Title: Fluffing Extremal Kerr, semi-classical microstates of Kerr black hole

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Abstract: <p>I present a one-function family of solutions to 4D vacuum Einstein equations. While all diffeomorphic to the extremal Kerr black hole, they are labeled by well-defined conserved charges and are hence distinct geometries. Out of the appropriate combination of these charges, we construct a Virasoro algebra and consistency conditions lead us to a proposal for identifying extreme Kerr black hole microstates, dubbed as extreme Kerr fluff. Counting these microstates, we correctly reproduce the extreme Kerr black hole entropy.</p>

Fluffing Extreme Kerr

Semi-Classical Microstates of Kerr Black Holes

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Hamiltonian Formalism

- The construction of Hamiltonian formulation involves an explicit choice of time direction. This breaks the covariant form of general relativity.
- *From the relativistic point of view we are thus singling out one particular observer and making our whole formalism refer to the time for this observer. That, of course, is not really very pleasant to a relativist, who would like to treat all observers on the same footing. However, it is a feature of the present formalism which I do not see how one can avoid if one wants to keep to the generality of allowing the Lagrangian to be any function of the coordinates and velocities.* P. A. M. Dirac

Hamiltonian Formalism, A very brief review

- The cotangent bundle of the configuration manifold Γ, is naturally endowed with a canonical 1-form θ, which $\Omega = d\theta$ is non-degenerate, and therefore defines a symplectic form on Γ.
- Given a Hamiltonian function H on Γ, Hamiltonian vector field X_H is defined by $dH = \Omega(X_H, \ )$.
- Darboux's theorem: *There is no local invariant in symplectic geometry.*

$$\Omega = \sum_i dp^i \wedge dq^i$$

- In the Darboux base $X_H = \frac{\partial H}{\partial p_i}\partial_{q_i} - \frac{\partial H}{\partial q_i}\partial_{p_i}$ and therefore canonical equations reduce to

$$\dot{q}^i = \partial_{p_i} H \qquad \dot{p}^i = -\partial_{q_i} H$$

Hamiltonian Formalism, A very brief review

- Ω is non-degenerate $\Longrightarrow \Omega^{ab} = (\Omega_{ab})^{-1}$
- Poisson bracket between two functions:
  $\{f, g\} = \Omega(X_f, X_g) \Longrightarrow \{f, g\} = \Omega^{ab}\partial_a f \partial_b g$
- Dynamical equations: $\frac{df}{dt} = \{f, H\}$
- Liouville's theorem: $\mathcal{L}_{X_H}(\Omega \wedge \Omega \cdots \Omega) = 0$

Hamiltonian Formalism, Symplectic symmetry

- If $\mathcal{L}_Y\Omega = 0$, Y is generator of a symplectic symmetry.

$$\mathcal{L}_Y\Omega = Y \cdot d\Omega + d(Y \cdot \Omega) = d(Y \cdot \Omega) = 0 \longrightarrow Y \cdot \Omega = dH_Y$$

$$Y^a = \Omega^{ab}\partial_b H_Y \equiv \partial^a H_Y$$

- On-shell value of H_Y is called charge of Y over that solution.
- Conserved charges $\{H, H_Y\} = 0$.

Hamiltonian Formalism, Algebra of Symplectic Symmetries

- Symplectic Symmetries form an algebra

$$\mathcal{L}_{Y_i}\Omega = \mathcal{L}_{Y_j}\Omega = 0 \quad \Rightarrow \quad \mathcal{L}_{[Y_i,Y_k]}\Omega = \mathcal{L}_{Y_i}\mathcal{L}_{Y_j}\Omega - \mathcal{L}_{Y_j}\mathcal{L}_{Y_i}\Omega = 0$$

- Fundamental Theorem of symplectic symmetries:
  The algebra of generators of symplectic symmetries (H_{Y_i}) through the Poisson bracket is the same as the algebra of symplectic symmetries (Y_i) through the Lie bracket, up to a central extension. $\{H_{Y_i}, H_{Y_j}\} = H_{[Y_i,Y_j]} + C, \quad dC = 0.$

$$\partial_a H_{[Y_i,Y_j]} = \Omega_{ab}[Y_i, Y_j]^b = \Omega_{ab}\mathcal{L}_{Y_i} Y_j^b = \mathcal{L}_{Y_i}\Omega_{ab} Y_j^b = \mathcal{L}_{Y_i}(\mathrm{d}H_{Y_j})_a$$

$$(\mathrm{d}(Y_i \cdot \mathrm{d}H_{Y_j})_a = \partial_a(\Omega^{bc}\partial_b H_{Y_i}\partial_c H_{Y_j} = \partial_a(\{H_{Y_i}, H_{Y_j}\})$$

$$\{H_Y, C\} = 0$$

Covariant Phase-Space Formalism, set up

- $\mathcal{F}$: space of all field configurations Φ with given b.c
- $\tilde{\mathcal{F}} \subset \mathcal{F}$: on-shell field configurations
- $\delta\Phi$: An infinitesimal field perturbation $\rightarrow$ tangent vector to $\mathcal{F}$
- The idea of *CPSM* is taking $\tilde{\mathcal{F}}$ as a phase-space (symplectic manifold).
- How to construct a symplectic structure?

