Title: Approximate Operator Algebra Quantum Error Correction (Decoding the Hologram in AdS/CFT)

Date: Nov 22, 2017 04:00 PM

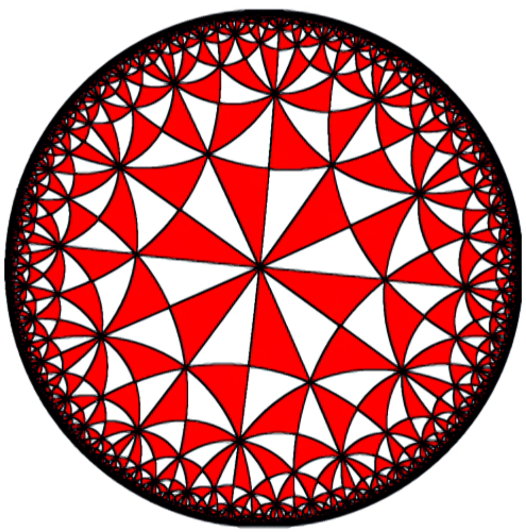
URL: http://pirsa.org/17110136

Abstract: <p>Quantum error correction -- originally invented for quantum computing -- has proven itself useful in a variety of non-computational physical systems, as the ideas of QEC are broadly applicable. In this talk, I'll mention a few examples of error correction in the wild, including the recent discovery that the AdS/CFT correspondence implements quantum error correction. We will then study the hypothesis that any local bulk operator in AdS can be reconstructed using only a causally disconnected subregion of the CFT. This hypothesis has been proven under the assumption that error correction in AdS/CFT is exact, but this assumption is not expected to be true. Fortunately, recent advances in the theory of approximate quantum error correction have emerged. We will review these results on recoverability and approximate quantum error correction, as well as AdS/CFT and the so-called entanglement wedge reconstruction hypothesis. We will then prove the entanglement wedge hypothesis robustly and find an explicit formula for reconstructed bulk operators. If time permits, we will explore a generalization of the theory of universal recovery channels to the case of finite-dimensional von Neumann algebras.</p>

Approximate Operator Algebra Quantum Error Correction

(Decoding the Hologram in AdS/CFT)

arXiv:1704.05839



Grant Salton

Stanford University

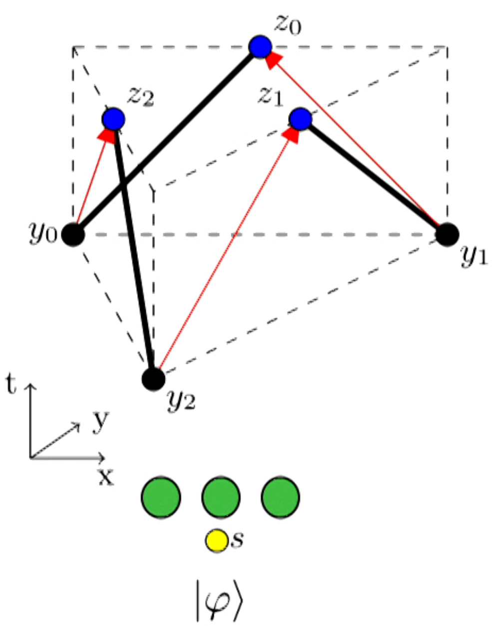
With: Jordan Cotler, Patrick Hayden, Brian Swingle, and Michael Walter

PERIMETER INSTITUTE

November 22, 2017

Smorgasbord of QEC in the wild

Information Replication



arXiv:1601.02544

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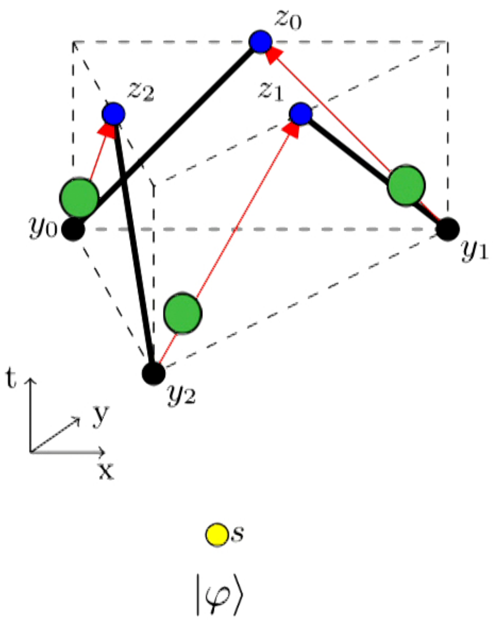

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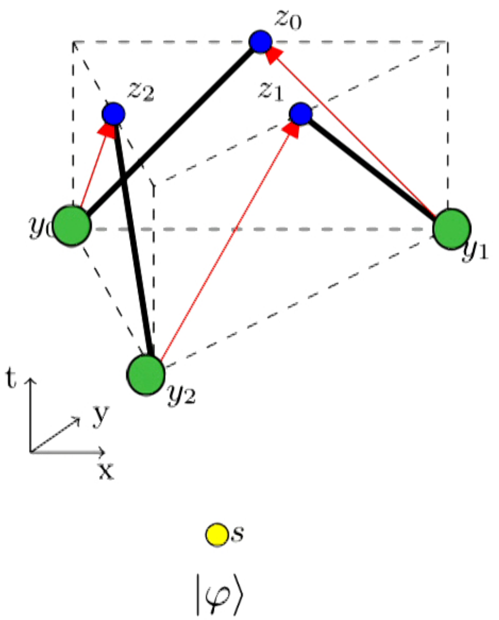

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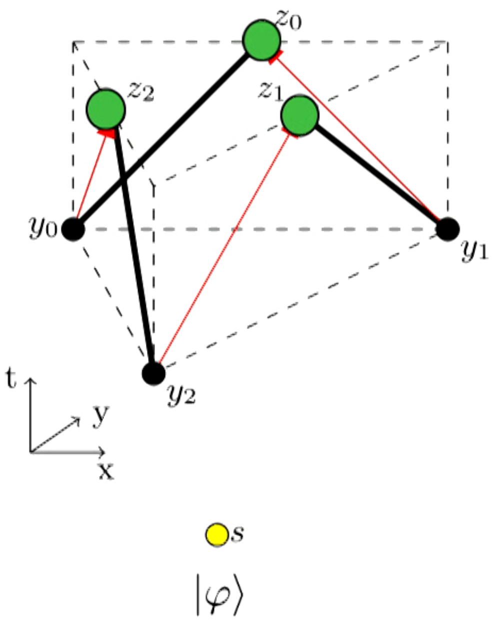

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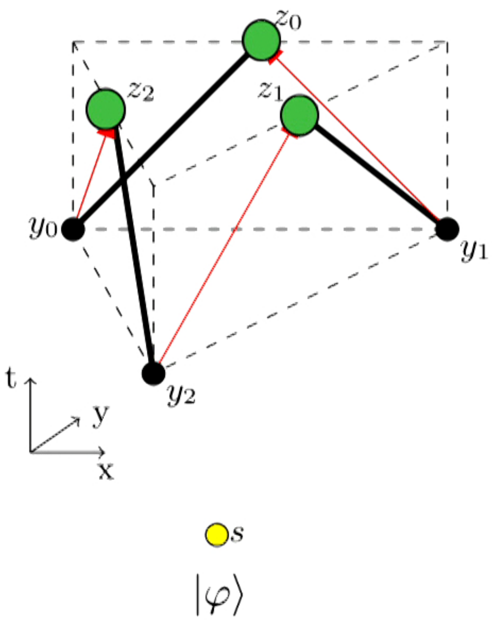

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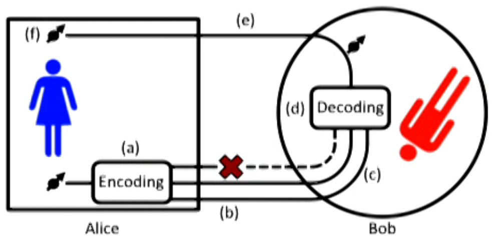
Covariant QEC



$$U_1 \otimes U_2 \otimes U_3 \, \mathcal{E}(U_{\text{in}}^\dagger \rho_{\text{in}} U_{\text{in}}) \, U_1^\dagger \otimes U_2^\dagger \otimes U_3^\dagger = \mathcal{E}(\rho_{\text{in}})$$

No-go Theorem:
No *finite dimensional* covariant codes for continuous symmetries

arXiv:1709.04471

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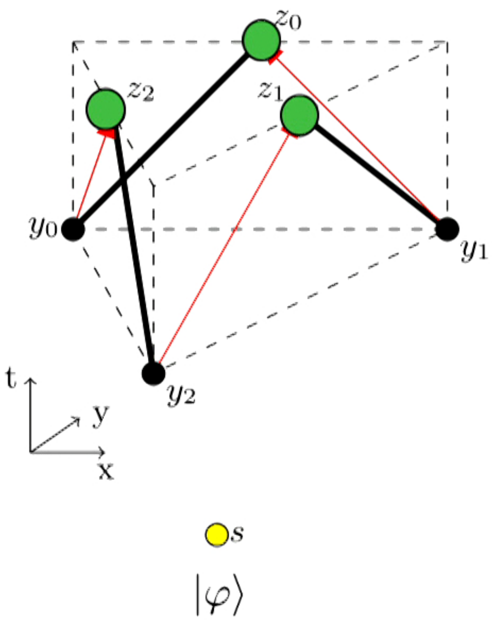

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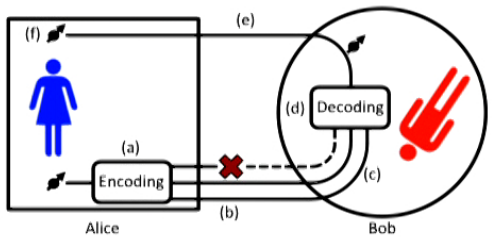

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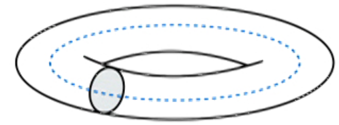
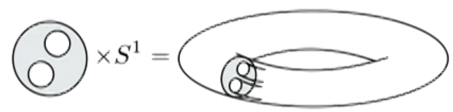
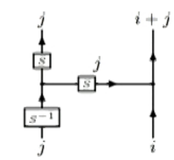
TQFT States



$$Z(D \times S^1; j) = |j\rangle \in \mathcal{H}_{T^2}$$





= CNOT

Abelian CS: $U(1)_k$

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Stabilizer States

arXiv:1611.01516

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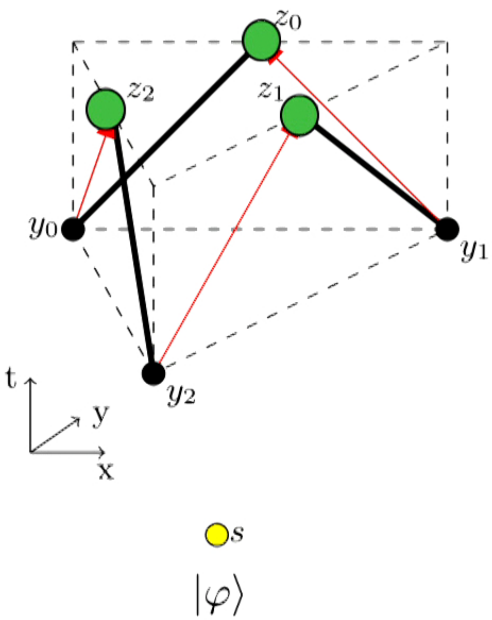

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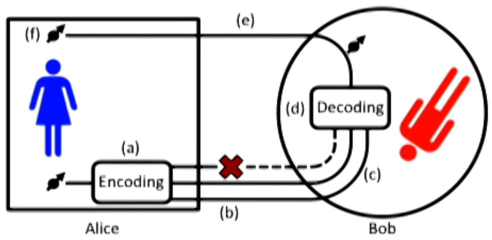
Covariant QEC



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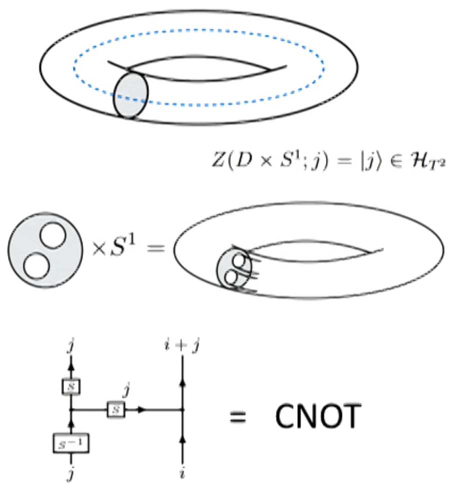

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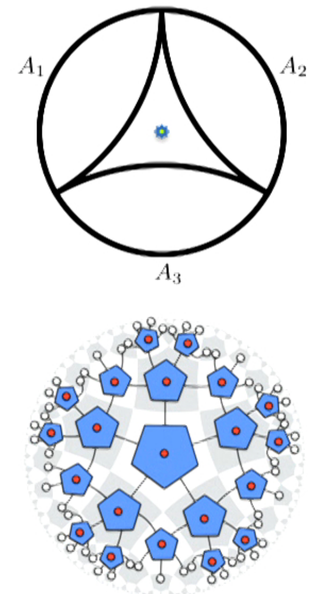
Holography



Taken from Pastawski, Yoshida, Harlow, Preskill. JHEP 2015

arXiv:1704.05839

Where are we headed?

