Title: Scattering Amplitudes and the Associahedron

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Abstract: <p>We establish a direct connection between scattering amplitudes for bi-adjoint scalar theories and a classic polytope--the "associahedron"--known to mathematicians since the 1960s. We find an associahedron naturally living in kinematic space. The tree level scattering amplitude is simply a geometric invariant of the associahedron called its "canonical form" [2], which is a differential form on kinematic space with logarithmic singularities on the boundaries of the associahedron. We show that basic physical principles like locality and unitarity are "rediscovered" as properties of the geometry, and certain "soft" limits can be converted to geometric statements as well.</p>

<p> </p>

<p>The associahedron in kinematic space plays an important role in the context of scattering equations. We discover that the scattering equations act as a diffeomorphism between the interior of the open string moduli space (yet another associahedron) and the associahedron in kinematic space. This observation provides the key to a novel derivation of the bi-adjoint CHY formula. Finally, we emphasize the importance of the scattering amplitude as a differential form, as suggested by the "canonical form" construction. We argue on general grounds that every color-dressed amplitude is dual to a form on kinematic space. This is motivated by the surprising observation that "projective" forms on kinematic space satisfy Jacobi identities and therefore have novel implications for color-kinematics duality.</p>

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<p>References:</p>

<p>[1] Nima Arkani-Hamed, Yuntao Bai, Song He & Gongwang Yan. In preparation.</p>

<p>[2] Nima Arkani-Hamed, Yuntao Bai & Thomas Lam. Positive Geometries and Canonical Forms. arXiv 1703.04541</p>

Scattering Amplitudes and the Associahedron

Yuntao Bai

with Nima Arkani-Hamed, Song He & Gongwang Yan, to appear;
and Nima Arkani-Hamed & Thomas Lam arXiv:1703.04541

Princeton University

Positive Geometries and Canonical Forms

- We introduce **positive geometries and canonical forms** as a new framework for thinking about a class of scattering amplitudes.
- Loosely speaking, a positive geometry $\mathcal{A}$ is a closed geometry with boundaries of all co-dimensions (e.g. polytopes).

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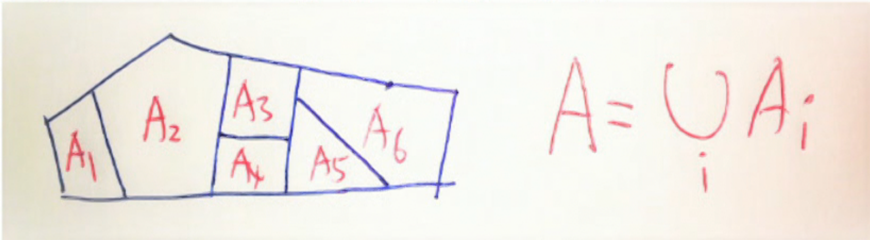
- We introduce **positive geometries and canonical forms** as a new framework for thinking about a class of scattering amplitudes.
- Loosely speaking, a positive geometry $\mathcal{A}$ is a closed geometry with boundaries of all co-dimensions (e.g. polytopes).
- **Each positive geometry has a unique differential form** $\Omega(\mathcal{A})$ **called its canonical form** defined by the following properties:
  1. It has logarithmic (i.e. $d\log z$) singularities on the boundaries of $\mathcal{A}$.
  2. Its singularities are recursive: At every boundary $\mathcal{B}$, we have $\text{Res}_{\mathcal{B}}\Omega(\mathcal{A}) = \Omega(\mathcal{B})$.
  3. $\Omega(\mathcal{A}) = \pm 1$ if $\mathcal{A}$ is a point.

Positive Geometries and Canonical Forms

- The canonical form has two remarkable properties.
- **Triangulation**: Given a subdivision of $\mathcal{A}$ by finitely many pieces $\mathcal{A}_i$, we have $\Omega(\mathcal{A}) = \sum_i \Omega(\mathcal{A}_i)$.



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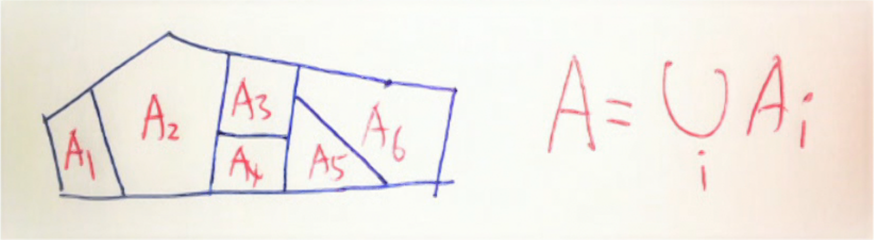

- **Pushforward**: Given a diffeomorphism mapping $\mathcal{A}$ to $\mathcal{B}$, the map pushes $\Omega(\mathcal{A})$ to $\Omega(\mathcal{B})$.

$$\text{If} \quad \mathcal{A} \xrightarrow{\text{diffeomorphism } \phi} \mathcal{B}$$

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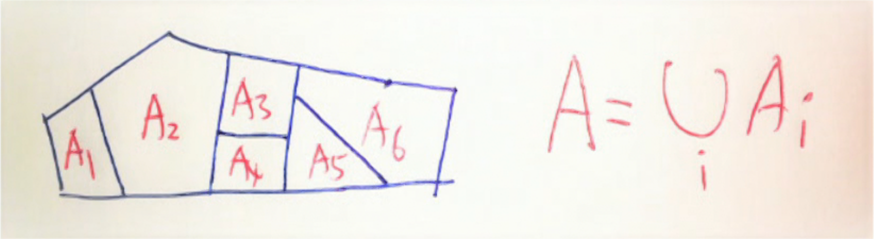

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- **For positive geometries that appear in physics, the canonical form is a physical quantity (e.g. scattering amplitude).**

Positive Geometries and Canonical Forms

- For instance, the amplituhedron $\mathcal{A}(k,n;L)$ is a positive geometry. The canonical form $\Omega(\mathcal{A}(k,n;L))$ is conjectured to be the n-particle N^{k}MHV tree level amplitude for $L=0$ and the L-loop integrand for $L>0$.

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- **Our focus today:** The $(n-3)$-dimensional associahedron $\mathcal{A}_n$ is a positive geometry, and its canonical form $\Omega(\mathcal{A}_n)$ is the n-particle tree level amplitude of planar bi-adjoint scalar theory, which we refer to as "bi-adjoint amplitudes" in this talk.

