Title: Surprises in the Path Integral for Gravity

Date: Nov 14, 2017 09:00 AM

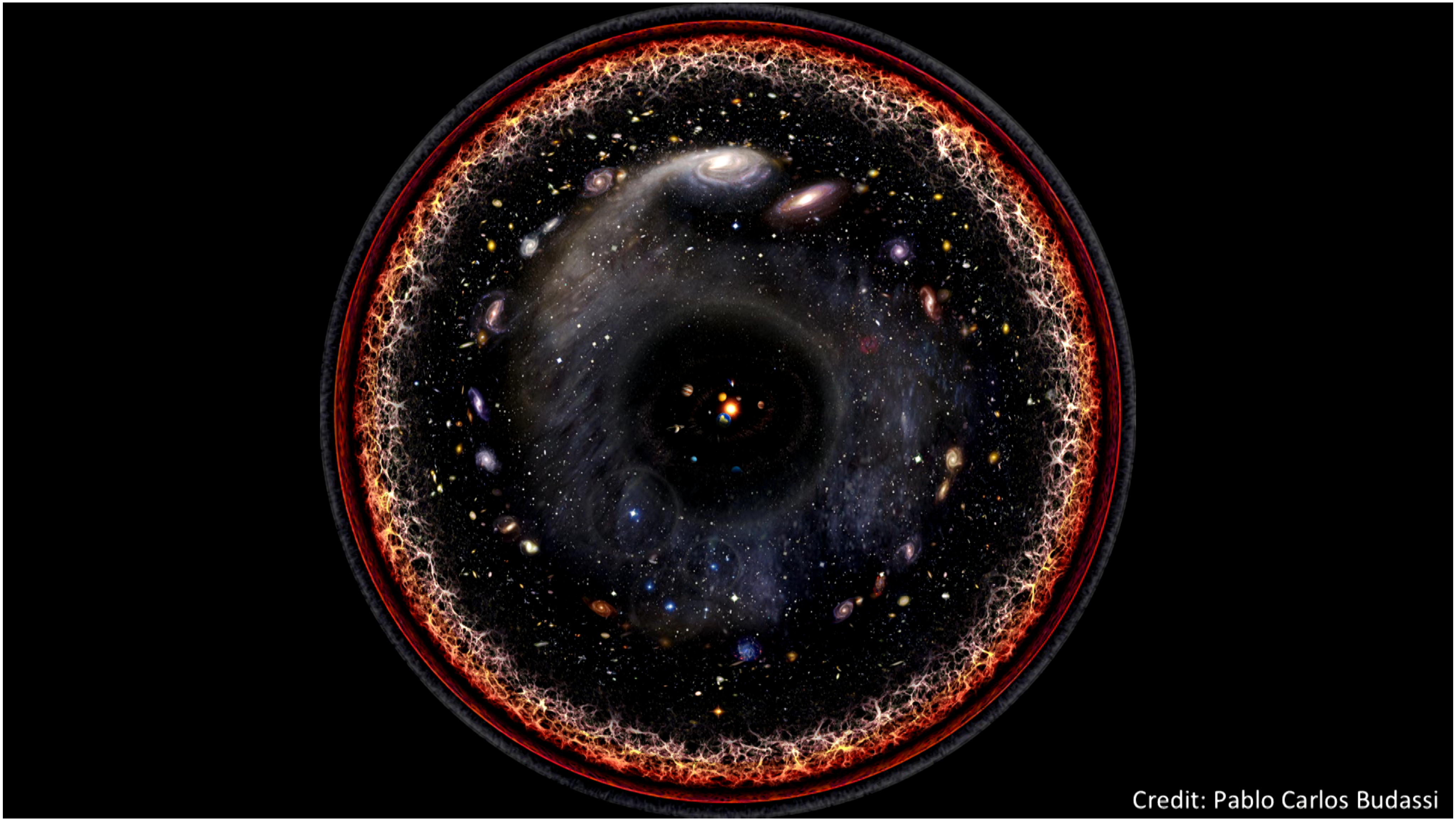
URL: http://pirsa.org/17110120

Abstract:

Surprises in the Path Integral for Gravity

Neil Turok
Perimeter Institute

work with A. Di Tucci, J. Feldbrugge, A. Fertig,
J-L. Lehners, L. Sberna


Credit: Pablo Carlos Budassi

astonishing simplicity: just 5 numbers

			Measurement Error
today	Expansion rate:	67.8±0.9 km s^{-1} Mpc^{-1}	1%
	(Temperature)	2.728 ± 0.004 K	.1%
	(Age)	13.799 ±0.038 bn yrs	.3%
energy	Baryon-entropy ratio	6±.1x10^{-10}	1%
	Dark matter-baryon ratio	5.4± 0.1	2%
	Dark energy density	0.69±0.006 x critical	2%
geometry	Scalar amplitude	4.6±0.006 x 10^{-5}	1%
	Scalar spectral index n_s (scale invariant = 0)	-.033±0.004	12%

$+m_\nu 's;$ but $\Omega_k,\ 1+w_{DE},\ \frac{dn_s}{d\ln k},\ \langle\delta^3\rangle,\langle\delta^4\rangle..,r=\frac{A_{gw}}{A_s}$ consistent with zero

Behind it all is surely an idea so simple, so beautiful, that when we grasp it - in a decade, a century or a millenium - we will all say to each other, how could it have been otherwise? How could we have been so stupid?

*John A. Wheeler, How Come the Quantum? Ann. N.Y.A.S., **480**, 304-316 (1986).*

Nature has found a way to create an enormous hierarchy of scales, more economically than in any current theory

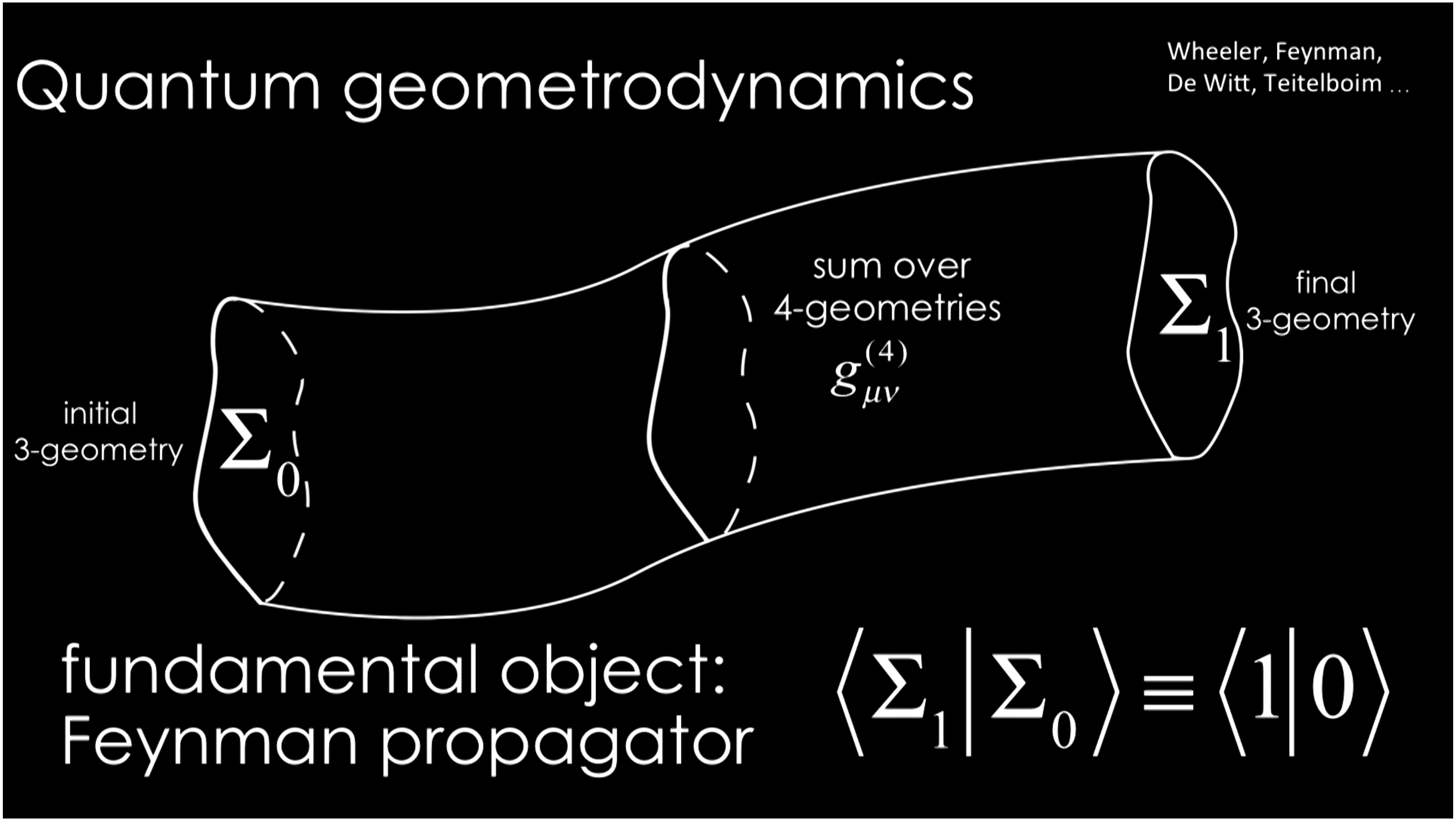
A fascinating situation, demanding new ideas

One of the most minimal is to revisit quantum cosmology

The simplest cosmological model is de Sitter, relevant interesting both for today's dark energy and for inflation

Perhaps the most impressive fact which emerges from a study of the quantum theory of gravity is that it is an extraordinarily economical theory. It gives one just exactly what is needed in order to analyze a particular physical situation, but not a bit more. Thus it will say nothing about time unless a clock to measure time is provided, and it will say nothing about geometry unless a device (either a material object, gravitational waves, or some other form of radiation) is introduced to tell when and where the geometry is to be measured.[50] In view of the strongly operational foundations of both the quantum theory and general relativity this is to be expected. When the two theories are united the result is an operational theory *par excellence*.[51]

B.S. DeWitt, *Phys. Rev.* 160, 1967 (p 1140)


Quantum geometrodynamics
Wheeler, Feynman,
De Witt, Teitelboim ...
sum over
4-geometries
$g_{\mu\nu}^{(4)}$
final
3-geometry
Σ_1
initial
3-geometry
Σ_0
fundamental object:
Feynman propagator
$\langle \Sigma_1 | \Sigma_0 \rangle \equiv \langle 1 | 0 \rangle$

phase space Lorentzian path integral

$$ADM : ds^2 \equiv (-N^2 + N_i N^i)dt^2 + 2N_i dt dx^i + h_{ij}^{(3)} dx^i dx^j$$

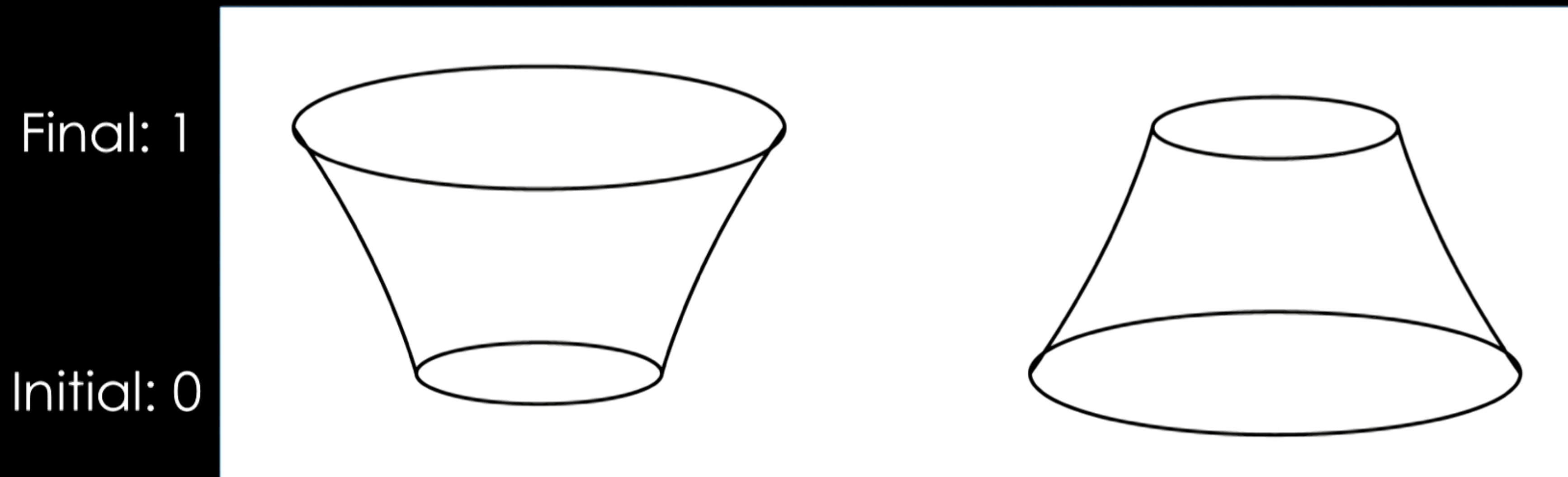
$$\langle 1|0\rangle = \int DN \int DN^i \int_{\Sigma_0}^{\Sigma_1} Dh_{ij}^{(3)} \int D\pi_{ij}^{(3)} e^{\frac{i}{\hbar}S}$$

$$S = \int_0^1 dt \int d^3x (\pi_{ij}^{(3)} \dot{h}_{ij}^{(3)} - N_i H^i - NH)$$

Basic references:

C. Teitelboim (now Bunster), "Causality and Gauge Invariance in Quantum Gravity and Supergravity," Phys. Rev. Lett. 50, 705 (1983); see also Phys. Rev. D25, 3159 (1983); D28, 297 (1983).

Some basic points: Feynman propagator, defined by integrating only over positive lapse N allows you to distinguish an expanding from a contracting universe.



Teitelboim, …

theories of initial conditions for inflation

Wave function of the Universe

J. B. Hartle
Enrico Fermi Institute, University of Chicago, Chicago, Illinois 60637
and Institute for Theoretical Physics, University of California, Santa Barbara, California 93106

S. W. Hawking
Department of Applied Mathematics and Theoretical Physics, Silver Street, Cambridge, England
and Institute for Theoretical Physics, University of California, Santa Barbara, California 93106
(Received 29 July 1983)

The quantum state of a spatially closed universe can be described by a wave function which is a functional on the geometries of compact three-manifolds and on the values of the matter fields on these manifolds. The wave function obeys the Wheeler-DeWitt second-order functional differential equation. We put forward a proposal for the wave function of the "ground state" or state of minimum excitation: the ground-state amplitude for a three-geometry is given by a path integral over all compact positive-definite four-geometries which have the three-geometry as a boundary.

*Physical Review **D28** (12) (1983) 2960*

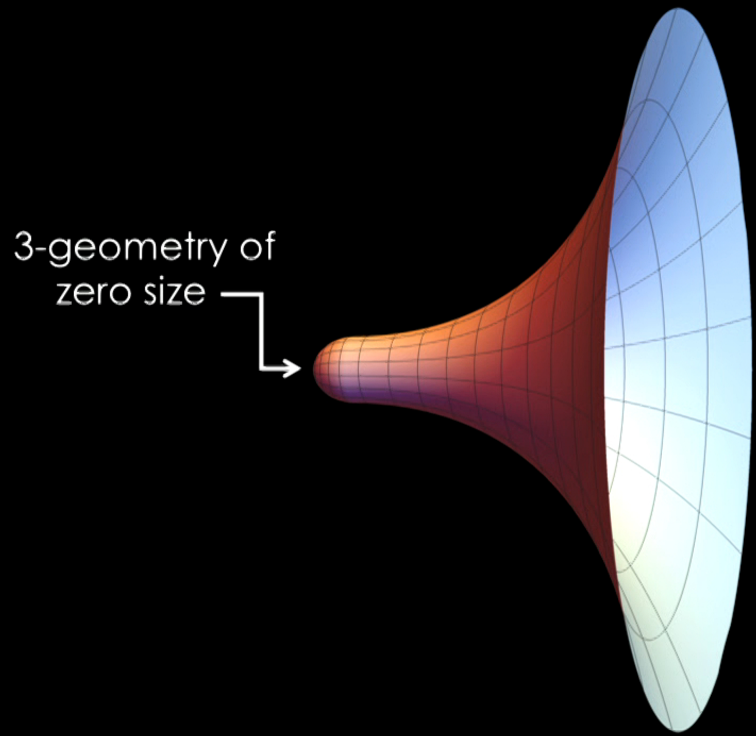
One can interpret the functional integral over all compact four-geometries bounded by a given three-geometry as giving the amplitude for that three-geometry to arise from a zero three-geometry, i.e., a single point. In other words, the ground state is the amplitude for the Universe to appear from nothing.[4] In the following we shall elaborate on this construction and show in simple models that it indeed supplies reasonable wave functions for a state of minimum excitation.

[4]For related ideas, see A. Vilenkin, Phys. Lett. 117B, 25 (1982); Phys. Rev. D 27, 2848 (1983).

Revised Vilenkin proposal (framed in terms of Lorentzian path integral):
Phys Rev. D30, 509 (1984); Phys Rev D50, 2581 (1994), gr-qc/9403010

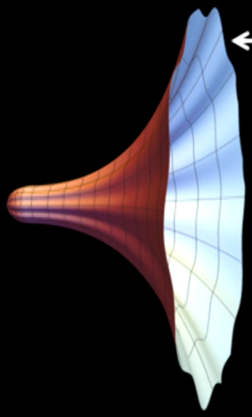
Earlier versions: Lemaitre, Fomin, Tryon, Brout-Englert-Gunzig …

no boundary proposal (path integral version)



A very beautiful idea: the laws of physics determine their own initial conditions simplest model: Einstein gravity plus cosmological constant

$$S = \int \left(\tfrac{1}{2} R - \Lambda\right) + \text{surface terms} \quad (8\pi G \equiv 1)$$



Usual claim:

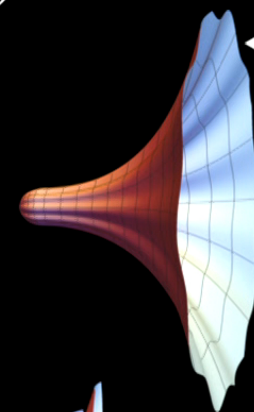
$$\Psi \propto e^{+\frac{12\pi^2}{\hbar\Lambda}(1 - l(l+1)(l+2)h_T^2)}$$

Our claim:

$$\Psi \propto e^{-\frac{12\pi^2}{\hbar\Lambda}(1 - l(l+1)(l+2)h_T^2)}$$

simplest model: Einstein gravity plus cosmological constant

$$S = \int \left(\tfrac{1}{2}R - \Lambda\right) + \text{surface terms} \quad (8\pi G \equiv 1)$$

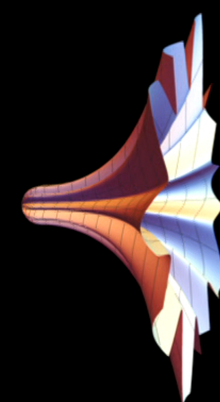


Usual claim:

$$\Psi \propto e^{+\frac{12\pi^2}{\hbar\Lambda}(1 - l(l+1)(l+2)h_T^2)}$$

Our claim:

$$\Psi \propto e^{-\frac{12\pi^2}{\hbar\Lambda}(1 - l(l+1)(l+2)h_T^2)}$$



Perturbations out
of control: there is
no meaningful
one-point amplitude
for a 3-geometry
"persistence of nothing"

Some overlap with previous work: Vilenkin (bg), Rubakov (perts), Ambjorn/Loll (bg), Sorkin (bg)...

