Title: B-model for knot homology.

Date: Nov 20, 2017 11:00 AM

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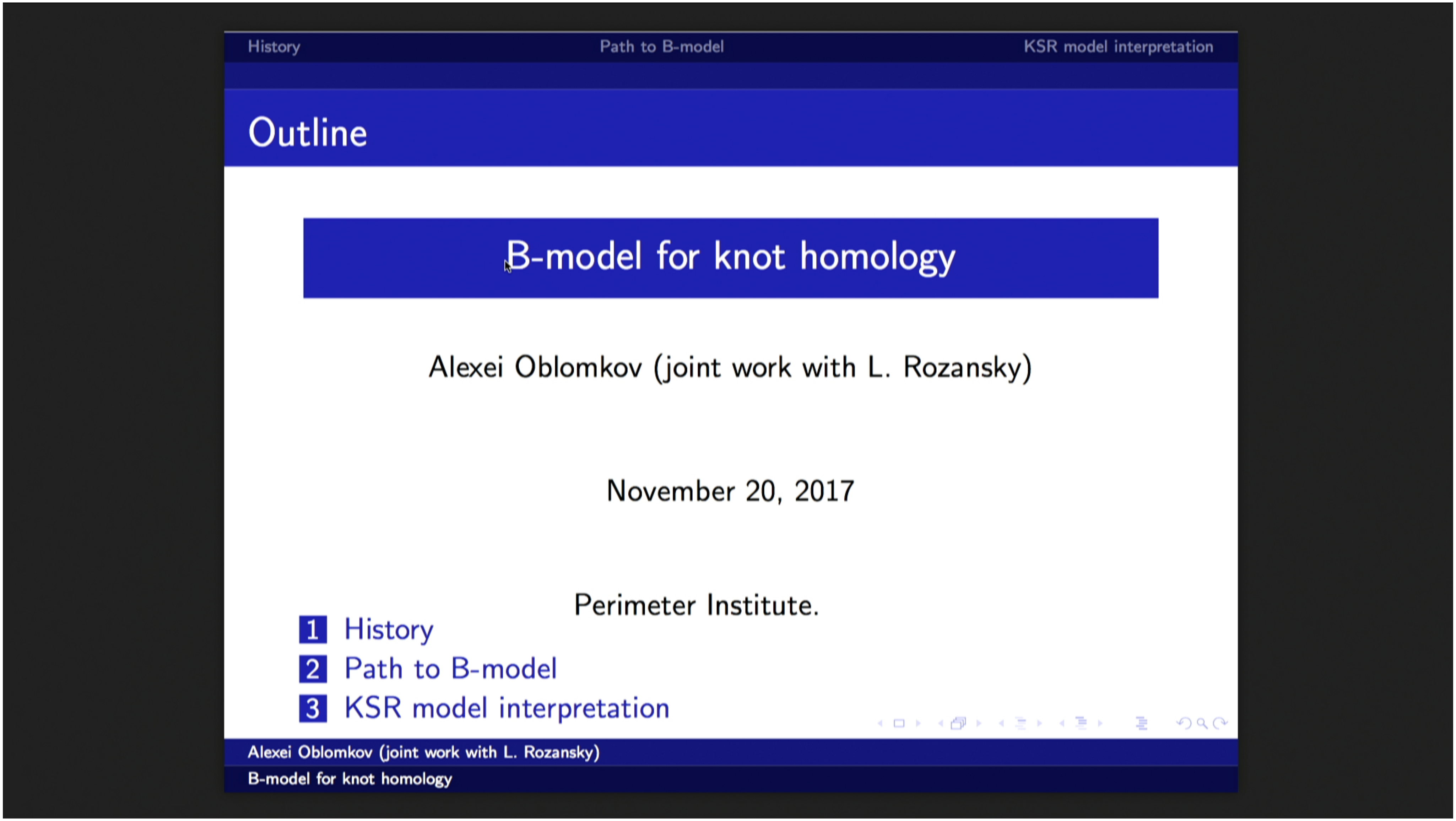
Abstract: <p>Talk is based on the joint work with Lev Rozansky. In my talk will outline a construction that provides complex C_b of coherent sheaves on the Hilbert scheme of n points on the plane for every n-stranded braid b. The space of global sections of C_b is a categorification of the HOMFLYPT polynomial of the closure $L(b)$ of the braid. I will also present a physical interpretation of our construction as a particular case of Kapustin-Saulina-Rozansky 3D topological field theory.</p>


