Title: Covariant observables and (quantum) extension theorems

Date: Nov 16, 2017 10:00 AM

URL: http://pirsa.org/17110134

Abstract:

Covariant observables and (quantum) extension theorems

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Path Integral for Gravity Workshop, Perimeter Institute, November, 2017

$$Z[g] = \int d[g] \exp\left(\frac{iS[g]}{\hbar}\right)$$

What are the Covariant Beables or Observables in Quantum Gravity?

- Intrinsic teleology in GR
  - Blackholes
  - Cauchy hypersurface: (Σ, h)
  - Global Causality Conditions

  $\vdots$

Covariant observables seem to require a "fully-evolved" spacetime

What are the Covariant Beables or Observables in Quantum Gravity?

- Intrinsic teleology in GR suggests we need "fully-evolved" spacetimes
- Access only to limited spacetime regions
- *"Acorn's intrinsic telos is to become a fully grown oak tree"* — Aristotle
- What are the covariant "acorns" of quantum gravity?

Quantum measure formulation

Quantum Theory is a Generalisation of Classical Stochastic Dynamics

Sorkin (1994)

Classical Stochastic Dynamics given by a Probability Measure Space: $(\Omega, \mathcal{A}, \mu)$

- Sample Space Ω: space of histories.
- Event Algebra $\mathcal{A}$: set of all beables or propositions about the system.

  (These are the "acorns" – to be identified)

  - $\alpha \in \mathcal{A}$, $\alpha \subseteq \Omega$
  - $\mathcal{A}$ is closed under finite set union, intersection and complementation.
  - $\emptyset, \Omega \in \mathcal{A}$

- Probability Measure $\mu : \mathcal{A} \to [0, 1]$: finitely additive

  Kolmogorov Sum Rule: $\mu(\alpha_1 \sqcup \alpha_2) = \mu(\alpha_1) + \mu(\alpha_2)$

Quantum Dynamics is given by a Quantum Measure Space $(\Omega, \mathcal{A}, \mu)$

- Sample Space Ω: space of histories.
- Event Algebra $\mathcal{A}$: set of all beables or propositions about the system.

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  - $\alpha \in \mathcal{A}$, $\alpha \subseteq \Omega$
  - $\mathcal{A}$ is closed under finite set union, intersection and complementation.
  - $\emptyset, \Omega \in \mathcal{A}$

- Quantum Measure $\mu : \mathcal{A} \to \mathbb{R}^+$:

  Quantum Sum Rule:

  $$\mu(\alpha_1 \sqcup \alpha_2 \sqcup \alpha_3) = \mu(\alpha_1 \sqcup \alpha_2) + \mu(\alpha_1 \sqcup \alpha_3) + \mu(\alpha_2 \sqcup \alpha_3) - \mu(\alpha_1) - \mu(\alpha_2) - \mu(\alpha_3).$$

The Quantum Measure

Decoherence Functional $D : \mathcal{A} \times \mathcal{A} \to \mathbb{C}$

Gellmann, Hartle, Halliwell, Omnes, Griffiths, Sorkin ..

- $D(\alpha, \beta) = D^*(\beta, \alpha)$: Hermitian
- $D(\alpha, \sqcup_i^k \beta_i) = \sum_i^k D(\alpha, \beta_i)$: finitely biadditive
- $D(\Omega, \Omega) = 1$
- $M_{ij} \equiv D(\alpha_i, \alpha_j)$ for any finite collection $\{\alpha_i\}$ is positive semi-definite: D is strongly positive.

$$\mu(\alpha) \equiv D(\alpha, \alpha)$$

An example: particle mechanics in $\mathbb{R}^N$

$$D(\gamma,\gamma') = \psi^*(\gamma(0))\exp(-iS[\gamma])\delta(\gamma(T)-\gamma'(T))\exp(iS[\gamma'])\psi(\gamma'(0))$$

