gamma\lambda} |\alpha)(\beta|(\gamma|\,|\lambda)$$

$$\mathcal{C}(\langle i| \otimes_{\mathfrak{g}} |j\rangle) = \delta_{i,j}$$

$$\mathcal{C}(|i\rangle \otimes_{\mathfrak{g}} \langle j|) = (-1)^{|i||j|}\mathcal{C}(\langle j| \otimes_{\mathfrak{g}} |i\rangle) = (-1)^{|i|}\delta_{i,j}$$

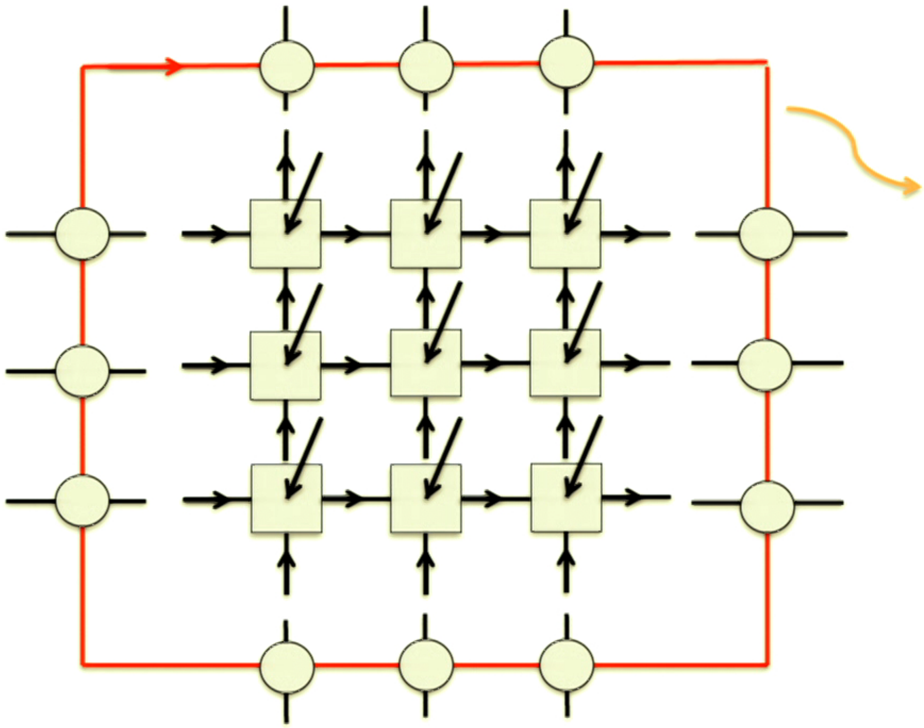
Fermionic PEPS and spin structures



$$\mathcal{P} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$$