History
Path to B-model
KSR model interpretation
Outline
B-model for knot homology
Alexei Oblomkov (joint work with L. Rozansky)
November 20, 2017
Perimeter Institute.
1 History
2 Path to B-model
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B-model for knot homology

Jones polynomial

Jones polynomial, V. Jones 1984

$$V(O) = q^{1/2} + q^{-1/2},$$

$$q^{-1}V(L_+) - qV(L_-) = (q^{1/2} - q^{-1/2})V(L_0),$$

Physics of knots

- Witten provided an interpretation of Jones polynomial as $2+1$ quantum Yang Mills, 1989

Khovanov homology

$$\mathrm{Kh}(4_1) = q^{-5}t^{-2} + q^{-1}t^{-1} + q^1t^0 + qt + q^5t^2$$

Theorem (Khovanov 2000)

For every link L there are graded spaces $H^*_{Kh}(L)$ such that

$$\sum_i (-1)^i \dim_q(H^i_{Kh}(L)) = V(L).$$

HOMFLY-PT homology

J. Hoste, A. Ocneaunu, K. Millet, P. Freyd, W. Lickorich; J. Przytycki, P. Traczyk; V. Jones 1985

HOMFLY-PT polynomial

$$P(O) = (a^{-1} - a)/(q^{1/2} - q^{-1/2}),$$

$$a^{-1}P(L_+) - aP(L_-) = (q^{1/2} - q^{-1/2})P(L_0).$$

Theorem (Khovanov-Rozansky, 2007, 2008)

For every link L there are doubly graded spaces $H^*_{KhR}(L)$ such that

$$P(L) = \sum_i (-1)^i dim_{q,a} H^i_{KhR}(L).$$

Braids and links

Elements σ_i, $i = 1, \ldots, n-1$ generate Br_n

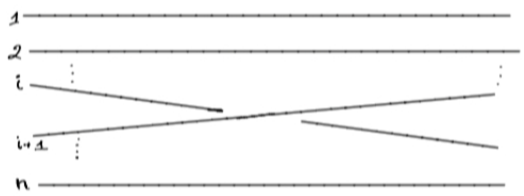

Figure: Generator σ_i

Geometric construction for KhR homology

Geometric version of the construction of KhR homology, Williamson, Webster 2009

$$Br_n \ni \beta \mapsto \Phi_\beta \in Perv(B_n \backslash GL_n / B_n),$$

$$H^*_{KhR}(L(\beta)) = (H^*(GL_n/B_n^\Delta, \Phi_\beta), d_{chr}).$$

Here the homology $H^*(GL_n/B_n^\Delta)$ acquires double grading from two weight filtration and d_{chr} is the chromotagraphic differential.

A/B model?

$$Pevr(B_n \backslash GL_n / B_n) = B_n\text{—equivariant constructible sheaves on } Fl_n =$$

$$\mathrm{Fuk}_{B_n}(T^* Fl_n) = A - \text{model for } T^* Fl_n$$

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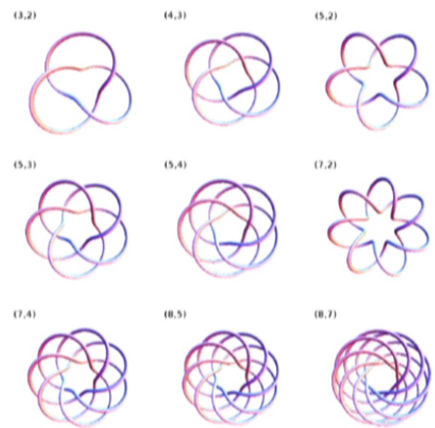
Question

What is the B-model version of the construction? How we can compute KhR homology with coherent instead of constructible sheaves?

Torus knots

Mathematicians need some examples to make a reasonable guess for B-model. Torus links are in some sense exactly solvable and provide lots of data for a guess.