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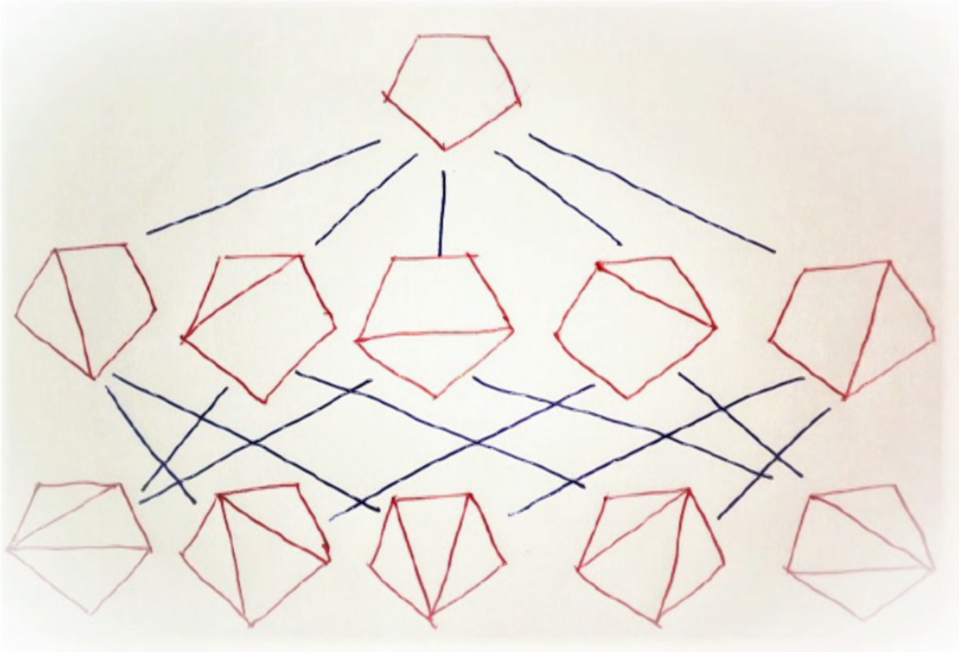
- The associahedron knows a lot about physics! It knows about amplitudes, locality, unitarity, "soft" limits, scattering equations and color-kinematics duality.

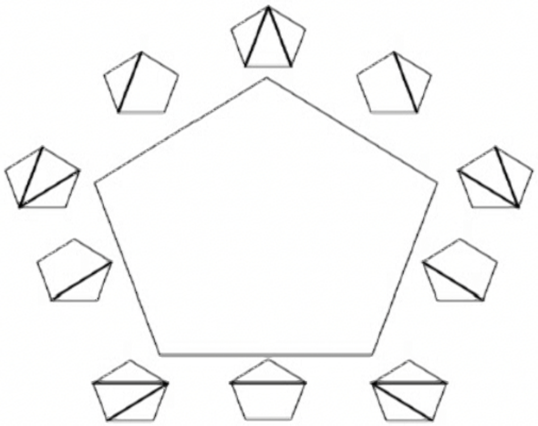
The Associahedron

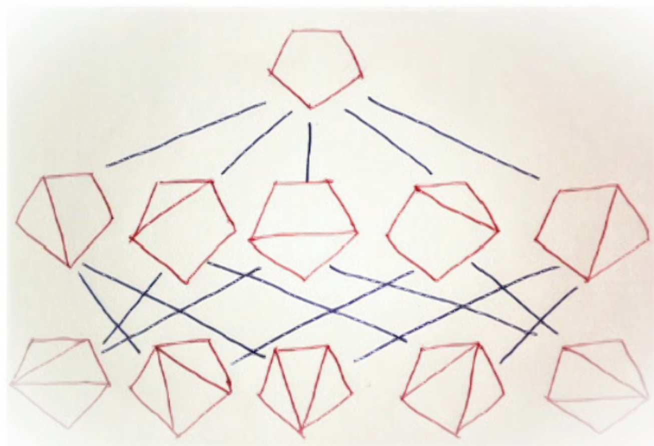
- A partial triangulation of the regular n-gon is a set of non-crossing diagonals. The set of all partial triangulations of the n-gon can be organized in a hierarchical web:



The Associahedron

- The associahedron of dimension $(n-3)$ is a polytope whose codimension d boundaries are in 1-1 correspondence with the partial triangulations with d diagonals. And the lines connecting partial triangulations tell us how the boundaries are glued together.





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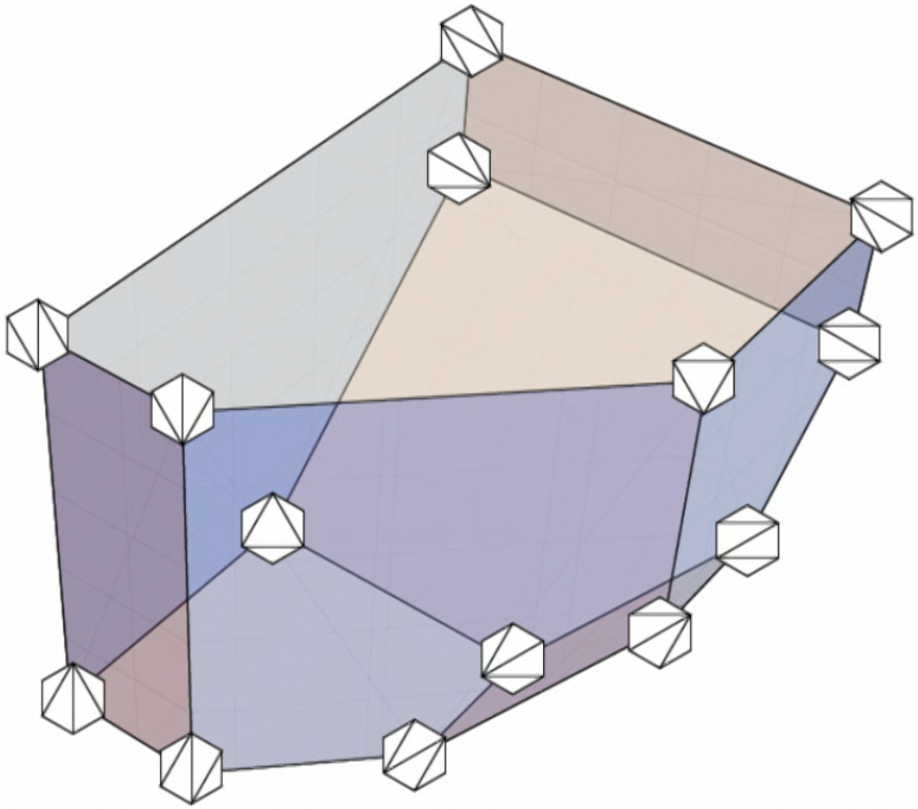

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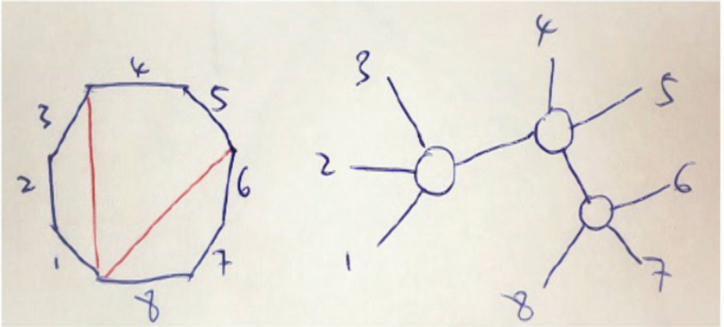
- Recall that partial triangulations are dual to cuts on planar cubic diagrams, with each diagonal corresponding to a cut. So the codimenion d boundaries of the associahedron are dual to d-cuts.