- Context: AdS/CFT
- Main question: What in the bulk is dual to a subregion of the boundary?
- For the experts: our end goal is an explicit formula for entanglement wedge reconstruction in AdS/CFT
- We generalize results on universal recovery channels to prove the entanglement wedge hypothesis robustly, and we give an explicit formula

Recovery: Classical case – Bayes' Rule

Problem: Given a classical, stochastic map $p(y|x)$ and an observation of y, try to infer x.

In other words, we want to find a different stochastic map that reverses the original map.

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Bayes' Rule:

$$p(x|y) = \frac{p(x)p(y|x)}{p(y)}$$

We think of the map $p(y|x)$ as a channel, with $p(x)$ as input.

Sanity check: input $p(y)$ into the new channel $p(x|y)$. This gives

$$\sum_y p(x|y)p(y) = \sum_y \frac{p(x)p(y|x)}{p(y)} p(y)$$
$$= \sum_y p(x)p(y|x)$$
$$= p(x)$$

which is the random variable X, as expected.

...ase – Bayes' Rule

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A trivial rewriting

$$p(x|y) = \frac{d}{dt}\bigg|_{t=0} \log\left(\frac{p(y)}{p(x)} + t\,p(y|x)\right)$$

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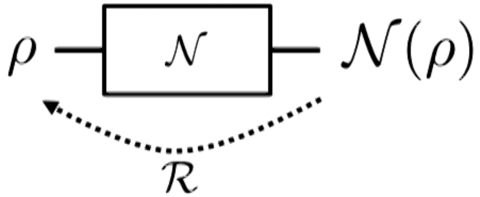

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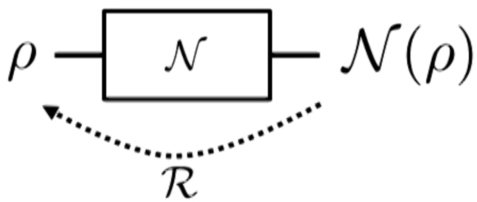

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Relative Entropy: $D(\rho\|\sigma) = \mathrm{Tr}(\rho \log \rho) - \mathrm{Tr}(\rho \log \sigma)$

Equal to zero iff $\rho = \sigma$.

A measure of distinguishability.

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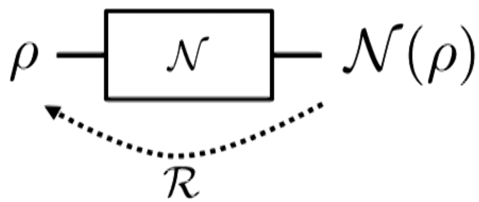

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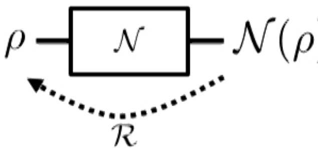
Exact recovery: Petz map



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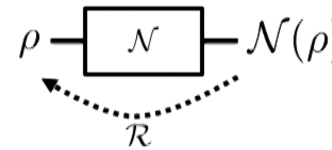


$$D(\rho\|\sigma) \overset{!}{=} D\left(\mathcal{N}(\rho)\|\mathcal{N}(\sigma)\right)$$

$$\Longleftrightarrow$$

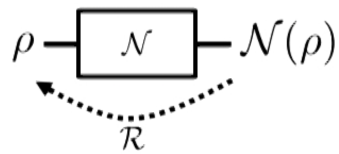
$$\exists\ \mathcal{P} \text{ such that}, \forall\, \rho, \sigma, \quad \mathcal{P} \circ \mathcal{N}(\rho) = \rho \qquad \mathcal{P} \circ \mathcal{N}(\sigma) = \sigma$$

$\mathcal{P}$ is called the **Petz map**, and it is given by

$$\mathcal{P}_{\sigma,\mathcal{N}}(\,\cdot\,) = \sigma^{1/2}\mathcal{N}^{\dagger}\left(\mathcal{N}(\sigma)^{-1/2}(\,\cdot\,)\mathcal{N}(\sigma)^{-1/2}\right)\sigma^{1/2}$$

M. Ohya and D. Petz. Quantum Entropy and Its Use. Springer-Verlag, 1993.

Approximate recovery: universal channels



$$\mathcal{R}_{\sigma,\mathcal{N}} \circ \mathcal{N}(\rho) \approx \rho$$

For any channel $\mathcal{N}$ there exists a recovery channel $\mathcal{R}_{\sigma,\mathcal{N}}$ depending only on σ and $\mathcal{N}$ such that

$$D(\rho\|\sigma) - D(\mathcal{N}(\rho)\|\mathcal{N}(\sigma)) \geq -2\log F(\rho, (\mathcal{R}_{\sigma,\mathcal{N}} \circ \mathcal{N}(\rho))$$

Fidelity $F(\rho,\sigma) = \|\sqrt{\rho}\sqrt{\sigma}\|_1$

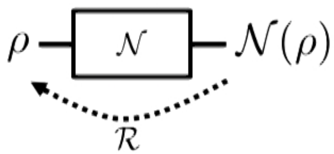
Explicit form of recovery channel

$$\mathcal{R}_{\sigma,\mathcal{N}}(\,\cdot\,) = \int_{\mathbb{R}} dt\, \beta_0(t) \sigma^{\frac{1-it}{2}} \mathcal{N}^\dagger \left(\mathcal{N}(\sigma)^{-\frac{1-it}{2}} (\,\cdot\,) \mathcal{N}(\sigma)^{-\frac{1+it}{2}} \right) \sigma^{\frac{1+it}{2}}$$

$$\beta_o(t) = \frac{\pi}{2}(\cosh(\pi t) + 1)^{-1}$$

M. Junge, R. Renner, D. Sutter, M. M. Wilde, and A. Winter, arXiv preprint arXiv:1509.07127 (2015).

Approximate recovery: universal channels



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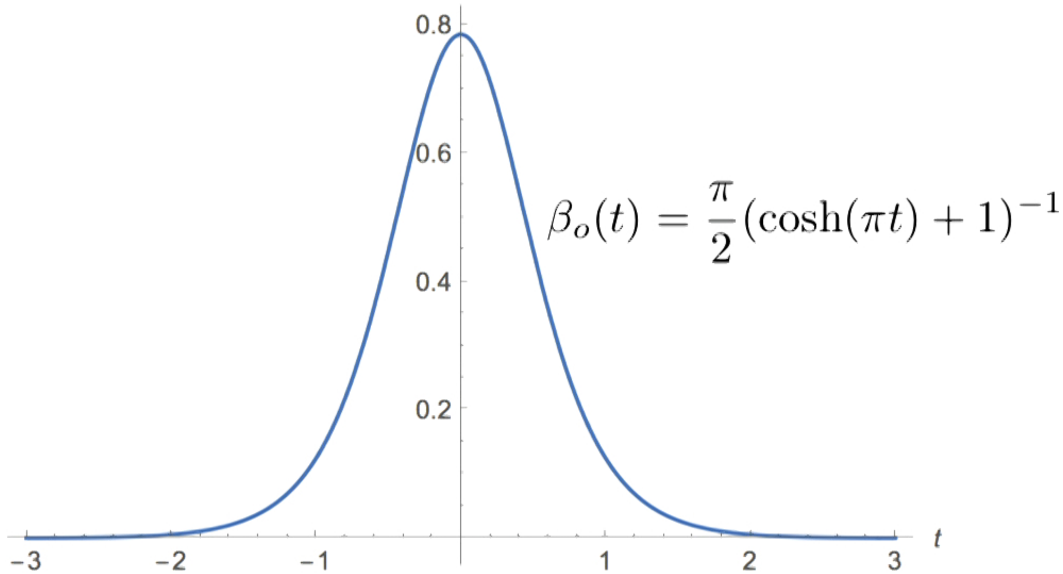
Tools we use

In[8]:= `Plot[π/2 (Cosh[π t] + 1)^-1, {t, -3, 3}, PlotRange → All, AxesLabel → Automatic]`

Out[8]=



In[9]:= `Integrate[π/2 (Cosh[π t] + 1)^-1, {t, -∞, ∞}]`

Out[9]= 1

A non-commutative version of Bayes' Rule

Petz map: $\mathcal{P}_{\sigma,\mathcal{N}}(\,\cdot\,) = \sigma^{1/2}\mathcal{N}^{\dagger}\left(\mathcal{N}(\sigma)^{-1/2}(\,\cdot\,)\mathcal{N}(\sigma)^{-1/2}\right)\sigma^{1/2}$

Provided $D(\rho\|\sigma) = D\left(\mathcal{N}(\rho)\|\mathcal{N}(\sigma)\right)$

The Petz map reduces to Bayes' rule in the classical case:

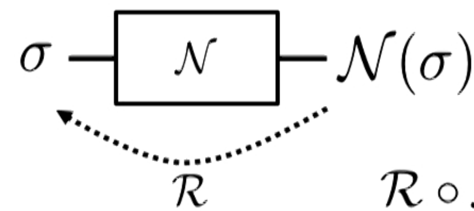
- $\mathcal{N} \sim p(y|x)$, and $\mathcal{N}^{\dagger} = \mathcal{N}$
- $\sigma \sim p(x)$
- $\mathcal{N}(\sigma) \sim p(y) \quad \left(\sum_x p(y|x)p(x) \equiv p(y)\right)$

Since all terms commute, the Petz map reduces to Bayes' rule as follows:

$$\begin{aligned}\mathcal{P}_{\sigma,\mathcal{N}}(\,\cdot\,) &= \sigma^{1/2}\mathcal{N}^{\dagger}\left(\mathcal{N}(\sigma)^{-1/2}(\,\cdot\,)\mathcal{N}(\sigma)^{-1/2}\right)\sigma^{1/2} \\ &= \sigma\mathcal{N}\left((\,\cdot\,)\mathcal{N}(\sigma)^{-1}\right) \sim \frac{p(x)p(y|x)}{p(y)}\end{aligned}$$

A non-commutative version of Bayes' Rule

Classical case: $p(x|y) = \frac{d}{dt}\Big|_{t=0} \log\left(\frac{p(y)}{p(x)} + t\, p(y|x)\right)$



$\mathcal{R} \circ \mathcal{N}(\sigma) = \sigma$

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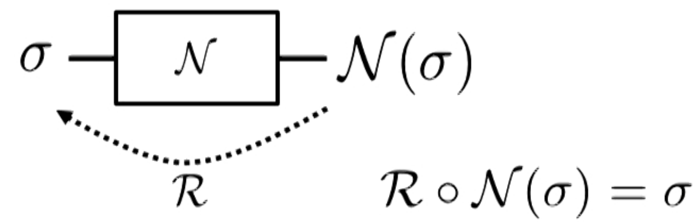
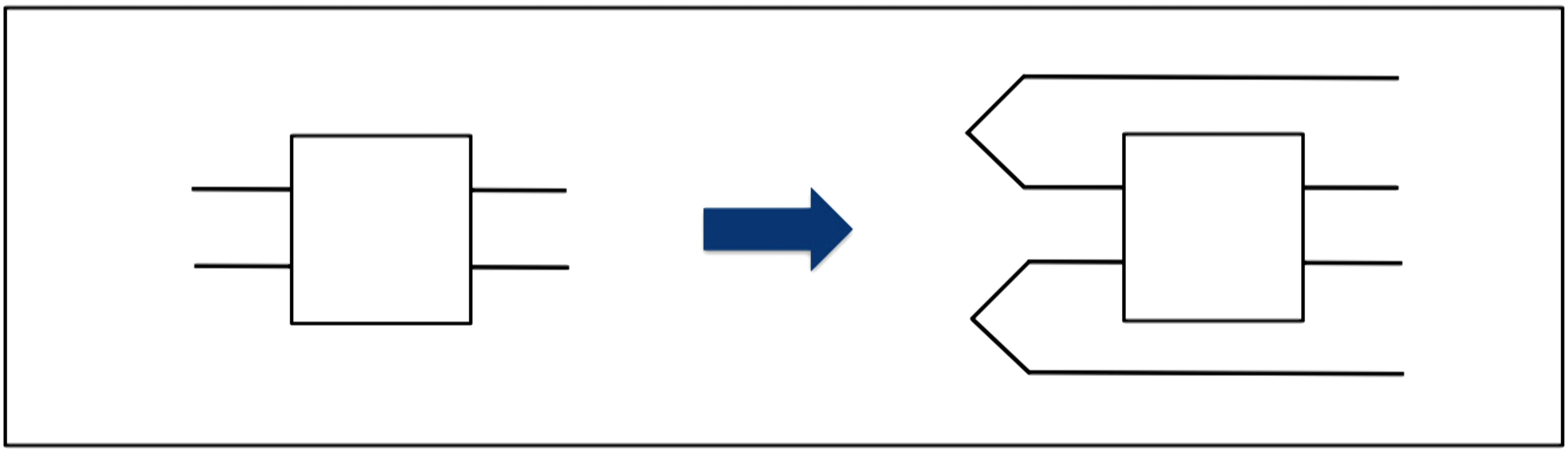


Approximate recovery:



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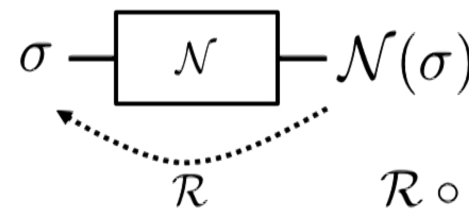