Torus knots

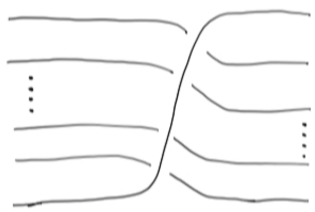


Figure: $cox_n \in Br_n$

$$cox_n = \sigma_1 \cdot \sigma_2 \cdots \sigma_{n-1}.$$

$$T_{m,n} = L(cox_n^m).$$

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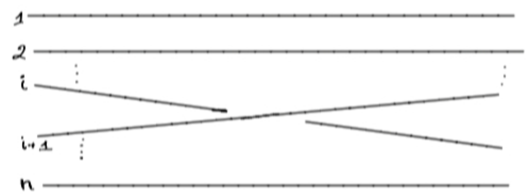

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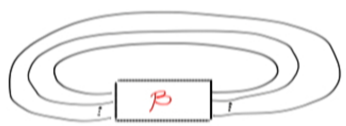

Figure: Closure $L(\beta)$ of the braid β

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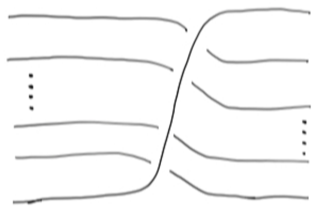

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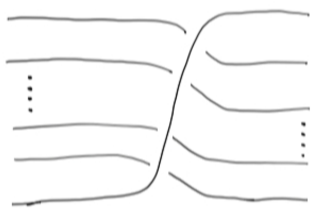

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Conjecture (Gorsky, O. Rasmussen, Shende, 2012, Aganagic Shakirov, 2011)

$$H^*_{KhR}(T_{n,1+nk}) = H^0(Z, \Lambda^\bullet \mathcal{B} \otimes L^k), \quad Z \subset Hilb_n(\mathbb{C}^2).$$

Hilbert schemes

Definition

$Hilb_n(\mathbb{C}^2)$ is the manifold that parameterizes ideals $I \subset \mathbb{C}[x, y]$ of codimension n.

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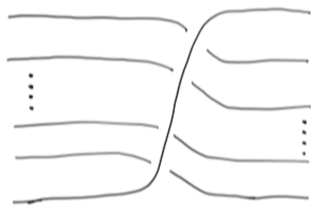
$$HC : Hilb_n(\mathbb{C}^2) \to Sym^n\mathbb{C}^2, \quad HC(I) = supp(\mathbb{C}[x,y]/I).$$

$$Z = HC^{-1}(n \cdot (0,0)), \quad \mathcal{B}^\vee|_I = \mathbb{C}[x,y]/I, \quad L = \det(\mathcal{B}).$$

The $T_{sc} = \mathbb{C}^* \times \mathbb{C}^*$ action on $\mathbb{C}^2$ induces the action on $Hilb_n(\mathbb{C}^2)$, hence double grading on $H^i(Z, L^k \otimes \Lambda^m \mathcal{B})$.

Torus knots



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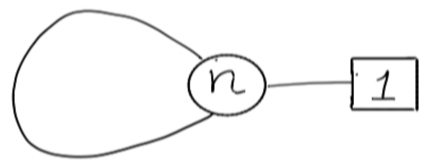
Quiver



$Hilb_n(\mathbb{C}^2)$ as quiver variety

$$\mu : T^*\mathfrak{gl}(n) \to \mathfrak{gl}(n), \quad \mu(X, Y) = XY - YX,$$
$$Hilb_n(\mathbb{C}^2) = \mu^{-1}(0)^{stab}/GL_n.$$

Conjecture (Gorsky,O., Rasmussen, Shende 2012; Gorsky, Negut, Rasmussen 2016)

There is (a canonical) way to construct for $\beta \in Br_n$ there is $\mathcal{F}_\beta \in D^{coh}_{T_{sc}}(FHilb_n(\mathbb{C}^2))$ such that

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Theorem (O., Rozansky 2016)

The conjecture is true if we adjust slightly $D^{coh}_{T_{sc}}(\dots)$

Free Hilbert scheme

$\mathfrak{b}, \mathfrak{n}$ are upper and strictly upper triangular matrices.

$$FHilb_n^{free}(\mathbb{C}) = \{(X, Y, v) \in \mathfrak{b} \times \mathfrak{n} \times V | \mathbb{C}\langle X, Y \rangle v = V\}/GL_n.$$

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$$D^{per}_{T_{sc}} = \{\dots \xrightarrow{d_1} \mathcal{C}_0 \xrightarrow{d_0} \mathcal{C}_1 \xrightarrow{d_1} \mathcal{C}_0 \xrightarrow{d_0} \dots | \mathcal{C}_i \text{ are coherent } \}.$$

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Theorem (O., Rozansky 2016)

For every $\beta \in Br_n$ there is $\mathcal{F}_\beta \in D^{per}_{T_{sc}}(FHilb_n^{free}(\mathbb{C}))$ such that

$$supp(\mathcal{H}^\bullet(\mathcal{F})) \subset FHilb_n(\mathbb{C}) \text{ and}$$

$$H^*(\beta) = \mathbb{H}(\mathcal{F}_\beta \otimes \Lambda^* \mathcal{B}) \text{ is HOMFLY-PT homology of } L(\beta).$$

Partial twists

Figure: Element $\delta_i \in Br_n$

$$Tw_k = \delta_n \cdots \delta_k$$

is a full twist on last $n-k$ strands.

