Title: Nets vs. factorization algebras: lessons from the comparison

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Abstract: <p>Making perturbative quantum field theory (QFT) mathematically rigorous is an important step towards understanding how the non-perturbative framework should look like. Recently, two approaches have been developed to address this issue: perturbative algebraic quantum field theory (pAQFT) and the factorization algebras approach developed by Costello and Gwilliam. The former works primarily in Lorentzian signature, while the later in Euclidean, but there are many formal parallels between them. In this talk I will show the recent results relating these two frameworks and discuss the consequences for future research (joint work with Gwilliam).</p>

Nets vs. factorization algebras

AQFT, Haag-Kastler ['64]

LC QFT, Brunetti - Fredenhagen - Verch

BV version of the above: 2011, Fredenhagen & R.

$\gtrsim$ 2011

Costello - Gwilliam

1) <u>Def</u>: A QFT model on a spacetime $\mathcal{M}$ (Lorentzian) is a functor from $\mathrm{Caus}(\mathcal{M})$ (family of opens that are causally convex)

↳ contains all its causal curves

to $\mathrm{Alg}^*(\mathrm{Nuc}_{\hbar})^{inj}$ → with injective morphisms

$\hbar$ → formal power series

Nuc ↳ top. v. spaces w/ nice topology and <u>unique</u> tensor product

Such that Einstein causality holds:

$\mathcal{O}_1, \mathcal{O}_2 \in \mathrm{Caus}(\mathcal{M})$ that are spacelike $\mathcal{O}_1 \times \mathcal{O}_2$

$$[\mathfrak{A}(\mathcal{O}_1), \mathfrak{A}(\mathcal{O}_2)]_{\mathcal{O}} = \{0\}, \quad \text{where } \mathcal{O}_1, \mathcal{O}_2 \subset \mathcal{O}.$$

<u>Def</u>: Classical theory: $\mathfrak{P} : \mathrm{Caus}(\mathcal{M}) \to \mathrm{Poi}^*(\mathrm{Nuc})^{(\mathrm{in})}$

s.t. Einstein causality holds.

Assume that $\mathcal{M}$ has Cauchy surfaces (globally hyperbolic)

<u>Def</u>: $\mathfrak{A}(\mathfrak{P})$ satisfies the time-slice axiom if

for $\mathcal{N} \in \mathrm{Caus}(\mathcal{M})$ then $\mathfrak{A}(\mathcal{N}) \cong \mathfrak{A}(\mathcal{O})$

Σ Cauchy surface of $\mathcal{O}$ $\left(\mathfrak{P}(\mathcal{N}) \cong \mathfrak{P}(\mathcal{O})\right)$

→ To introduce interaction it is useful to have time-ordered products.

→ Related to the Dyson formula for interacting fields.

Def: Time-ordered prod. is a functor

$$\mathfrak{A}_T : \mathrm{Caus}(\mathcal{M}) \longrightarrow \mathrm{CAlg}^*(\mathrm{Nuc}_{\mathbb{R}})^{inj}$$

such that:

1) $\mathfrak{A}_T(\mathcal{O}) \cong \mathfrak{A}(\mathcal{O})$ as vector spaces
2) $\mathfrak{A}_T(\mathcal{O}) \cong \mathfrak{P}(\mathcal{O})$ as commutative algebras
3) For $F \in \mathfrak{A}(\mathcal{O}_1)$, $G \in \mathfrak{A}(\mathcal{O}_2)$ such that $\mathcal{O}_1$ is in the future of $\mathcal{O}_2$

Def: Time-ordered prod. is a functor

$$\mathfrak{A}_T : \mathrm{Caus}(\mathcal{M}) \longrightarrow \mathrm{CAlg}^*(\mathrm{Nuc}_{tn})^{(in)}$$

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3) a) For $F \in \mathfrak{A}(\mathcal{O}_1)$, $G \in \mathfrak{A}(\mathcal{O}_2)$ such that $\mathcal{O}_1$ is in the future of $\mathcal{O}_2$

$$F \cdot_T G = F \star G$$

belongs to $\mathfrak{A}_T$

belongs to $\mathfrak{A}$

b) $\mathcal{O}_2$ is in the future of $\mathcal{O}_1$, then

$$F \cdot_T G = G \star F$$

II. Costello-Gwilliam approach

<u>Def:</u> A classical field theory is 1-shifted Poisson algebra $\mathcal{P}$ in fact. alg.

<u>Def:</u> A quantum theory $\mathcal{A}$ is Beilinson-Drinfeld (BD) algebra in fact. alg.

III Statement of the results
(scalar field without renormalization)

Thm $\mathfrak{P}|_{\mathrm{Caus}(\mathcal{M})} \Rightarrow \mathfrak{f}$ is a quasi-isomorphism up to completion

$\overline{H^*(\mathfrak{P})|_{\mathrm{Caus}(\mathcal{M})}} \simeq \mathfrak{f}$ is a natural iso scalar field wi... ...alization)

Thm 1. $\mathcal{P}|_{\mathrm{Caus}(M)} \Rightarrow \mathcal{F}$ is a quasi-isomorphism up to completion

$H^*\left(\overline{\mathcal{P}|_{\mathrm{Caus}(M)}}\right) \cong \mathcal{F}$ is a natural iso as commutative alg.