We evaluate the Lorentzian gravitational path integral using cosmological perturbation theory and P-L/Cauchy to determine relevant saddles

Integrate out background (zero mode) first, then fluctuations (if nonzero), then N

Background:

$$ds^2 = -\bar{N}^2 dt^2 + a^2 d\Omega_3^2; \qquad S = 2\pi^2 \int_0^1 dt \left[-N^{-1} 3a\dot{a}^2 + N(3a - \Lambda a^3) \right]$$

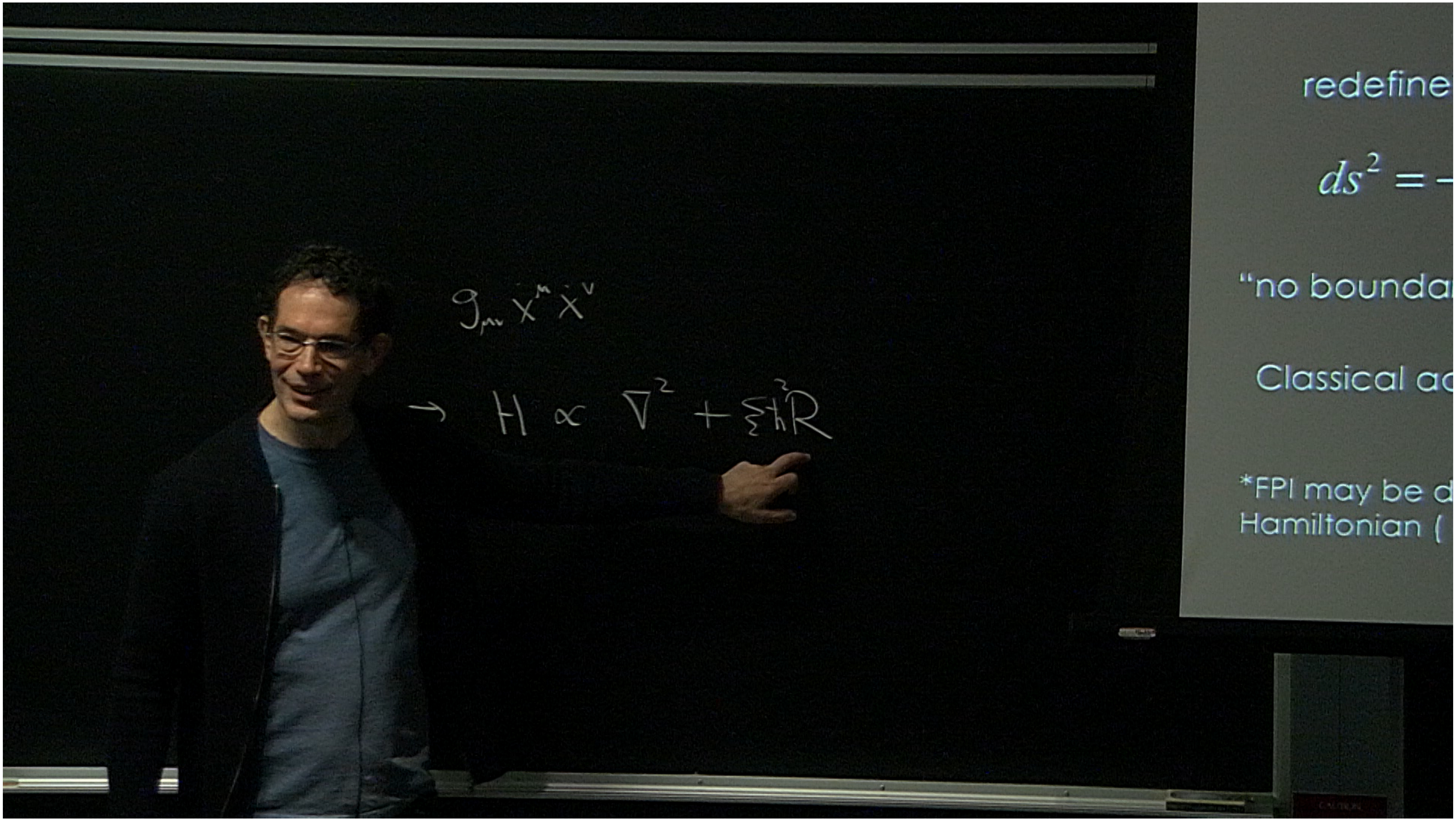
redefine* $\bar{N} \equiv N a^{-1},\ q \equiv a^2 \Rightarrow\ S = 2\pi^2 \int_0^1 dt \left[-N^{-1} \tfrac{3}{4} \dot{q}^2 + N(3 - \Lambda q) \right]$ quadratic in q (Halliwell)

$ds^2 = -N^2 q^{-1} dt^2 + q\ d\Omega_3^2$; work in gauge $N =$ const

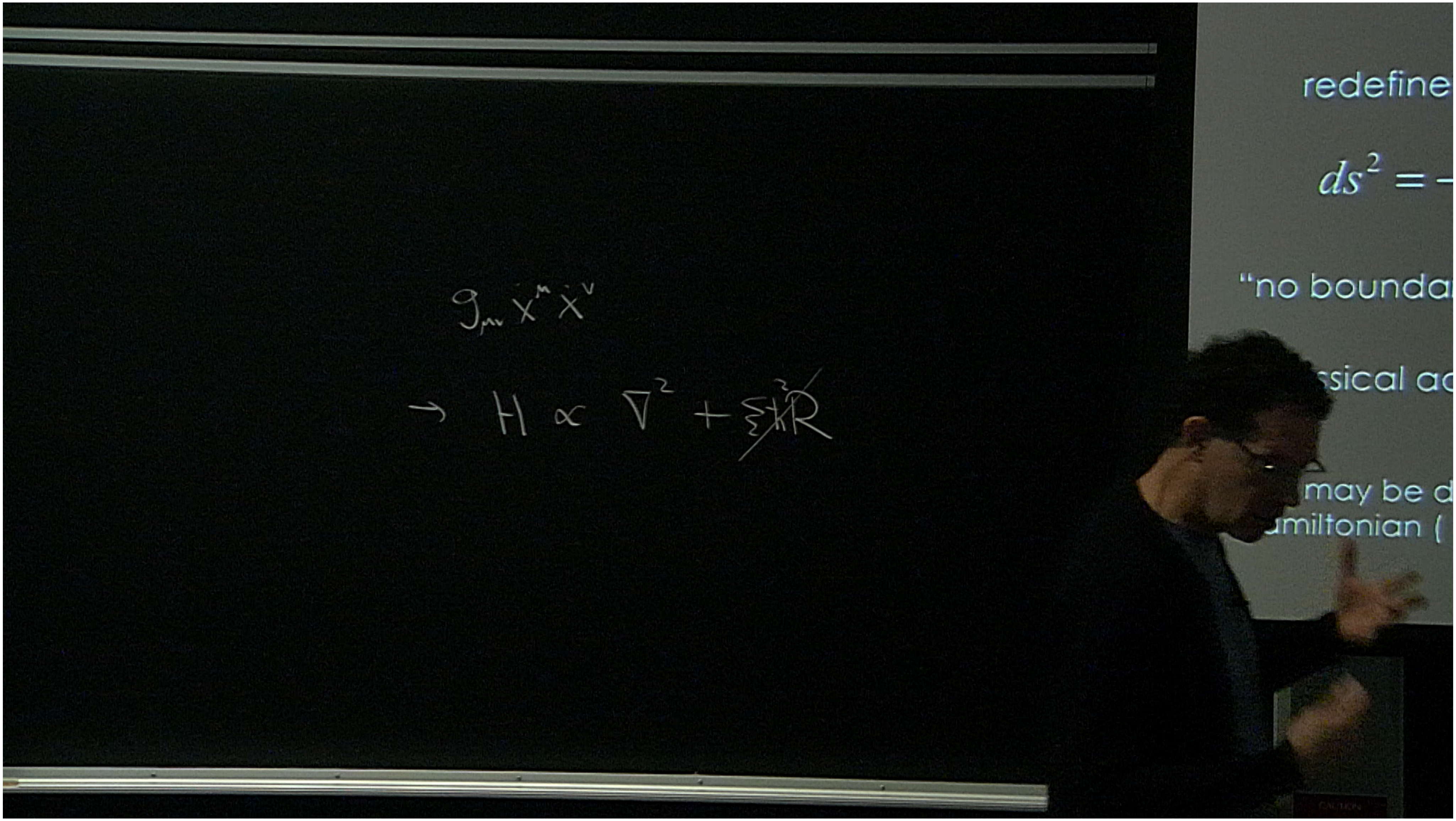
"no boundary" classical solution: $q_{cl}(t) = \tfrac{1}{3}\Lambda N^2 t^2 + (-\tfrac{1}{3}\Lambda N^2 + q_1)t :\quad q_{cl}(0) = 0,\ q_{cl}(1) = q_1$

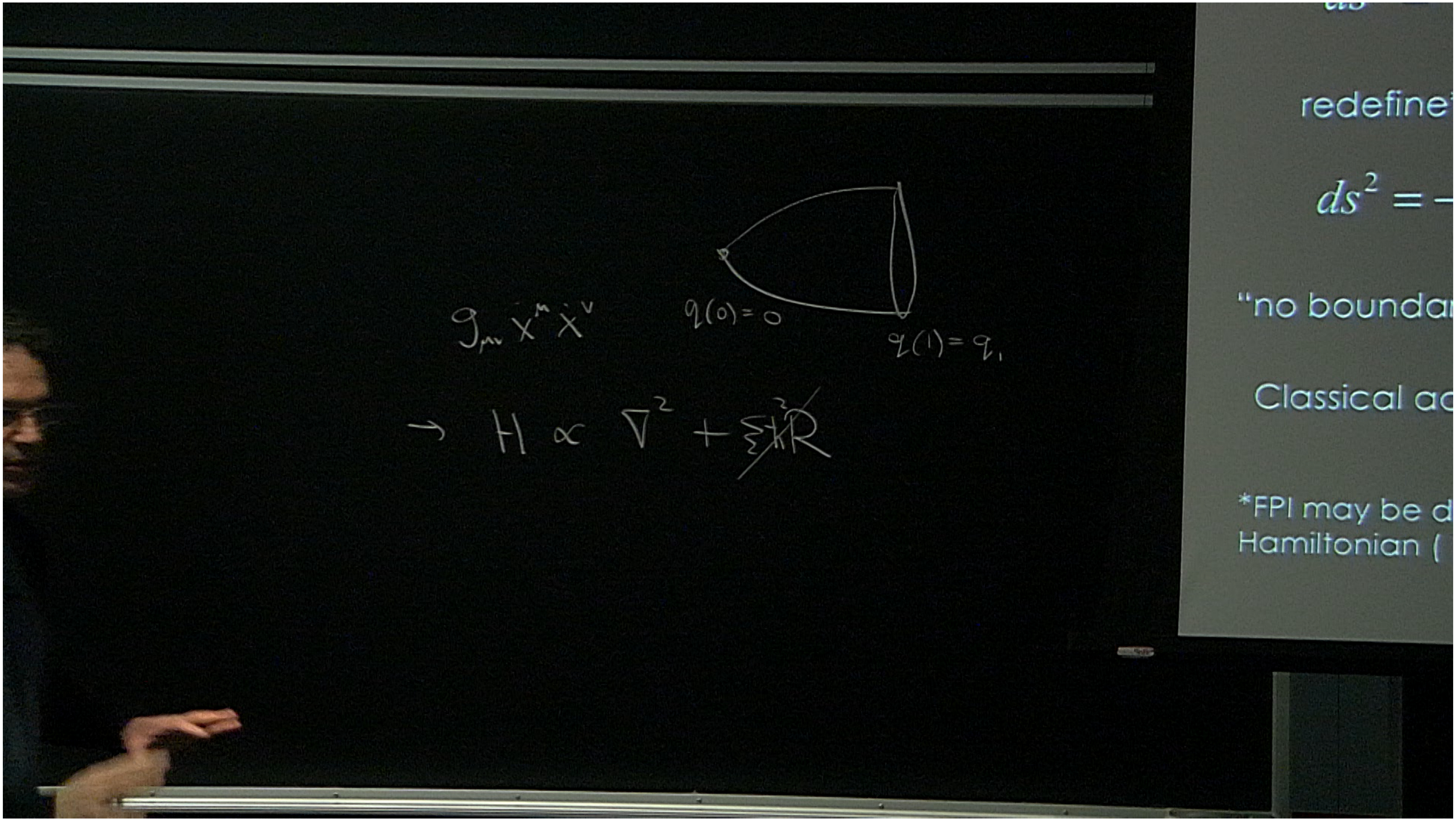
Classical action: $$S_{cl}(q_1; N) = 2\pi^2 \left[\tfrac{1}{36}\Lambda^2 N^3 + (3 - \tfrac{1}{2}\Lambda q_1)N - \tfrac{3}{4} q_1^2 N^{-1} \right]$$

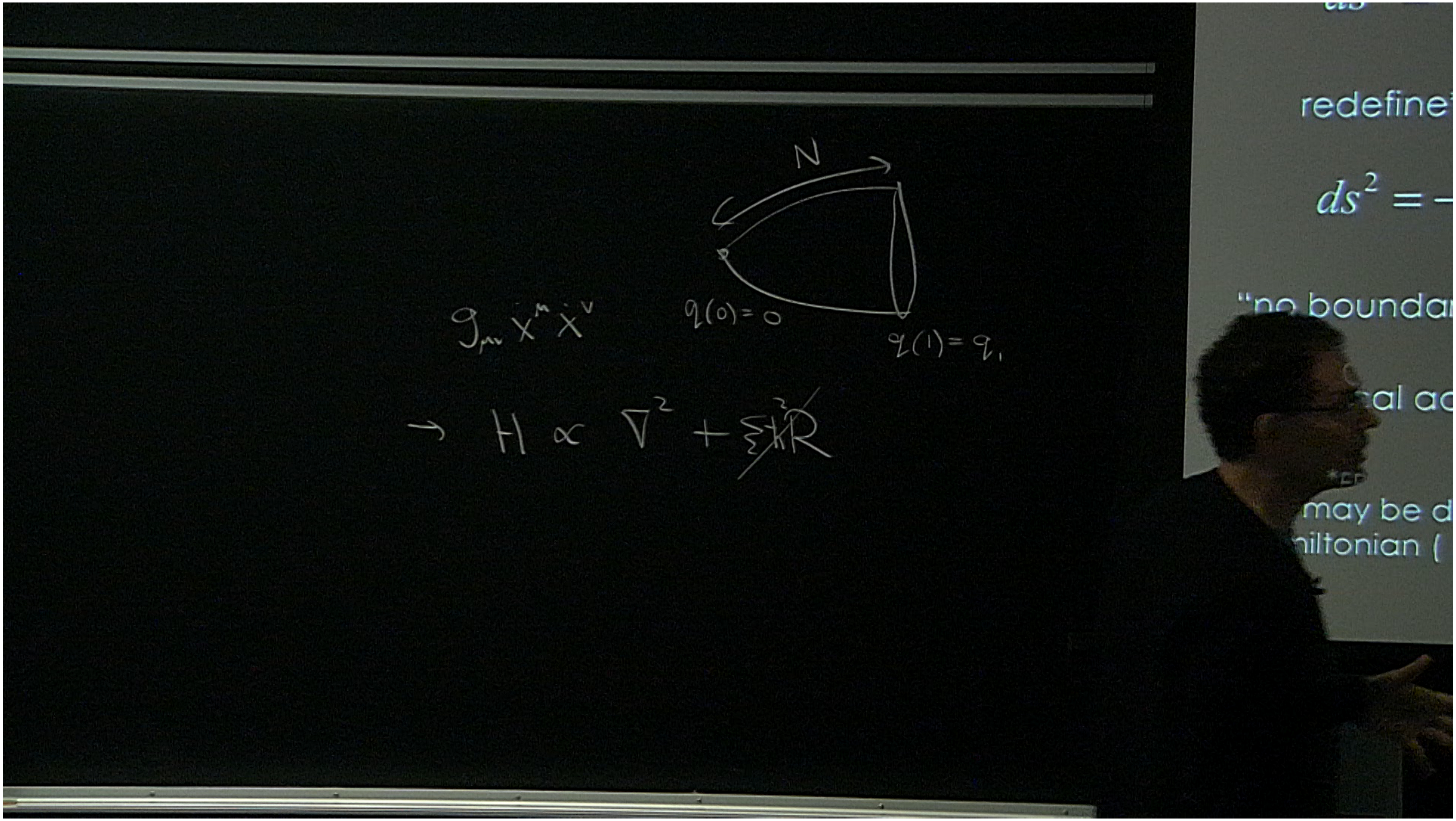
*FPI may be defined to be invariant under such redefinitions: this requires $O(\hbar^2)$ quantum corrections to the Hamiltonian ($\xi R_{\text{superspace}}$) which can be important at small q: for simplicity we neglect it here. (see Gielen+NT)