Pre-Symplectic Structure

- Starting with an action
  $S = \int L, \qquad L(\Phi) = \mathcal{L}(\Phi)\ \frac{\sqrt{-g}}{d!}\epsilon_{\mu_1,\cdots\mu_d}\ dx^{\mu_1} \wedge \cdots dx^{\mu_d}$

$$\delta S = 0 \to \int E(\Phi)\delta\Phi + d\Theta(\Phi, \delta\Phi) = 0$$

$$\Theta = \sqrt{-g}\theta^{\mu}\frac{1}{(d-1)!}\epsilon_{\mu\mu_1,\cdots\mu_{d-1}}\ dx^{\mu_1} \wedge \cdots dx^{\mu_{d-1}}$$

- Example:

$$\mathcal{L} = \frac{R}{16\pi G_N} \to E(\Phi)\delta\Phi = \frac{\sqrt{-g}}{16\pi G_N d!}\epsilon_{\mu_1..\mu_d}(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R)\delta g^{\mu\nu}dx^{\mu_1}\cdots\wedge dx^{\mu_d}$$

$$\to \theta^{\mu} = \frac{1}{16\pi G_N}(\nabla_{\alpha}\delta g^{\mu\alpha} - \nabla^{\mu}\delta g^{\alpha}_{\alpha})$$

Pre-Symplectic Structure

- Ambiguities:
  1. under $L \to L + d\mu$
     E.o.M does not change but $\Theta \to \Theta + \delta\mu$
  2. $\Theta \to \Theta + dY$

Pre-Symplectic Structure

- pre-symplectic current ω:

$$\omega \equiv \omega[\delta_1\Phi, \delta_2\Phi] = \delta_1\Theta(\delta_2\Phi, \Phi) - \delta_2\Theta(\delta_1\Phi, \Phi)$$

$$L \to L + d\mu, \quad \omega \to \omega + (\delta_1\delta_2 - \delta_2\delta_1)\mu = \omega$$

$$\Theta \to \Theta + dY, \ \omega \to \omega + d(\delta_1 Y(\delta_2\Phi, \Phi) - \delta_2 Y(\delta_1\Phi, \Phi))$$

- $d\omega \approx 0$

$$d\omega = \delta_1 d\Theta(\delta_2) - (1 \leftrightarrow 2) = \delta_1(\delta_2 L - E\delta_2\Phi) - (1 \leftrightarrow 2)$$

$$= (\delta_1\delta_2 - \delta_1\delta_1)L + \delta_1 E\delta_2\Phi - \delta_2 E\delta_1\Phi \approx 0$$

- Pre-symplectic form: $\Omega = \int_\Sigma \omega$

Residual Gauge Transformation

- Residual gauge transformations $\longrightarrow$ Conserved surface charges
- It plays an important role in holographic description of gravity
- $\delta_\chi \Phi$ denotes a gauge transformation by gauge parameter χ.
  example: in E.M $\delta_\chi A = \mathrm{d}\,\chi$ or in gravitational theoreis $\delta_\chi g_{\mu\nu} = \mathcal{L}_\chi g_{\mu\nu}$
- For an infinitesimal gauge transformation $\delta_\chi \Phi$, the generator of gauge transformation H_χ is defined by $\delta_\chi \Phi = \{\Phi, H_\chi\}$.

$$H_\chi = \int^\Phi \delta H_\chi, \qquad \delta H_\chi = \int_\Sigma \omega[\delta\Phi, \delta_\chi \Phi]$$

- On-shell value of H_χ is called charge of the gauge transformation

Residual Gauge Transformation

Example:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \quad \Rightarrow \quad \theta^{\mu} = F^{\mu\nu}\delta A_{\nu}$$

$$\Omega[\delta_1 A, \delta_2 A; A] = \int_{\Sigma}(\delta_1 F^{\mu\nu}\delta_2 A_{\nu} - \delta_2 F^{\mu\nu}\delta_1 A_{\nu})ds_{\mu}$$

Taking $\delta_{\epsilon}A_{\mu} = \partial_{\mu}\epsilon(x) \quad \Rightarrow$

$$\delta H_{\epsilon} = \int_{\Sigma}(\delta F^{\mu\nu}\delta_{\epsilon}A_{\nu} - \delta_{\epsilon}F^{\mu\nu}\delta A_{\nu})ds_{\mu} = \int_{\Sigma}(\delta F^{\mu\nu}\partial_{\nu}\epsilon)ds_{\mu}$$

I.B.P $\Rightarrow \delta_{\epsilon}H = \int \epsilon\delta\partial_{\nu}F^{\mu\nu} + \oint \epsilon\delta F^{\mu\nu}$. If ϵ is A-independent
$\Rightarrow H_{\epsilon} = \int_{\Sigma}\epsilon\partial_{\nu}F^{\mu\nu} + \oint_{\partial\Sigma}\epsilon F^{\mu\nu}$.
Charges are on-shell value of $H_{\epsilon} \Rightarrow Q_{\epsilon} = \oint_{\partial\Sigma}\epsilon(x)F^{\mu\nu}$