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Choi states of the channels:

$$|\Phi\rangle = \sum_j |j\rangle\,|j\rangle$$

$$\Phi_{\mathcal{N}} = (\mathrm{id} \otimes \mathcal{N})\big[\,|\Phi\rangle\langle\Phi|\,\big]$$

$$\Phi_{\mathcal{R}} = (\mathrm{id} \otimes \mathcal{R})\big[\,|\Phi\rangle\langle\Phi|\,\big]$$

Choi state of universal recovery channel

$$\Phi_{\mathcal{R}} = \frac{d}{dt}\Big|_{t=0} \log\left(\overline{\mathcal{N}(\sigma)} \otimes \sigma^{-1} + t\,\Phi_{\mathcal{N}^*}\right)$$

Tools we use

$$D\log(A)[B] := \frac{d}{dt}\Big|_{t=0} \log(A + t\,B) = \int_{-\infty}^{\infty} dt\, \beta_0(t) A^{-\frac{1+it}{2}} B A^{-\frac{1-it}{2}}$$

$$\text{Schwarz inequality: } \mathcal{M}[X^\dagger]\mathcal{M}[X] \leq \mathcal{M}[X^\dagger X]$$

D. Sutter, M. Berta, and M. Tomamichel, Commun. Math. Phys. 352, 37 (2017), 1604.03023

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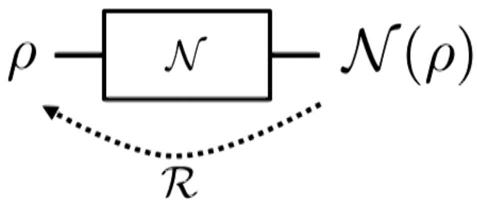

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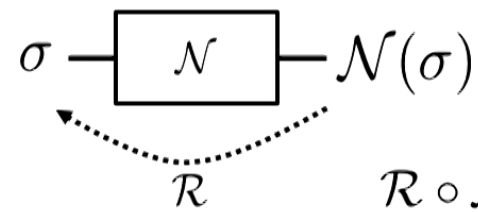

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Approximate universal recovery for algebras

Lemma:

Let

- $\mathcal{A}$ and $\mathcal{B}$ be **finite-dimensional von Neumann algebras**
- $\mathcal{N} : S(\mathcal{A}) \to S(\mathcal{B})$ be a quantum channel
- $\rho, \sigma \in S(\mathcal{A})$ be states such that $\operatorname{supp}\rho \subseteq \operatorname{supp}\sigma$

Then $$D(\rho\|\sigma) - D(\mathcal{N}[\rho]\|\mathcal{N}[\sigma]) \geq -2\log F(\rho, (\mathcal{R}_{\sigma,\mathcal{N}} \circ \mathcal{N})[\rho])$$

$$\mathcal{R}_{\sigma,\mathcal{N}}[\gamma] := \int dt\, \beta_0(t)\sigma^{-it/2}\mathcal{P}_{\sigma,\mathcal{N}}\left[\mathcal{N}[\sigma]^{it/2}\gamma\mathcal{N}[\sigma]^{-it/2}\right]\sigma^{it/2}$$

$$\beta_o(t) = \frac{\pi}{2}(\cosh(\pi t) + 1)^{-1}$$

AdS/CFT

Gravity in anti-de Sitter space in d+1 dimensions

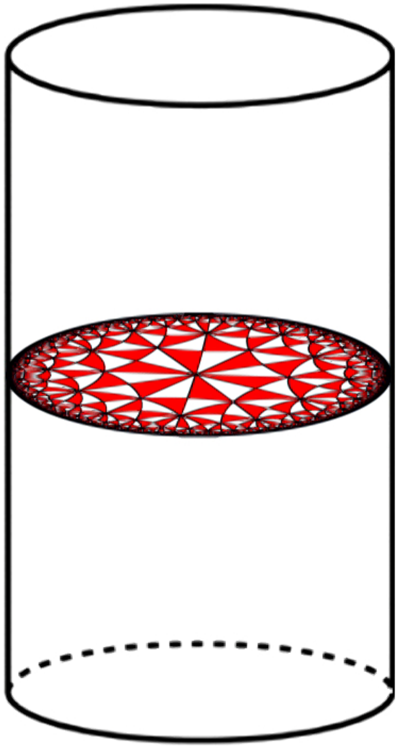
Conformal field theory in d dimensions

AdS/CFT

Gravity in anti-de Sitter space in d+1 dimensions

dual to

Conformal field theory in d dimensions



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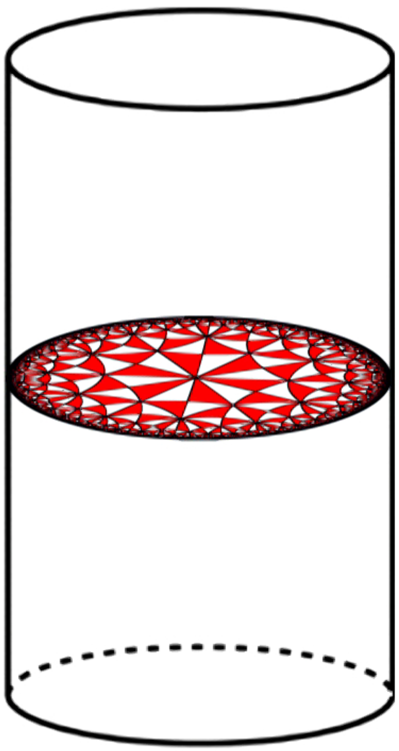
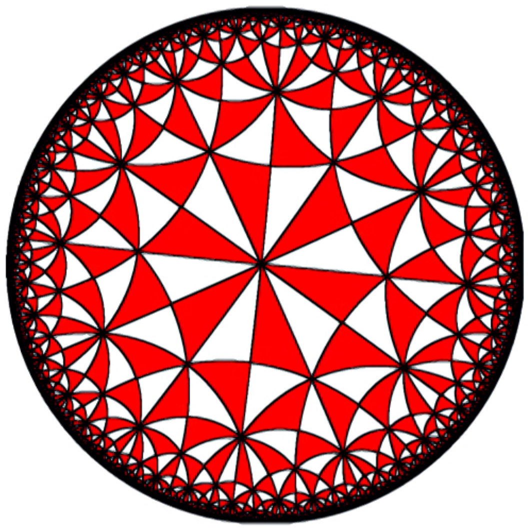




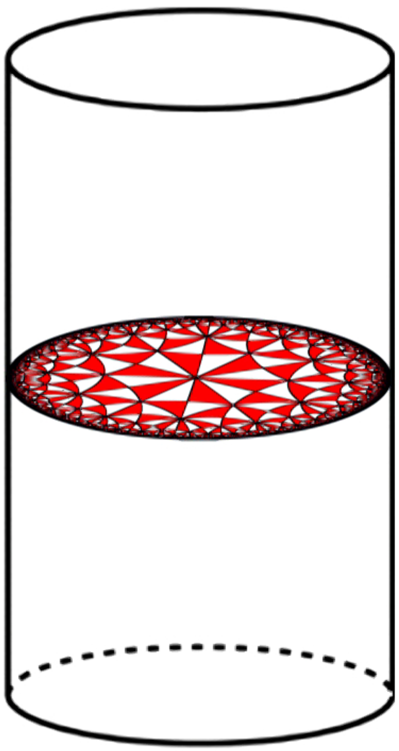
We often think of the CFT as living on the boundary of AdS

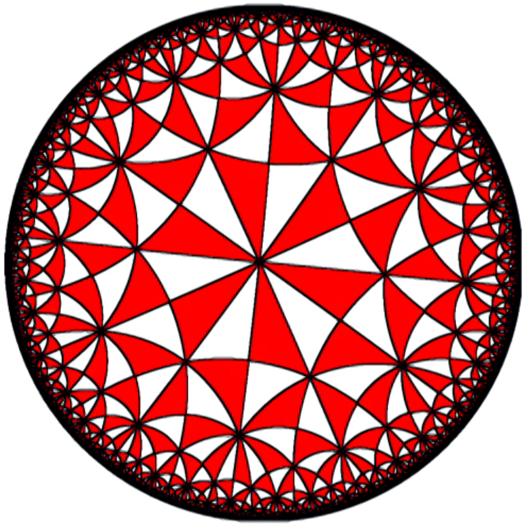
AdS/CFT

Gravity in anti-de Sitter space in d+1 dimensions

dual to

Conformal field theory in d dimensions





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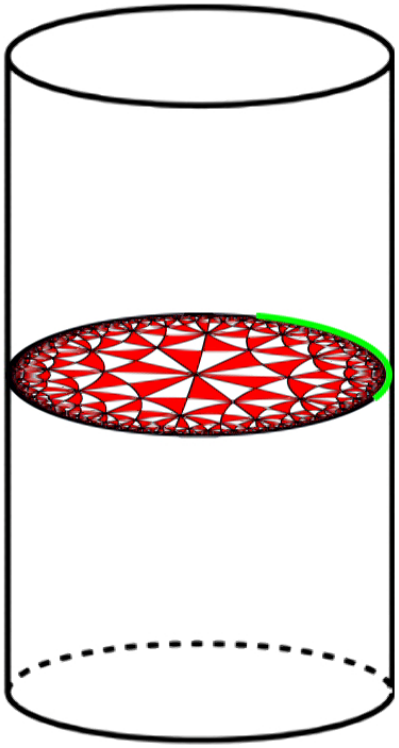

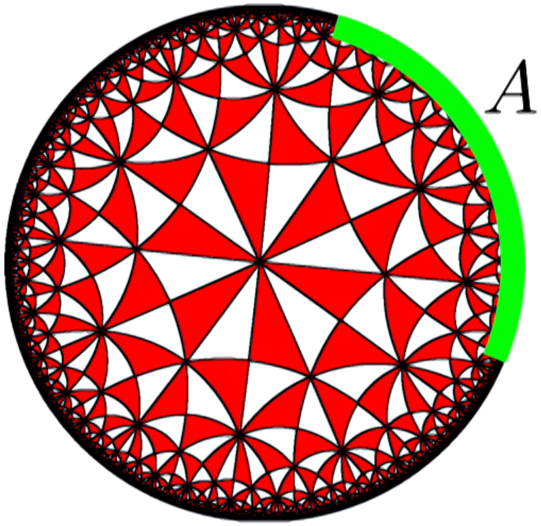

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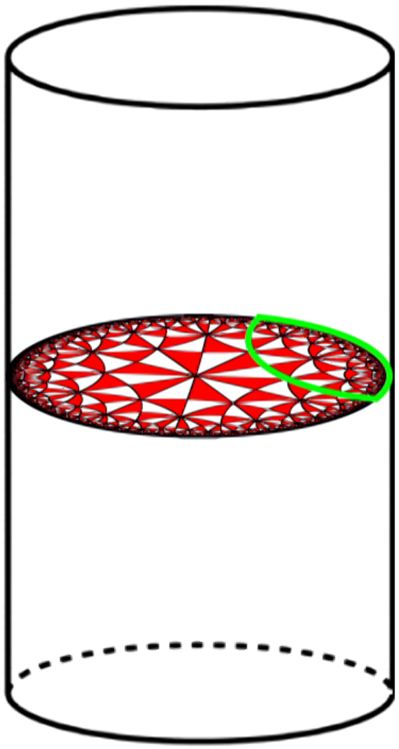

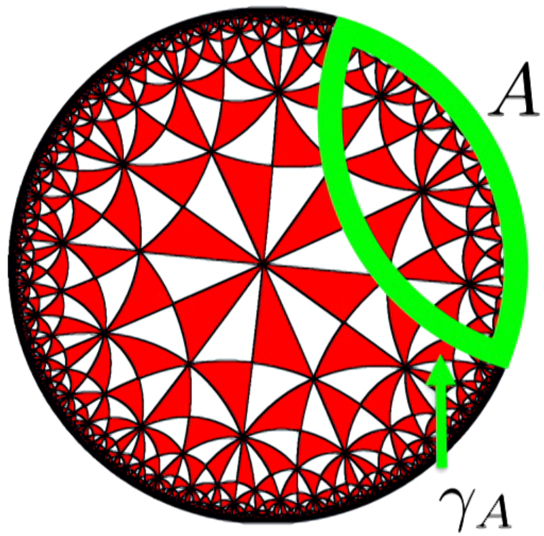

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$$S(A) = \frac{|\gamma_A|}{4G_N} + \ldots$$

Bulk reconstruction and HKLL

Gravity in anti-de Sitter space in d+1 dimensions

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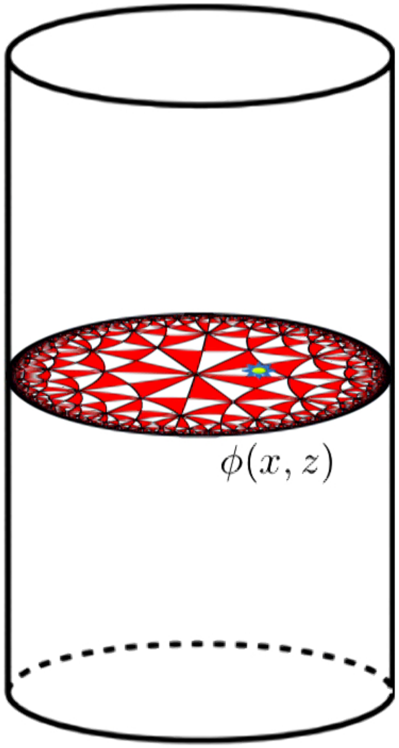