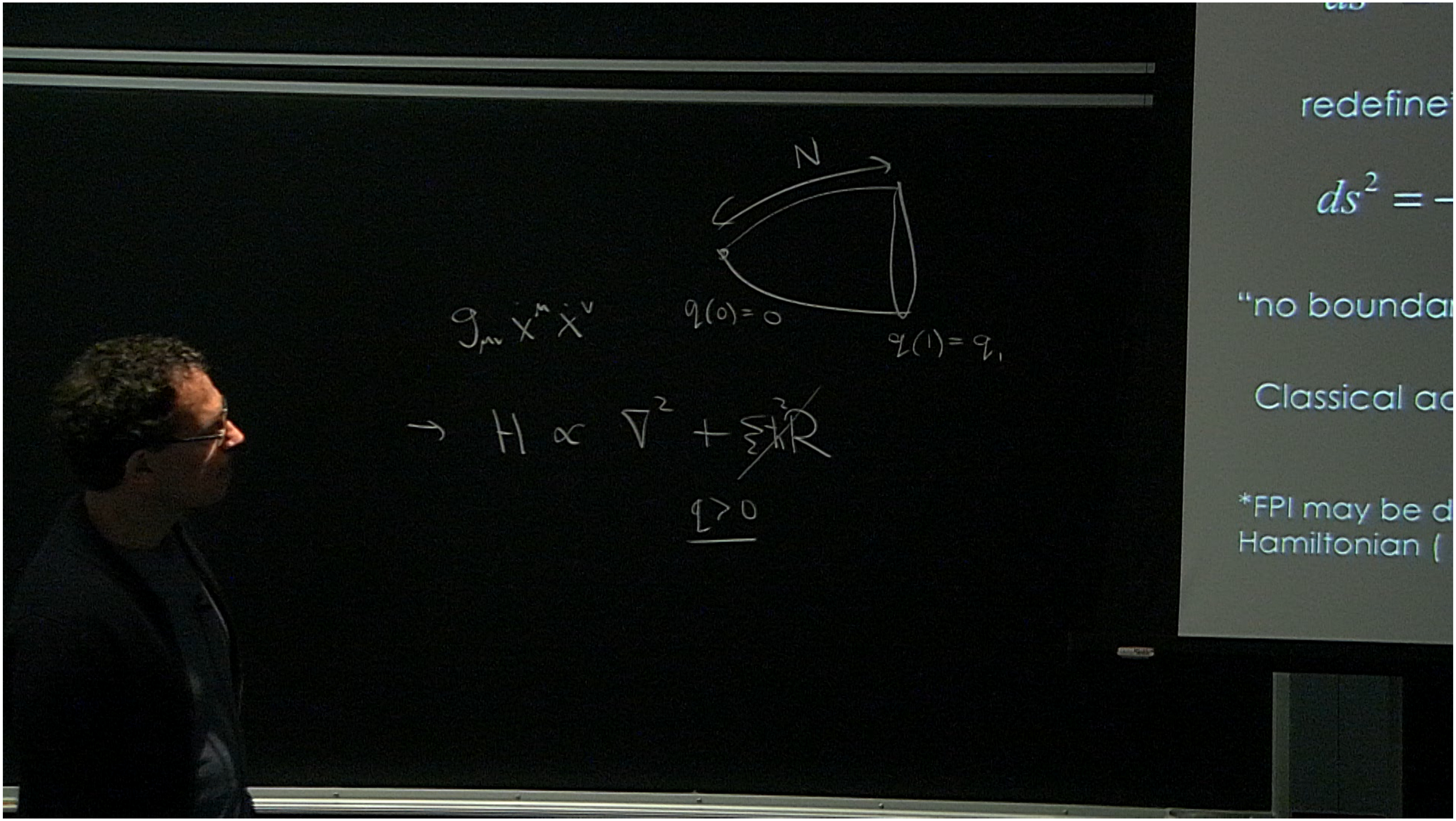


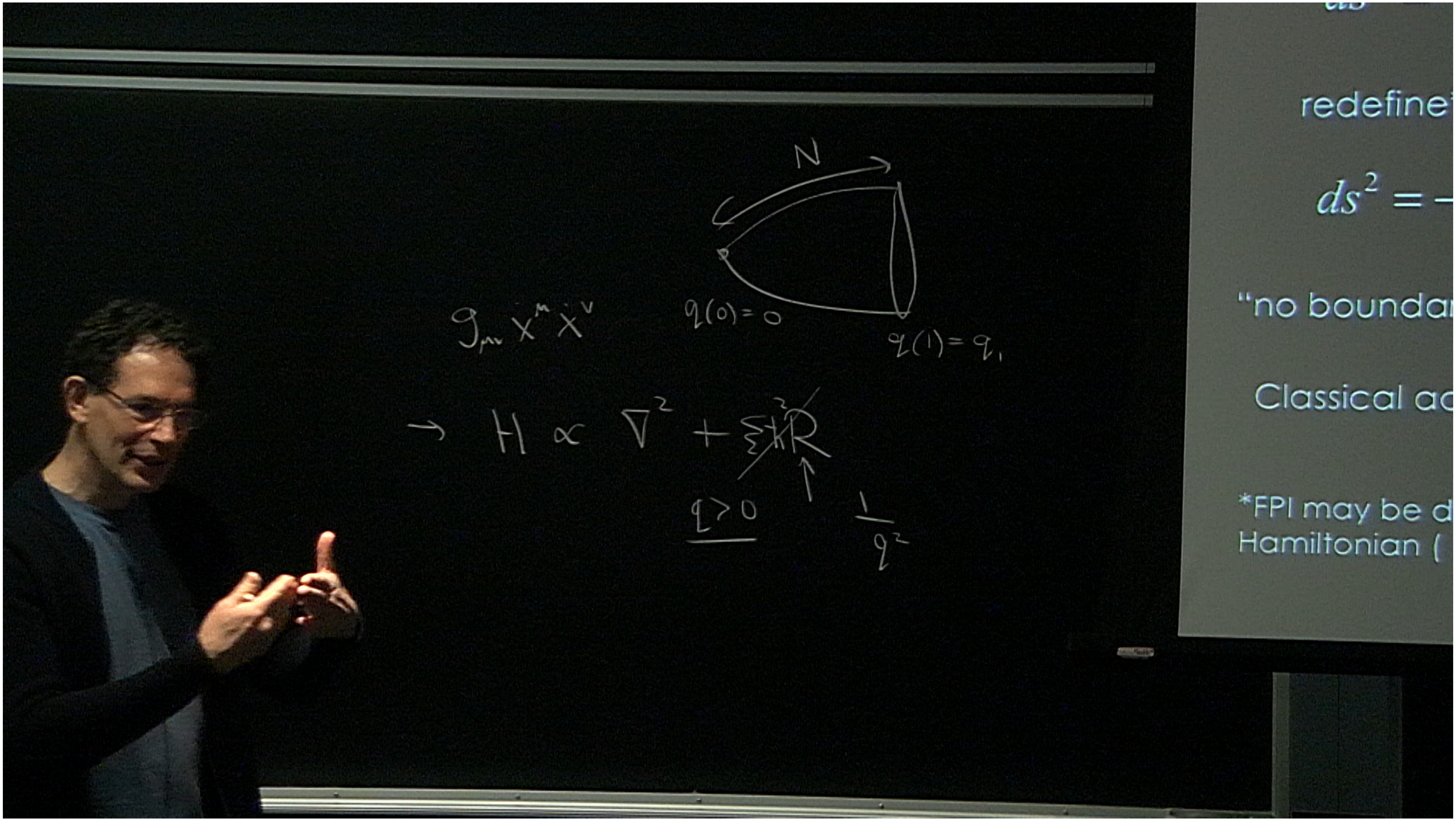
$g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu$
$\rightarrow H \propto \nabla^2 + \xi\hbar^2 R$
redefine
$ds^2 =$
"no bounda
Classical ac
*FPI may be d
Hamiltonian (

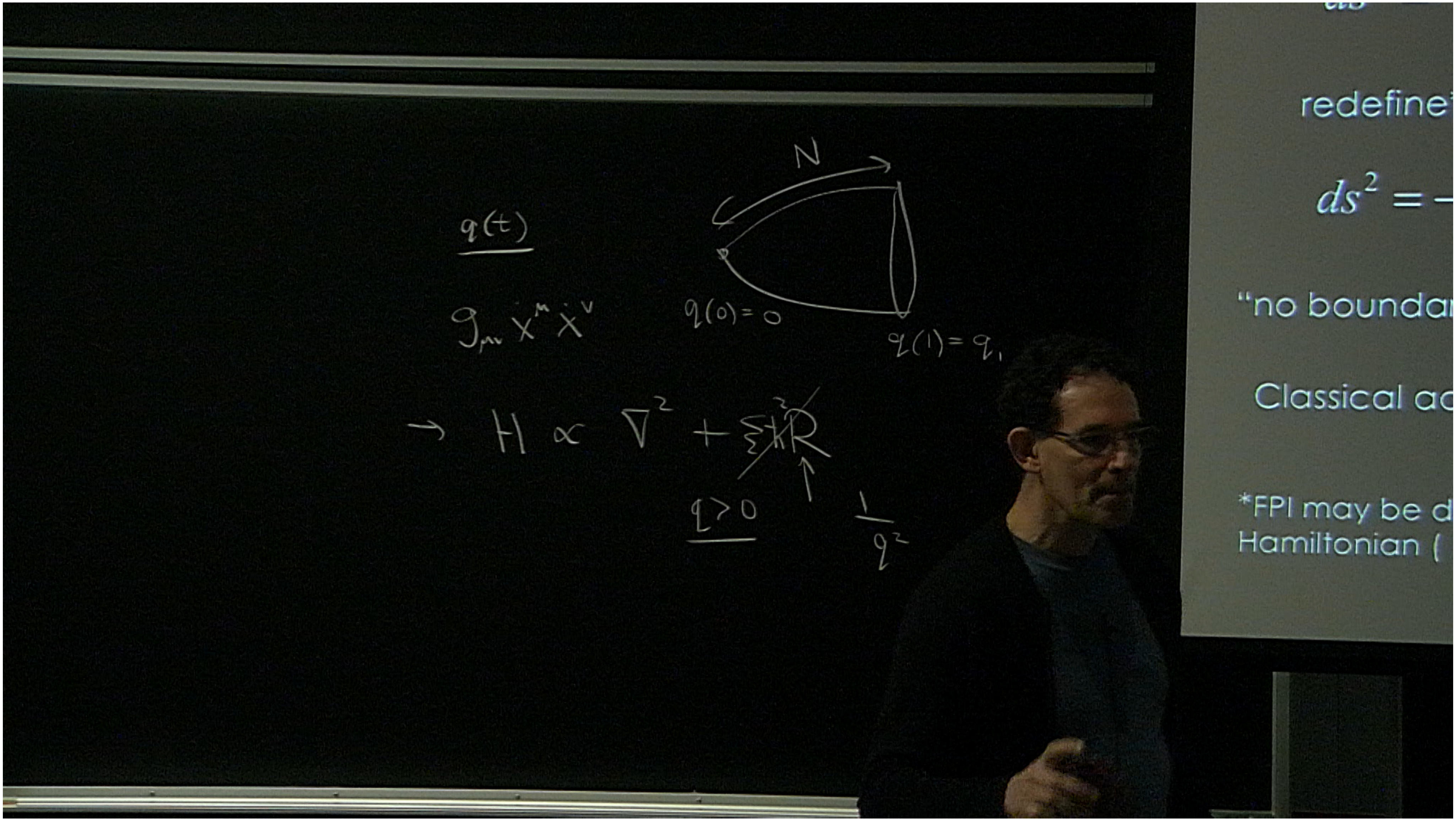

$g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu$
$\rightarrow H \propto \nabla^2 + \xi \hbar^2 R$
redefine
$ds^2 =$
"no bounda
ssical ac
may be d
amiltonian (

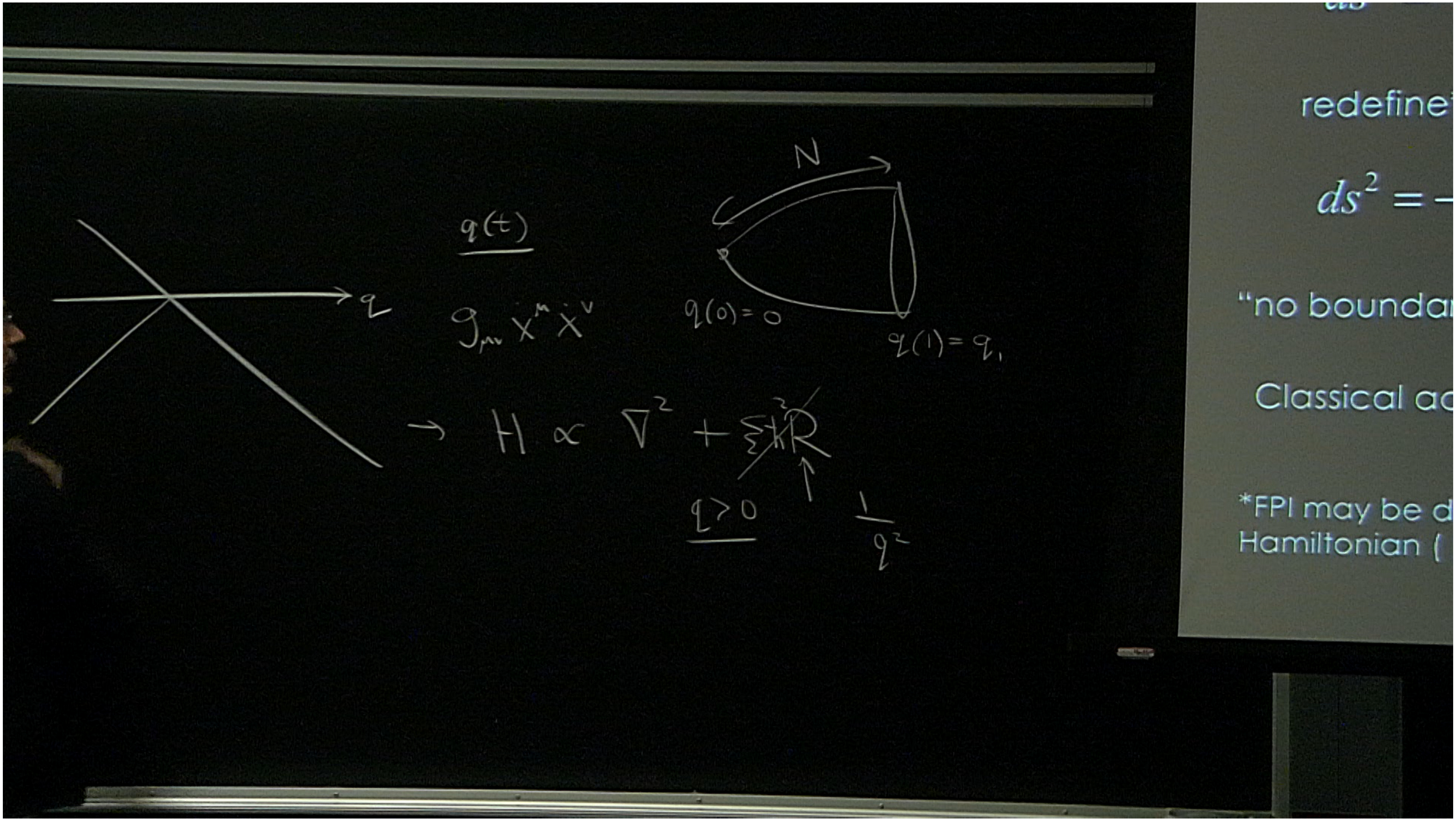

redefine
ds² = −
“no boundar
Classical ac
*FPI may be d
Hamiltonian (
g_{μν} ẋ^μ ẋ^ν
q(0) = 0
q(1) = q₁
→ H ∝ ∇² + ξħ²R


N
q(0) = 0
q(1) = q1
gμν ẋμ ẋν
→ H ∝ ∇2 + ξħ2R
redefine
ds2 = −
"no boundar
al ac
may be d
iltonian (

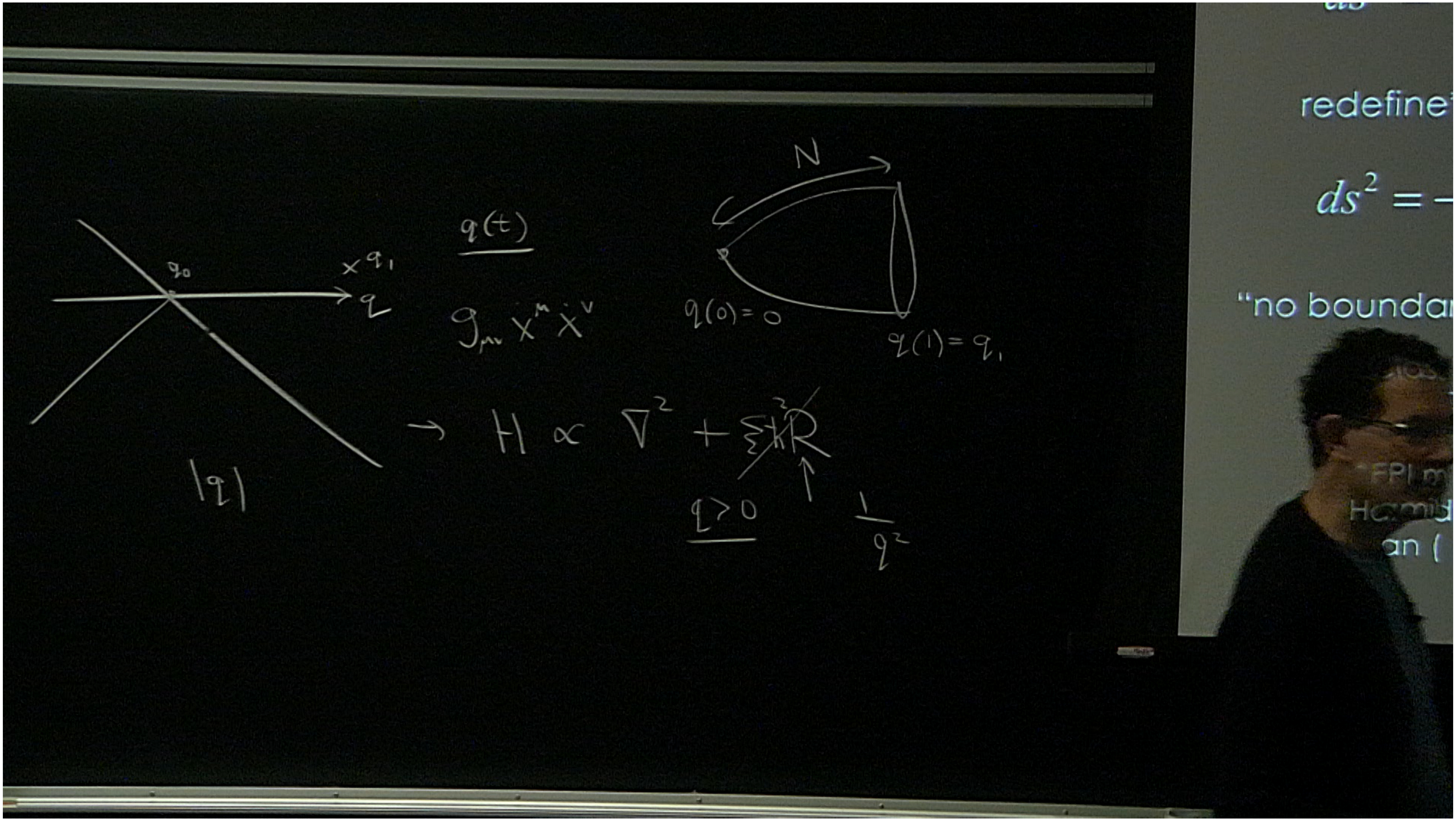

N
q(0)=0
q(1)=q1
gμν x^μ x^ν
→ H ∝ ∇² + ξħ²R
q>0
redefine
ds² = −
"no boundar
Classical ac
*FPI may be d
Hamiltonian (


N
q(0) = 0
q(1) = q_1
redefine
"no bounda
Classical ac
*FPI may be d
Hamiltonian (


q(t)
N
q(0) = 0
q(1) = q1
q > 0
redefine
"no boundar
Classical ac
*FPI may be d
Hamiltonian (



q(t)
q
$g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu$
N
q(0) = 0
q(1) = q₁
→ H ∝ ∇² + ξħ²R
q > 0
1/q²
redefine
$ds^2 =$
"no bounda
Classical ac
*FPI may be d
Hamiltonian (


q(t)
$g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu$
N
q(0) = 0
q(1) = q₁
|q|
$H \propto \nabla^2 + \xi \hbar^2 R$
q > 0
$\frac{1}{q^2}$
redefine
$ds^2 =$
“no boundar

$$e.g.\ I = \int_{-\infty}^{+\infty} e^{ix^2}\, dx$$ is a conditionally convergent integral