Conserved Charges in Gravitational Theories

- Lagrangian is "Invariant" under gauge trans. up to a total derivative $\delta_\chi L = dM_\chi[\Phi]$
- χ is generator of *local coordinate transformation*, which is a vector field.
- Gauge transformation is given by Lie derivative $\delta_\chi L = \mathcal{L}_\chi L = \chi \cdot dL + d(\chi \cdot L) = d(\chi \cdot L) \quad \Rightarrow \quad M_\chi = \chi \cdot L$
- $J_\chi \equiv \Theta[\delta_\chi \Phi, \Phi] - M_\chi[\Phi] \implies dJ_\chi \approx 0 \implies J_\chi = d\bar{Q}_\chi - S_\chi$
- Fundamental identity of C.P.S.M for gravity:

$$\omega[\delta\Phi, \delta_\chi\Phi; \Phi] = G_\chi + d\kappa_\chi$$

where $G_\chi = \delta S_\chi + \chi \cdot E[\Phi]\delta\Phi$ and $\kappa_\chi = \delta\bar{Q}_\chi - \chi \cdot \Theta + d(possible\ b.t)$
Therefore generator of gauge transformation H_χ obtained from

$$\delta H_\chi = \int_\Sigma \omega[\delta\Phi, \delta_\chi, \Phi] = \int_\Sigma G_\chi + \oint_{\partial\Sigma} \kappa_\chi \implies$$

$$\delta Q_\chi = \oint_{\partial\Sigma} \kappa_\chi[\delta\Phi, \Phi]$$

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Example:

- Einstein gravity in 4d: $L = \frac{R}{16\pi G_N}\frac{\sqrt{-g}}{4!}\epsilon_{\mu_1\mu_2\mu_3\mu_4}dx^{\mu_1}\wedge\cdots dx^{\mu_4}$
  defining $\delta g_{\mu\nu} = h_{\mu\nu}$ we get

$$E[\Phi]\delta\Phi = \frac{\sqrt{-g}}{16\pi G_N}(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R)h^{\mu\nu}\frac{1}{4!}\epsilon_{\mu_1\mu_2\mu_3\mu_4}dx^{\mu_1}\wedge\cdots dx^{\mu_4}$$

$$\theta^\mu = \frac{1}{16\pi G_N}(\nabla_\nu h^{\mu\nu} - \nabla^\mu h) \implies *Q^{\mu\nu}_\chi = \frac{1}{16\pi G_N}(\nabla^\nu\chi^\mu\nabla^\mu\chi^\nu)$$

$$\kappa_\chi = \frac{1}{4}\sqrt{-g}\,\kappa^{\mu\nu}_\chi\,\epsilon_{\mu\nu\alpha\beta}dx^\alpha\wedge dx^\beta$$

where

$$\kappa^{\mu\nu}_\chi = \frac{1}{16\pi G_N}(\chi^\nu\nabla^\nu h - \chi^\nu\nabla_\sigma h^{\mu\sigma} + \chi_\sigma\nabla^\nu h^{\mu\sigma} - h^{\rho\nu}\nabla_\rho\chi^\mu + \frac{1}{2}h\nabla^\nu\chi^\mu)$$

Integrability Conditions :

- It is not obvious that the infinitesimal charge defined in is really a variation of a function Q_χ over the phase space.
- Example from Thermodynamics: heat transformation δQ vs. $\delta S = \int_\gamma \frac{\delta Q}{T}$
- Theorem: The necessary and sufficient condition for charge integrability:

$$\delta_1 \delta_2 Q_\chi - \delta_2 \delta_1 Q_\chi = 0$$

- Finiteness of charges
- Conservation of charges

Examples :

- $\chi = \partial_t$ for Schwarzschild B.H metric $\Longrightarrow\ \delta Q_{\partial_t} = \delta M\ \Rightarrow Q_\chi = M$
- $\chi = \partial_\phi$ for Kerr B.H metric $\Longrightarrow\ \delta Q_{\partial_\chi} = \delta(ma)\ \Rightarrow Q_\chi = -J$

BTZ Black Hole arXiv:1607.00009, 1608.01293, 1705.06257

- Starting with an extremal BTZ B.H solution

$$ds^2 = -\frac{r^2 - 2r_0^2}{\ell^2}dt^2 + \frac{\ell^2 r^2}{(r^2 - r_0^2)^2}dr^2 - 2r_0^2\, dtd\varphi + r^2 d\varphi^2, \quad M = \ell j = \frac{r_0^2}{4G}$$

$$ds^2 = \ell^2\frac{dz^2}{z^2} - \left(rdx^+ - \ell^2\frac{f_- dx^-}{r}\right)\left(rdx^- - \ell^2\frac{f_+ dx^+}{r}\right), \quad x^\pm = \frac{t}{\ell} \pm \varphi$$

Extremal BTZ corresponds to $f_- = 0$ (or $f_+ = 0$) and $f_+ > 0$.