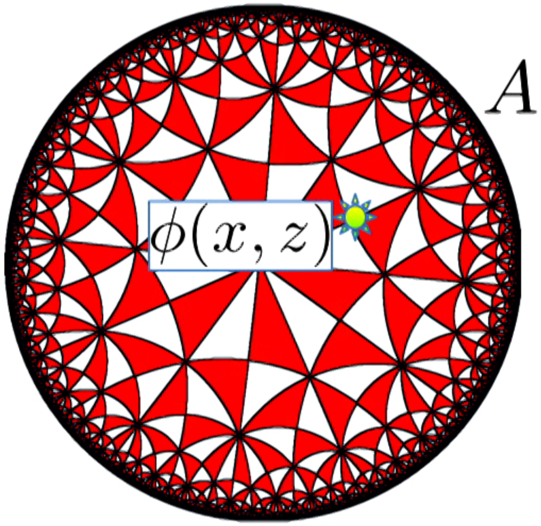

HKLL: express local bulk operators as smeared boundary operators

A. Hamilton, D. Kabat, G. Lifschytz, and D. A. Lowe, Phys. Rev. D 74, 066009 (2006).

Bulk reconstruction and HKLL

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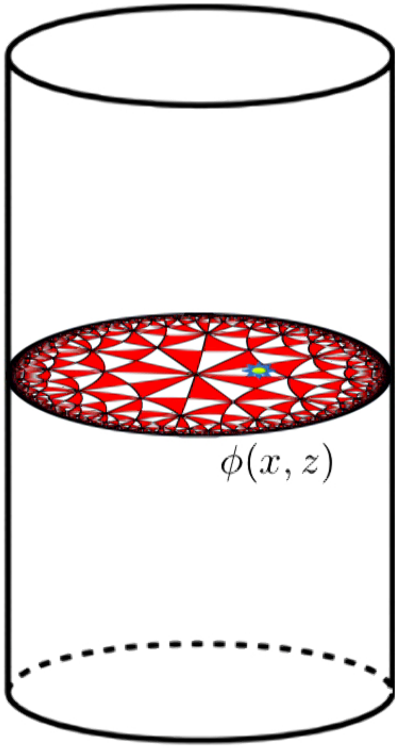

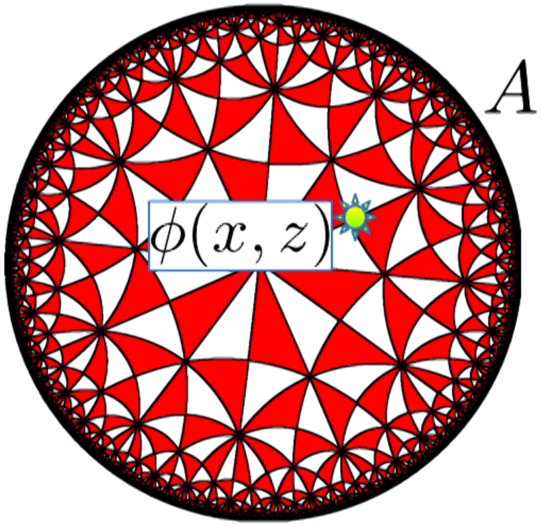

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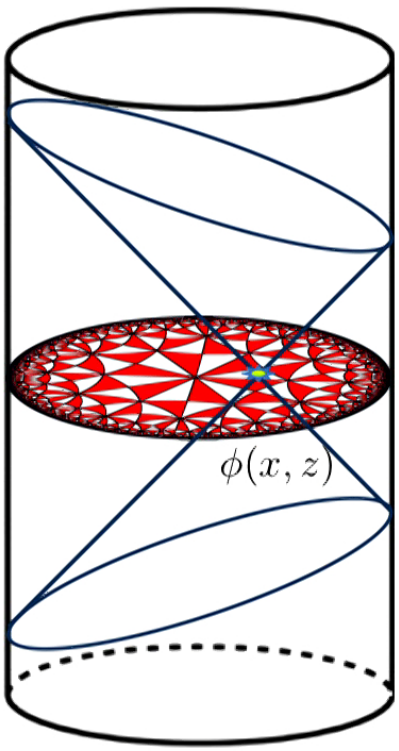

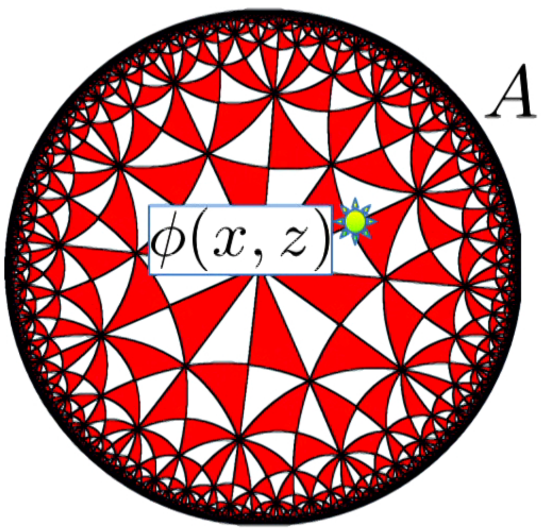

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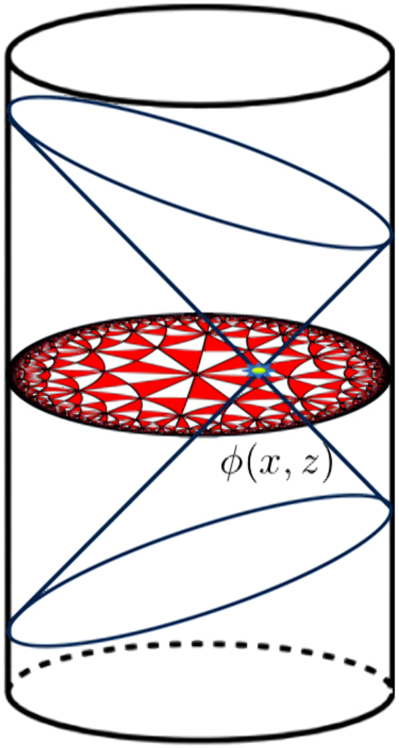

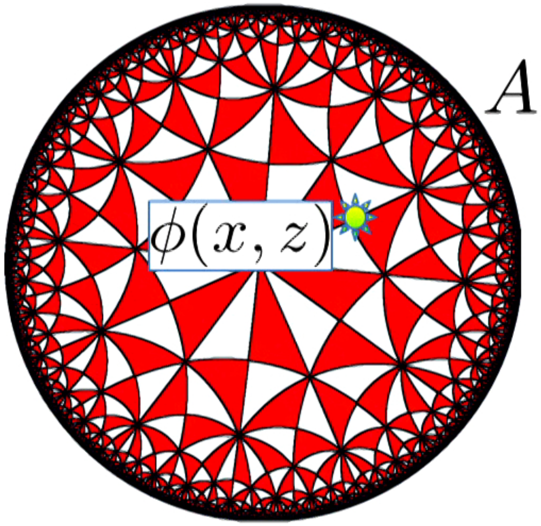

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$$\mathrm{Tr}\,\phi_{\text{bulk}}\rho_{\text{bulk}} = \mathrm{Tr}\,\mathcal{O}_{\text{bdy}}\rho_{\text{bdy}}$$

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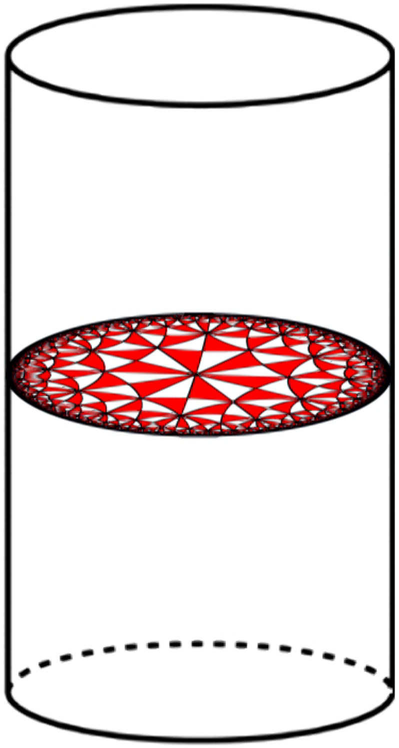
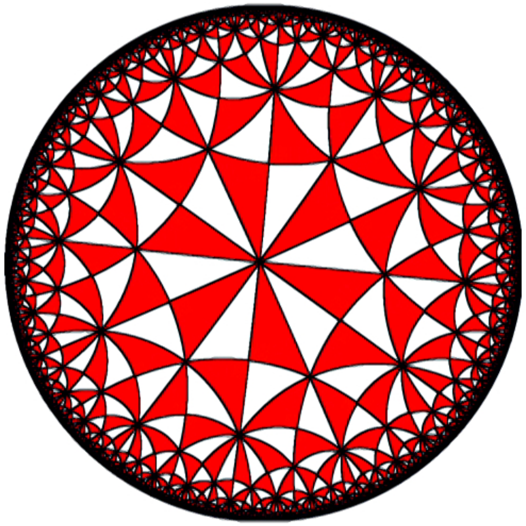
Causal wedge reconstruction





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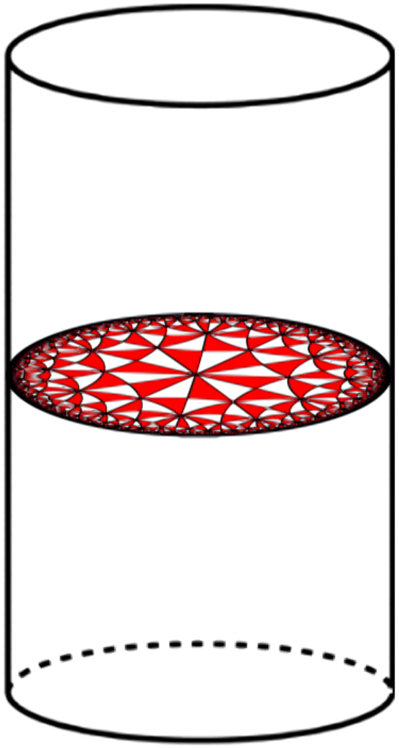

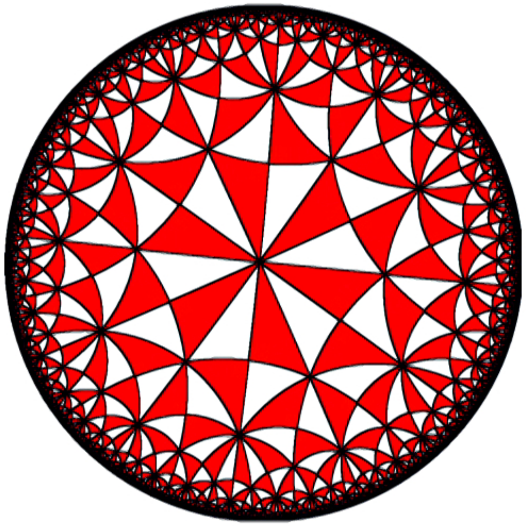

Consider a boundary subregion A,
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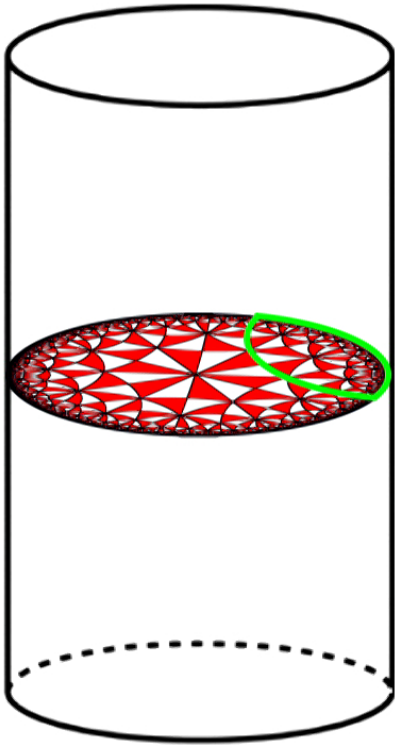