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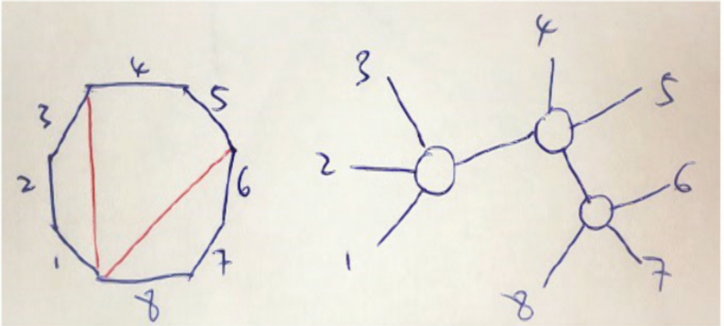

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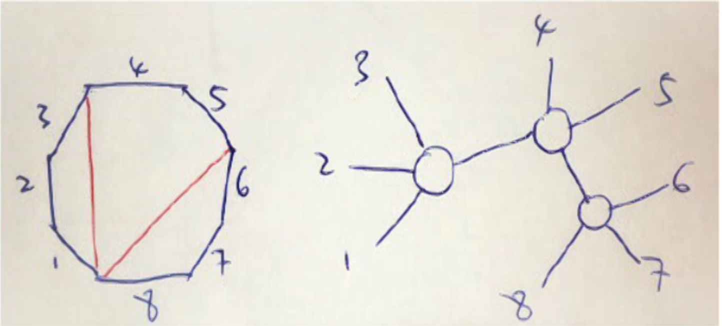

- The boundaries of the associahedron are therefore dual to the singularities of a cubic scattering amplitude.
- It appears that the associahedron knows about the structure of planar cubic amplitudes. It is therefore natural to look for an explicit construction of an associahedron within kinematic space.

The Associahedron in Kinematic Space

- We consider scattering n massless particles with momenta k_i^μ for $i = 1, \ldots, n$ in any number of dimensions.

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- We define the Mandelstam variables as usual: $s_{ij} = (k_i + k_j)^2 = 2k_i \cdot k_j$. There are $n(n-1)/2$ of these.
- More generally, we have $s_{i_1 \ldots i_m} = (k_{i_1} + \cdots + k_{i_m})^2$.
- There are n kinematic constraints: $\sum_j s_{ij} = 0$ for each i. So kinematic space has dimension $\frac{n(n-1)}{2} - n = \frac{n(n-3)}{2}$.

Cutting Out the Associahedron

- We first require all **planar** propagators to be positive: $s_{i,i+1,i+2,\ldots,i+m} \geq 0$. Hence the codimension 1 boundaries correspond to individual planar cuts.

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- We set s_{ij} to be a **negative constant** for all **non-adjacent** index pairs $1 \leq i < j \leq n-1$. Let, $c_{ij} \equiv -s_{ij}$.
- The number of constants is: $(n-3)+(n-2)+\cdots+1 = \frac{(n-2)(n-3)}{2}$. This cuts the space down to the required dimension: $\frac{n(n-3)}{2} - \frac{(n-2)(n-3)}{2} = n-3$.

The Kinematic Associahedron

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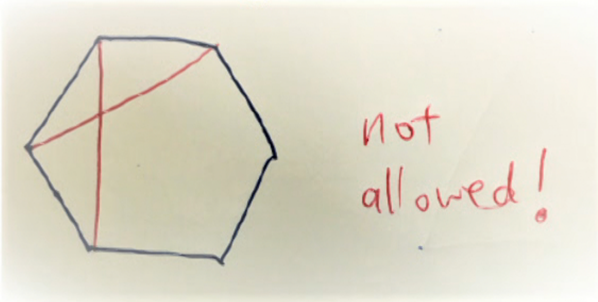
- **The intersection between the big simplex and these equations is an associahedron!**
- To show this, we argue that the boundaries of the polytope correspond to cuts on planar cubic diagrams. But this is true by construction, since the boundaries are defined by cuts.
- However, we need to argue that cuts with crossing-diagonals are forbidden. This follows from the negative constants and some kinematic algebra.



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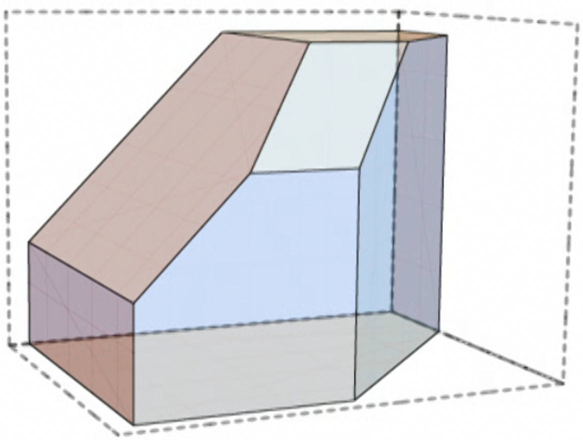
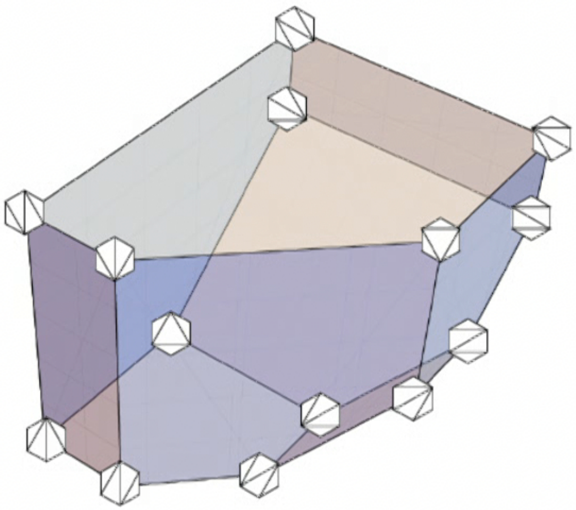
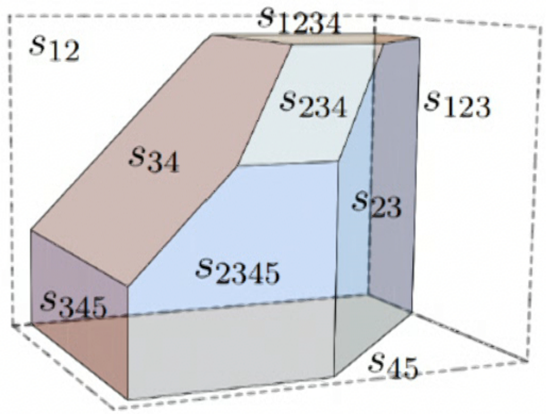
- Here we show a numerical plot (left) for $n = 6$, which is equivalent to the 3D associahedron (right).