$\lfloor x$

$$-iR\int_0^{\frac{\pi}{4}} d\phi e^{i\phi} e^{iR^2 e^{2i\phi}}$$

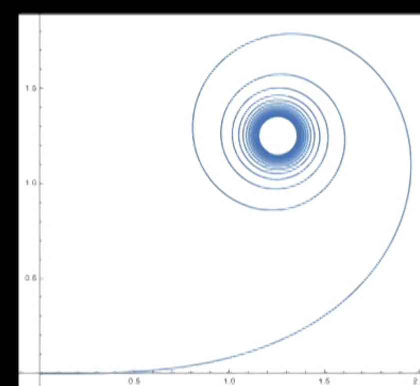
$$x = e^{i\frac{\pi}{4}} s$$

but $\left|\int d\phi e^{iR^2 e^{2i\phi}}\right| < \frac{1}{R^2}$

$-R$

$$e^{i\frac{\pi}{4}} \int_{-R}^{R} ds e^{-s^2}$$

$+R$

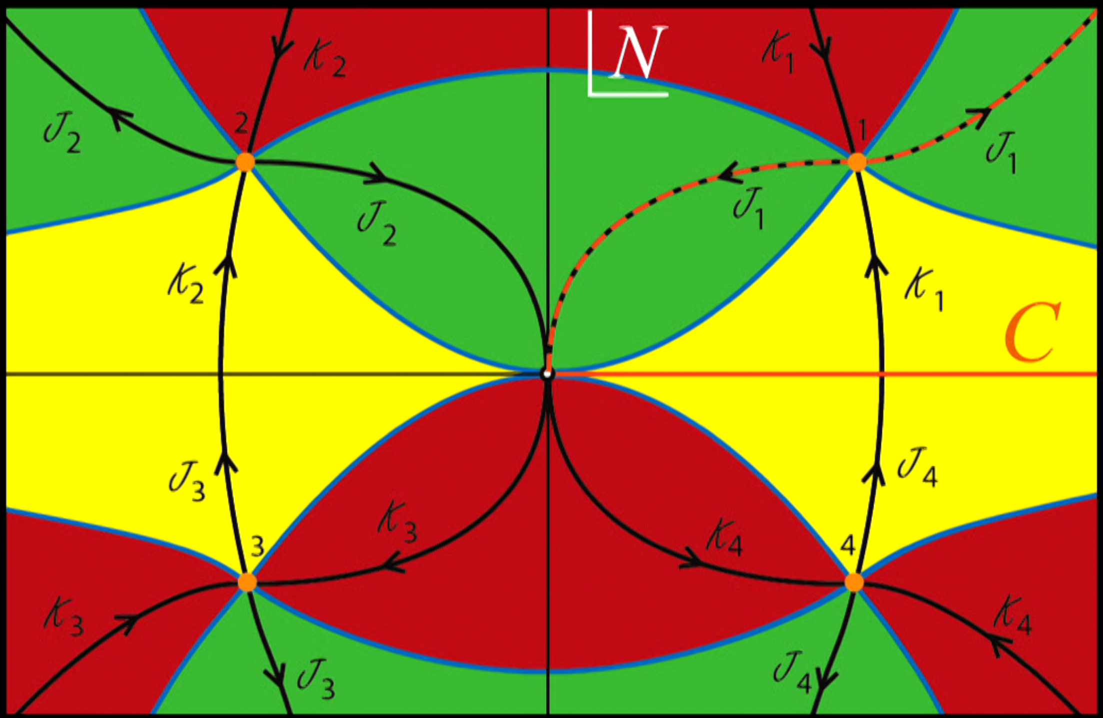


Take limit $R \rightarrow \infty$;

$$I \rightarrow e^{i\frac{\pi}{4}} \sqrt{\pi}$$

$$\langle 1|0\rangle_F = \int_{0^+}^{\infty} dN \sqrt{\frac{3\pi i}{2\hbar N}}\, e^{\frac{i}{\hbar}S_{cl}(q_1;N)}$$

4 saddles, related by
$N \rightarrow -N$ and
complex conjugation



P-L theory:

every saddle σ defines a "Lefshetz thimble" J_σ (complete steepest descent contour) upon which integral is absolutely convergent. Generically, each J_σ intersects a steepest ascent contour K_σ

intersection number $\rightarrow \langle J_\sigma K_{\sigma'}\rangle = \delta_{\sigma\sigma'}$

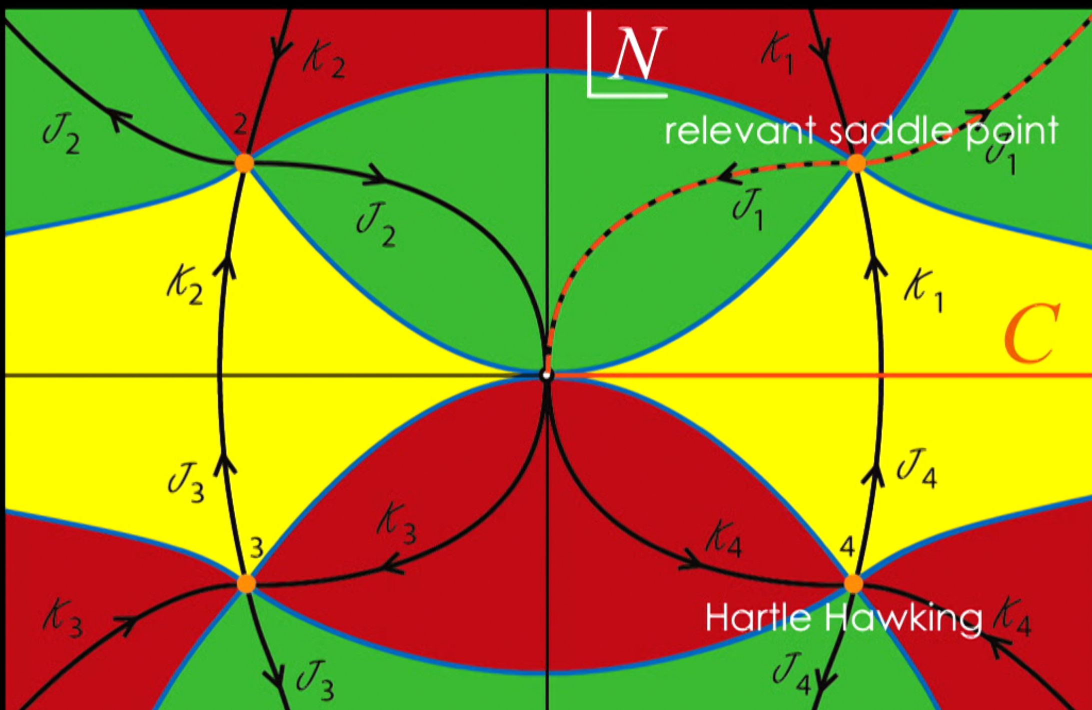
One can deform the defining contour C into one passing along a number of thimbles,

$$C = \sum_\sigma n_\sigma J_\sigma \Leftrightarrow n_\sigma = \langle C K_\sigma \rangle$$

i.e., a saddle contributes iff its steepest *ascent* contour intersects C

$$\langle 1|0\rangle_F = \int_{0^+}^{\infty} dN \sqrt{\frac{3\pi i}{2\hbar N}} e^{\frac{i}{\hbar} S_{cl}(q_1;N)}$$

4 saddles, related by
$N \rightarrow -N$ and
complex conjugation



P-L theory:

every saddle σ defines a "Lefshetz thimble" J_σ (complete steepest descent contour) upon which integral is absolutely convergent. Generically, each J_σ intersects a steepest ascent contour K_σ

intersection number → $\langle J_\sigma K_{\sigma'} \rangle = \delta_{\sigma\sigma'}$

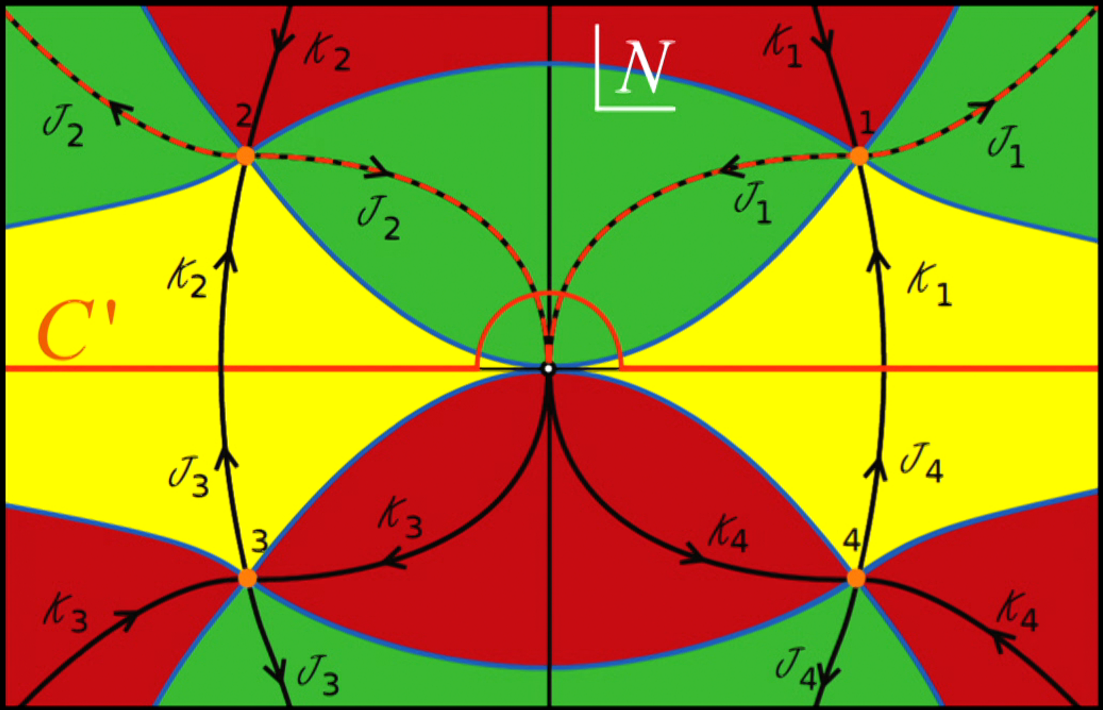
One can deform the defining contour C into one passing along a number of thimbles,

$$C = \sum_\sigma n_\sigma J_\sigma \Leftrightarrow n_\sigma = \langle C K_\sigma \rangle$$

i.e., a saddle contributes iff its steepest *ascent* contour intersects C

Above gives the Feynman propagator: one can also integrate over $C' = (-\infty, \infty)$ passing above 0 which just gives the real part of the Feynman propagator

From $\hat{H}\langle 1|0\rangle_F = -i\hbar\delta(\Sigma_1 - \Sigma_0)$, it follows that $\hat{H}\,\mathrm{Re}[\langle 1|0\rangle_F] = 0$. So the contour integral over C' gives a solution of the homogeneous WdW equation



basic issues with the Euclidean path integral

Usual Wick rotation $N = -iN_E$ renders exponent $\frac{i}{\hbar}S \equiv -\frac{1}{\hbar}S_E$ real but it is an odd function of N_E so, semiclassically, the integral over $-\infty < N_E < \infty$ diverges (in any dimension). Conversely, integrating over a half-line does not provide a "wavefunction of the universe" satisfying the homogeneous WdW equation.

Furthermore, in D=4, divergences at $N_E \to 0^{\pm}$ and $N_E \to \pm\infty$ have opposite signs so that (for $q_1 > 0$) the half-line integral diverges.

Perturbations (to quadratic order in action with no backreaction):

$$ds^2 = -N^2 q^{-1} dt^2 + q(g_{ij}^{S_3} + h_{ij}^T) dx^i dx^j;$$

$$S = S^{(0)} + S^{(2)}; \quad S^{(2)} = \pi^2 \int_0^1 dt [N^{-1} q^2 \dot{h}_{Tl}^2 - Nl(l+2) h_{Tl}^2]$$

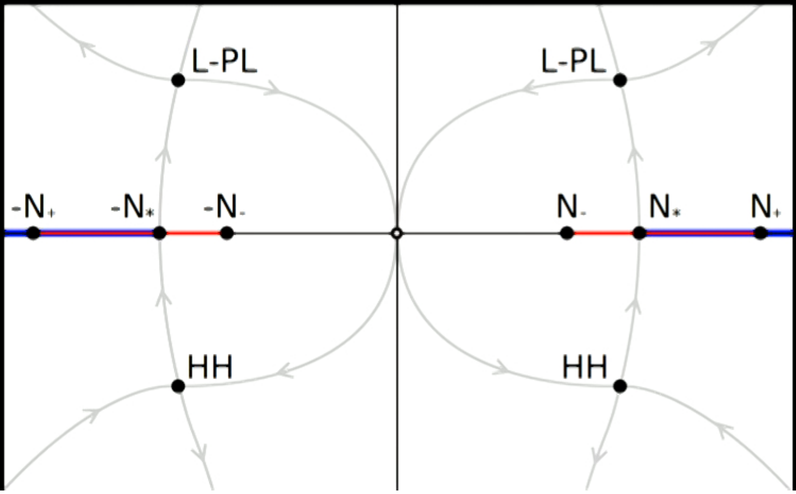
redefine: $\chi_l = q\ h_{Tl} \Rightarrow eom \quad -\ddot{\chi}_l + \frac{1}{4t^2}(\gamma^2 - 1)\chi_l == 0,\ t \to 0$

Can show $\mathrm{Re}[\gamma] > 0$ everywhere in complex N-plane (ensures finite action) except on two branch cuts (arise only because of infinite dimensionality)

$-N_+ < N < -N_-,\ N_- < N < N_+$ where

$$N_\pm = \frac{3}{\Lambda}\sqrt{2l(l+2) + q_1 \frac{\Lambda}{3} \pm 2\sqrt{l(l+2)(l(l+2) + q_1 \frac{\Lambda}{3})}}$$

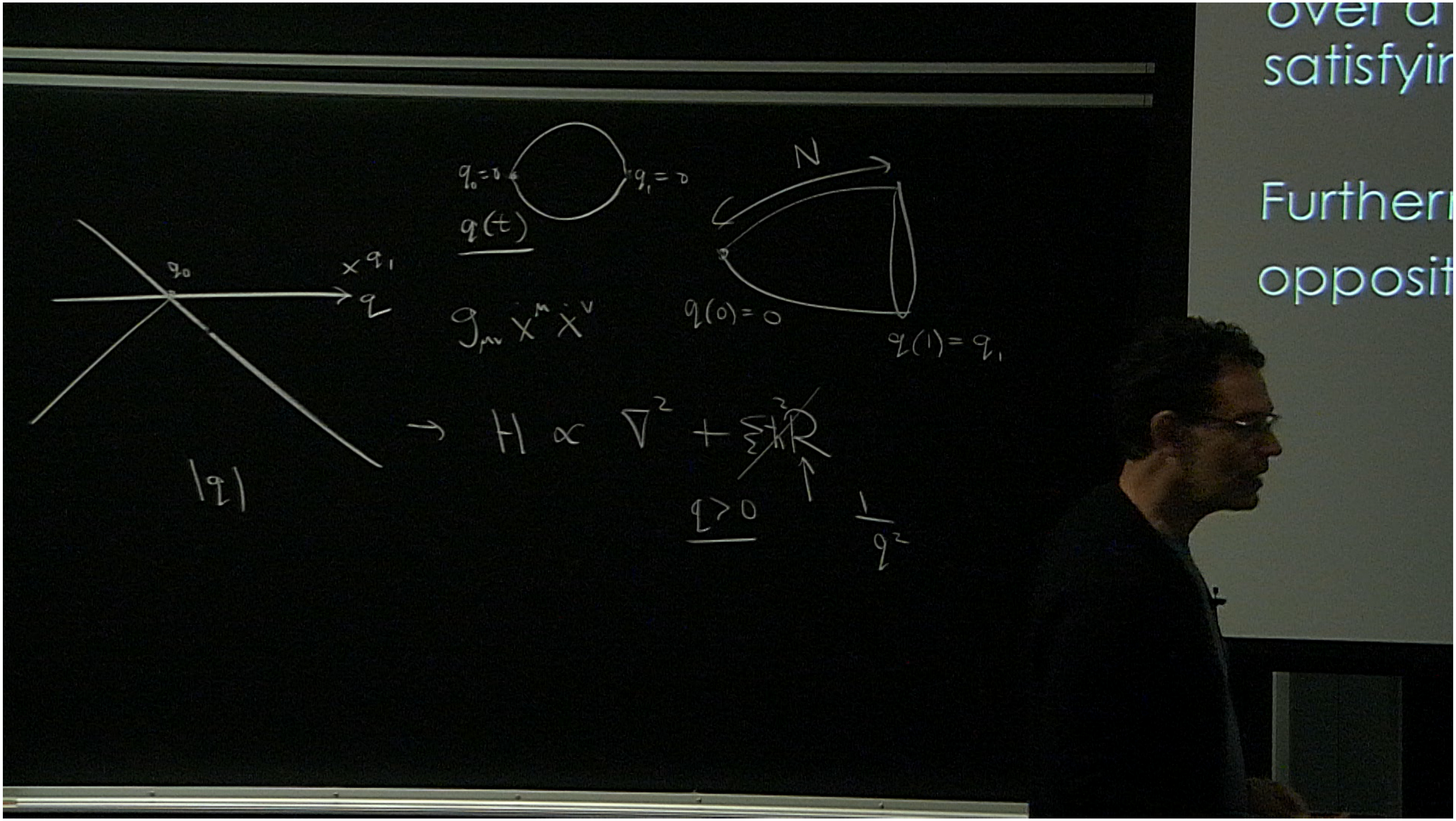
$$N_* = \sqrt{N_+ N_-} = \sqrt{\frac{3}{\Lambda} q_1}$$



basic issues with the Euclidean path integral

Usual Wick rotation $N = -iN_E$ renders exponent $\frac{i}{\hbar} S \equiv -\frac{1}{\hbar} S_E$ real but it is an odd function of N_E so, semiclassically, the integral over $-\infty < N_E < \infty$ diverges (in any dimension). Conversely, integrating over a half-line does not provide a "wavefunction of the universe" satisfying the homogeneous WdW equation.