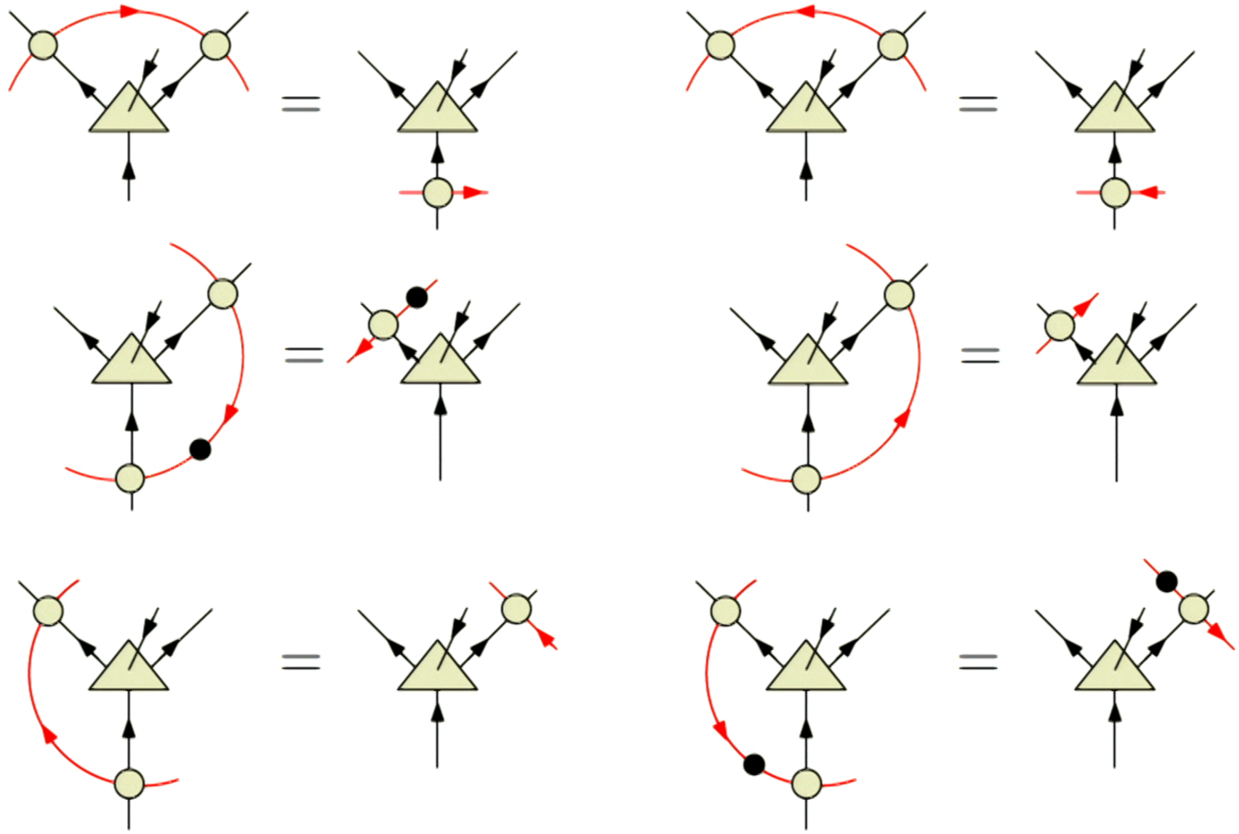
Choice of arrows =
Choice of spin structure

Topological information in fermionic PEPS



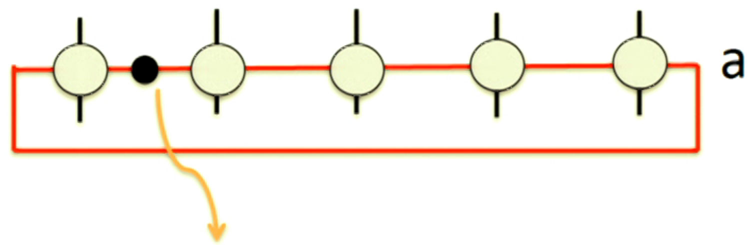
Fermionic Matrix Product Operator (fMPO); two irreducible types: trivial and Majorana

Topological information in PEPS



Topological information in PEPS

$O_a =$



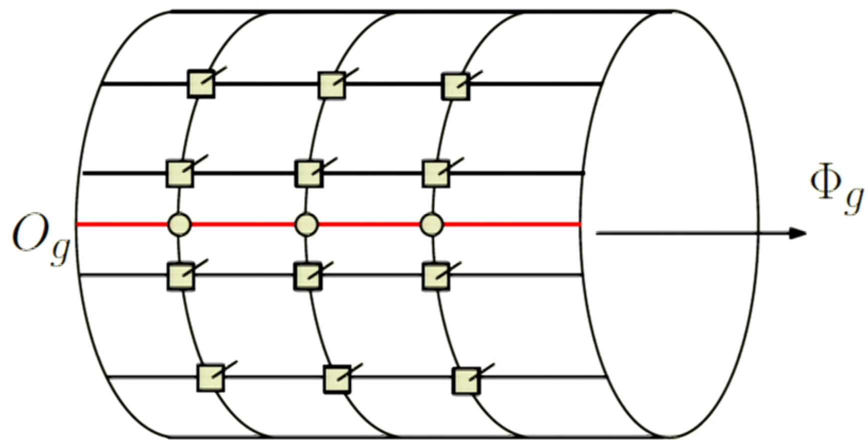
fMPOs appear with an odd number of parity matrices in the PEPS