The Kinematic Associahedron

- This picture already contains a lot of information about amplitudes.



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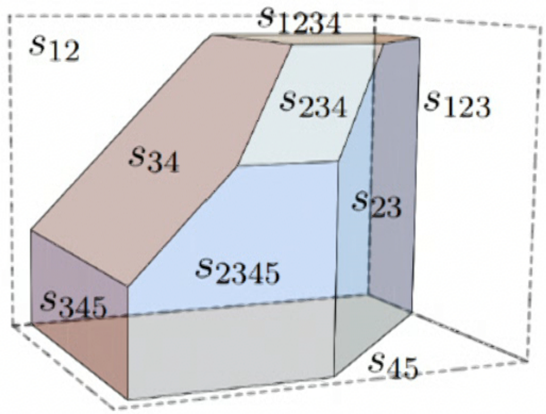

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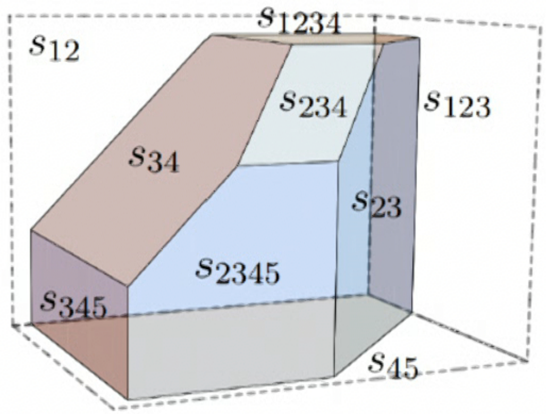

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- For any pair of **intersecting** facets, the double cut is non-vanshing. (e.g. $s_{23} = 0$ and $s_{234} = 0$)

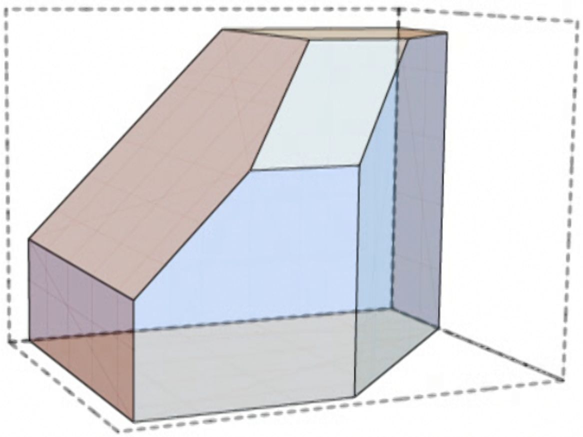
The Canonical Form of the Associahedron

- Now that we have constructed a **positive geometry**, the next step in our program is to study its **canonical form** and look for a physical interpretation.

$$\text{Positive Geometry} \rightarrow \text{Canonical Form} = \text{Physical Quantity}$$

The Canonical For of the Associahedron

- We first observe that the associahedron is a **simple polytope**. i.e. Each vertex is adjacent to exactly D facets, where D is the dimension of the polytope.



The Canonical For of the Associahedron

- The canonical form for a simple polytope ($D > 1$) is:

$$\sum_{\text{vertex } Z} \prod_{a=1}^{D} d\log(F_{a,Z})$$

where $F_{a,Z} = 0$ are the equations of the facets adjacent to Z ordered by orientation.

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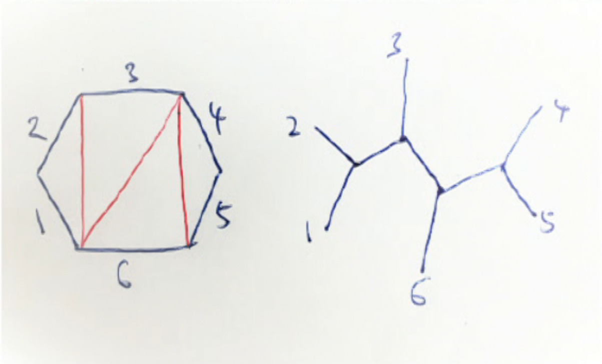

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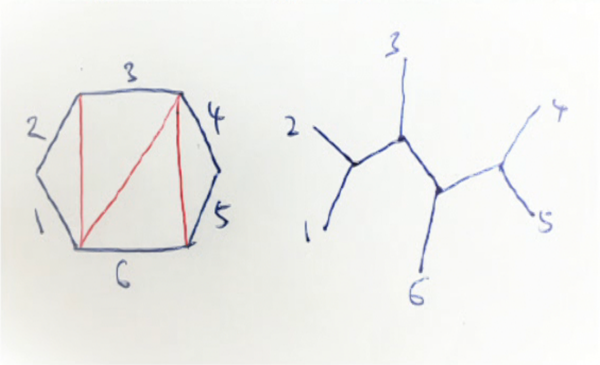
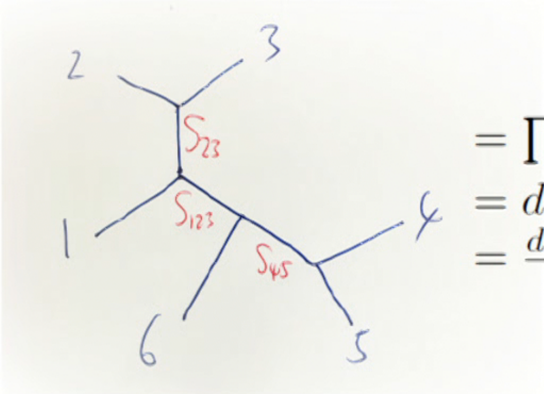


- The canonical form is therefore a sum over planar cubic diagrams. This looks like an amplitude.

The Canonical Form of the Associahedron

- The expression for each vertex is the product of d-log of the equations for the $(n-3)$ adjacent facets. But the facets are given by cutting the propagators $s_{I_a} = 0$ on the diagram. So the expression is just the d-log of the propagators.



$$= \prod_{a=1}^{n-3} \frac{ds_{I_a}}{s_{I_a}}$$
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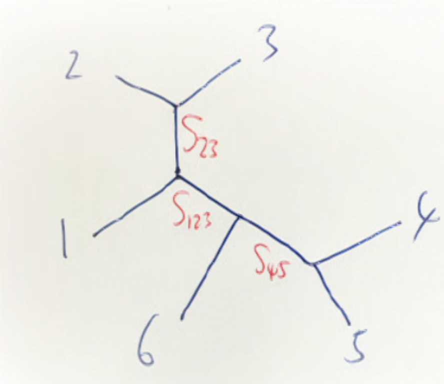

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- The numerator $\prod_{a=1}^{n-3} ds_{I_a}$ is identical for each diagram when pulled back onto the $(n-3)$-subspace containing the associahedron.