Furthermore, in D=4, divergences at $N_E \to 0^{\pm}$ and $N_E \to \pm\infty$ have opposite signs so that (for $q_1 > 0$) the half-line integral diverges.


over a
satisfyin
Furtherr
opposit

Perturbations (to quadratic order in action with no backreaction):

$$ds^2 = -N^2 q^{-1} dt^2 + q(g_{ij}^{S_3} + h_{ij}^T) dx^i dx^j;$$

$$S = S^{(0)} + S^{(2)}; \quad S^{(2)} = \pi^2 \int_0^1 dt [N^{-1} q^2 \dot{h}_{Tl}^2 - Nl(l+2) h_{Tl}^2]$$

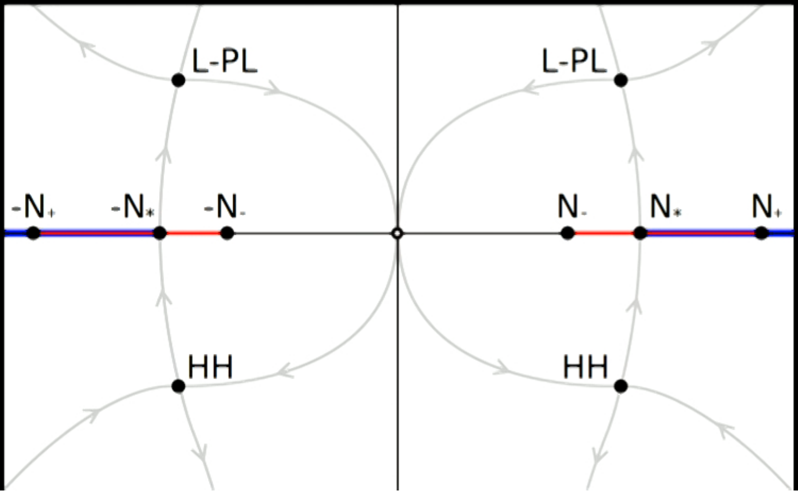
redefine: $\chi_l = q\ h_{Tl} \Rightarrow eom \quad -\ddot{\chi}_l + \frac{1}{4t^2}(\gamma^2 - 1)\chi_l == 0,\ t \to 0$

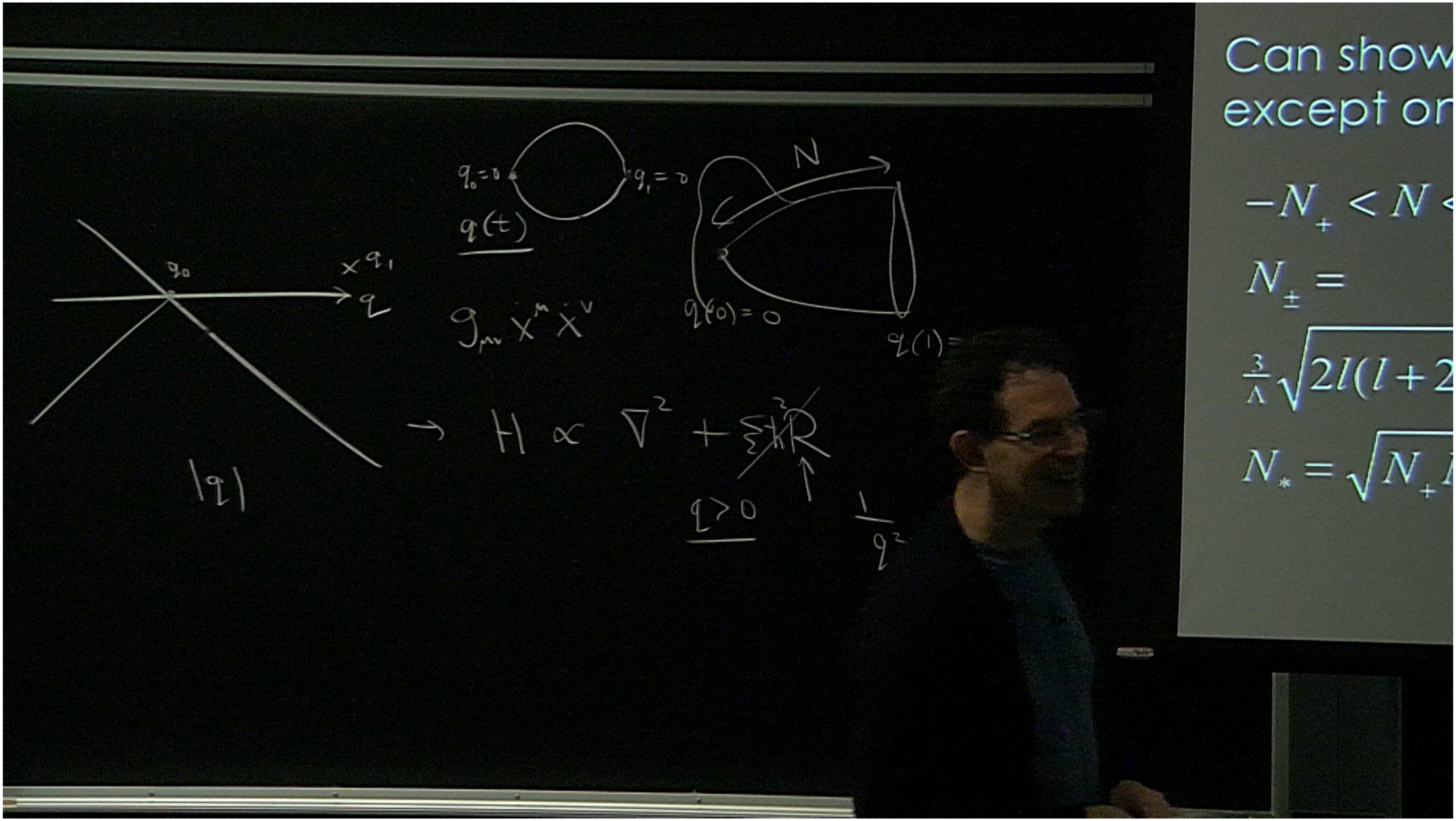
Can show $\mathrm{Re}[\gamma] > 0$ everywhere in complex N-plane (ensures finite action) except on two branch cuts (arise only because of infinite dimensionality)

$-N_+ < N < -N_-, \ N_- < N < N_+$ where

$$N_\pm = \frac{3}{\Lambda}\sqrt{2l(l+2) + q_1 \frac{\Lambda}{3} \pm 2\sqrt{l(l+2)(l(l+2) + q_1 \frac{\Lambda}{3})}}$$

$$N_* = \sqrt{N_+ N_-} = \sqrt{\frac{3}{\Lambda} q_1}$$





Perturbations (to quadratic order in action with no backreaction):

$$ds^2 = -N^2 q^{-1} dt^2 + q(g_{ij}^{S_3} + h_{ij}^T) dx^i dx^j;$$

$$S = S^{(0)} + S^{(2)}; \quad S^{(2)} = \pi^2 \int_0^1 dt [N^{-1} q^2 \dot{h}_{Tl}^2 - N l(l+2) h_{Tl}^2]$$

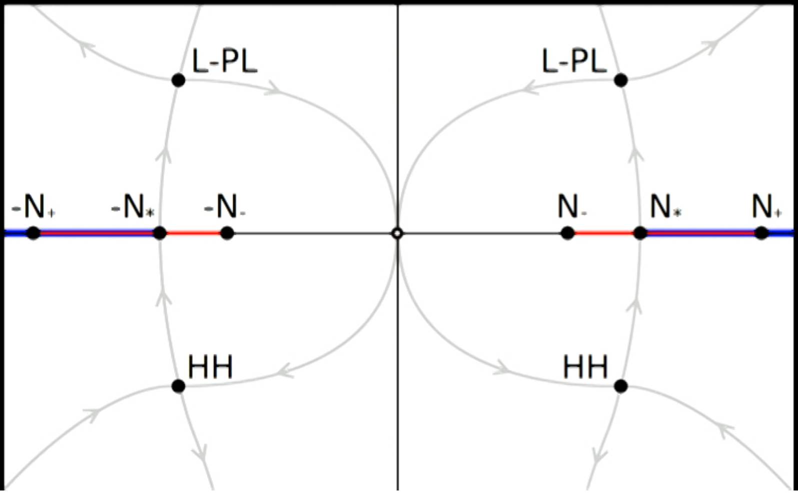
redefine: $\chi_l = q \; h_{Tl} \Rightarrow eom \quad -\ddot{\chi}_l + \frac{1}{4t^2}(\gamma^2 - 1)\chi_l == 0, \; t \to 0$

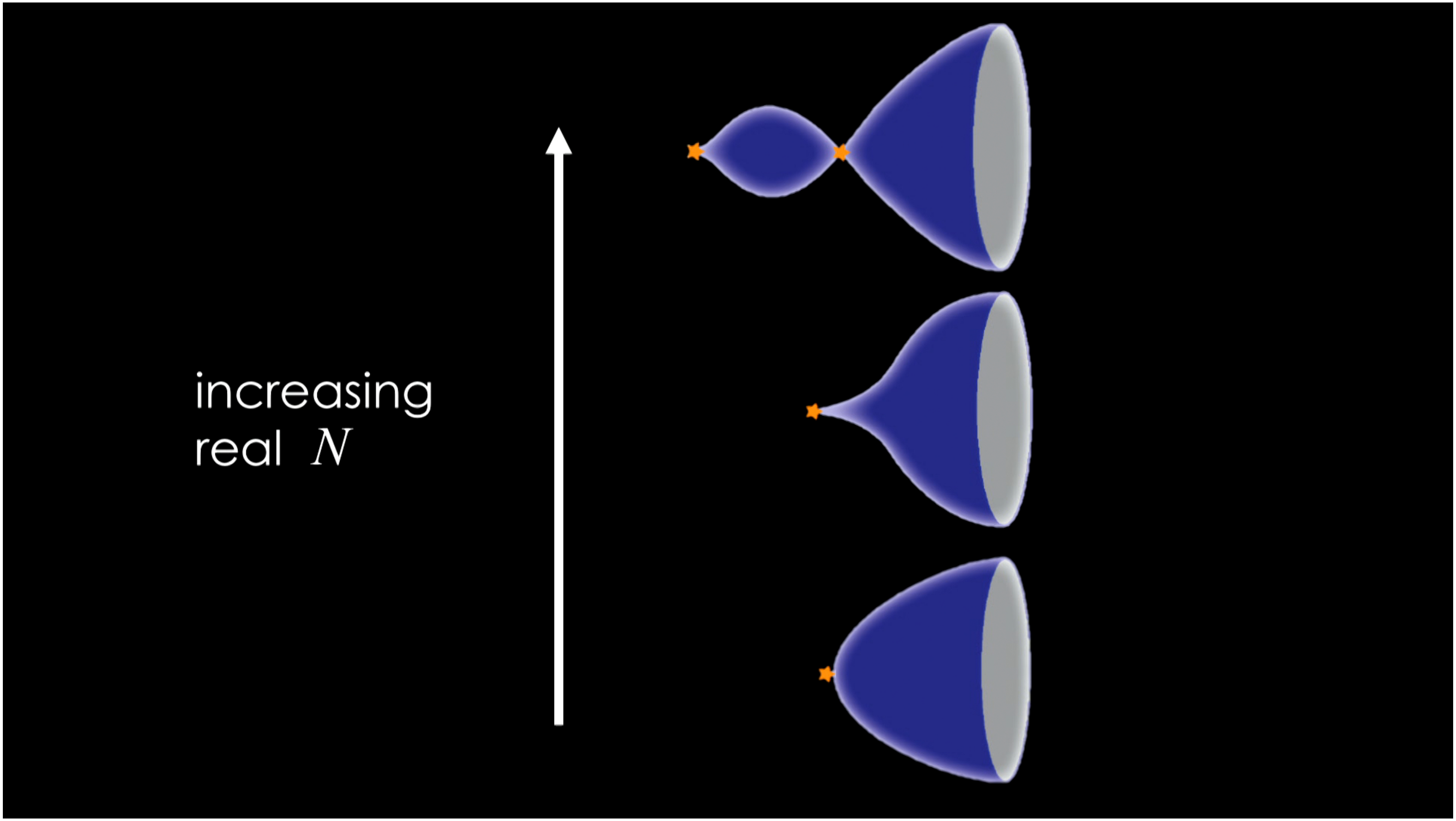
Can show $\mathrm{Re}[\gamma] > 0$ everywhere in complex N-plane (ensures finite action) except on two branch cuts (arise only because of infinite dimensionality)

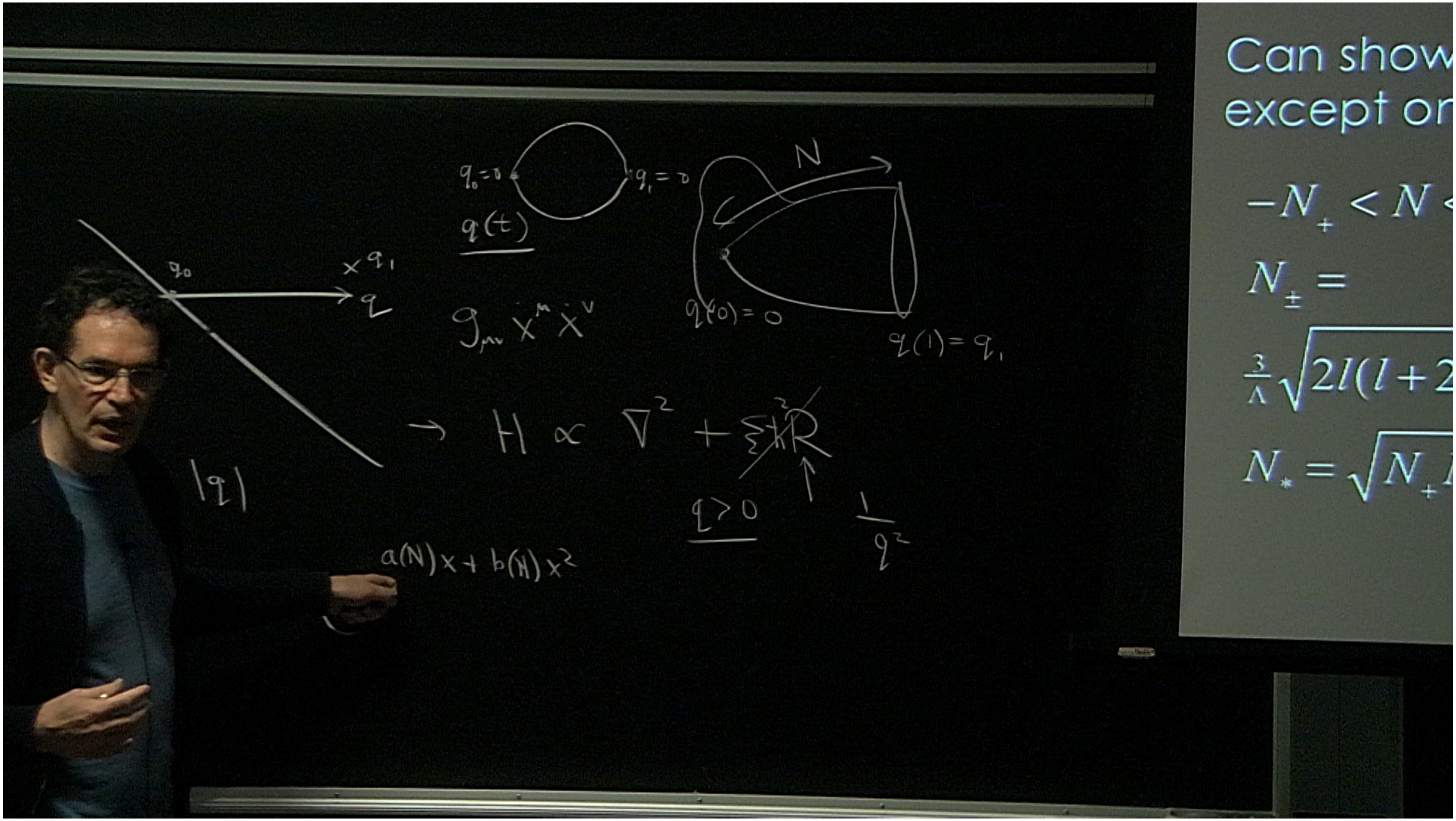
$-N_+ < N < -N_-, \; N_- < N < N_+$ where

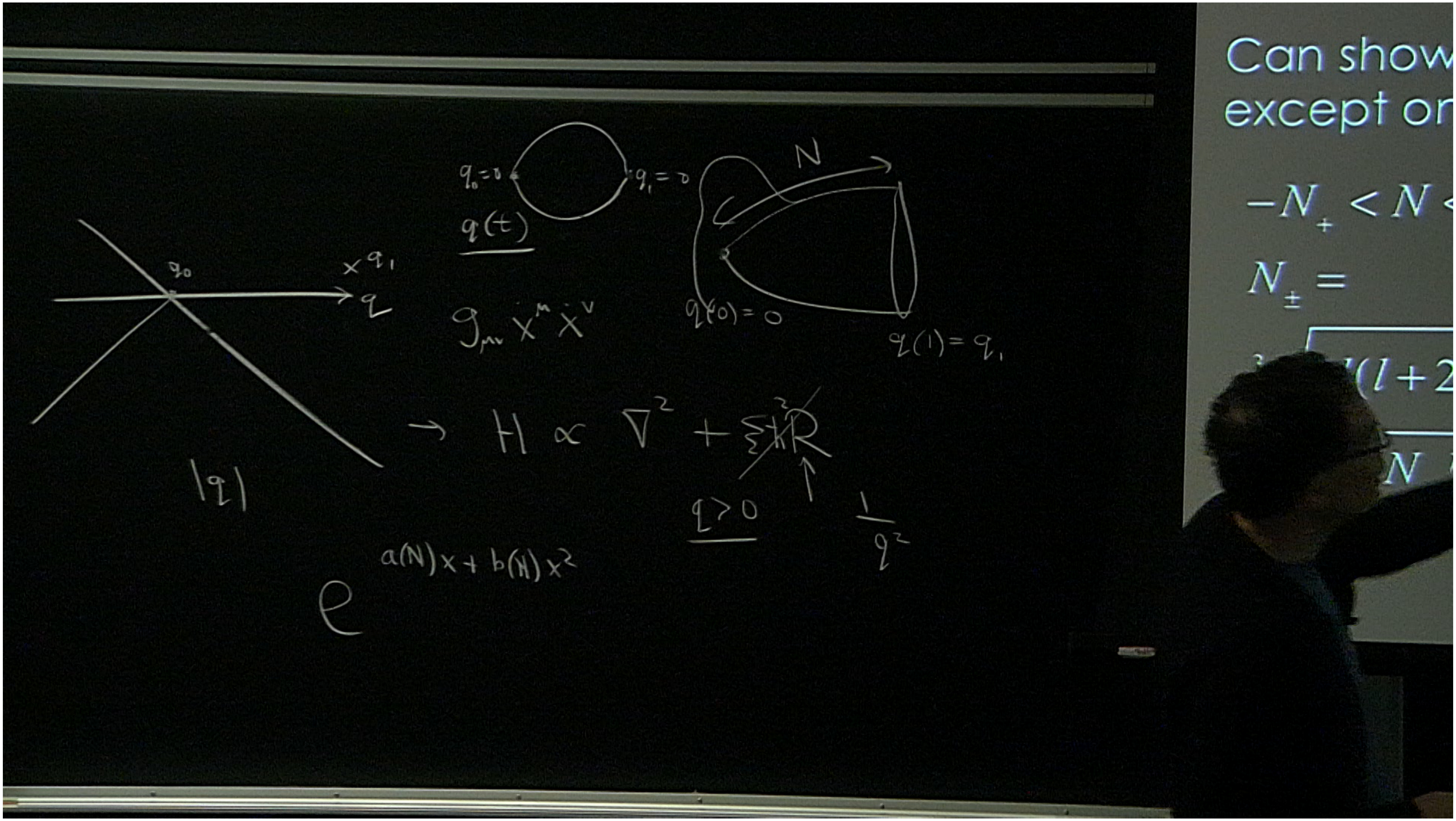
$$N_\pm = \frac{3}{\Lambda}\sqrt{2l(l+2) + q_1 \frac{\Lambda}{3} \pm 2\sqrt{l(l+2)(l(l+2) + q_1 \frac{\Lambda}{3})}}$$