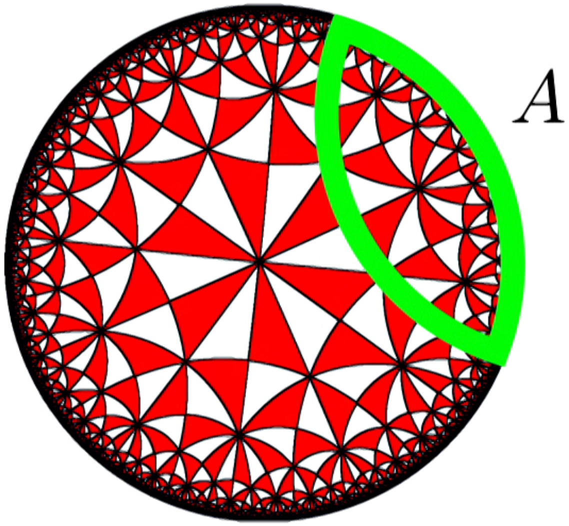

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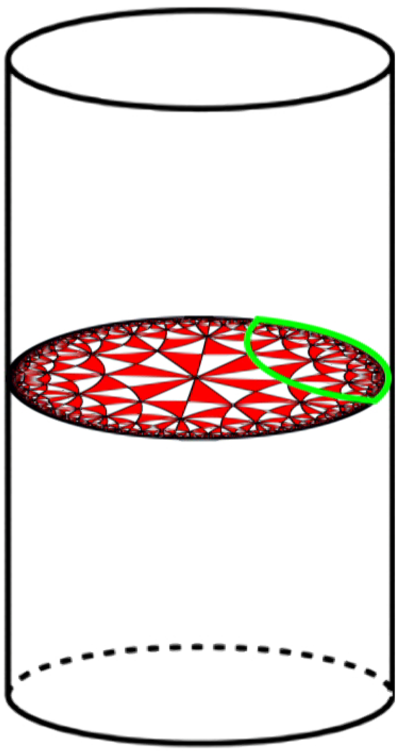
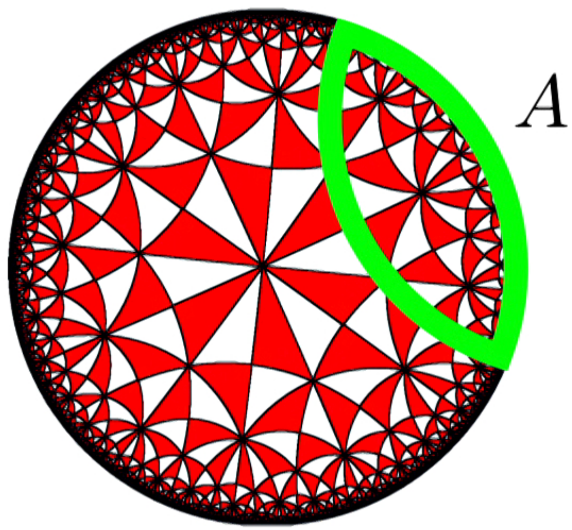
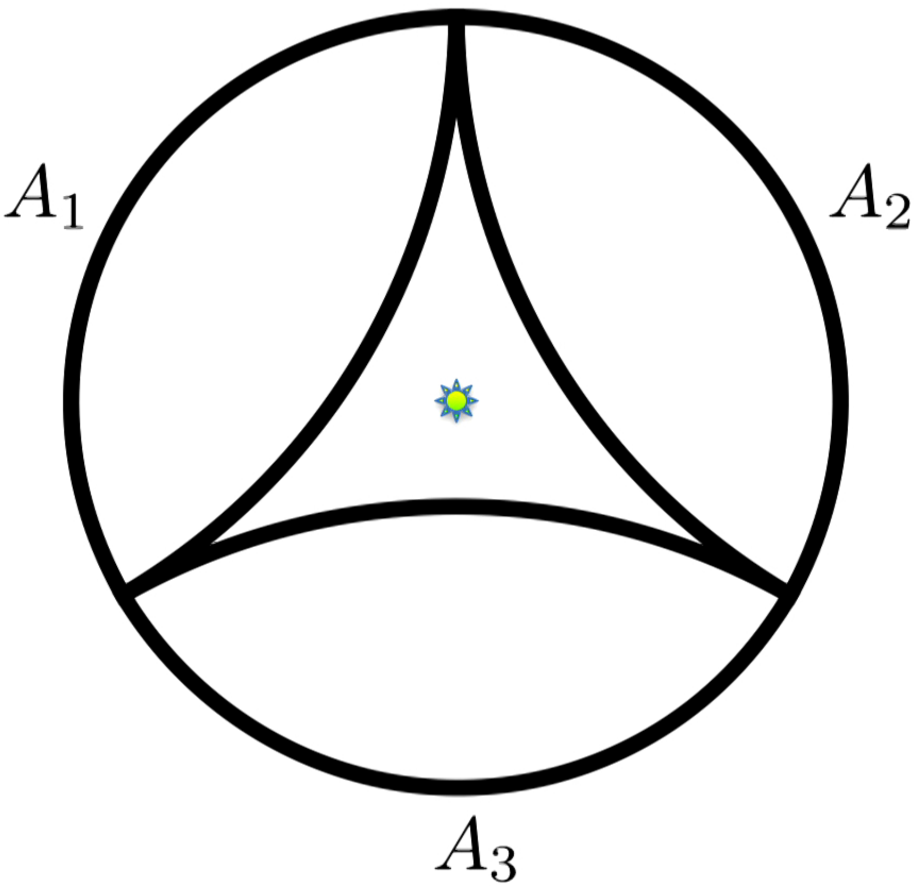




Consider a boundary subregion A, constructed by tracing over $\bar{A}$

We can try to find all bulk operators expressible with support only on A

Holographic quantum error correction



Use causal wedge reconstruction for each of

$$A_1 \cup A_2,\ A_1 \cup A_3,\ A_2 \cup A_3$$

A. Almheiri, X. Dong, and D. Harlow, JHEP 04, 163 (2015), arXiv:1411.7041

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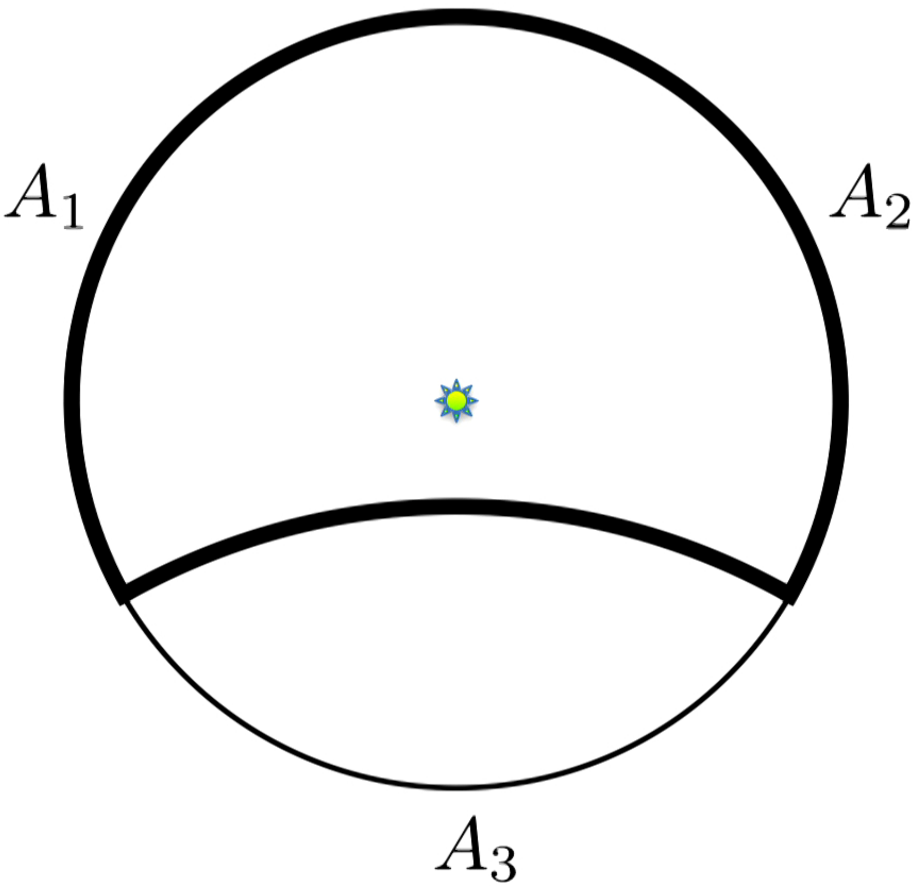

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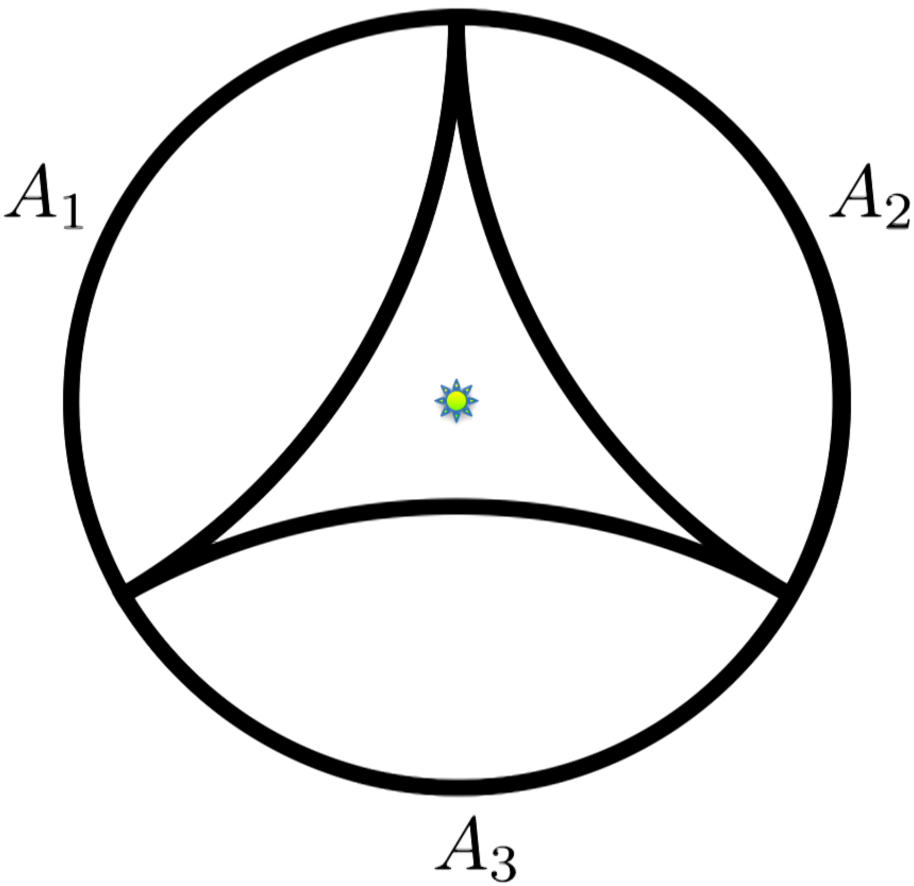

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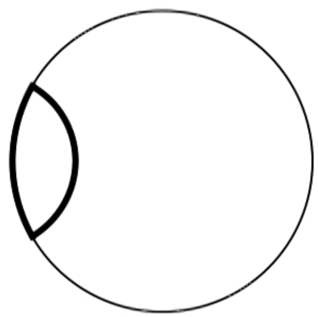
This is a (2,3)-threshold secret sharing quantum error correcting code!

Capable of correcting for loss of any 1 out of the three regions

$$\mathcal{H}_{\rm code} \subseteq \mathcal{H}_{\rm CFT}$$

A. Almheiri, X. Dong, and D. Harlow, JHEP 04, 163 (2015), arXiv:1411.7041

When might HKLL fail?



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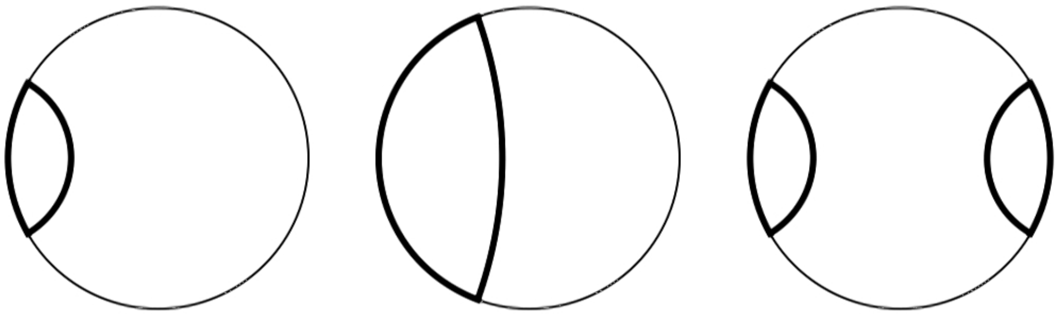

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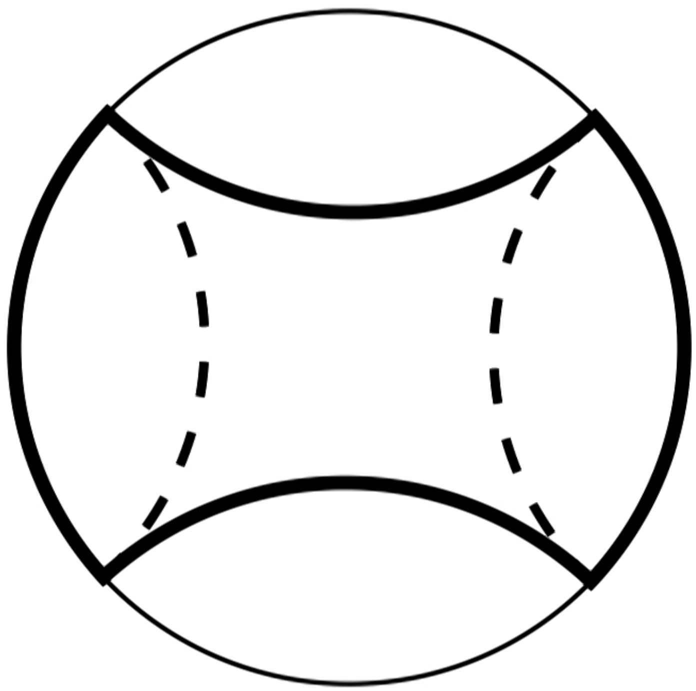