The Canonical Form of the Associahedron

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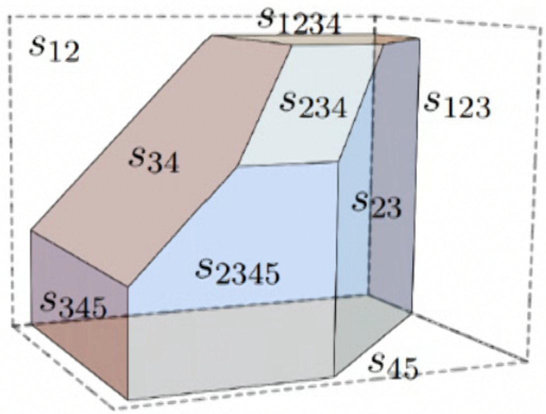

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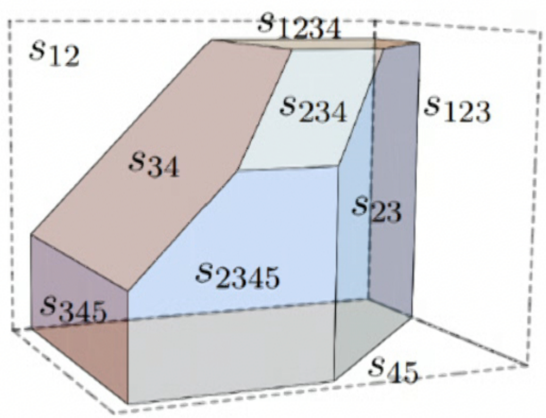

- It follows that the numerator $\prod ds_I$ is identical for each vertex.

The Canonical Form of the Associahedron

- The expression for each diagram is therefore:

$$\text{Planar cubic diagram} = \left(\prod_{a=1}^{n-3} \frac{1}{s_{I_a}}\right) d^{n-3}s$$

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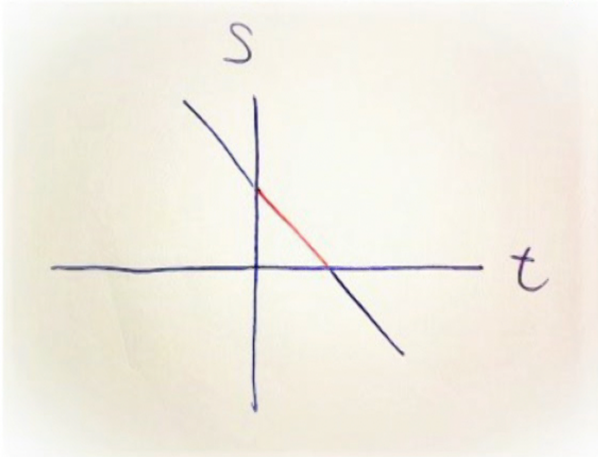
- The terms add to form the bi-adjoint amplitude:

$$\begin{aligned} \text{Canonical form} &= \sum \text{Planar cubic diagram} \\ &= (\text{Amplitude}) d^{n-3}s \end{aligned}$$

- It is crucial that we pull back to the $(n-3)$-subspace where the form reduces to a top form, otherwise we cannot factor out the $d^{n-3}s$ to get an amplitude.

An Example for $n = 4$

- For $n = 4$, we have s, t, u satisfying $s + t + u = 0$.
  We impose $s, t \geq 0$ and $u < 0$ constant (hence $ds = -dt$).
  The associahedron is the line segment (red).
  The canonical form is the 4 point amplitude.



$$\text{Canonical form} = \frac{ds}{s} - \frac{dt}{t} = \left(\frac{1}{s} + \frac{1}{t}\right) ds = (\text{4pt Amplitude})\, ds$$

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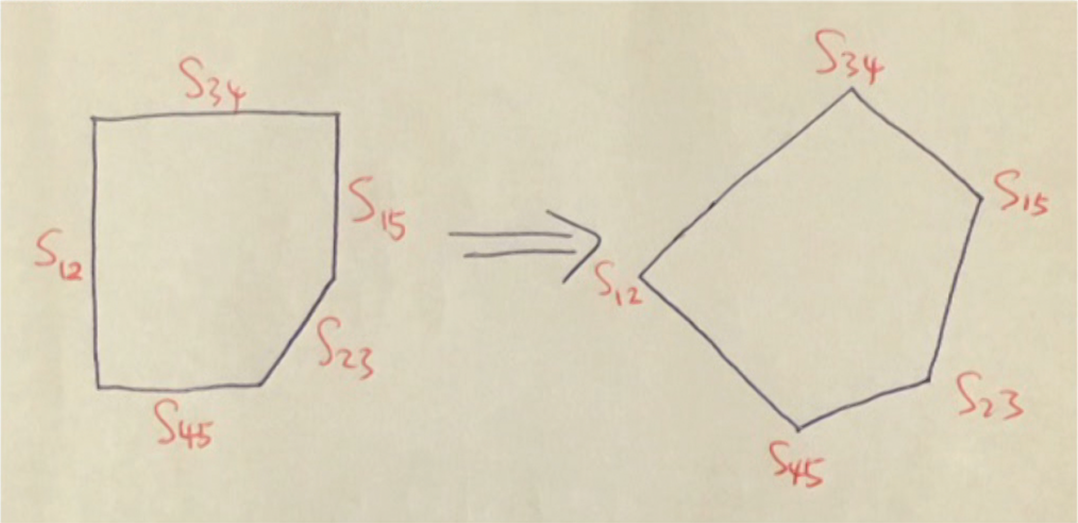
- So we have established the bi-adjoint amplitude as the canonical form of a polytope.
- In practice, this is useful for computing the amplitude because there are many useful theorems on computing canonical forms.
- We present two novel ways of computing the amplitude:
  - Dual associahedron
  - Direct triangulation of the associahedron

The Dual Associahedron

- Recall that every polytope has a **dual polytope**. So there is a **dual associahedron**.



The Dual Associahedron

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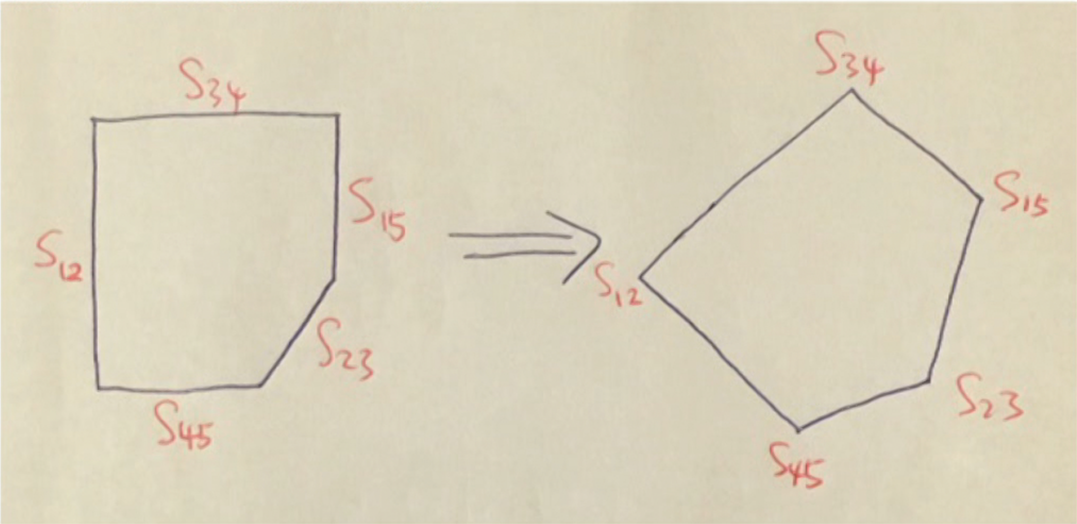

- The polytope and its dual have “inverse” combinatorics. That is, every codimension d boundary of the polytope is dual to a dimension $d-1$ boundary of the dual polytope.