Thm 2

$$\mathcal{A}|_{\mathrm{Caus}(M)} \Rightarrow \mathfrak{A} \quad \text{— " —}$$

$$H^*(\mathcal{A}|_{\mathrm{Caus}(M)}) \cong \mathfrak{A} \quad \text{as vector spaces}$$

Thm 3. Restrict $\mathcal{A}|_{\mathrm{Caus}(M)} \Rightarrow \mathfrak{A}|_{\mathrm{Caus}(M)}$ as algebras

$\mathrm{Caus}(M)$ ↓ tubular neigh. of a Cauchy surface

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IV. Proof of comparison theorems $\mathcal{M}=(M,g)$ (joint with Gwilliam)

configuration space: $\mathcal{E}(\mathcal{M}) = C^\infty(M,\mathbb{R})$

$\mathcal{D}(\mathcal{M}) = C^\infty_c(M,\mathbb{R})$

Equations of motion: $P\varphi = 0$, $\varphi \in \mathcal{E}(\mathcal{M})$

$P = -(\Box + m^2)$

→ Space of distributional solutions $V \subset \mathcal{D}'(\mathcal{M})$

→ Observables: $\mathrm{Sym}_{alg.}(V^*)$ ⇝ do it in dg algebras

Rem: The AQFT definitions generalize to dg versions:

1) replace everything by chain complexes and drop "inj".

2) axioms up to quasi-iso.

is a functor

2) axioms ...

Def: Linear observables $f \in \mathcal{D}(M)$

$$\mathcal{O}_f(\varphi) = \int_M \varphi(x) f(x) \, d\mu_g(x)$$

Polynomials: $\mathrm{Sym}_{alg}(\mathcal{D}(M))$

Regular polynomial polyvector fields: $\widetilde{PV} = \mathrm{Sym}_{alg}(\mathcal{D}(M) \oplus \mathcal{D}(M)[1])$

1st completion: $PV^{-*}(U) = \bigoplus_{n \geq 0} \mathcal{D}(U^n \times U^k)_{S_n \times S_k}$

2^{nd} completion: Consider the space $\mathfrak{P}\mathcal{V}_{reg}(\mathcal{M})$ which contains also infinite series convergent in a certain topology.

... $\mathcal{F}V_{reg}(\mathcal{M})$

which contains also infinite series convergent in a certain topology.

Differential: $\delta_S(f_1 \dots f_n \, g_1 \wedge \dots \wedge g_m) =$

$$\mathrm{Sym}^n_{alg}(\mathcal{D}(\mathcal{M})) \otimes \Lambda^m_{alg}(\mathcal{D}(\mathcal{M}))$$

$$= \sum_{i=1}^{m} (-1)^{i-1} P g_i \, f_1 \dots f_n \, g_1 \wedge \dots \wedge \hat{g}_i \wedge \dots g_m$$

$$C^\bullet \mathcal{P}(\mathcal{M}) = (\mathcal{P}\mathcal{V}_{reg}, \delta_s)$$

$$C^\bullet \mathcal{P}(\mathcal{O}) = (PV, \delta_s)$$

forgetfull to commutative algebras

"proves" Thm 1

The algebraic structures

1) $\mathfrak{A}_T(\mathcal{O})$ is related to $\mathcal{A}(\mathcal{O})$, $\mathcal{O} \in \mathrm{Caus}(\mathcal{M})$

Note: we have Green functions for P of a particular type:

$G^A / G^R \rightarrow$ advanced / retarded

$$G^D = \frac{1}{2}\left(G^A + G^R\right), \quad G^c = G^R - G^A$$

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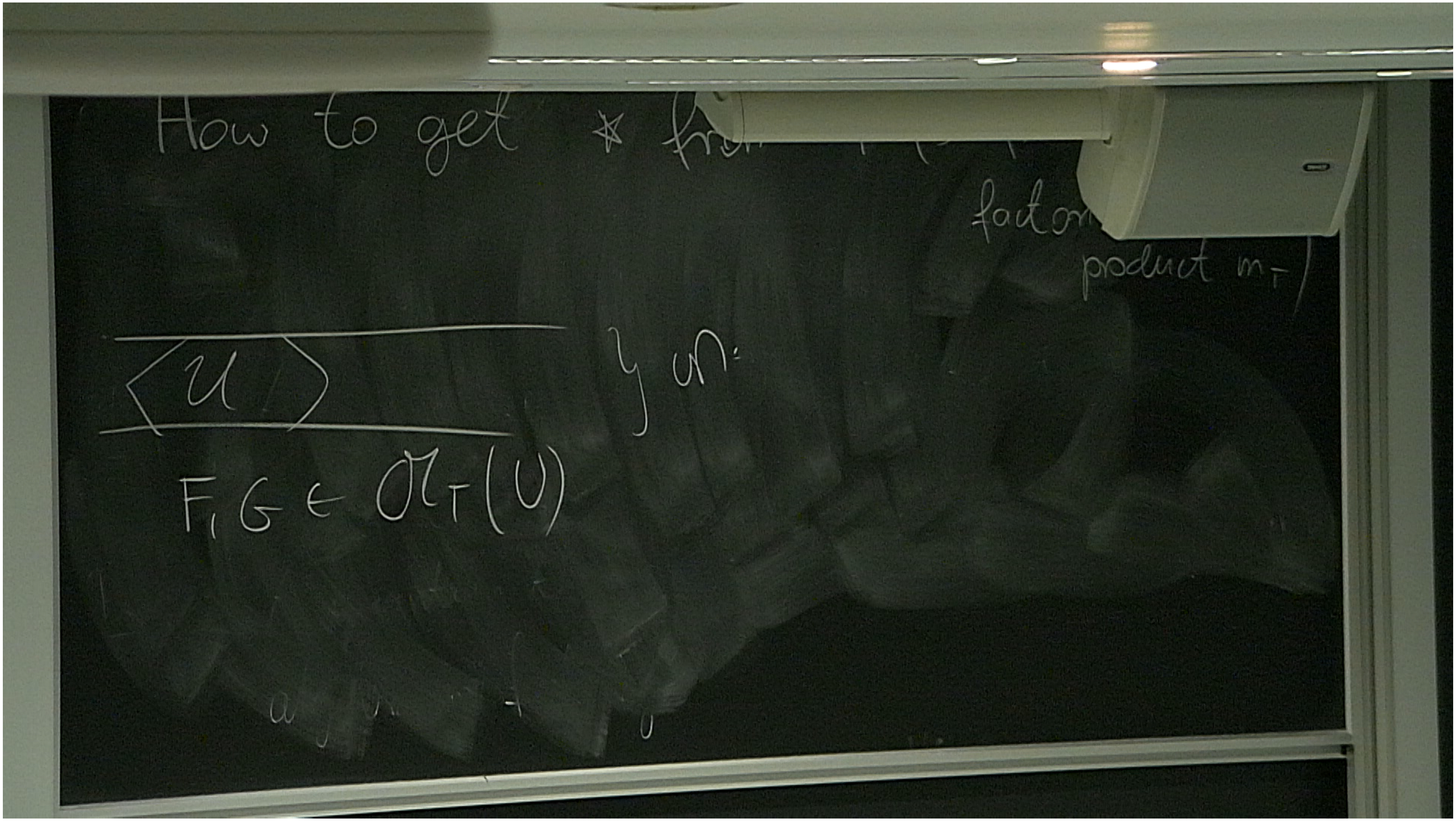
2) $\ldots = e^{\ldots}$, $\delta_{GD} \ldots (\delta, \delta\varphi \ldots$

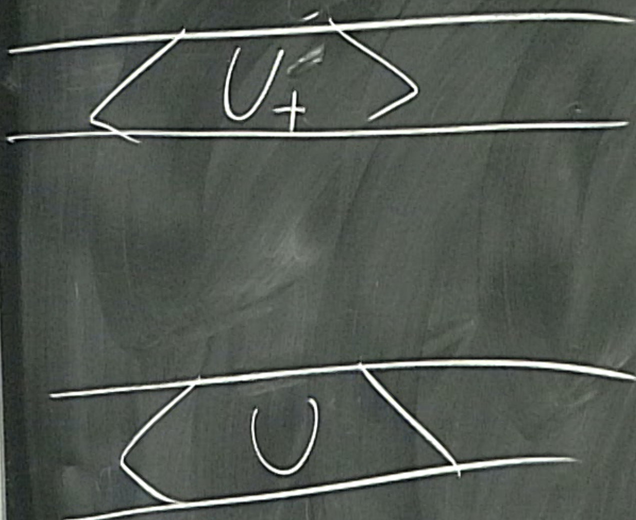
$$(PV[[\hbar]], \delta_s) \xleftarrow{J} (PV[[\hbar]], \delta_s + i\hbar \ldots$$

" " " " BV L…

<u>products:</u> $\cdot_T$

$$\mathcal{A}|_{\mathrm{Caus}(M)} \Rightarrow \mathfrak{A}_T \quad \text{quasi iso}$$


How to get ⋆ from
factori
product m_T)
$\langle \mathcal{U} \rangle$
$F, G \in \mathfrak{A}_T(U)$



there is a map $\beta_+ : (\mathbb{R}\mathcal{V}_{reg}^{\mathcal{U}}[[\hbar]], \delta_S)$

$$\downarrow$$

$$(\mathbb{R}\mathcal{V}_{reg}(\mathcal{U}_+)[[\hbar]], \delta_S)$$

which is quasi iso.

$$(F \star G) \doteq m_1(\beta_+ F, \ G) = \dots = F \star G$$

mod image of δ_S.