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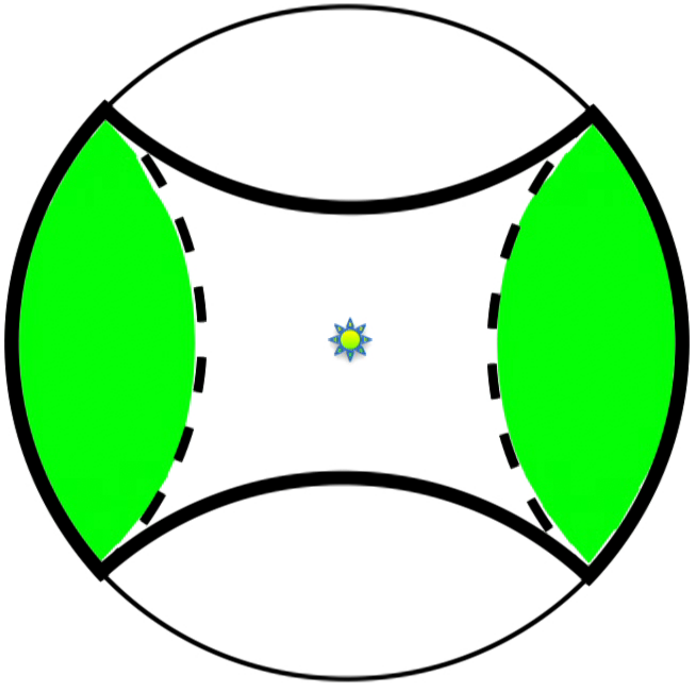

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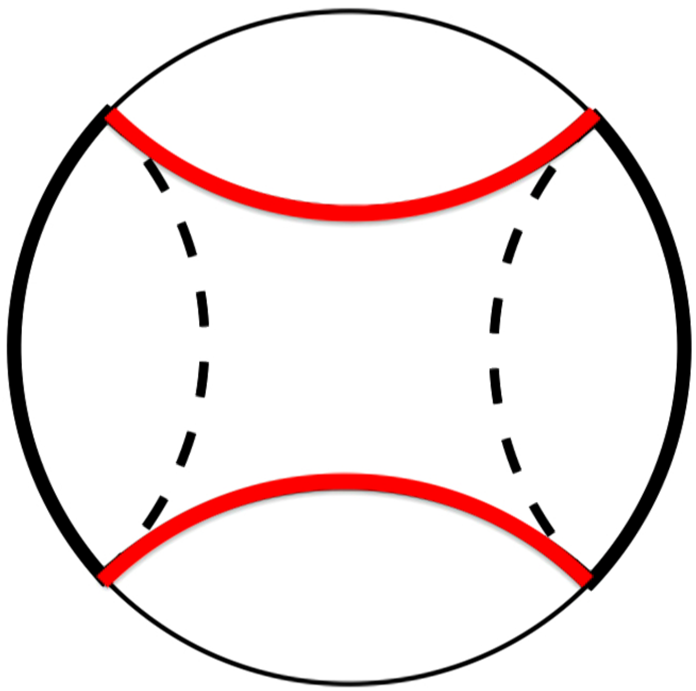

The entropy of the boundary reduced density matrix is given by

$$S(A) = \frac{|\gamma_A|}{4G_N} + \ldots$$

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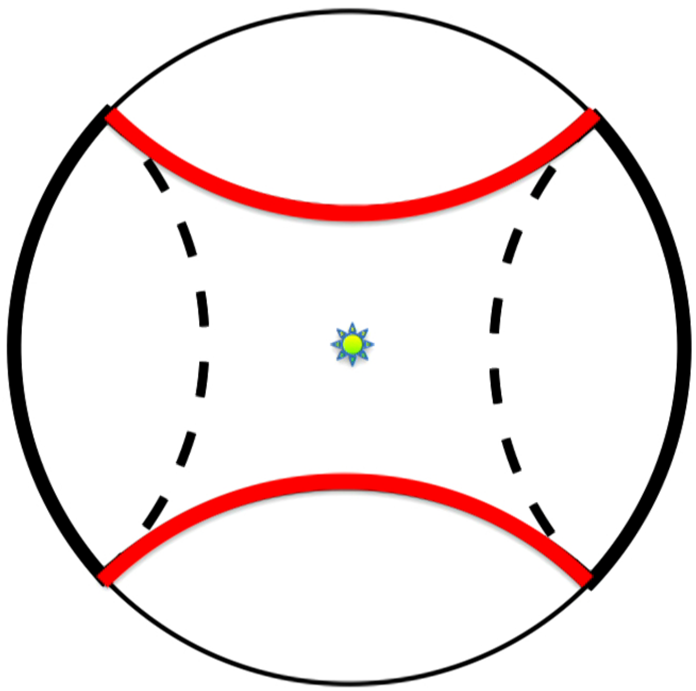

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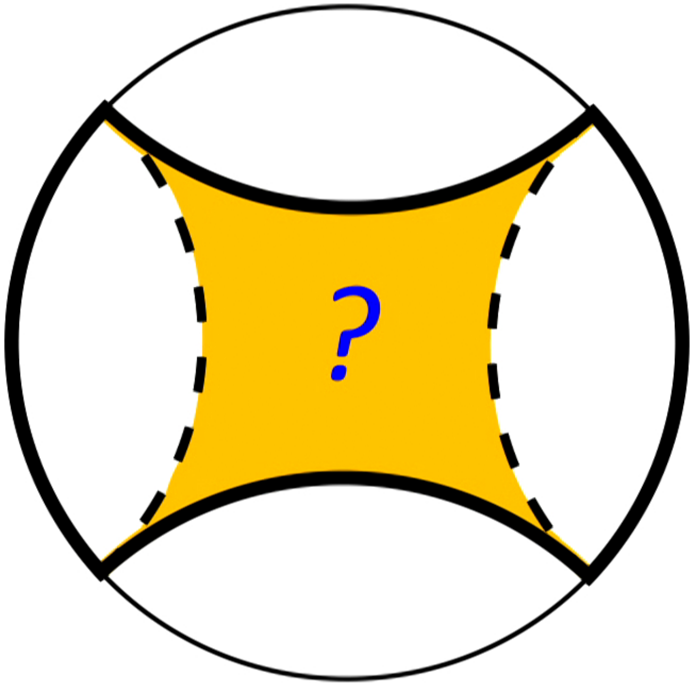


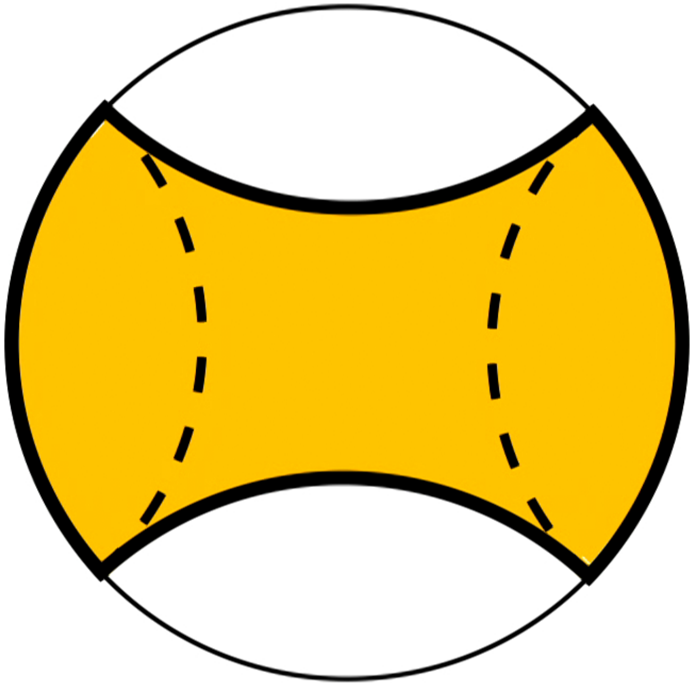
The entropy of the boundary reduced density matrix is given by

$$S(A) = \frac{|\gamma_A|}{4G_N} + S(a) + \ldots$$

Properties of the density matrix on A are sensitive to operators living outside of the causal wedges

This leads to the entanglement wedge hypothesis

Entanglement Wedge Hypothesis



The entanglement wedge is the domain of dependence of the orange region

The full region is a codimension zero spacetime region in the bulk

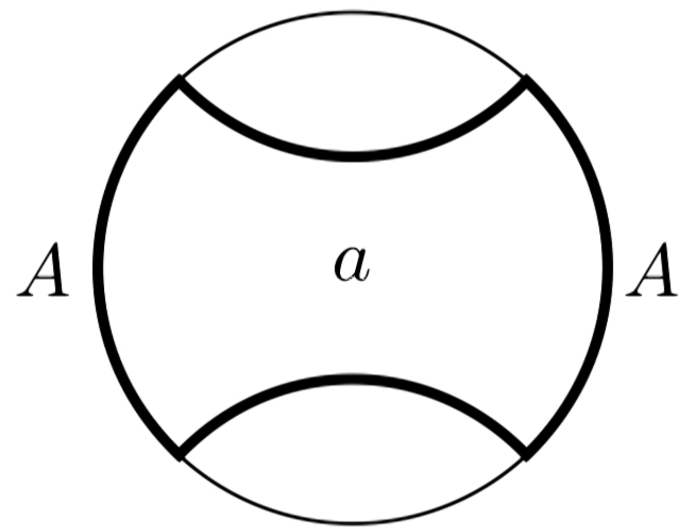
Hypothesis: any bulk operators in the entanglement wedge can be reconstructed on the associated boundary subregion

Entanglement wedge reconstruction

JLMS: boundary relative entropy ≈ bulk relative entropy

$$D(\rho_A\|\sigma_A) = D(\rho_a\|\sigma_a) + O(1/N)$$

$$D(\rho_A\|\sigma_A) = D(\rho_a^{\{\sigma\}}\|\sigma_a) + \left[\mathrm{Tr}\left(\rho_a^{\{\sigma\}}\mathcal{A}_{\mathrm{loc}}^{\{\sigma\}}\right) - \mathrm{Tr}\left(\rho_a^{\{\rho\}}\mathcal{A}_{\mathrm{loc}}^{\{\rho\}}\right) + S\left(\rho_a^{\{\sigma\}}\right) - S\left(\rho_a^{\{\rho\}}\right)\right]$$



Relative Entropy:

$$D(\rho\|\sigma) = \mathrm{Tr}(\rho\log\rho) - \mathrm{Tr}(\rho\log\sigma)$$

A measure of distinguishability.

Equal to zero if and only if $\rho = \sigma$.

D. L. Jafferis, A. Lewkowycz, J. Maldacena, and S. J. Suh, JHEP 2016, 1 (2016)

X. Dong, D. Harlow, and A. C. Wall, Phys. Rev. Lett. 117, 021601 (2016)

Exact entanglement wedge reconstruction

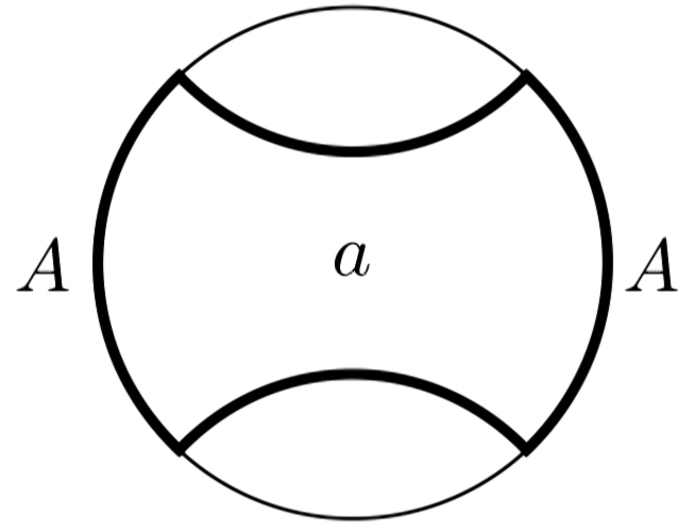
$$D(\rho_A \| \sigma_A) = D(\rho_a \| \sigma_a) + O(1/N)$$

When this condition holds exactly (e.g., $N \to \infty$), the entanglement wedge hypothesis has been argued to be true.