The Dual Associahedron

- **Claim**: The canonical form of a polytope is determined by the volume of the dual.

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- We can therefore compute the amplitude by triangulating the dual associahedron.
- **We "rediscover" Feynman diagram expansion as a particular triangulation of the dual associahedron volume!**

An Example for $n = 5$

- Consider the 5-term Feynman expansion for the 5-point amplitude:

$$\text{Amplitude} = \frac{1}{s_{12}s_{45}} + \frac{1}{s_{45}s_{23}} + \frac{1}{s_{23}s_{15}} + \frac{1}{s_{15}s_{34}} + \frac{1}{s_{34}s_{12}}$$

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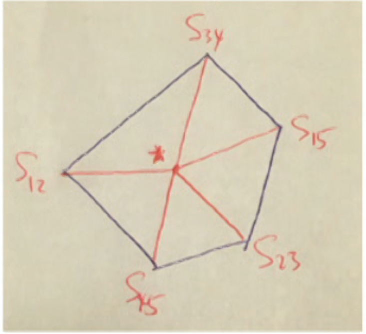
- This is a triangulation of the dual pentagon volume with a reference point * on the interior.



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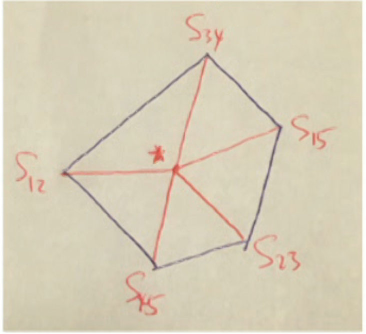
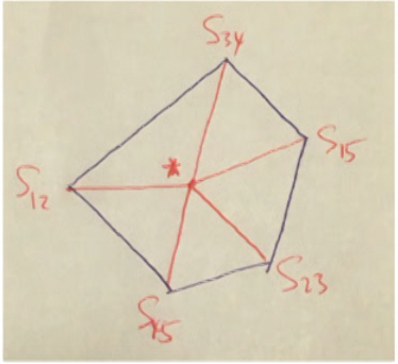
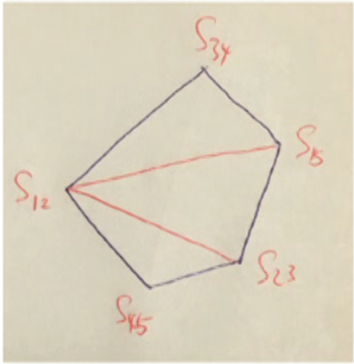


- For general n, a similar kind of triangulation provides the Feynman expansion.

An Example for $n = 5$

- But the Feynman expansion is slightly redundant, since we can get away with a 3-piece triangulation.





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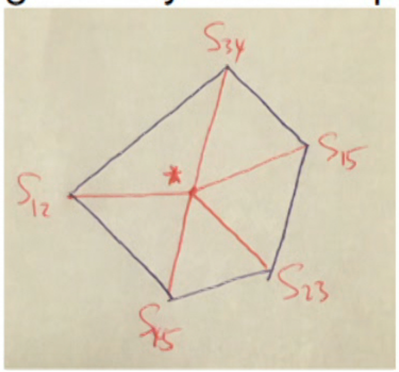

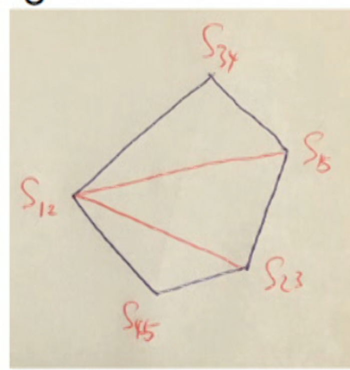

- This leads to a novel 3-term expansion for the 5-point amplitude:

$$\frac{-(s_{13}+s_{14})}{s_{12}s_{15}s_{34}} + \frac{-(s_{13}+s_{14})}{s_{12}s_{23}s_{15}} + \frac{-s_{13}}{s_{12}s_{45}s_{23}}$$

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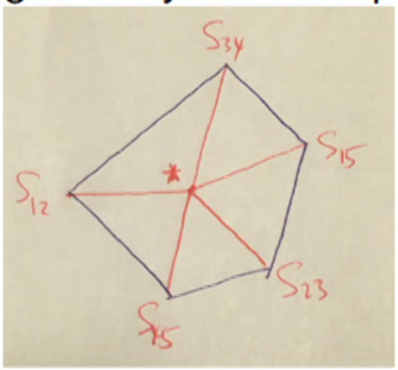

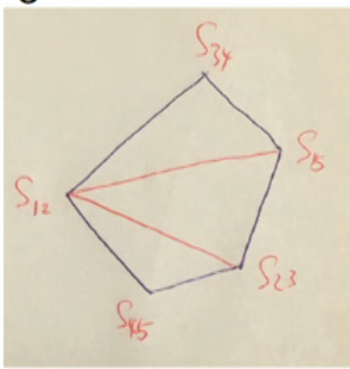

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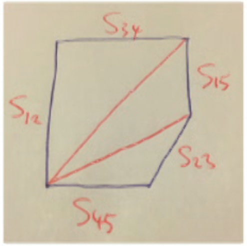
- This can be extended to all n, leading to novel formulas for the amplitude.

Triangulating the Associahedron

- Recall that canonical forms are **triangulation independent**.



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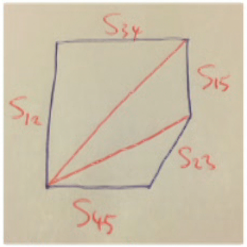

- Thus for $n=5$, the canonical form for the pentagon is the sum over the canonical forms for the three pieces.

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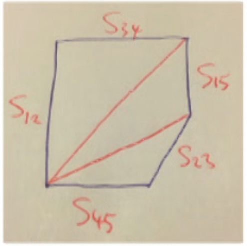

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- This gives yet another formula for the amplitude, but with **spurious poles** (colored).