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Coxeter links

$$\pi : FHilb_n \to Hilb_n, \quad FZ = \pi^{-1}(Z).$$

Theorem (O., Rozansky 2017)

$$\Phi_{cox_n} = [\mathcal{O}_{FZ}]^{vir}.$$

There is an extension of the theorem to the case of

$$cox_S = \vec{\prod}_{i \in S} \sigma_i, \quad S \subset \{1, \dots, n-1\}.$$

Thus homology of the *Coxeter link* $L(cox_S \cdot \delta^{\vec{k}})$ is given by the homology of $[\mathcal{O}_{FZ_S}]^{vir} \otimes \mathcal{L}^{\vec{k}}$.

Coxeter links

How large is the class of Coxeter links?

Powers of the full twist

If $S = \emptyset$ then $cox_S = 1$ and $L(\prod_i \delta_i^m) = T_{n,mn}$.

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If $S=\{1,\dots,n-1\}$ then $cox_S=cox_n$. Set $k_i=\lfloor\frac{im}{n}\rfloor-\lfloor\frac{(i-1)m}{n}\rfloor$ then $L(cox_n\prod_i \delta_i^{k_i})=T_{m,n}$, for $(m,n)=1$.

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We expect to have some explicit formulas for links that are obtained from unknot by the Coxeter cabling procedure.

KSR outline

Kapustin, Saulina and Rozansky proposed a realization of the 3D topological field theory, 2008.

Three-category $3Cat_{sym}$

$$Obj(3Cat_{sym}) = \{\text{holomorphic symplectic manifolds}\}$$

$$Hom(X, Y) = \{(F, L, f : F \to L), L \subset X \times Y \text{ is Lagrangian}\}.$$

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$(F, L, f) \in Hom(X, Y)$ $(G, L', g) \in Hom(Y, W)$ compose to

$$(H, L'', h), \quad H := (F \times W) \times_{X \times Y \times W} (X \times G)$$

and $h : H \to X \times W$, $L'' = h(H)$.

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For $(F, L, f), (F', L', f') \in Hom(X, Y)$ we have

$$Hom((F, L, f), (F', L', f')) := D^{per}(F \times_{X \times Y} F'),$$

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For $(Z, w) \in Hom(X, Y)$, $(Z', w') \in Hom(Y, W)$:

$$(Z, w) \circ (Z', w') = (Z \times Y \times Z', w' - w) \in Hom(X, W).$$

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For $(Z, w), (Z', w') \in Hom(X, Y)$ we have

$$Hom((Z, w), (Z', w')) = MF(X \times Z \times Z' \times Y, w' - w).$$

$3Cat_{sym}$ vs $3Cat_{man}$

Functor $3Cat_{man} \to 3Cat_{sym}$

$$X \mapsto T^*X,$$
$$(Z, w) \mapsto (F_w, L_w, \pi)$$

$$Z \times T^*X \supset F_w \ni (z, x, p) \text{ if } \partial_z w(z, x) = 0, \quad p = \partial_x w(z, x).$$

Let impose condition on (Z_i, w_i): $Crit_w \subset \{w = 0\}$, then we have

$$MF(X \times Z_1 \times Z_2 \times Y, w_1 - w_2) \to D^{per}(F_{w_1} \times_{T^*(X \times Y)} F_{w_2}).$$

Main example

$3Cat\ \mathfrak{gl}$

$$Obj = \{\mathfrak{gl}_n, n \in \mathbb{Z}_{\geq 0}\},$$

$$Hom(\mathfrak{gl}_n, \mathfrak{gl}_m) = \{Z \text{ with Hamiltonian } GL_n \times GL_m \text{ action}\}$$

$\mathfrak{gl} \to 3Cat_{man}$

$$Hom(\mathfrak{gl}_n, \mathfrak{gl}_m) \ni Z \mapsto (Z, w(x, z, y) = \mu_n(z)(x) - \mu_m(z)(y)).$$

Moment maps: $\mu_n : Z \to \mathfrak{gl}_n^*$, $\mu_m : Z \to \mathfrak{gl}_m^*$