The proof relies on algebraic consequences of exact equality in the relative entropy condition

These algebraic consequences are known not to hold when the relative entropy condition is only approximately satisfied

No explicit expression for reconstruction



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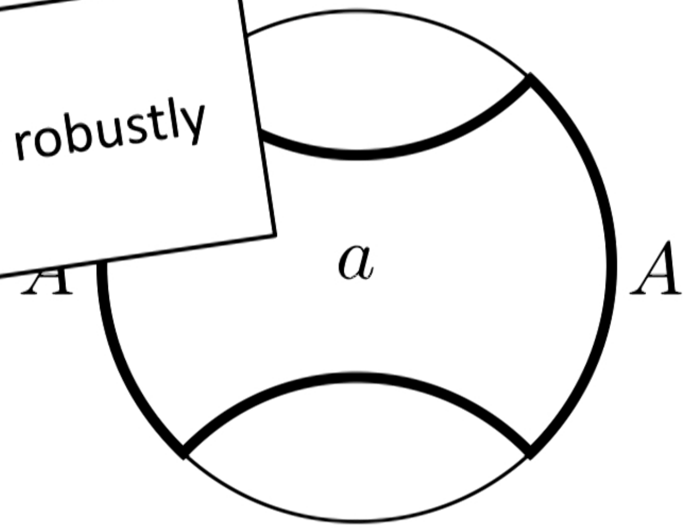
Our goal:

- Prove entanglement wedge reconstruction robustly
- Give an explicit formula



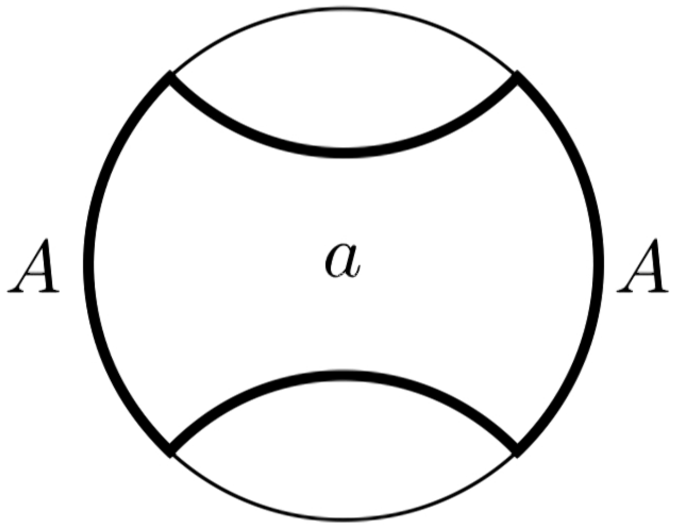
X. Dong, D. Harlow, and A. C. Wall, Phys. Rev. Lett. 117, 021601 (2016)

Remember: JLMS

$$D(\rho\|\sigma) - D(\mathcal{N}(\rho)\|\mathcal{N}(\sigma)) \geq -2\log F(\rho, (\mathcal{R}_{\sigma,\mathcal{N}} \circ \mathcal{N}(\rho))$$

JLMS:

$$D(\rho_A\|\sigma_A) = D(\rho_a\|\sigma_a) + O(1/N)$$



JLMS implies that the difference between the bulk and boundary relative entropies should be small (zero to leading order in $1/N$).

This means we expect we can find a recovery channel that works with high fidelity

D. L. Jafferis, A. Lewkowycz, J. Maldacena, and S. J. Suh, JHEP 2016, 1 (2016).

Entanglement wedge reconstruction

- Goal: reconstruct the entanglement wedge on A
  - Define the bulk to boundary channel $\mathcal{N} : S(\mathcal{H}_{\text{code}}) \to S(\mathcal{H}_A)$

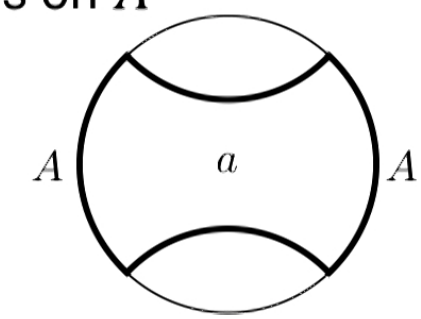
- Find a recovery channel $\mathcal{R}$ such that $\mathcal{R} \circ \mathcal{N}(\rho_a) \approx \rho_a$

- $\mathcal{R}$ is then a map from boundary to bulk states $\mathcal{R} : S(\mathcal{H}_A) \to S(\mathcal{H}_a)$

- The adjoint, $\mathcal{R}^*$, is then a map from bulk operators in a to operators on A

- JLMS then suggests that the recovery will work with high fidelity

$$D(\rho\|\sigma) - D(\mathcal{N}(\rho)\|\mathcal{N}(\sigma)) \geq -2\log F(\rho, (\mathcal{R}_{\sigma,\mathcal{N}} \circ \mathcal{N}(\rho))$$



Entanglement wedge reconstruction

Philosophy: the bulk-to-boundary map is reversible, but by tracing over $\bar{A}$ we introduce noise

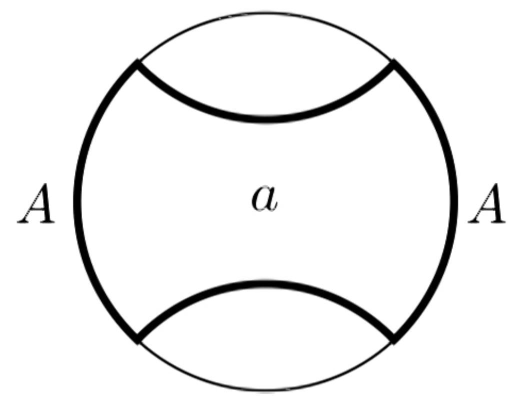
We want to recover from this noise process and recover the information on the boundary subregion

$$\mathcal{N}(\rho) := \mathrm{Tr}_{\bar{A}}\, J\,\rho\, J^\dagger \qquad \rho \in \mathcal{H}_{\text{code}}$$

Assumptions:

1) There is a global quantum channel from bulk-to-boundary: an isometry from the code space to the CFT

$$J : \mathcal{H}_{\text{code}} \to \mathcal{H}_{\text{CFT}} \qquad \rho_{\text{code}} \mapsto J\,\rho_{\text{code}}\,J^\dagger$$



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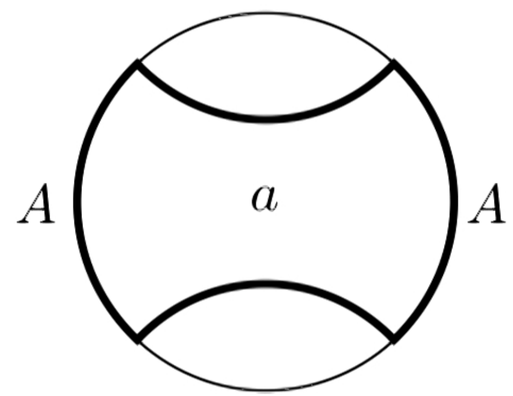
2) JLMS $D(\rho_A \| \sigma_A) = D(\rho_a \| \sigma_a) + O(1/N)$

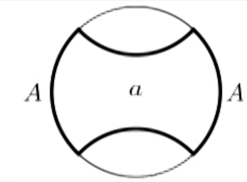
0) [Not needed] Tensor factorization

$$\mathcal{H}_{\text{code}} = \mathcal{H}_a \otimes \mathcal{H}_{\bar{a}}$$
$$\mathcal{H}_{\text{CFT}} = \mathcal{H}_A \otimes \mathcal{H}_{\bar{A}}$$

Only for simplicity! We do not have to make this assumption

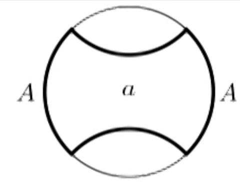


Subregion Subtlety



- JLMS tells us that, for $\rho, \sigma \in \mathcal{H}_{\rm code}$, we have $\left|D(\mathrm{Tr}_{\bar{a}}\,\rho \| \mathrm{Tr}_{\bar{a}}\,\sigma) - D\big(\mathcal{N}(\rho)\|\mathcal{N}(\sigma)\big)\right| \leq \epsilon$
- i.e., $D\big(\mathcal{M}(\rho)\|\mathcal{M}(\sigma)\big) \approx D\big(\mathcal{N}(\rho)\|\mathcal{N}(\sigma)\big)$, where $\mathcal{M}$ is bulk partial trace
- This is the wrong form! **Monotonicity of relative entropy:** $D(\rho\|\sigma) \geq D\left(\mathcal{N}(\rho)\|\mathcal{N}(\sigma)\right)$
- Introduce the "local channel": $\mathcal{T}: S(\mathcal{H}_a) \to S(\mathcal{H}_A)$ $\boxed{\mathcal{T}(\rho_a) = \mathcal{N}(\rho_a \otimes \tau_{\bar{a}})}$
- JLMS then says $D(\rho_a\|\sigma_a) \approx D\big(\mathcal{T}(\rho_a)\|\mathcal{T}(\sigma_a)\big)$
- Find recovery map for $\mathcal{T}$ and then prove that it still works for $\mathcal{N}$

Subregion Subtlety



$$\mathcal{T}(\rho_a) = \mathcal{N}(\rho_a \otimes \tau_{\bar{a}})$$

$$\Big| D(\rho_a \| \sigma_a) - D\big(\mathcal{N}(\rho) \| \mathcal{N}(\sigma)\big) \Big| \leq \epsilon$$

$$\|\mathcal{T}(\rho_a) - \mathcal{N}(\rho)\|_1 = \|\mathcal{N}(\rho_a \otimes \tau_{\bar{a}}) - \mathcal{N}(\rho)\|_1$$

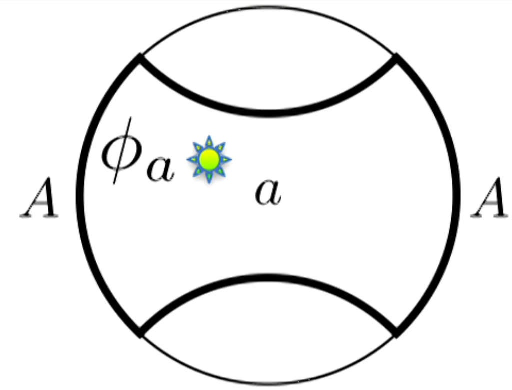
Pinsker's inequality $\leq \sqrt{2 \ln 2 D\big(\mathcal{N}(\rho_a \otimes \tau_{\bar{a}}) \| \mathcal{N}(\rho)\big)}$

JLMS $\leq \sqrt{2 \ln 2 \epsilon}$

So we conclude that the recovery map for $\mathcal{T}$ will work well for $\mathcal{N}$

Entanglement wedge reconstruction (finally!)

Let ϕ_a be an operator in the entanglement wedge



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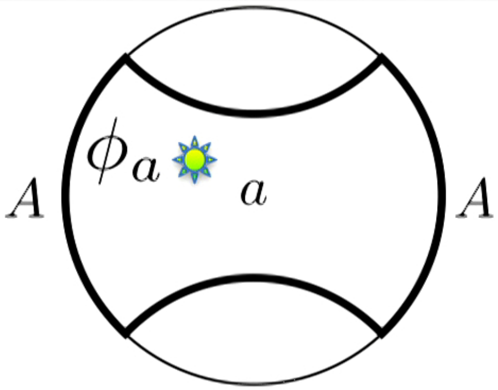

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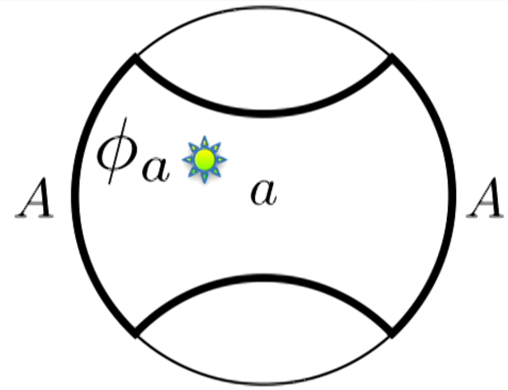

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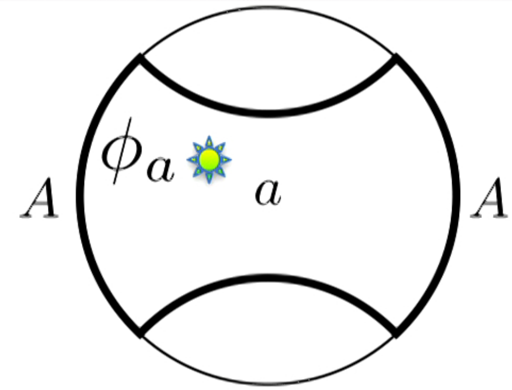

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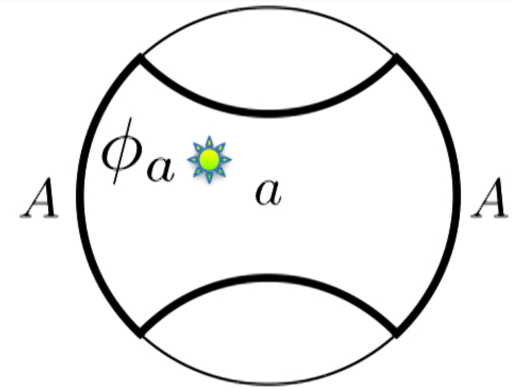

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The boundary operator corresponding to ϕ_a can be computed as a response in the boundary modular Hamiltonian H_A to a perturbation of the maximally mixed code state in the direction of ϕ_a

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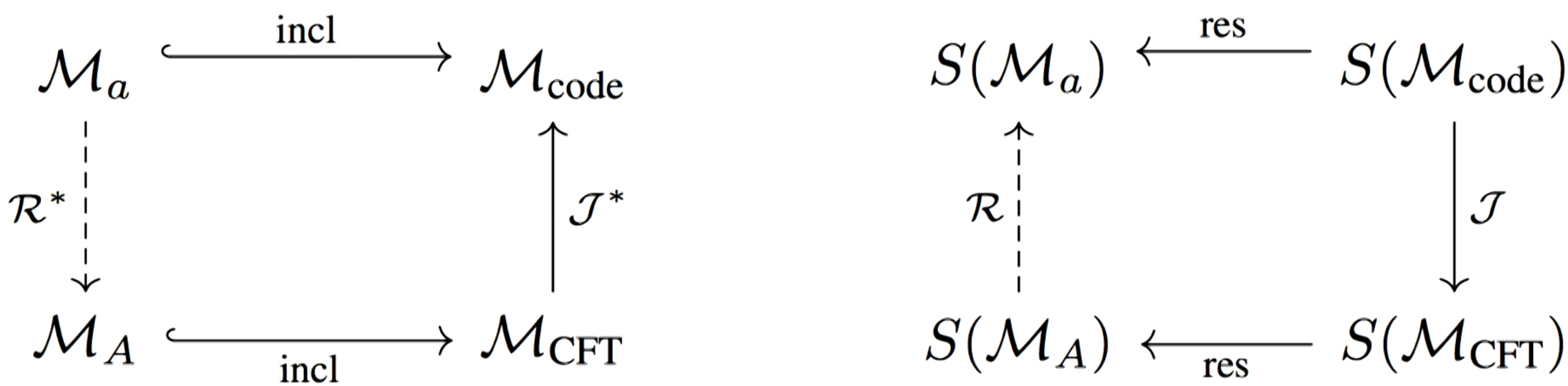
In the approximate case, our proof does not generalize to higher-point functions… ☹

Algebraic formalism

We work with algebras of observables in the bulk and boundary

We prove an analog of the recovery channel theorem of Junge *et al.* for algebras



Algebraic formalism

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Definition: *A* ***von Neumann algebra on*** $\mathcal{H}$ *is a set* $M \subseteq \mathcal{L}(\mathcal{H})$ *such that:*

- $\forall \lambda \in \mathbb{C}, \lambda I \in M$
- $\forall x \in M, x^\dagger \in M$
- $\forall x, y \in M, xy \in M$
- $\forall x, y \in M, x + y \in M$

In other words, it is a set of linear operators on $\mathcal{H}$ which is closed under hermitian conjugation, addition, multiplication, and which contains all scalar multiples of the identity operator.