All Ordering Pairs

- Our discussion so far applies only to bi-adjoint amplitudes $m[\alpha|\beta]$ with $\alpha = \beta = (1, \ldots, n)$. For general orderings, again there is an associahedron whose canonical form is the amplitude as a form.

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- So the associahedron is the underlying geometry for all bi-adjoint tree amplitudes.

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  Every positive geometry $\mathcal{A}$ has a unique form $\Omega(\mathcal{A})$ called its canonical form.
- **The associahedron is a positive geometry whose canonical form is the bi-adjoint amplitude.**
- We also established two new ways of computing the amplitude geometrically.

Lessons from the Associahedron

- I promised earlier that the associahedron knows a lot about physics:
  - Locality & Unitarity
  - "Soft" Limits
  - Scattering Equations
  - Color-kinematics Duality

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  - Color-kinematics Duality
- We are now on our way to exploring these properties from a new perspective.

Scattering Equations

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- This is a bit tricky to see in σ variables. The associahedron becomes more transparent if we think about cross ratios.

Scattering Equations

- We define cross ratios:

$$u_{i,j} := \frac{(\sigma_i - \sigma_{j-1})(\sigma_{i-1} - \sigma_j)}{(\sigma_i - \sigma_j)(\sigma_{i-1} - \sigma_{j-1})}$$

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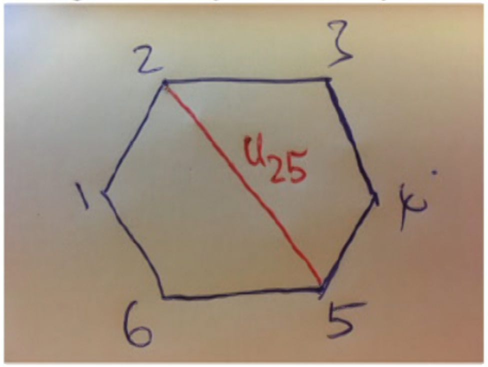
- The subscript (i, j) can be identified with the diagonal of a n-gon between vertices i and j. So these cross ratios correspond to diagonals just like planar Mandelstam variables.



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- The cross ratios satisfy the **non-crossing identity**

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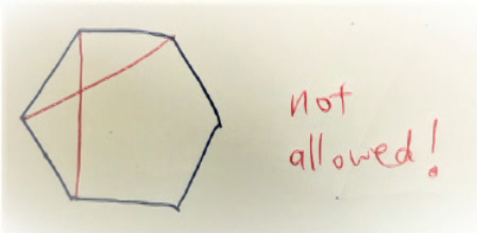
- Crossing diagonals violate the non-crossing identity and are thus forbidden.



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- Pulling back to σ-space, we find the Parke-Taylor form.
- **Therefore the Parke-Taylor form is the canonical form of the worldsheet associahedron.**

Scattering Equations

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- The two spaces are related by the **scattering equations**: $E_d(\{\sigma_a, s_{bc}\}) = 0$. So it is natural to expect that the two associahedra are also related.

Scattering Equations as a Diffeomorphism

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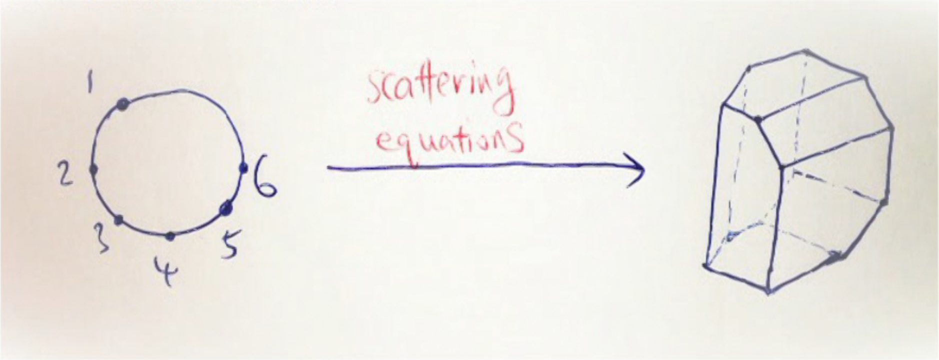
- **Hence the scattering equations act as a diffeomorphism from the worldsheet associahedron to the kinematic associahedron.**



Scattering Equations as a Diffeomorphism

- One motivation for the diffeomorphism is provided by rewriting the scattering equations:

$$s_{a\cdots b-1} = -\sum_{\substack{1\leq i<a \\ a<j<b}} \sigma_{a,j}\,\frac{s_{ij}}{\sigma_{ij}} - \sum_{\substack{a\leq i<b \\ b<j<n}} \sigma_{i,b-1}\,\frac{s_{ij}}{\sigma_{ij}} - \sum_{\substack{1\leq i<a \\ b\leq j<n}} \sigma_{a,b-1}\,\frac{s_{ij}}{\sigma_{ij}}\,,$$

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- Recalling that the s_{ij} appearing on the right are negative constants, this provides the desired map:

$$\begin{array}{ccc} \text{Worldsheet Assoc.} & \longrightarrow & \text{Kinematic Assoc.} \\ \{\sigma_1 < \cdots < \sigma_n\} & \longrightarrow & \{s_{a\cdots b-1>0}\} \end{array}$$

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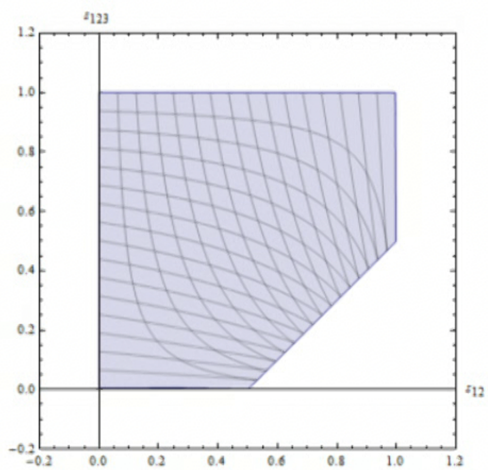
- Proving that this is a diffeomorphism is more involved.

Scattering Equations as a Diffeomorphism

Kinematic space



Moduli space:
$(\sigma_1, \sigma_4, \sigma_5) = (0, 1, \infty)$
$0 < \sigma_2 < \sigma_3 < 1$

Scattering equations:
$s_{12} = -\frac{\sigma_2}{\sigma_3}(s_{13} + s_{14}\sigma_3)$
$s_{123} = -\frac{1}{1-\sigma_2}(s_{24}(\sigma_3 - \sigma_2) + s_{14}\sigma_3(1 - \sigma_2))$

Image created using Mathematica 9

Scattering Equations as a Diffeomorphism

- Recall: Diffeomorphisms push canonical forms to canonical forms.

$$\text{If} \quad \mathcal{A} \xrightarrow{\text{diffeomorphism } \phi} \mathcal{B}$$

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- Hence, $\frac{d^n\sigma/\text{Vol } SL(2)}{\prod_{i=1}^{n}(\sigma_i - \sigma_{i+1})} \xrightarrow{\text{pushforward by } \phi} (\text{Amplitude}) d^{n-3}s$

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- **The scattering equations push the Parke-Taylor form to the amplitude form!**

CHY Formula as a Pushforward

- In other words,

$$\sum_{\text{sol. } \sigma} \frac{d^n\sigma/\text{Vol } SL(2)}{\prod_{i=1}^{n}(\sigma_i - \sigma_{i+1})} = (\text{Amplitude}) d^{n-3}s$$

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- This is equivalent to the CHY formula for the bi-adjoint amplitude:

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- The second Parke-Taylor factor in CHY should be thought of as a normalization factor for the delta function.