- Consider one-function class of solutions generated by

$$d\varphi \to J(\varphi)d\varphi$$

BTZ Black Hole Phase-Space

$$ds^2_{[J]} = -\frac{r^2 - 2r_0^2}{\ell^2}dt^2 + \frac{\ell^2 r^2}{(r^2 - r_0^2)^2}dr^2 - 2J(\varphi)r_0^2\, dtd\varphi + r^2 J^2(\varphi)d\varphi^2,$$

- In above *phase-space* consider transformations $\hat{\chi} = \epsilon(\varphi)\partial_\varphi$ and phase-dependent transformation $\eta = \frac{J_0\epsilon(\varphi)}{2\beta J(\varphi)}\partial_\varphi$
- Their Fourier modes $\hat{\chi}_n$ and η_n forms algebras

$$[\hat{\chi}_m, \hat{\chi}_n] = -i(m-n)\hat{\chi}_{m+n}, \quad [\eta_m, \eta_n]_* = 0, \quad [\hat{\chi}_m, \eta_n]_* = in\, \eta_{m+n}$$

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Twisted Sugawara Construction

- Their corresponding charges form similar algebra up to central extension

$$\{\hat{L}_m, \hat{L}_n\}_* = -i(m-n)\hat{L}_{m+n}, \quad \{\mathbb{J}_m, \mathbb{J}_n\}_* = \frac{inc}{12}\delta_{m+n}, \quad \{\hat{L}_m, \mathbb{J}_n\}_* = in\mathbb{J}_{m+n}$$

- A combination $\mathbb{L}_n = \hat{L}_n + in\mathbb{J}_n$ give

$$\{\mathbb{L}_m, \mathbb{L}_n\}_* = -i(m-n)\mathbb{L}_{m+n} - im^3\frac{c}{12}\delta_{m+n}, \quad c = \frac{3\ell}{2G_N}$$

- Using the Dirac quantisation rules $\{\ \} \to -i\,[\ ]$

$$[\mathbb{L}_m, \mathbb{L}_n] = (m-n)\mathbb{L}_{m+n} + m^3\frac{c}{12}\delta_{m+n}, \quad [\mathbb{J}_m, \mathbb{J}_n] = \frac{mc}{2}\delta_{m+n}$$

- Adjusted Lie bracket

$$[\chi(\epsilon_1;\Phi),\chi(\epsilon_2;\Phi)]_* \equiv [\chi(\epsilon_1;\Phi),\chi(\epsilon_2;\Phi)] - (\delta_{\epsilon_1}\chi(\epsilon_2,\Phi) - \delta_{\epsilon_2}\chi(\epsilon_1,\Phi))$$

Conclusion

- We have shown how the horizon fluff idea can be worked through for the BTZ and 4D extremal Kerr black hole.
- Our analysis expands upon the Kerr/CFT analysis in three important ways:
  1. our symmetry algebra is defined over the whole extremal Kerr geometry and not only in the near horizon region
  2. we have introduced the extremal Kerr phase space, our symmetries are symplectic (and not just asymptotic)
  3. besides the Vir. we have a current algebra and our symmetry generator diffeos. are all along the azimuthal angle ∂_ϕ
- Our preliminary analysis shows that similar features can be extended to other extremal black holes in higher dimensions.
- Whether similar analysis and horizon fluff proposal work for generic non-extremal Kerr geometry?

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$$\hat{L}_n = \bigcirc \sum_{p \in \mathbb{Z}} \bar{J}_{n-p} \bar{J}_p$$

Hilbert Space

- Given the sym. algebra one can construct Hilbert space of unitary representations of the algebra.

$$\forall n > 0 \quad \mathbb{J}_n|0; J_0\rangle = 0, \quad \mathbb{J}_0|0; J_0\rangle = J_0$$

$$|\{n_i\}, J_0\rangle = \prod_{\{n_i>0\}} J_{-n_i}|0, J_0\rangle$$

- For real J_0, these are black hole states.

Twisted Sugawara Construction

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Another Construction of Virasoro Algebra

- Positivity of norm implies $\mathbb{J}_n^\dagger = \mathbb{J}_{-n}$ for $n \neq 0$ but $\mathbb{J}_0$ can be either Hermitian or Anti-Hermitian.
- The Hilbert space includes states with $\mathbb{J}_0 = \pm i\nu/2$ for $\nu \in (0, 1]$.
- It is more convenient to work with operators $\mathbb{W} =: \exp(-2\int \mathbb{J}) :$
- One expects ν to take discrete values, $\nu = r/c, r = 1, 2, \cdots, c$.
- Recalling $\mathbb{W}(\varphi + 2\pi) = e^{2i\pi\nu}\mathbb{W}(\varphi)$, we have c independent $\mathbb{W}$ fields.
- In the large c limit, these fields provide a free field rep. for the Vir. algebra.
- It is more conveniently to work with $\mathcal{J}_n$ operators which are the collection of Fourier modes of the c independent $\mathbb{W}$-fields, into a single operator $\mathcal{J}_{pc+r} = \mathbb{W}^r_p$,

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$$\hat{L}_n = O \sum_{p \in \mathbb{Z}} J_{n-p} J_p$$

$J_0 \in \mathbb{R}$ BTZ (Horizon)