See, for example, arXiv:1607.03901 by Daniel Harlow

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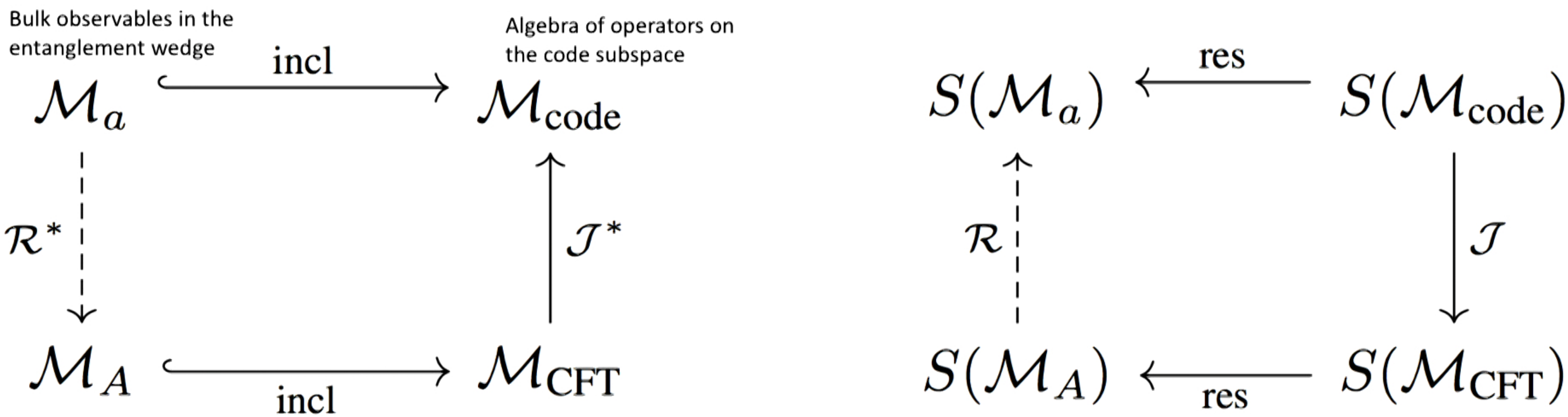

Algebraic formalism

Assumptions:

1) There is a global quantum channel from bulk-to-boundary (not necessarily an isometry)

$$\mathcal{J} : S(\mathcal{M}_{\text{code}}) \to S(\mathcal{M}_{\text{CFT}})$$

2) JLMS For all $\rho, \sigma \in S(\mathcal{M}_{\text{code}}),\ |D(\rho_a \| \sigma_a) - D(\mathcal{J}(\rho)_A \| \mathcal{J}(\sigma)_A)| \leq \epsilon$

where we denote by ρ_X the restriction $\rho|_{\mathcal{M}_X}$ of a state ρ to some subalgebra $\mathcal{M}_X$.

We extend the universal recovery theorem of Junge et al. to the case of von Neumann algebras

Given two VN algebras and a quantum channel between their state spaces, we prove a recovery theorem for this channel

Algebraic formalism

For all $\phi_a \in \mathcal{M}_a$ $(\sigma = \tau_{\text{code}})$

$$\mathcal{R}^*(\phi_a) = \frac{1}{d_{\text{code}}} \int dt\, \beta_0(t) e^{\frac{1}{2}(1-it)H_A} \mathcal{J}[\phi_a]_A e^{\frac{1}{2}(1+it)H_A}$$

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$$H_A[\rho] := -\log \mathcal{J}[\rho]_A$$

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- We proved entanglement wedge reconstruction robustly
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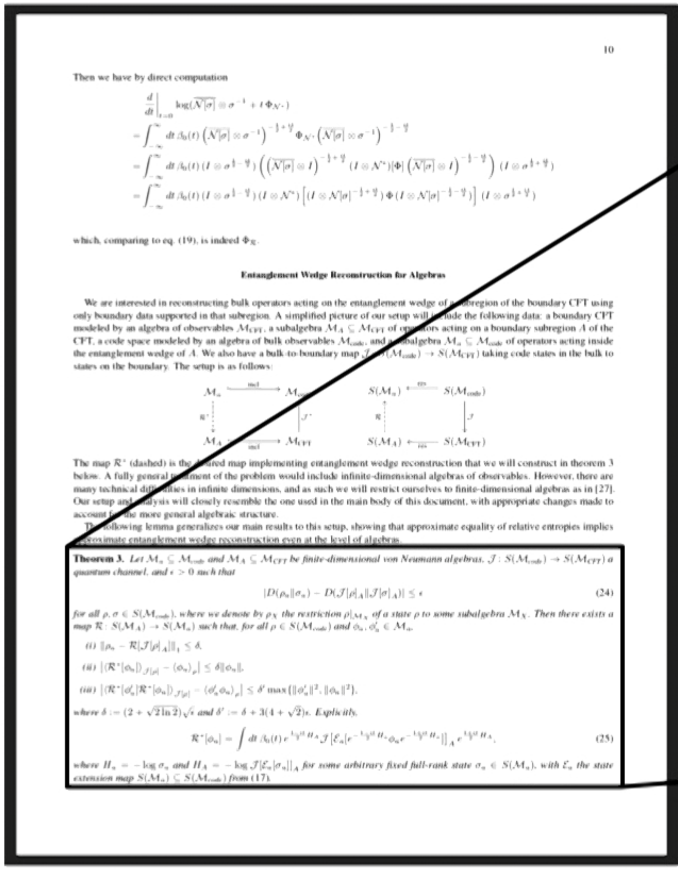

10

Then we have by direct computation

$$\begin{aligned}
&\frac{d}{dt}\bigg|_{t=0} \log(\overline{\mathcal{N}[\sigma]} \otimes \sigma^{-1} + t\,\Phi_{\mathcal{N}^*}) \\
&= \int_{-\infty}^{\infty} dt\, \beta_0(t) \left(\overline{\mathcal{N}[\sigma]} \otimes \sigma^{-1}\right)^{-\frac{1}{2}+\frac{it}{2}} \Phi_{\mathcal{N}^*} \left(\overline{\mathcal{N}[\sigma]} \otimes \sigma^{-1}\right)^{-\frac{1}{2}-\frac{it}{2}} \\
&= \int_{-\infty}^{\infty} dt\, \beta_0(t) \,(I \otimes \sigma^{\frac{1}{2}-\frac{it}{2}}) \left(\left(\overline{\mathcal{N}[\sigma]} \otimes I\right)^{-\frac{1}{2}+\frac{it}{2}} (I \otimes \mathcal{N}^*)[\Phi] \left(\overline{\mathcal{N}[\sigma]} \otimes I\right)^{-\frac{1}{2}-\frac{it}{2}}\right) (I \otimes \sigma^{\frac{1}{2}+\frac{it}{2}}) \\
&= \int_{-\infty}^{\infty} dt\, \beta_0(t) \,(I \otimes \sigma^{\frac{1}{2}-\frac{it}{2}}) (I \otimes \mathcal{N}^*) \left[(I \otimes \mathcal{N}[\sigma]^{-\frac{1}{2}+\frac{it}{2}}) \Phi (I \otimes \mathcal{N}[\sigma]^{-\frac{1}{2}-\frac{it}{2}})\right] (I \otimes \sigma^{\frac{1}{2}+\frac{it}{2}})
\end{aligned}$$

which, comparing to eq. (19), is indeed $\Phi_{\mathcal{R}}$.

Entanglement Wedge Reconstruction for Algebras

We are interested in reconstructing bulk operators acting on the entanglement wedge of a subregion of the boundary CFT using only boundary data supported in that subregion. A simplified picture of our setup will include the following data: a boundary CFT modeled by an algebra of observables $\mathcal{M}_{CFT}$, a subalgebra $\mathcal{M}_A \subseteq \mathcal{M}_{CFT}$ of operators acting on a boundary subregion A of the CFT, a code space modeled by an algebra of bulk observables $\mathcal{M}_{code}$, and a subalgebra $\mathcal{M}_a \subseteq \mathcal{M}_{code}$ of operators acting inside the entanglement wedge of A. We also have a bulk-to-boundary map $\mathcal{J}: S(\mathcal{M}_{code}) \to S(\mathcal{M}_{CFT})$ taking code states in the bulk to states on the boundary. The setup is as follows:

$$\begin{array}{ccccccc}
\mathcal{M}_a & \xrightarrow{\text{incl}} & \mathcal{M}_{code} & \qquad & S(\mathcal{M}_a) & \xleftarrow{\text{res}} & S(\mathcal{M}_{code}) \\
\downarrow \mathcal{R}^* & & \uparrow \mathcal{J}^* & & \uparrow \mathcal{R} & & \downarrow \mathcal{J} \\
\mathcal{M}_A & \xrightarrow{\text{incl}} & \mathcal{M}_{CFT} & & S(\mathcal{M}_A) & \xleftarrow{\text{res}} & S(\mathcal{M}_{CFT})
\end{array}$$

The map $\mathcal{R}^*$ (dashed) is the desired map implementing entanglement wedge reconstruction that we will construct in theorem 3 below. A fully general treatment of the problem would include infinite-dimensional algebras of observables. However, there are many technical difficulties in infinite dimensions, and as such we will restrict ourselves to finite-dimensional algebras as in [27]. Our setup and analysis will closely resemble the one used in the main body of this document, with appropriate changes made to account for the more general algebraic structure.

The following lemma generalizes our main results to this setup, showing that approximate equality of relative entropies implies approximate entanglement wedge reconstruction even at the level of algebras.

Theorem 3. *Let $\mathcal{M}_a \subseteq \mathcal{M}_{code}$ and $\mathcal{M}_A \subseteq \mathcal{M}_{CFT}$ be finite-dimensional von Neumann algebras, $\mathcal{J}: S(\mathcal{M}_{code}) \to S(\mathcal{M}_{CFT})$ a quantum channel, and $\epsilon > 0$ such that*

$$|D(\rho_a \| \sigma_a) - D(\mathcal{J}[\rho]_A \| \mathcal{J}[\sigma]_A)| \leq \epsilon \tag{24}$$

for all $\rho, \sigma \in S(\mathcal{M}_{code})$, where we denote by ρ_X the restriction $\rho|_{\mathcal{M}_X}$ of a state ρ to some subalgebra $\mathcal{M}_X$. Then there exists a map $\mathcal{R}: S(\mathcal{M}_A) \to S(\mathcal{M}_a)$ such that, for all $\rho \in S(\mathcal{M}_{code})$ and $\phi_a, \phi'_a \in \mathcal{M}_a$,

(i) $\|\rho_a - \mathcal{R}[\mathcal{J}[\rho]_A]\|_1 \leq \delta$,

(ii) $\left|\langle \mathcal{R}^*[\phi_a] \rangle_{\mathcal{J}[\rho]} - \langle \phi_a \rangle_\rho\right| \leq \delta \|\phi_a\|$,

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where $\delta := (2 + \sqrt{2 \ln 2})\sqrt{\epsilon}$ and $\delta' := \delta + 3(4 + \sqrt{2})\epsilon$. Explicitly,

$$\mathcal{R}^*[\phi_a] = \int dt\, \beta_0(t)\, e^{\frac{1-it}{2} H_A} \mathcal{J}\left[\mathcal{E}_a[e^{-\frac{1-it}{2} H_a} \phi_a e^{-\frac{1+it}{2} H_a}]\right]_A e^{\frac{1+it}{2} H_A}, \tag{25}$$

where $H_a = -\log \sigma_a$ and $H_A = -\log \mathcal{J}[\mathcal{E}_a[\sigma_a]]_A$ for some arbitrary fixed full-rank state $\sigma_a \in S(\mathcal{M}_a)$, with $\mathcal{E}_a$ the state extension map $S(\mathcal{M}_a) \subseteq S(\mathcal{M}_{code})$ from (17).

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