$$O_a^L O_b^L = \sum_{c=1}^{N} N_{ab}^c O_c^L$$

$$\left(O_a^L\right)^\dagger \equiv O_{a^*}^L$$

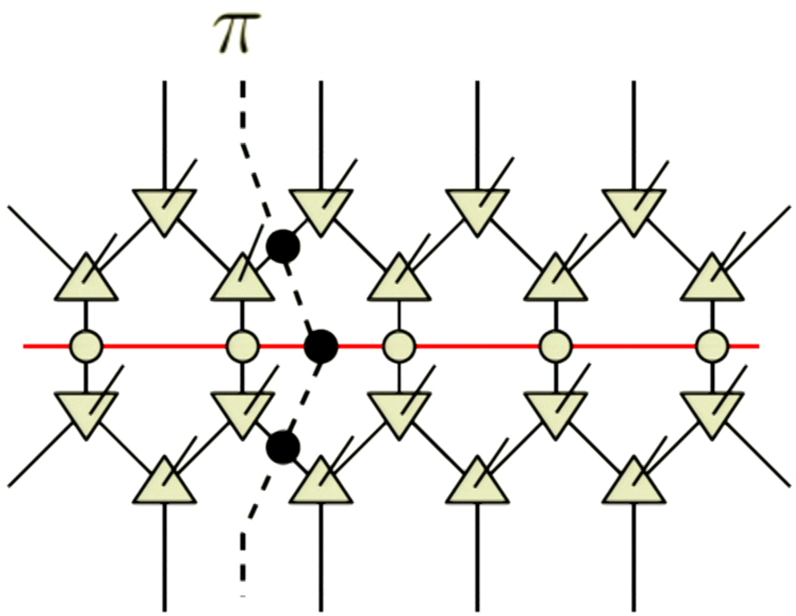
Fermionic algebra of one-dimensional operators

Symmetry defects in supercohomology SPT phases



$$\omega_g(h,k) = (-1)^{Z(h,k)(Z(hk,g)+Z(g,hk))}\frac{\alpha(h,g,k)}{\alpha(h,k,g)\alpha(g,h,k)}$$

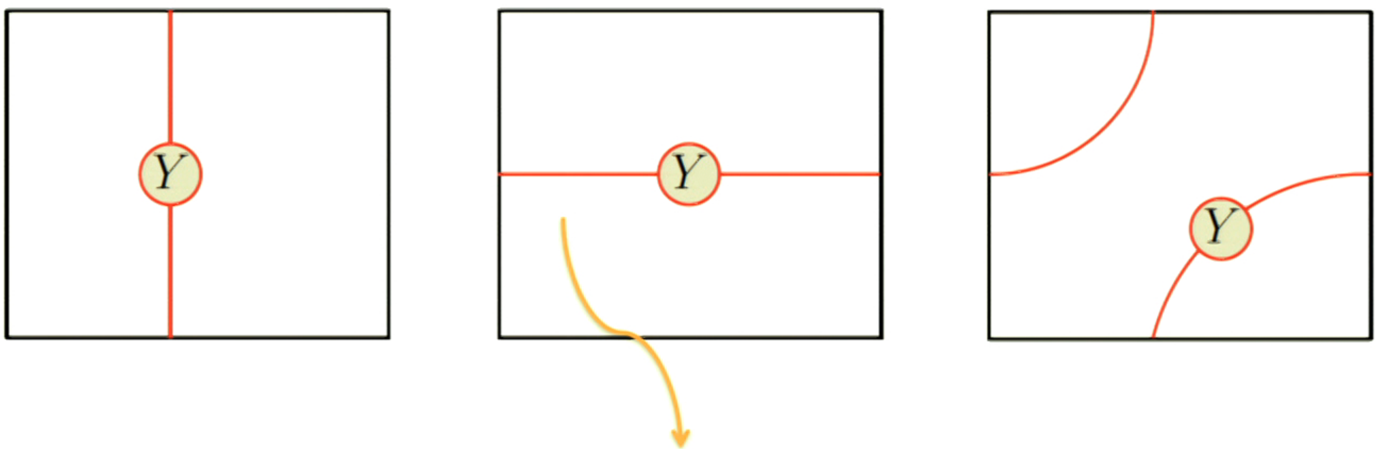
Symmetry defects in supercohomology SPT phases



$$\tilde{O}_g \tilde{O}_h = (-1)^{Z(g,h)} \tilde{O}_{gh}$$

Topological phases with Majorana-type anyons

Connection between ground state parity and spin structure for topological phases with Majorana fMPOs (similar to p+ip)



Majorana fMPO with periodic boundary conditions wrapping a non-contractible cycle of the torus

Conclusions

- Tensor network states can be used as trial wave functions to develop understanding of strongly interacting topological phases
- The codimension-one 'entanglement' operators contain all universal information, for both bosonic and fermionic systems
- There is a fruitful interplay between theory and numerics
- All these techniques apply similarly to three dimensional systems