Main example

Let a take $Z = T^*Fl$. $T^*Fl = \{(g, Y) \in GL_n \times \mathfrak{n}\}/B$

$$\mu(g, Y) = Ad_g(Y), \quad \phi = (Z, Tr(XAd_g Y)) \in Hom(\mathfrak{gl}_n, \mathfrak{gl}_0).$$

$Hom(\phi, \phi)$

$$MF_n = MF_{B^2}(\mathfrak{gl}_n \times G^2 \times \mathfrak{n}^2, W),$$

$$W(X, g_1, Y_1, g_2, Y_2) = Tr(X(Ad_{g_1} Y_1 - Ad_{g_2} Y_2)).$$

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Theorem (O.-Rozansky, 2017)

For any n there is group homomorphism:

$$\Psi : \quad Br_n^{aff} \to (MF_n, \star).$$

Affine Braid group

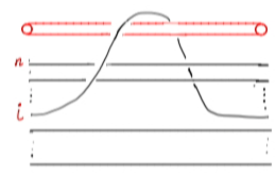

Figure: New element $\Delta_i \in Br_n^{aff}$

Affine Braid group

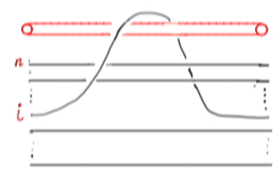

Figure: New element $\Delta_i \in Br_n^{aff}$

Forgetful map: $Br_n^{aff} \to Br_n$

$$\mathfrak{for} : \Delta_n \mapsto 1, \quad \Delta_k \mapsto \delta_k.$$

Framing

$$X = \mathfrak{gl}_n \times G^2 \times \mathfrak{n}^2.$$

Framed space

$$X^{fr} = \{(X, g_1, Y_1, g_2, Y_2, v) | \mathbb{C}\langle Ad_{g_1}^{-1}(X), Y_1\rangle v = \mathbb{C}^n\}$$

$$fr : X^{fr} \to X, \quad fr^* : MF_n \to MF_n^{fr}.$$

Theorem

$$fr^* \circ \Psi = \Psi \circ \mathfrak{for}.$$

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Knot homology

Embedding $j : FHilb^{free} \to X^{fr}$

$$(X, Y) \mapsto (X, 1, Y, 1, Y).$$

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Main construction

$$\mathcal{F}_\beta = j^*(\Psi(\beta)) \in MF(FHilb^{free}, 0) = D^{per}(FHilb^{free}).$$

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$$\mathcal{F}_\beta = j^*(\Psi(\beta)) \in MF(FHilb^{free}, 0) = D^{per}(FHilb^{free}).$$

Theorem (O.-Rozansky 2016)

The triply graded vector space

$$\mathbb{H}(\mathcal{F}_\beta \otimes \Lambda^* \mathcal{B}) \textit{ is an isotopy invariant of } L(\beta).$$

Where is *FHilb*?

$$\text{Knorrer reduction: } MF_n = MF_{B_n^2}(\mathfrak{b} \times G \times \mathfrak{n}, \bar{W}),$$
$$\bar{W}(X, g, Y) = Tr(XAd_g(Y)).$$

$$Crit(\bar{W}) : [X, Ad_g Y] = 0, \quad Ad_g Y \in \mathfrak{n}, \quad Ad_g X \in \mathfrak{b}.$$

Where is *FHilb*?

$$\text{Knorrer reduction: } MF_n = MF_{B_n^2}(\mathfrak{b} \times G \times \mathfrak{n}, \bar{W}),$$
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$\bar{j}^*(\bar{\mathcal{F}}_\beta) \in MF_B(FHilb_n^{free}, 0)$ has homology support in $FHilb_n$

$gl(m|n)$ homology

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Section

$\phi_{m|n} \in H^0(FHilb_n^{free}(\mathbb{C}), \mathcal{B}^\vee)$, $\phi_{m|n}(X,Y,v) = X^m Y^n v$

Differential

$i_{\phi_{m|n}} : \Lambda^i \mathcal{B} \to \Lambda^{i-1}\mathcal{B}$, $d_{m|n} : \mathcal{F}_\beta \otimes \Lambda^i \mathcal{B} \to \mathcal{F}_\beta \otimes \Lambda^{i-1}\mathcal{B}$.