- $\vec{\alpha} = (\alpha_1, \alpha_2 \ldots \alpha_T)$, α_i: Lebesgue measurable sets of $\mathbb{R}^N$.
- $D(\vec{\alpha},\vec{\beta}) = \int_{\vec{\alpha}} d[\gamma] \int_{\vec{\beta}} d[\gamma'] D(\gamma,\gamma') = \langle \psi_\alpha, \psi_\beta \rangle$
- $\psi_{\vec{\alpha}}(\vec{x}_T, T) =$
  $\chi_{\alpha_N} \int_{\alpha_{N-1}} d\vec{x}_{N-1} \ldots \ldots \int_{\alpha_1} d\vec{x}_0 K(\vec{x}_T, T|\vec{x}_{N-1}, t_{N-1}) \ldots K(\vec{x}_1, t_1|\vec{x}_0, 0)\psi(x,0)$

Observables or Beables are measurable sets $\alpha \in \mathcal{A}$.

$$\hat{O} \to \alpha$$

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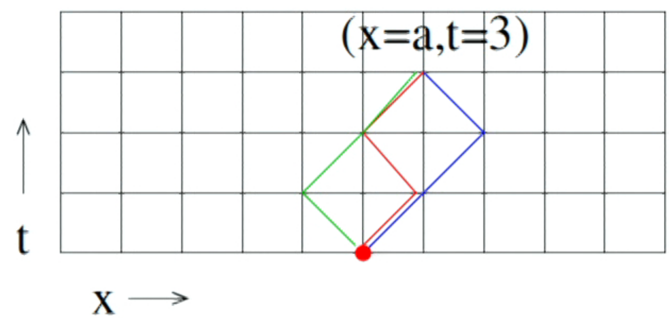
Hats off to Quantum Gravity!!

An Illustration: The Random Walk

- Ω: set of all infinite time trajectories on $\mathbb{Z} \times \mathbb{Z}^+$.
- $\mathcal{A}$: "measurable" events

Event: The walker is at $x = a$ at $t = 3$.

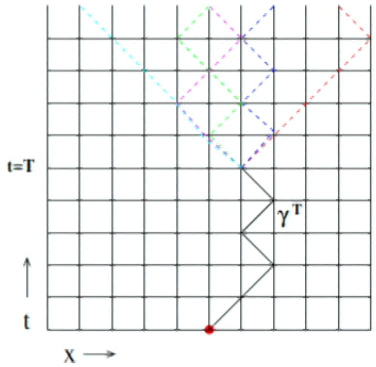
α : set of all paths which are at x=+a at time t=3.



- μ: Probability on $\mathcal{A}$.

For unbiased walker, $\mu(\alpha) = 3/8$ is the probability for this event.

Constructing $\mathcal{A}$ from Finite Time Paths



- γ^T: finite time path
- $\text{cyl}(\gamma^T) \equiv \{\gamma \in \Omega | \gamma|_{[0,T]} = \gamma^T\}$
- $\mathcal{A}$: generated by taking finite unions, intersections and complementations of cylinder sets.
- $\mu(\text{cyl}(\gamma^T)) \equiv Prob(\gamma^T) = 2^{-T}$.

What about infinite time events? These are analogs of covariant events.

- Return event
- First passage

  ⋮

- Infinite return event

- Infinite time events requires countable set operations
- Extension question: Are infinite time events measurable?
- Eg: Return event R
  - $R = \bigcup_{T \in \mathbb{Z}^+} R(T)$, $R(T) \in \mathcal{A}$: return for the first time at $t = T$.
  - $\mu(R(T))$ is defined but is $\mu(R) = \sum_T \mu(R(T))$ well defined?
- $R \in \mathcal{S}$, sigma algebra $\mathcal{S} \supset \mathcal{A}$:closed under countable set operations
- Kolmogorov-Caratheodary-Hanhn extension theorem:

  $(\Omega, \mathcal{A}, \mu)$ extends to $(\Omega, \mathcal{S}, \overline{\mu})$.

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- Remark: Wiener measure
  - $p(\Delta t, \Delta x) = (4\pi\Delta t)^{-\frac{n}{2}} \exp^{-\frac{(\Delta x)^2}{4\Delta t}}$

    defined only when the measure "extends" from $\mathcal{A}$ (generated by "window sets") to $\mathcal{S}$.

Extension of the quantum measure in gravity tied to covariance of beables.

Is there an analog of the KCH extension for the quantum measure?