$J_0^2 = -\frac{1}{4}$ G-AdS$_3$ NO-Horizon

$J_0^2 \in [-\frac{1}{4}, 0)$ Conic NO-Horizon

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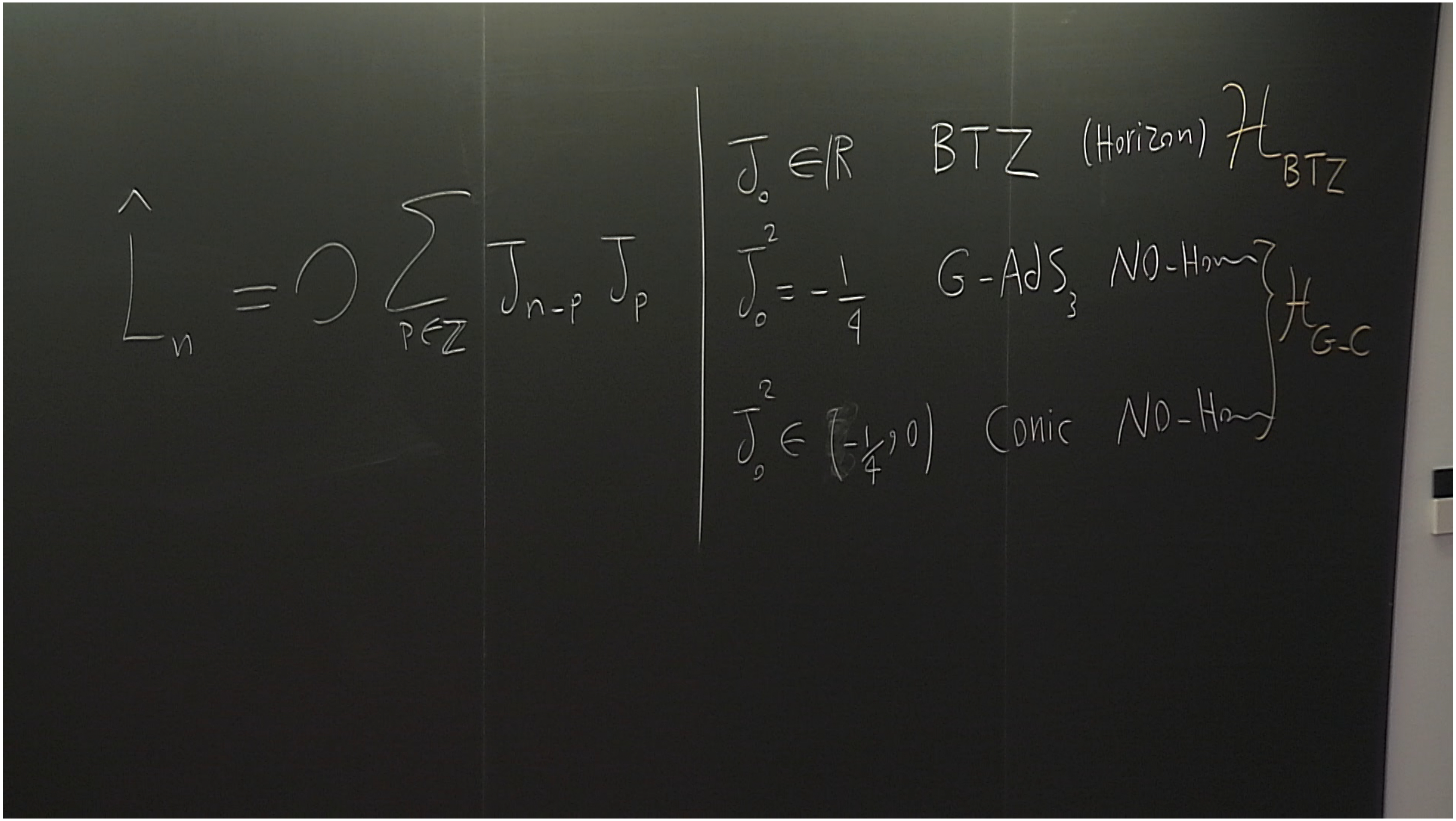
$\hat{L}_n = 0 \sum_{p \in \mathbb{Z}} J_{n-p} J_p$
$J_0 \in \mathbb{R}$ BTZ (Horizon) $\mathcal{H}_{BTZ}$
$J_0^2 = -\frac{1}{4}$ G-AdS$_3$ NO-Horizon
$J_0^2 \in (-\frac{1}{4}, 0)$ Conic NO-Horizon
$\mathcal{H}_{G-C}$

Another Construction of Virasoro Algebra

- In terms of $\mathcal{J}_n$ we get $L_n = \frac{1}{c}\sum_m : \mathcal{J}_{cn-m}\mathcal{J}_m :$

$$[L_m, L_n] = (m-n)L_{m+n} + m^3 \frac{c}{12}\delta_{m+n}$$

- To construct the Hilbert space for this Vir.

$$\forall n \geq 0, \quad \mathcal{J}_n|0\rangle = 0, \qquad |\{n_i\}\rangle = \prod_{n_i>0} \mathcal{J}_{-n_i}|0\rangle$$

- For any given state $|\psi\rangle$ we get $\mathcal{J}_0|\psi\rangle = 0$. Since $\mathcal{J}_0$ measures the energy from the near horizon viewpoint, this Hilbert space may be conveniently called Hilbert space of near horizon soft hairs.