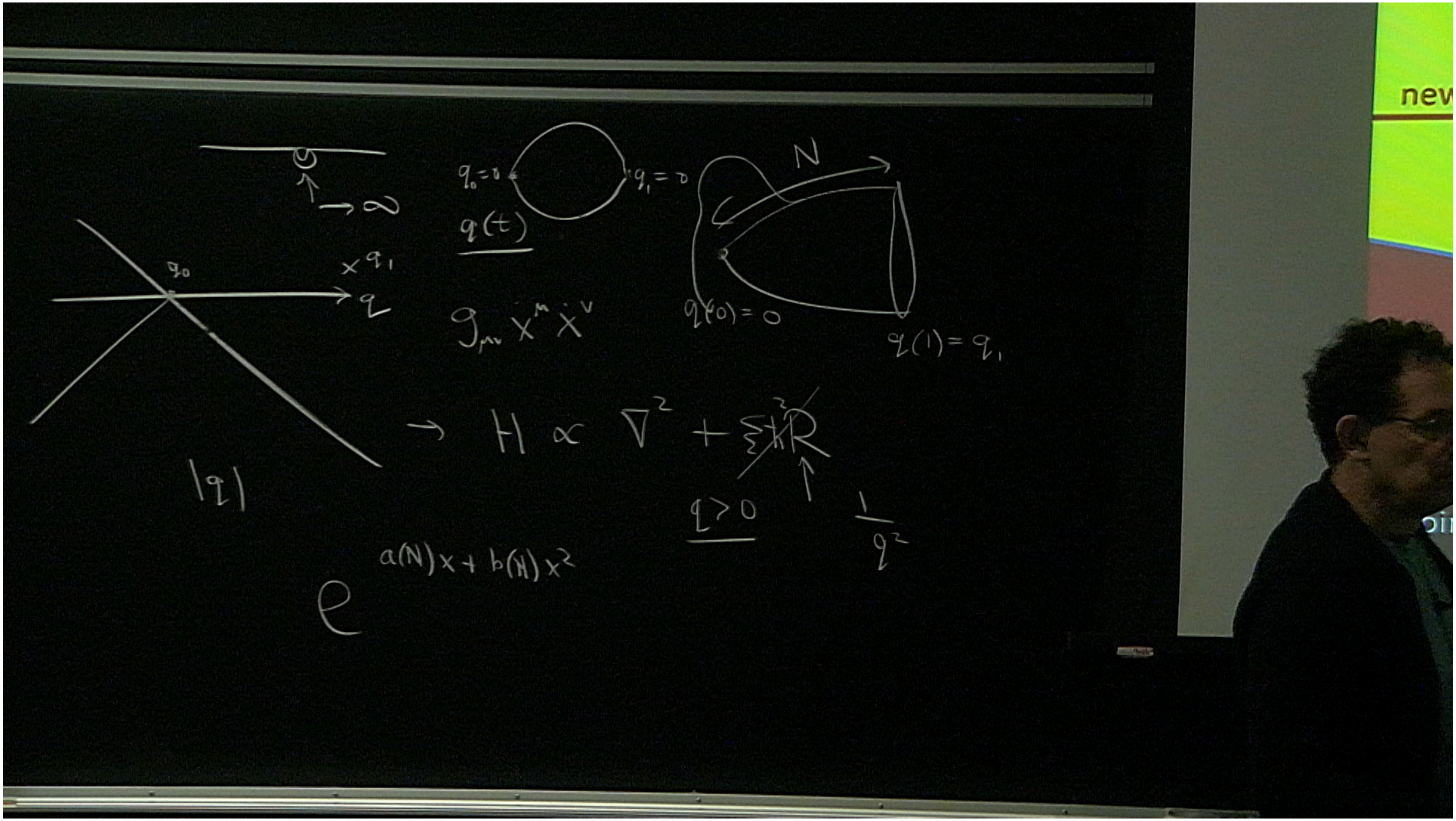
$$N_* = \sqrt{N_+ N_-} = \sqrt{\frac{3}{\Lambda} q_1}$$

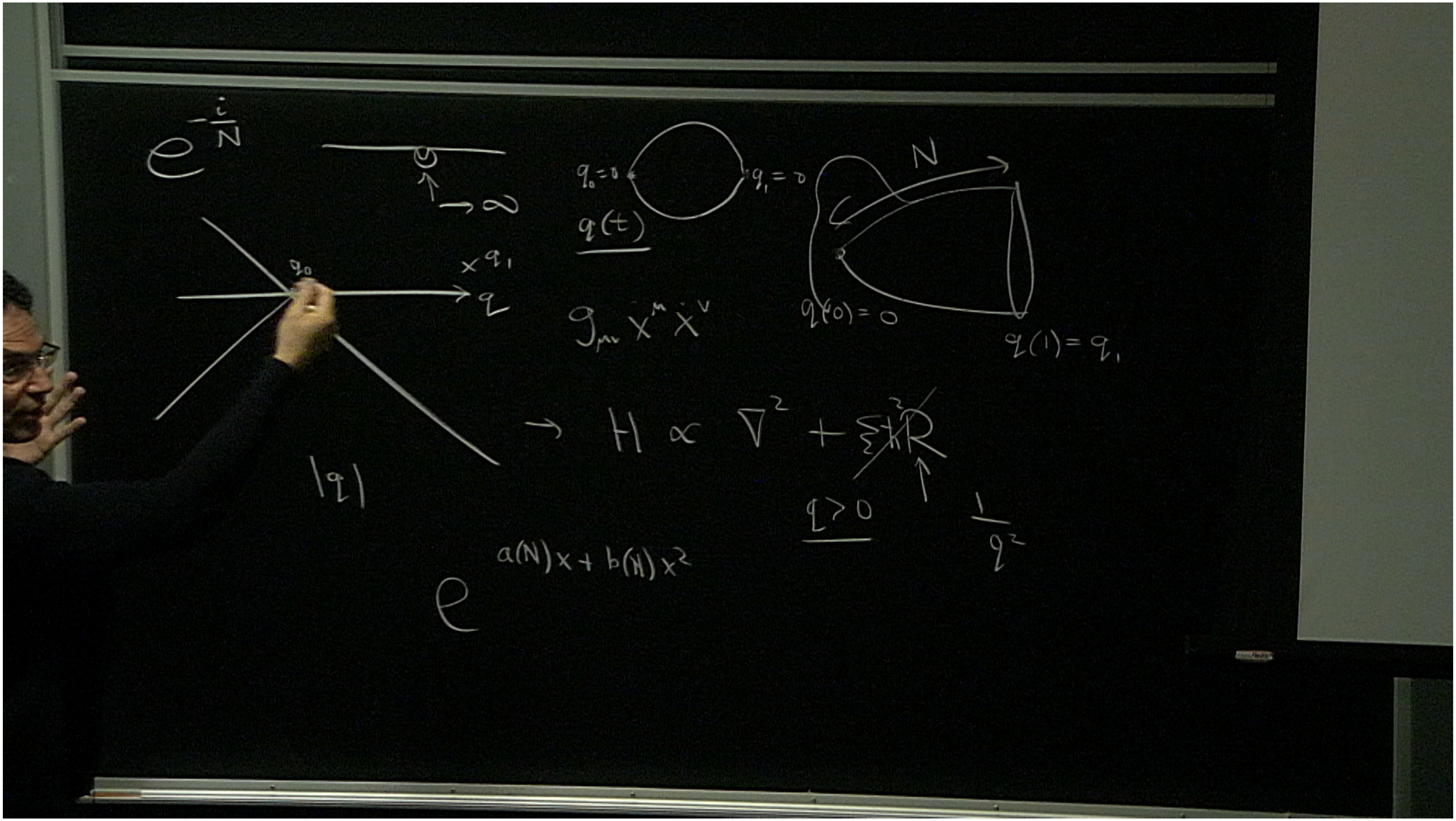



increasing
real N

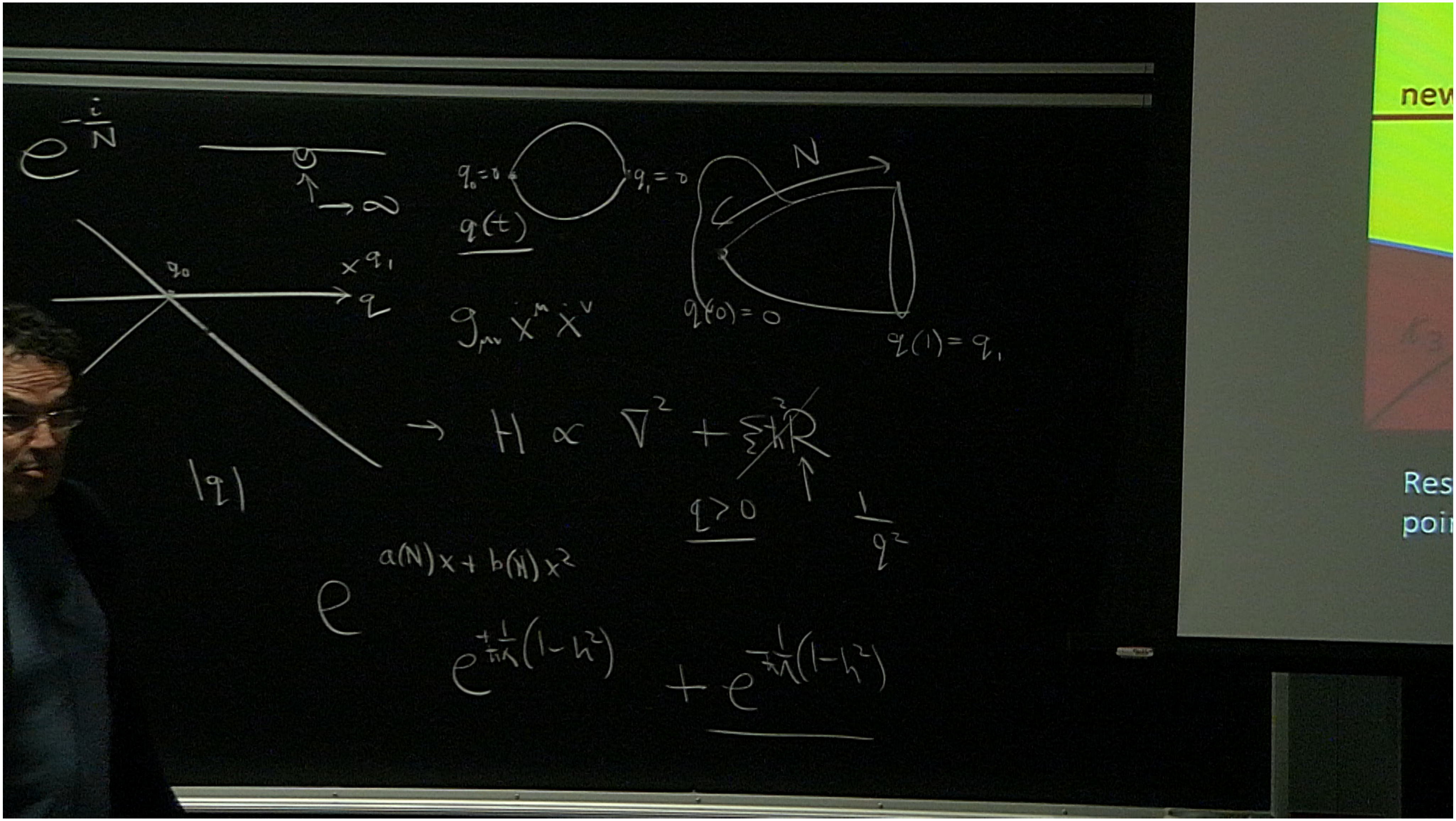

Can show
except or
$-N_+ < N <$
$N_\pm =$
$\frac{3}{\Lambda}\sqrt{2l(l+2}$
$N_* = \sqrt{N_+}$


Can show
except or
q(t)
N
q(0) = 0
q(1) = q₁
|q|
q > 0


q(t)
N
q(0) = 0
q(1) = q1
q > 0
e

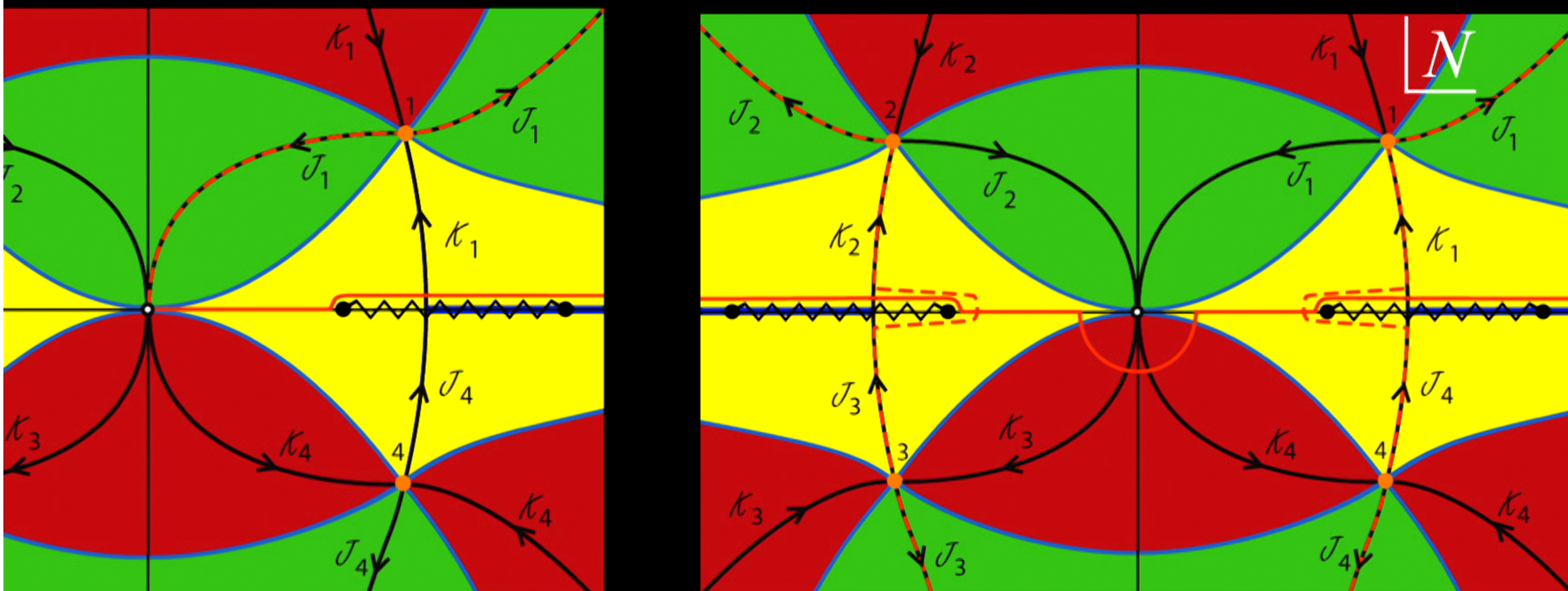


$e^{-\frac{i}{N}}$
$q_0 = 0$
$q_1 = 0$
$q(t)$
N
q_0
q_1
q
$g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu$
$q(0) = 0$
$q(1) = q_1$
$\rightarrow H \propto \nabla^2 + \xi \hbar^2 R$
$|q|$
$q > 0$
$\frac{1}{q^2}$
$e^{a(N)x + b(N)x^2}$



q(t)
q(0) = 0
q(1) = q_1
H ∝ ∇² + ...
q > 0
a(N)x + b(N)x²
new
Res
poi

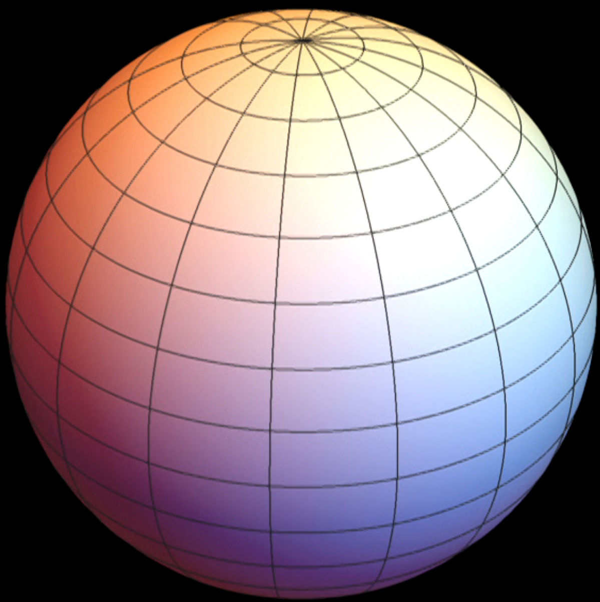
Non-analyticity arises in the exponent, from integrating out perturbations: cannot then apply Picard-Lefschetz flow for the remaining integral over N
However, Cauchy's theorem still applies: we just distort the contour in advance to avoid any branch cut which arises from integrating out fluctuations (to quadratic order)



Conclusion:

There is no meaningful one-point function
"Hartle-Hawking state" for a 3-geometry
(for 4d gravity with positive Λ)

Persistence of nothing



If we consider the limit $q_1 = q_0 = 0$, then the small N_E divergence disappears and the Euclidean path integral over the background becomes well defined

There is a saddle with $N_s = -\frac{6i}{\Lambda}; \quad S_E = -\frac{24\pi^2}{\hbar\Lambda}$

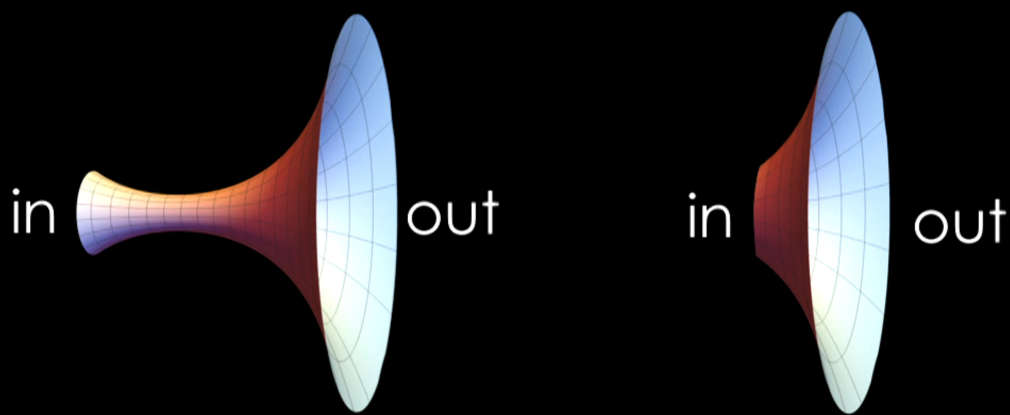
The Euclidean action for the tensor fluctuations is positive definite so that the nothing-nothing "self-energy" amplitude is real

We take this to mean that "nothing" is stable

quantum de Sitter

Lorentzian in-out amplitudes may be constructed semi-classically

For classically allowed q_0 and q_1, both larger than the de Sitter throat, there are always just two, real saddle point solutions



These interfere in interesting (and calculable) ways

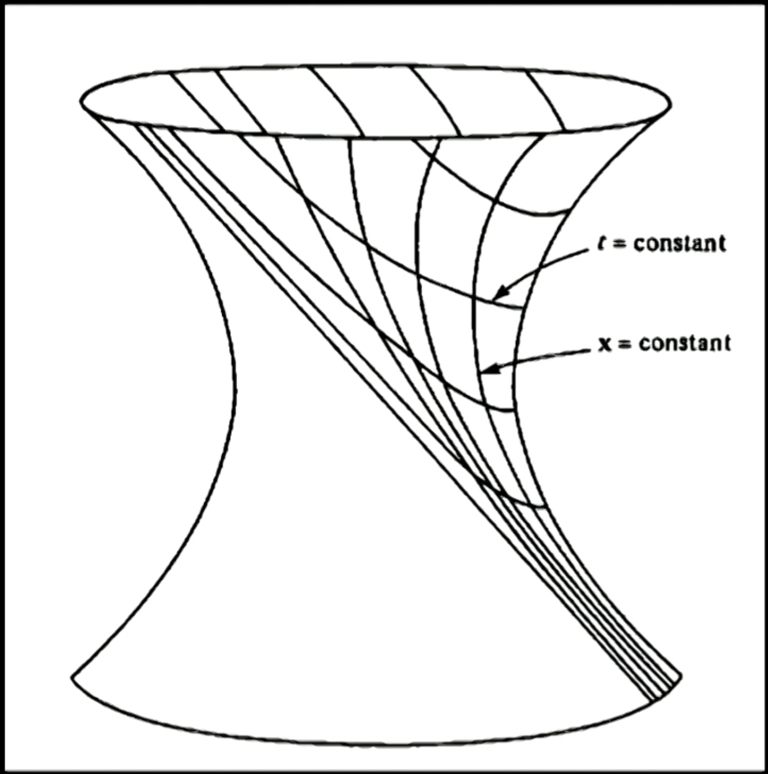
We have been able to find the linearized mode solutions analytically for general N, as well as to compute the corresponding classical action