Geroch, unpublished notes (available on Sorkin's webpage)

Complex Percolation Dynamics in Causal Set Quantum Gravity

Dowker, Johnston and Surya (2010), Sorkin and Surya (in prepn)

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Lets just focus on quantum mechanics for now..

The Quantum Measure as a Vector Measure

- A Histories Hilbert space $\mathcal{H}$: Dowker, Johnston and Sorkin, (2010)
  - V: free vector space of functions $f : \mathcal{A} \to \mathbb{C}$.
  - $\langle u, v \rangle_V \equiv \sum_{\alpha \in \mathcal{A}} \sum_{\beta \in \mathcal{A}} u^*(\alpha) v(\beta) D(\alpha, \beta)$
  - $\{u_i\} \sim \{v_i\} \quad \text{if} \quad \lim_{i \to \infty} \| u_i - v_i \|_V = 0,$
  - Hilbert Space $\mathcal{H} = V / \sim, \qquad [\{u_i\}] \in \mathcal{H}.$

The Quantum Measure as a Vector Measure

- A Histories Hilbert space $\mathcal{H}$: Dowker, Johnston and Sorkin, (2010)
- $\mu_\mathsf{v} : \mathcal{A} \to \mathcal{H}$ is a *vector measure* Dowker, Johnston and Surya (2010)

$$\mu_\mathsf{v}(\alpha) \equiv [\chi_\alpha] \in \mathcal{H}.$$

$$\chi_\alpha(\beta) = \begin{cases} 1 & \text{if } \beta = \alpha, \\ 0 & \text{if } \beta \neq \alpha. \end{cases}$$

$$\langle \mu_\mathsf{v}(\alpha), \mu_\mathsf{v}(\beta) \rangle = D(\alpha, \beta),$$

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  Finite additivity:

$$\mu_\mathrm{v}(\sqcup_{i=1}^{n} \alpha_i) = \sum_{i=1}^{n} \mu_\mathrm{v}(\alpha_i)$$

- Caratheodary-Hahn-Kluvanek theorem:
  An extension $(\Omega, \mathcal{A}, \mu_\mathrm{v}) \to (\Omega, \mathcal{S}_\mathcal{A}, \mu_\mathrm{v}^*)$ exists and is unique
  provided μ_v satisfies certain convergence conditions
- Unique extension of μ_v gives a unique extension of decoherence functional D.

Discrete finite unitary process

Extension of μ_{v} requires it to be of bounded total variation:

$$|\mu_{\mathsf{v}}|(\alpha) = \sup_{\Pi(\alpha)} \sum_{\alpha_i \in \Pi(A)} |\mu(\alpha_i)| < \infty$$

Discrete finite unitary process

Extension of μ_v requires it to be of bounded total variation:

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- $\mathcal{H} = \mathbb{C}^N$
- $\psi_m = U(m, m-1)U(m-1, m-2)\dots U(2,1)\psi$
- $D(\mathrm{cyl}(\gamma^T), \mathrm{cyl}(\gamma'^T)) = A^*(\gamma)A(\gamma)\delta(x(T) - x'(T))$
- $\widetilde{\psi}_\gamma = P_m U(m, m-1) P_{m-1} U(m-1, m-2)\dots U(2,1) P_1 \psi_{initial}$
- $\mu_v(\alpha) = \psi_\alpha$

The quantum vector measure for a "generic" finite unitary system with discrete time steps is not of bounded variation.

Dowker, Johnston and Surya (2010)

Is $\mathcal{S}$ too large?!

Use physics to determine which infinite time events we want..

Return Question: in $\bigvee \mathcal{A} \subset \mathcal{S}$. Can we restrict to an $\mathcal{S}'$ with $\mathcal{A} \subset \mathcal{S}' \subset \mathcal{S}$?

Even convergence (conditional convergence) of the quantum measure v/s unconditional convergence.

Measure on return event can be defined in discrete unitary processes. Sorkin (2011)

Open Question: What about the quantum particle in $\mathbb{R}^N$? Is there an analogue of the Wiener process?

Aspiration: quantum measure of gravity should evenly converge for a set of infinite time events of physical interest

Complex percolation dynamics for causal sets:
The quantum measure on the originary event is evenly convergent.
Sorkin and Surya (in preparation)