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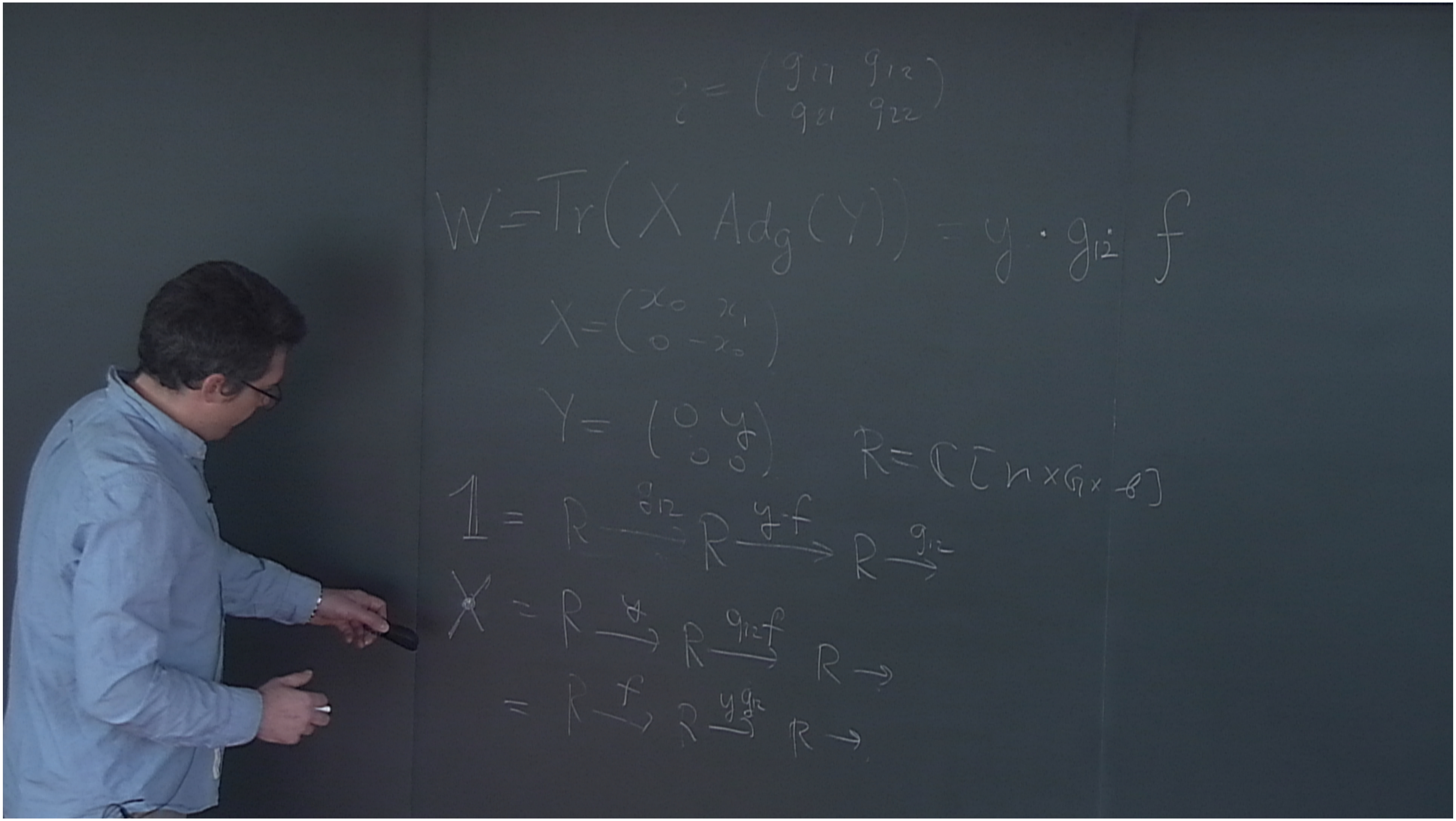
Let $d_{\mathcal{F}}$ be the differential of $\mathcal{F}_\beta \in D^{per}(FHilb_n^{free}(\mathbb{C}))$ and

$$\mathbb{H}_{m|n}(\beta) := H(\mathcal{F}_\beta \otimes \Lambda^\bullet \mathcal{B}, d_{\mathcal{F}} + d_{m|n})$$

Virtual sheaves

$FZ \subset \mathfrak{b} \times \mathfrak{n}$ and defined by the equations

$$X_{ii} = X_{i+1,i+1}, [X, Y]_{i,j} = 0, \quad j - i > 1.$$


W = Tr(X Ad_g(Y)) = y · g_12 f
X = (x_0 x_1 / 0 −x_0)
Y = (0 y / 0 0)
R = ℂ[n × G × B]