We also have developed numerical techniques to include nonlinear backreaction by iteratively improving the complex linear solutions

This provides a laboratory in which to study real-time quantum phenomena using semiclassical methods, for example the growth of perturbations in the contracting phase, leading to the creation of black holes which then evaporate in the expanding phase

By systematically enumerating relevant semiclassical saddles, we may explore whether de Sitter has a finite number of quantum states

Implications for inflation: well-known that flat slicing is geodesically incomplete

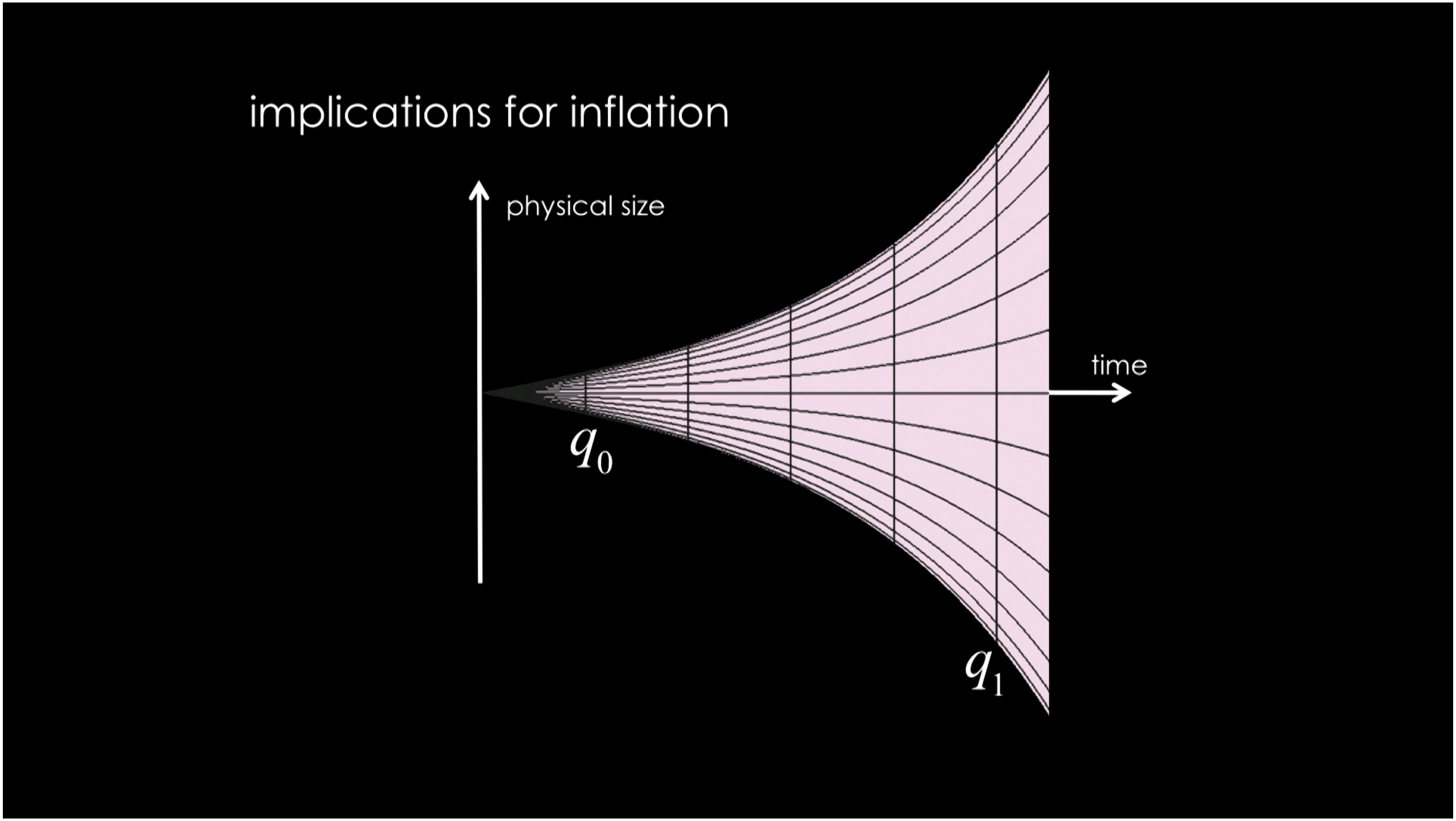


$$-dt^2 + e^{2Ht} d\vec{x}^2, \ -\infty < t < \infty$$

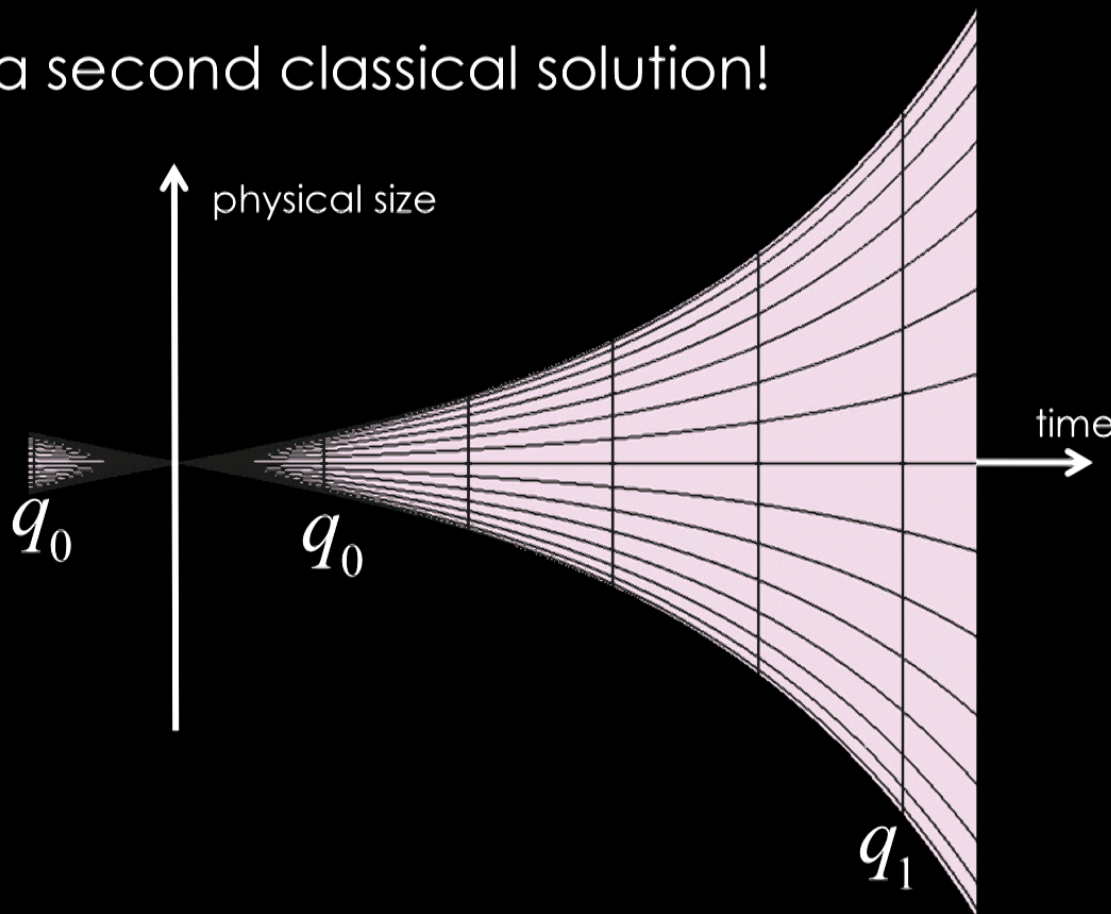
Global geometry is obtained by analytic continuation

New result: the semiclassical quantum description is similarly incomplete.

Quantising the background (via the semiclassical path integral) potentially ruins the "Bunch-Davies" vacuum.


implications for inflation
physical size
time
q_0
q_1

there is a second classical solution!



Quantum incompleteness of inflation

For inflation to be a complete theory, there should be a natural way of taking the the limit $q_0 \to 0$

The usual way to specify the "in" vacuum is to take $\eta_0 \to -\infty e^{-i\varepsilon}$

(where $a = e^{Ht_P} = -\frac{1}{H\eta}$)

e.g. S. Weinberg, arXiv: 0805.3781

However, we have $\eta_0 = -\frac{1}{H\sqrt{q_0}}$ so in quantum geometrodynamics this amounts to

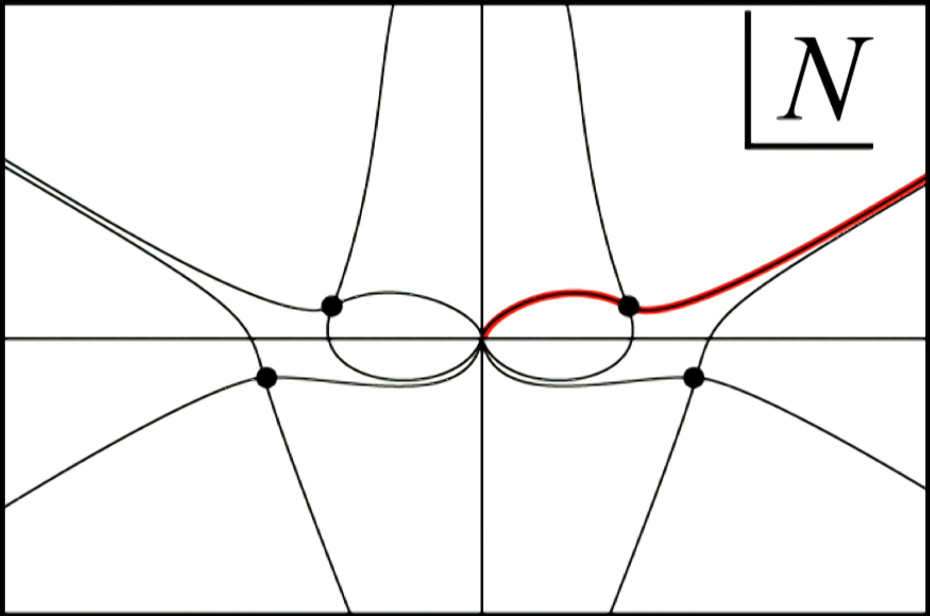
performing a small rephasing of q_0 in the opposite sense

Carrying this through, one finds that the relevant Lorentzian saddle (for the N-integral) is always the one in the upper-half N-plane, giving unbounded perturbations

So there is a tension between quantum geometrodynamics and inflation, meaning that the "Bunch-Davies" vacuum is potentially susceptible to large quantum gravitational effects

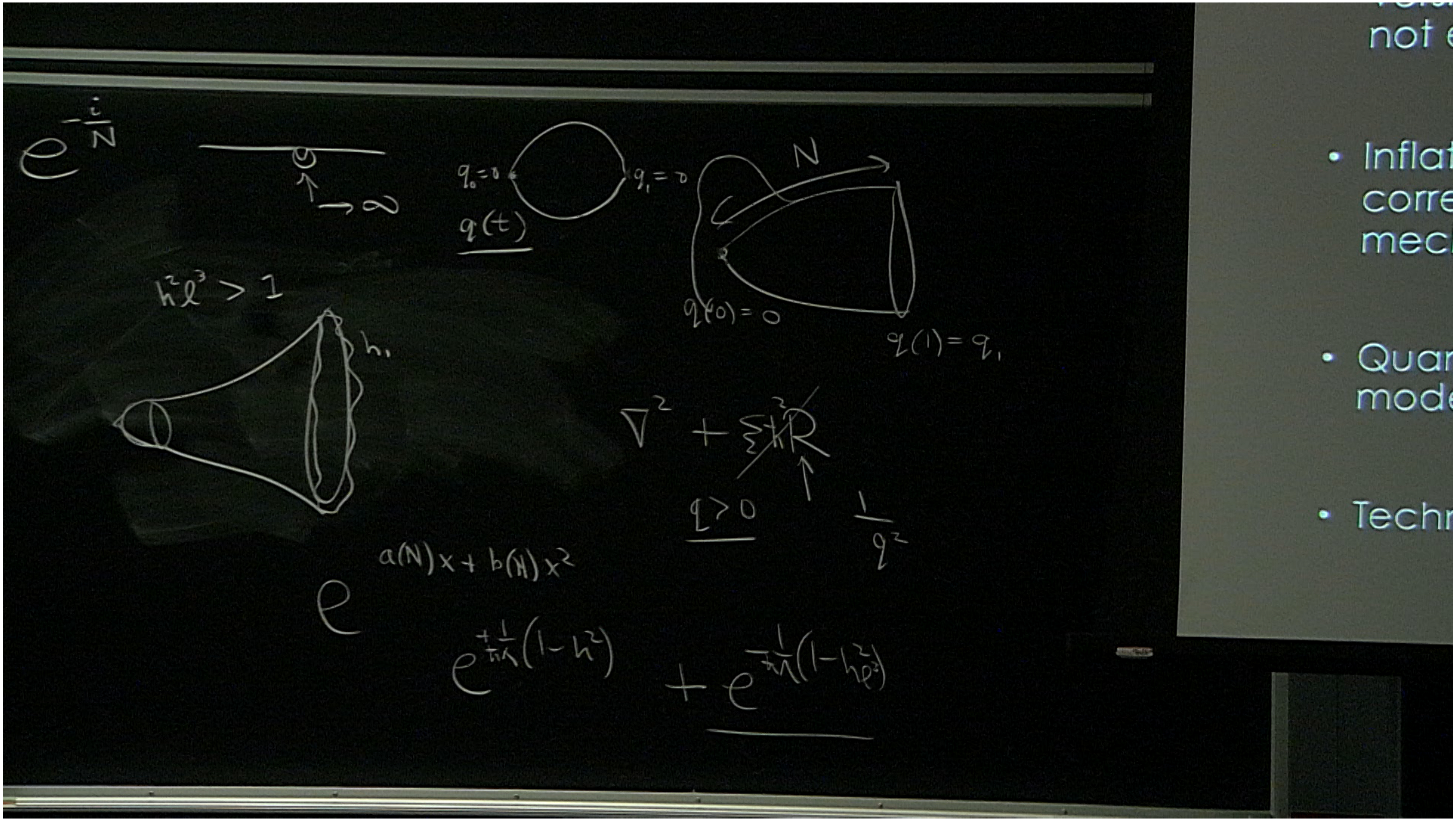
This quantum incompleteness is closely related to the classical, geodesic incompleteness of inflation

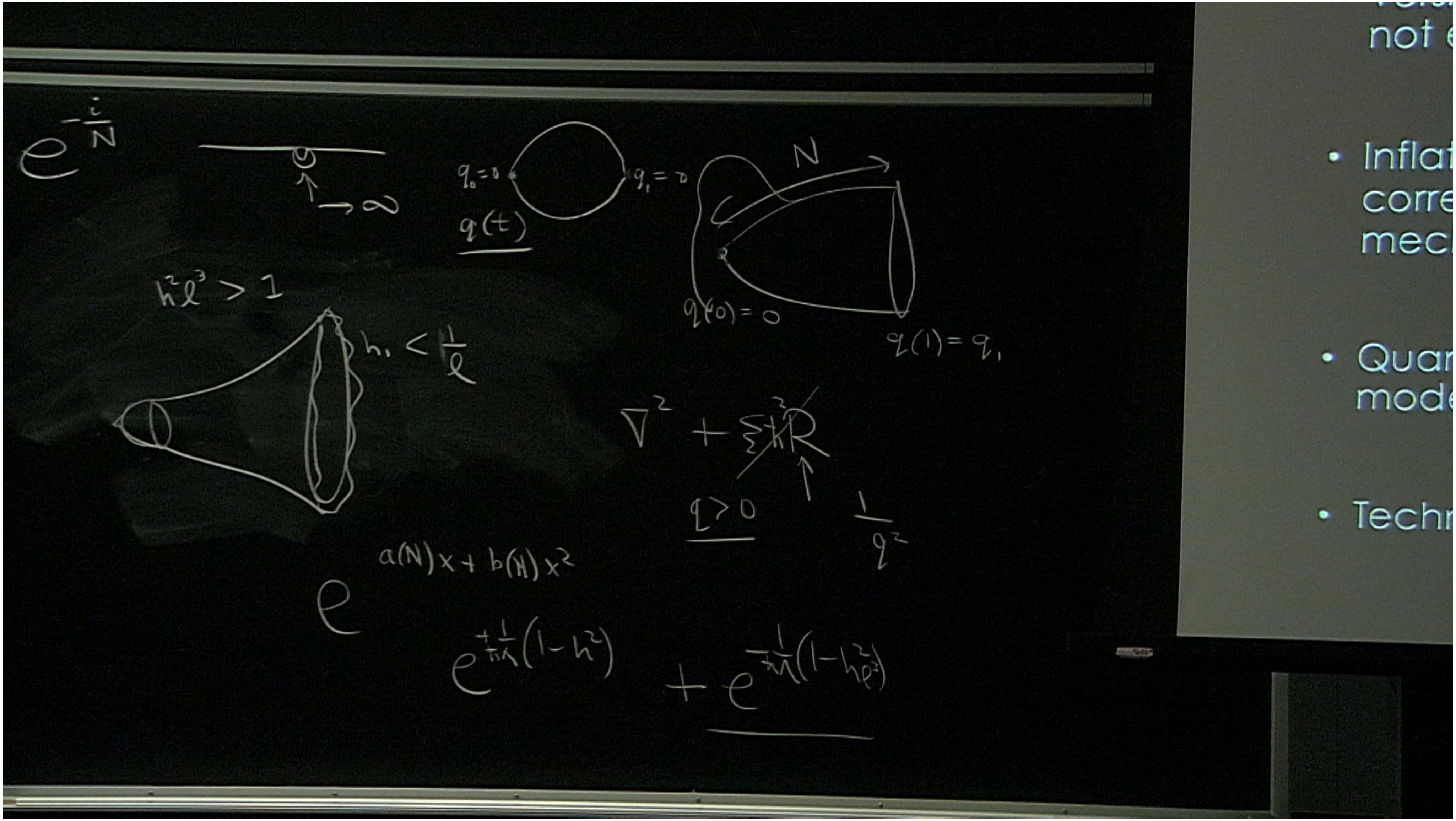
de Sitter flat slicing with $q_0 \to 0$ from uhp