Identifying Microstates

- Two different constructions for the same Vir. algebra at central charge c and their associated Hilbert spaces.
- Recalling that $\mathcal{J}_n$ was constructed from $\mathbb{W}_n$ which in turn is constructed from $\mathbb{J}_n$.
- We can use this equivalence to identify microstates of BTZ black hole.
- To identify the microstates, we propose that these two Vir. and the corresponding Hilbert spaces provide dual descriptions for the same physical system, i.e. we require $L_n = \mathbb{L}_n$, or more precisely

$$\frac{1}{c}\sum_p : \mathcal{J}_{nc-p}\mathcal{J}_p := in\mathbb{J}_n + \sum_p \mathbb{J}_{n-p}\mathbb{J}_p$$

Identifying Microstates

- BTZ black hole state $|0, J_0\rangle$ corresponds to set of $|\{n_i\}\rangle$ states in $\mathcal{H}_{\mathcal{J}}$ Hilbert space with same mass/angular momentum. Recalling $\ell M = J = \frac{c}{6} J_0^2$

$$\langle\{n_i\}|L_m|\{n'_i\}\rangle = \delta_{m,0}\delta_{n_i,n'_i}\frac{6}{c}J_0^2$$

- The solution turns out to be

$$|\{n'_i\}\rangle \quad \text{with} \sum_i n_i = c\ell M$$

Counting Microstates

- As the check for our proposal, we note that one can count number of states.
- For large $N = c\ell M$ this is the standard Hardy-Ramanujan problem.

$$P(N) \simeq \frac{\exp\left(2\pi\sqrt{\frac{N}{6}}\right)}{4\sqrt{3}\,N}$$

- The logarithm of this number gives the black hole entropy $S = 2\pi\sqrt{\frac{c\ell M}{6}}$ + log-corrections

BTZ Black Hole arXiv:1607.00009, 1608.01293, 1705.06257

- Starting with an extremal BTZ B.H solution

$$ds^2 = -\frac{r^2 - 2r_0^2}{\ell^2}dt^2 + \frac{\ell^2 r^2}{(r^2 - r_0^2)^2}dr^2 - 2r_0^2\, dt d\varphi + r^2 d\varphi^2, \quad M = \ell j = \frac{r_0^2}{4G}$$

$$ds^2 = \ell^2 \frac{dz^2}{z^2} - \left(r dx^+ - \ell^2 \frac{f_- dx^-}{r}\right)\left(r dx^- - \ell^2 \frac{f_+ dx^+}{r}\right), \quad x^\pm = \frac{t}{\ell} \pm \varphi$$

Extremal BTZ corresponds to $f_- = 0$ (or $f_+ = 0$) and $f_+ > 0$.

- Consider one-function class of solutions generated by

$$d\varphi \to J(\varphi) d\varphi$$

Extremal Kerr Black hole Phase-Space

1708.06378, 1711.xxxxx

$$ds^2 = -\frac{\Delta}{\Sigma}(dt + m\sin^2\theta\, d\phi)^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 + \frac{\sin^2\theta}{\Sigma}\Big((r^2+m^2)d\phi + m dt\Big)^2,$$

$$\Delta = (r-m)^2, \quad \Sigma = r^2 + m^2\cos^2\theta$$

where mass, angular momentum and entropy are given by

$$M = \frac{m}{G_N}, \quad j = \frac{m^2}{G_N}, \quad S_{B.H} = \frac{A_h}{4G_N} = 2\pi j$$

- To generate the Extremal Kerr Phase-Space we consider

$$d\phi \to J(\phi)d\phi, \qquad J_0 \equiv \frac{1}{2\pi}\int_0^{2\pi} J(\phi)d\phi = 1$$

- On this phase-space consider two trans., generated by vector fields

$$\hat{\chi} = \epsilon(\phi)\partial_\phi, \quad \eta = \frac{\epsilon(\phi)}{2J(\phi)}\partial_\phi$$

Extremal Kerr Black hole Phase-Space

$$\delta_{\hat{\chi}} g_{\mu\nu}[J] = \mathcal{L}_{\hat{\chi}} g_{\mu\nu} = g_{\mu\nu}[J + \delta_{\hat{\chi}} J] - g_{\mu\nu}[J], \quad \delta_\eta g_{\mu\nu}[J] = \mathcal{L}_\eta g_{\mu\nu} = g_{\mu\nu}[J + \delta_\eta J] - g_{\mu\nu}[J]$$

$$\delta_{\hat{\chi}} J \simeq (\epsilon J)', \qquad \delta_\eta J \simeq \epsilon'/2$$

- If we denote associated charges to $\hat{\chi}_n$ and η_n by $\hat{\mathbb{L}}_n$ and $\mathbb{J}_n$

$$[\hat{\chi}_m, \hat{\chi}_n] = -i(m-n)\hat{\chi}_{m+n}, \quad [\eta_m, \eta_n]_* = 0, \quad [\hat{\chi}_m, \eta_n]_* = in\eta_{m+n}$$

$$\{\hat{\mathbb{L}}_m, \hat{\mathbb{L}}_n\} = -i(m-n)\hat{\mathbb{L}}_{n+m} + \cdots, \quad \{\mathbb{J}_m, \mathbb{J}_n\} = 0 + \cdots, \quad \{\hat{\mathbb{L}}_m, \mathbb{J}_n\} = in\mathbb{J}_{m+n} + \cdots$$