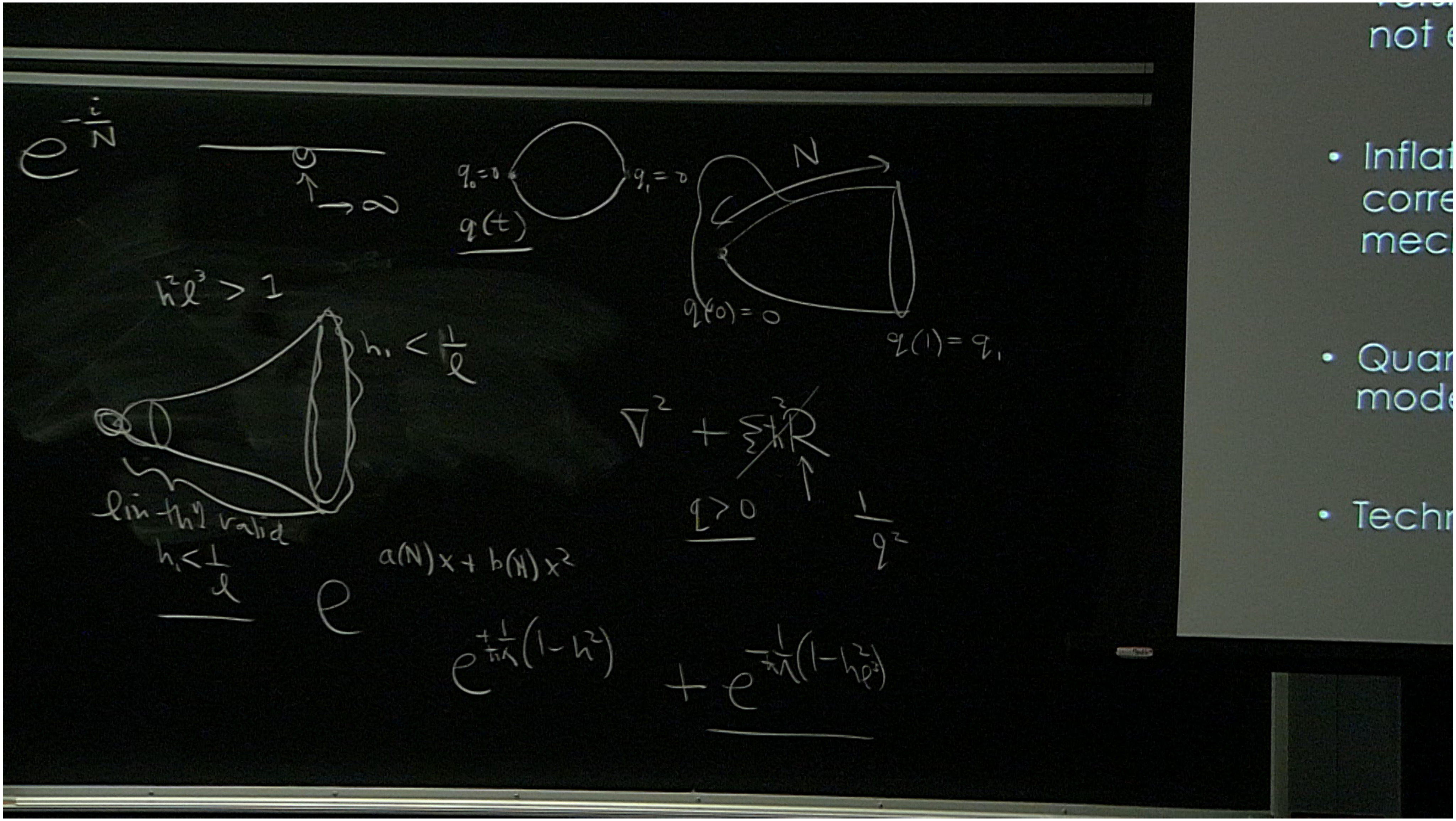
summary

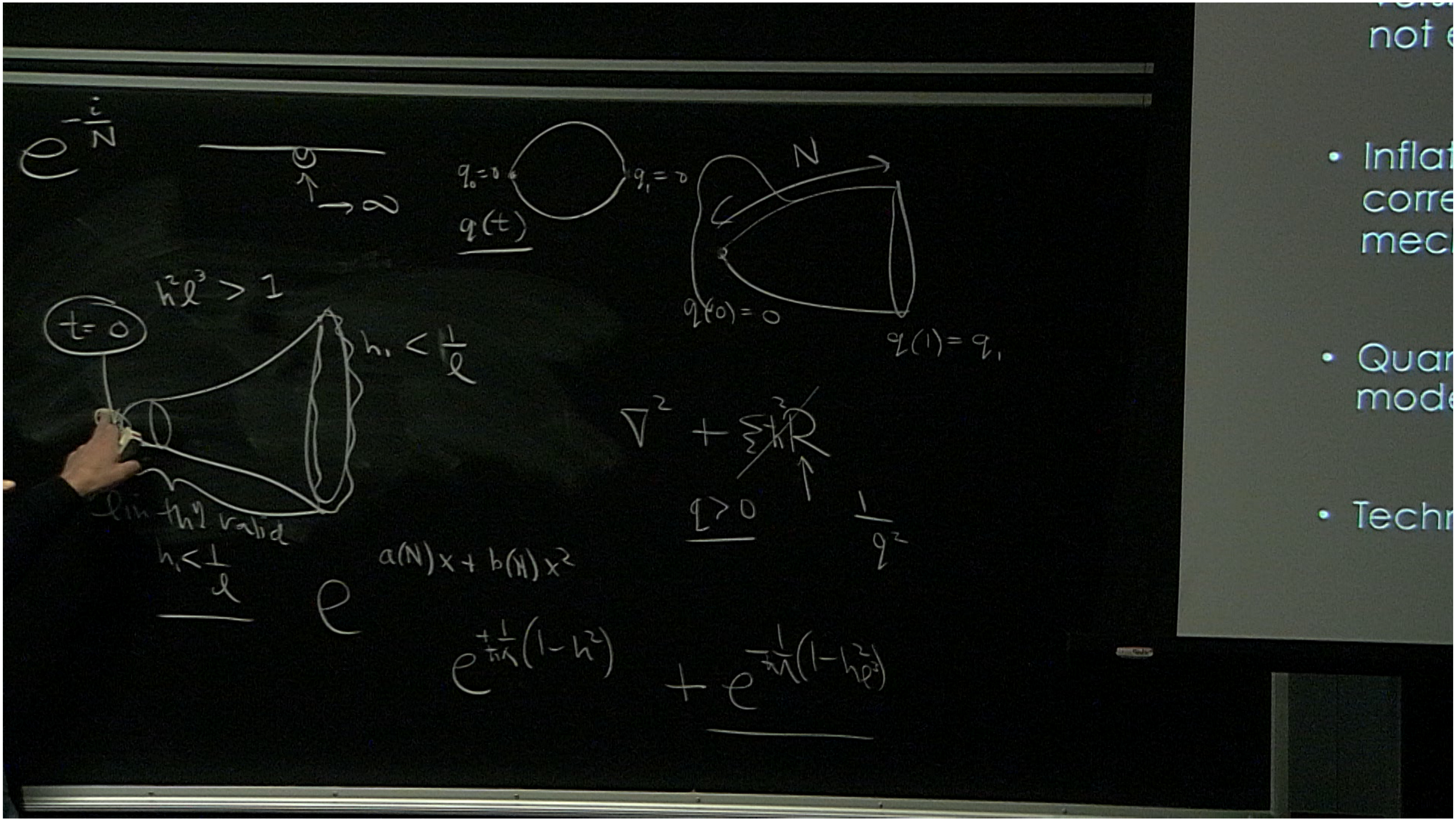
- Picard-Lefschetz-Cauchy deformation allows us to obtain unambiguous predictions from the Lorentzian path integral for gravity in the semiclassical limit.
- The (path integral formulation) of the no boundary proposal is an attractive idea but seems to be mathematically problematic. The Lorentzian semiclassical path integral version yields perturbations which are out of control. The Euclidean path integral does not exist.
- Inflation and the "Bunch-Davies" vacuum are subject to similar nonperturbative corrections, emphasizing that (without further definition) they are quantum mechanically incomplete
- Quantizing the background is important! Intriguing connection between the zero modes (IR) and the QFT vacuum for inhomogeneous perturbations (UV)
- Techniques potentially of wide applicability, *e.g.*, to black holes & holography

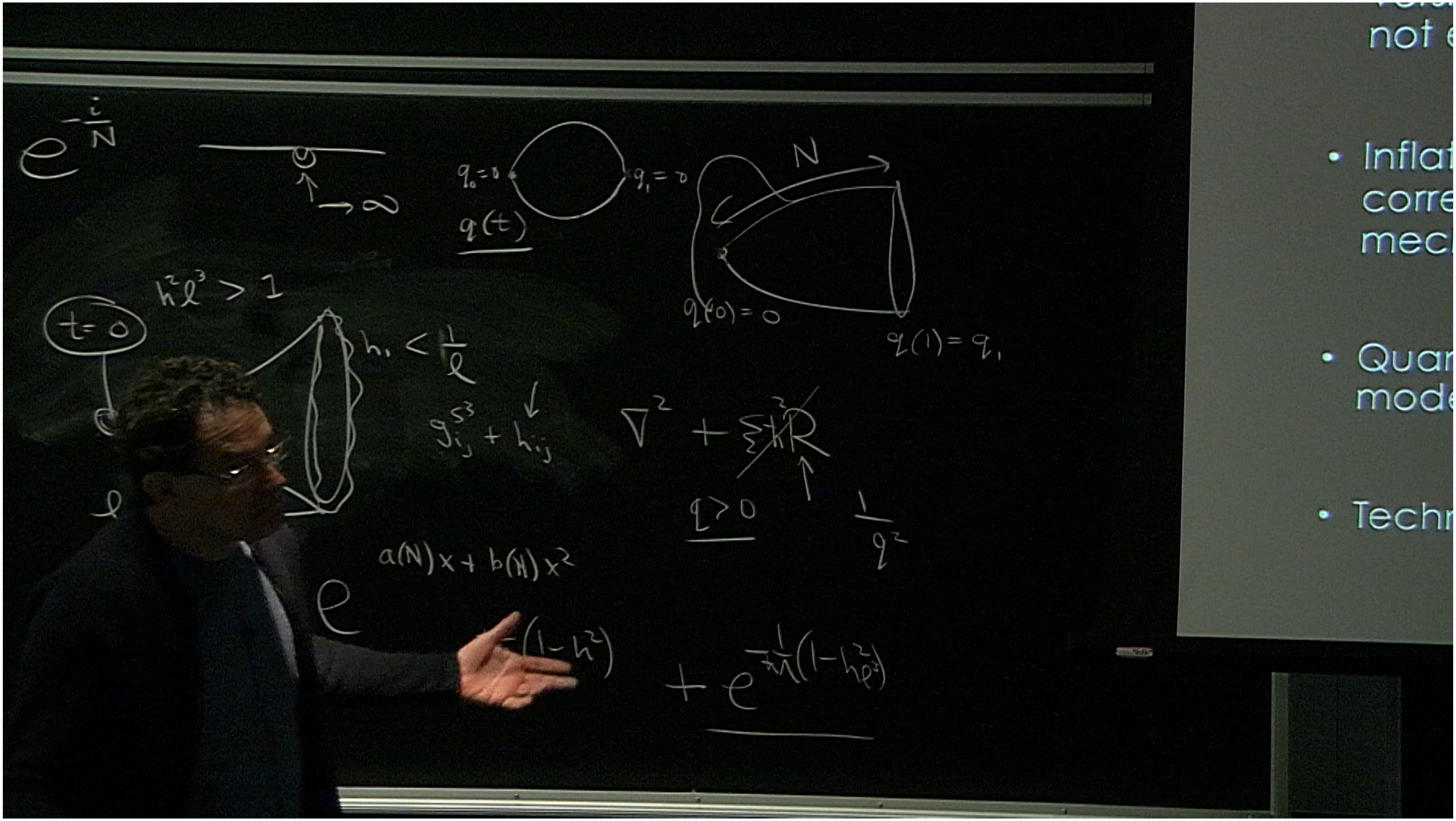

q(t)
N
q(0) = 0
q(1) = q₁
h₀
q > 0

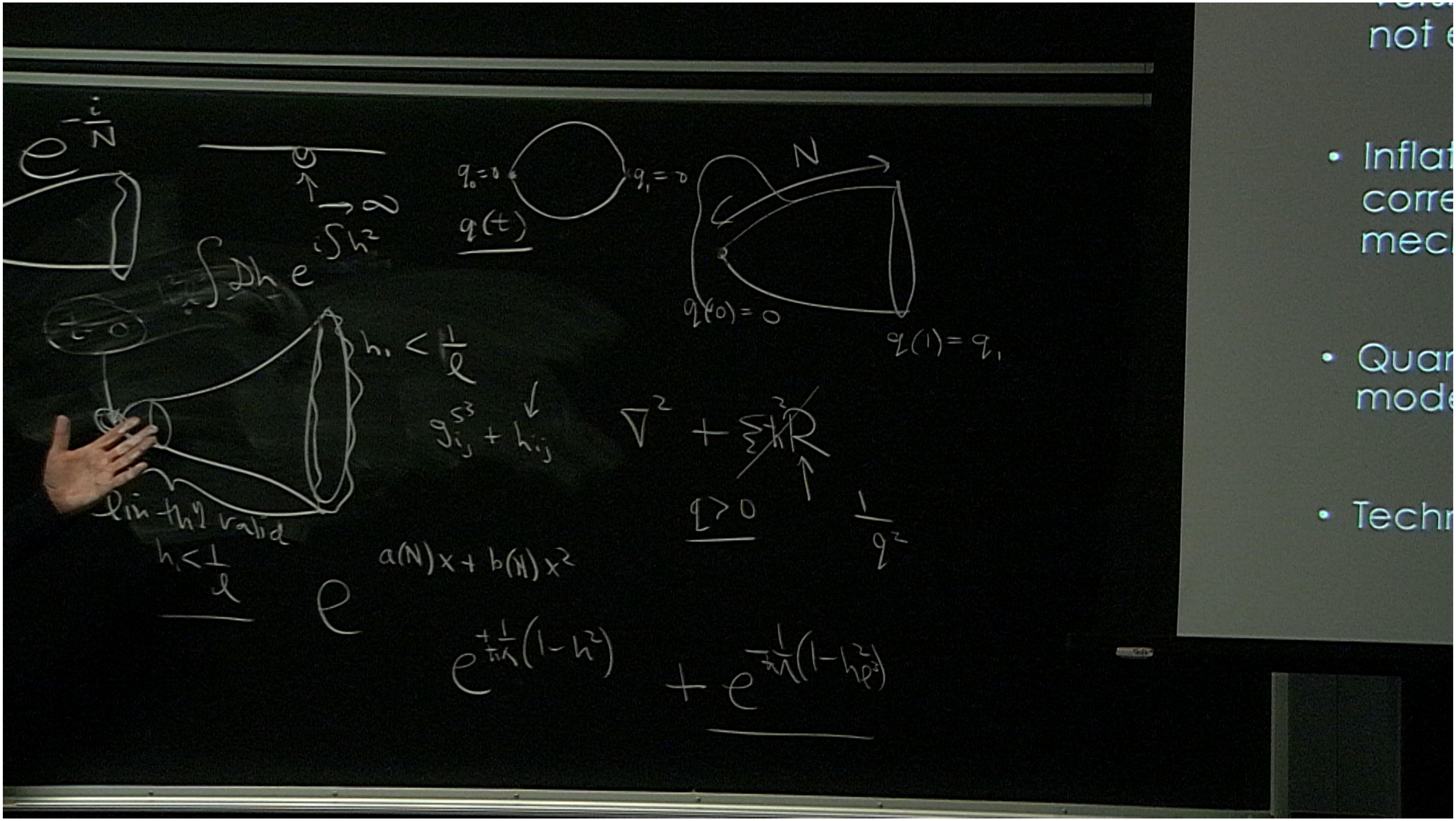


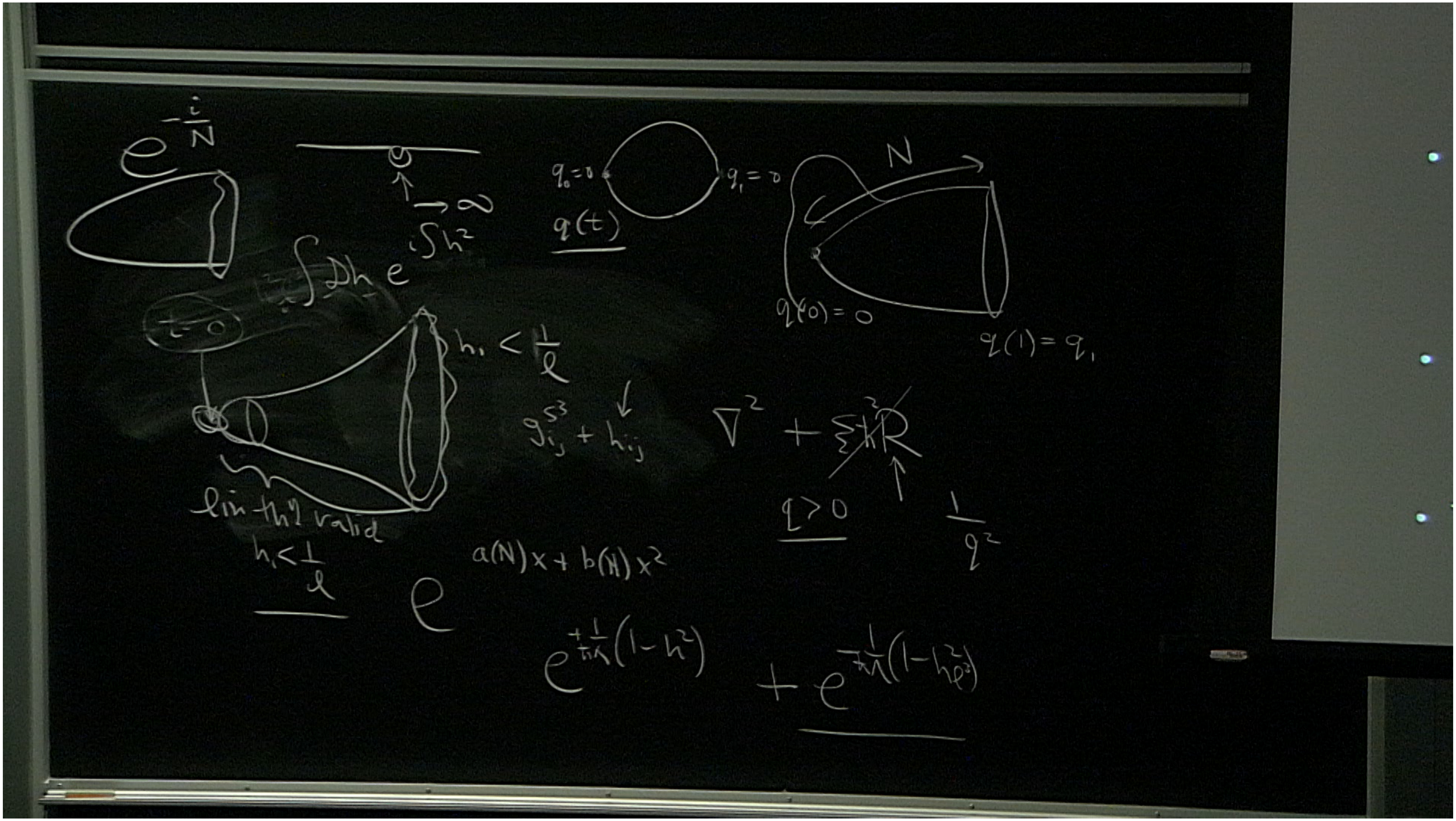
not
• Inflat
corre
mec
• Quar
mode
• Techr


q(t)
N
q(0) = 0
q(1) = q1
h1 < 1/l
lin-th'l valid
a(N) x + b(N) x²
not
Inflat
corre
mec
Quar
mode
Techr


N
q(t)
q(0) = 0
q(1) = q1
valid
a(N) x + b(N) x²
not


q(t)
N
q(0) = 0
q(1) = q1
t = 0
a(N)x + b(N)x2
q > 0


q(t)
q(0) = 0
q(1) = q1
N
a(N)x + b(N)x²


q(t)
N
q(0) = 0
q(1) = q₁
lin th'y valid
a(N)x + b(N)x²