- Standard CPSM analysis reveals that $\hat{\mathbb{L}}_n$ and $\mathbb{J}_n$ are both symplectic and integrable and that

$$\{\hat{\mathbb{L}}_m, \hat{\mathbb{L}}_n\} = -\frac{\mathrm{j}}{2\pi} \int \mathrm{d}\phi \, e^{i(m+n)\phi} \, (2J(imJ + J'))$$

Extremal Kerr Black hole Phase-Space

which gives

$$\hat{\mathbb{L}}_n = \frac{\mathrm{j}}{2\pi}\int \mathrm{d}\phi\, e^{in\phi} J^2, \qquad \mathbb{J}_n = \frac{\mathrm{j}}{2\pi}\int \mathrm{d}\phi\, e^{in\phi} J$$

We get

$$\{\mathbb{J}_m, \mathbb{J}_n\} = \frac{in}{2}\mathrm{j}\delta_{m+n,0}, \quad \hat{\mathbb{L}}_n = \frac{1}{\mathrm{j}}\sum_p \mathbb{J}_p \mathbb{J}_{n-p}, \quad \{\hat{\mathbb{L}}_m, \hat{\mathbb{L}}_n\} = -i(m-n)\hat{\mathbb{L}}_{m+n}$$

Twisted Sugawara Construction

- Given the symplectic symmetry generators $\hat{\chi}, \eta$, any linear combination of them is also a symplectic symmetry generator. In particular, let us consider

$$\chi[\epsilon(\phi)] = \left(\epsilon + \frac{\epsilon'}{2J}\right)\partial_\phi \equiv \hat{\chi}[\epsilon] + \eta[\epsilon'].$$

- The charge associated with $\epsilon = e^{in\phi}$, $\mathbb{L}_n$, is then

$$\mathbb{L}_n = \hat{\mathbb{L}}_n + in\mathbb{J}_n = \frac{1}{\mathrm{j}} \sum_p \mathbb{J}_p \mathbb{J}_{n-p} + in\mathbb{J}_n$$

and together with current $\mathbb{J}_n$ form a Kac-Moody algebra,

$$\{\mathbb{L}_m, \mathbb{L}_n\} = -i(m-n)\mathbb{L}_{m+n} - im^3 \frac{\mathrm{j}}{2} \delta_{m+n,0}.$$

- Quantising the algebra: $\{,\} \to -i[,]$.

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Extremal Kerr Hilbert space

- Vacuum state of $\mathcal{H}_{Kerr}$ can be defined by

$$\mathbb{J}_0|0;J_0\rangle = \mathrm{j}J_0|0;J_0\rangle, \qquad \mathbb{J}_n|0;J_0\rangle = 0, \quad n > 0,$$

- The other states in $\mathcal{H}_{Kerr}$ may then be constructed as

$$|\{n_i\};J_0\rangle = \mathcal{N}_{n_i} \prod_{n_i>0} \mathbb{J}_{-n_i}|0;J_0\rangle, \quad \mathcal{N}_{\{n_i\}}^{-2} = \prod n_i,$$

- $\mathcal{H}_{Kerr}$ includes states with imaginary $J_0 = \pm i\nu/2$ with $\nu \in (0,1]$

$$W(\phi) = e^{-2\int^{\phi} J}, \qquad \mathbb{W} = :e^{-\frac{2}{\mathrm{j}}\int^{\phi}\mathbb{J}}:,$$

- j is the angular momentum of the original Kerr black hole and it is expected to be quantised. Therefore, central charge $c = 6\mathrm{j}$ is also integer-valued.

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Microstate Counting

- For large j this is the standard Hardy-Ramanujan formula gives

$$S(\mathrm{j}) = 2\pi\mathrm{j} + \text{log-corrections},$$

reproducing the Bekenstein-Hawking entropy and its logarithmic corrections.

Extremal Kerr Black hole Phase-Space

1708.06378, 1711.xxxxx

$$ds^2 = -\frac{\Delta}{\Sigma}(\mathrm{d}t + \mathrm{m}\sin^2\theta\,\mathrm{d}\phi)^2 + \frac{\Sigma}{\Delta}\mathrm{d}r^2 + \Sigma\mathrm{d}\theta^2 + \frac{\sin^2\theta}{\Sigma}\Big((r^2+\mathrm{m}^2)\mathrm{d}\phi + \mathrm{m}\mathrm{d}t\Big)^2,$$

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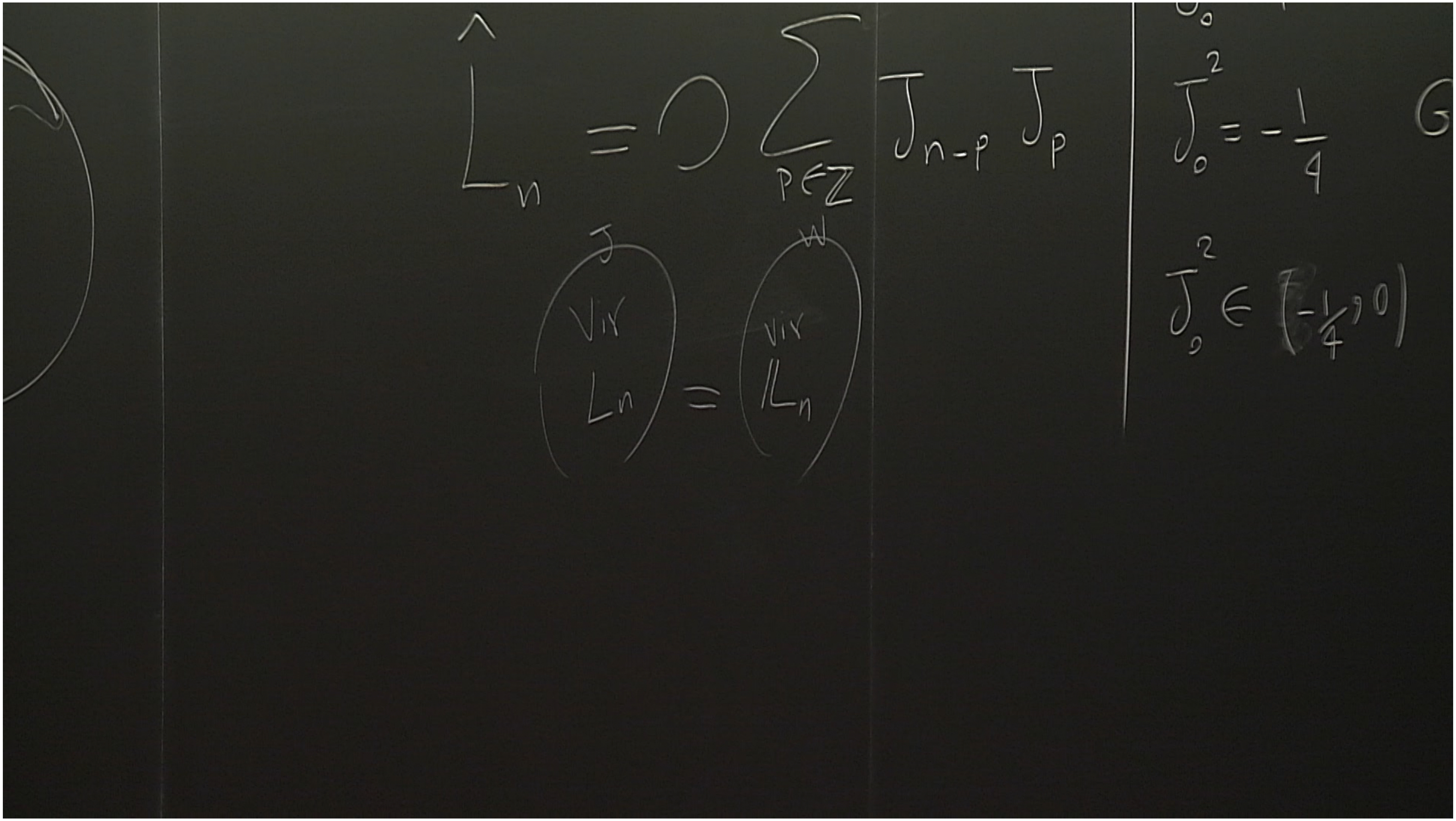
Conclusion

- We have shown how the horizon fluff idea can be worked through for the BTZ and 4D extremal Kerr black hole.
- Our analysis expands upon the Kerr/CFT analysis in three important ways:
  1. our symmetry algebra is defined over the whole extremal Kerr geometry and not only in the near horizon region
  2. we have introduced the extremal Kerr phase space, our symmetries are symplectic (and not just asymptotic)
  3. besides the Vir. we have a current algebra and our symmetry generator diffeos. are all along the azimuthal angle ∂_ϕ
- Our preliminary analysis shows that similar features can be extended to other extremal black holes in higher dimensions.
- Whether similar analysis and horizon fluff proposal work for generic non-extremal Kerr geometry?

- The phase space corresponding to generic Kerr will presumably have two or four independent functions and consequently one expects to see a larger algebra than $U(1)$ Kac-Moody.
- This symmetry algebra is inevitably a subalgebra of the asymptotic BMS_4 symmetry.
- The first check of our proposal was provided through reproducing the Bekenstein-Hawking area law.
- The non-trivial test, however, comes from the logarithmic corrections.
- The Hardy-Ramanujan counting gives $S = 2\pi \mathrm{j} - 2\ln \mathrm{j} +$ subleading.
- The Kerr/CFT analysis the log-corrections for the 4D extremal case is not yet available.
- Nonetheless, there are general analysis by Sen divides the log-corrections into "zero-mode" and "non-zero mode" contributions.

Thank You For Your Attention



$\hat{L}_n = \sum_{p \in \mathbb{Z}} J_{n-p} J_p$
$J_0^2 = -\frac{1}{4}$
$J_0^2 \in [-\frac{1}{4}, 0)$
J
W
Vir
L_n
=
Vir
$\mathbb{L}_n$
G