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- This provides a novel derivation of bi-adjoint CHY without reference to the usual arguments.
- We emphasize that the diffeomorphism/pushforward property is a completely general fact about positive geometries and canonical forms, and is true independently of its application to scattering equations and associahedra. In fact, it has appeared elsewhere in physics.

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- One surprising motivation is that **forms on kinematic space satisfy Jacobi identities** analogous to kinematic Jacobi identities for BCJ numerators.
- This implies a "**color-form duality**" analogous to color-kinematics duality.

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- To each ordered graph g, we assign the form

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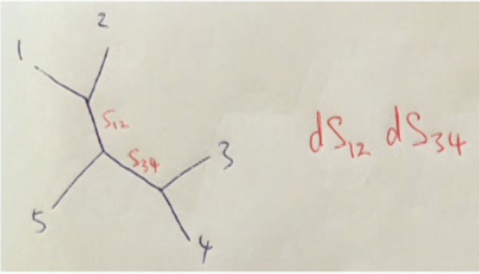
which is a product of ds_I over all the propagators s_I of the graph.



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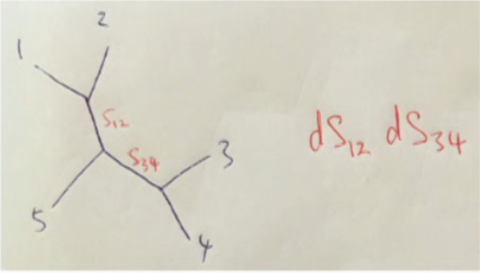

- The $\pm$ sign is determined by a prescription on the ordered graph.

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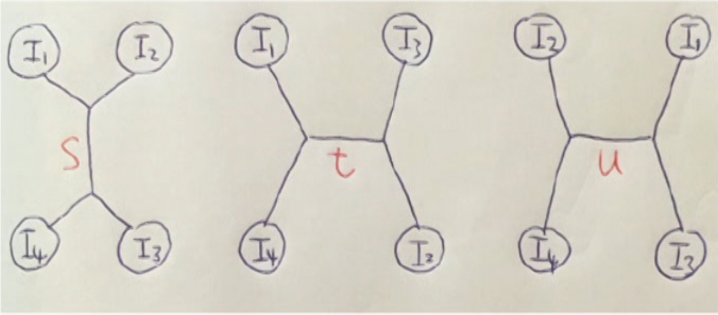

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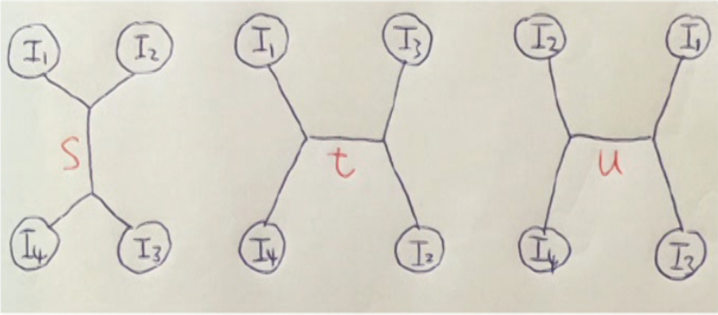

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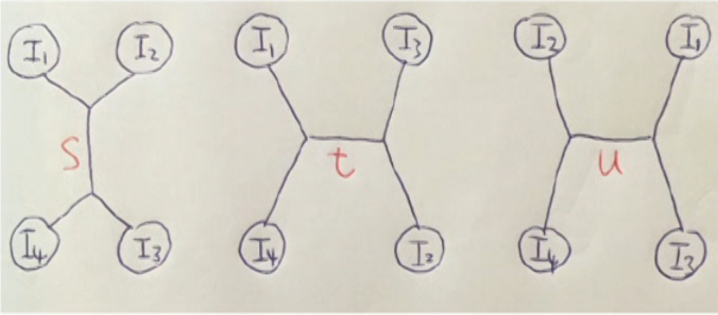

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- This is the **Jacobi identity for forms**.

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- While this looks unusual, it arises naturally from the "color-form" duality.

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- Let us apply the gauge transformation $s \to s'$ to the amplitude form:

$$\Omega(s) := \sum_{\text{cubic graph } g} \left(\frac{N_g}{\prod_g s_I} \right) \prod_g ds_I$$

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- We say therefore that the amplitude form is **projective** of degree D.

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- Consider a 4-point amplitude form with BCJ numerators N_s, N_t, N_u:

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- This argument extends to all n.

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Quick Summary So Far

- **Forms on kinematic space** satisfy **Jacobi relations** and are therefore **dual to color**.
- Every color-dressed amplitude can be transformed to a form, and vice versa. The form contains all information about the color factors even though no color factor appears.
- We introduced a **local** $GL(1)$ **gauge transformation** on kinematic space under which the amplitude form is **projective**.

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Main Idea 1 of 3:

- We introduced **positive geometries** and **canonical forms**.

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- This identity is analogous to the Jacobi identity $ff + ff + ff = 0$ for structure constants, and is ultimately at the heart of the duality.

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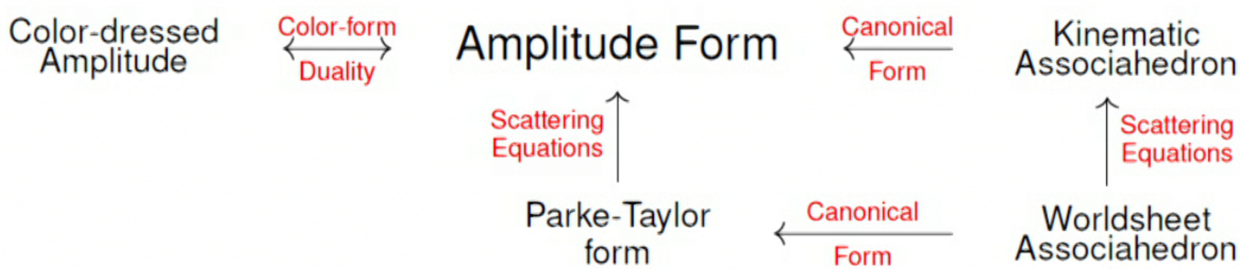
- The amplitude form ultimately connects all the ideas in this discussion.



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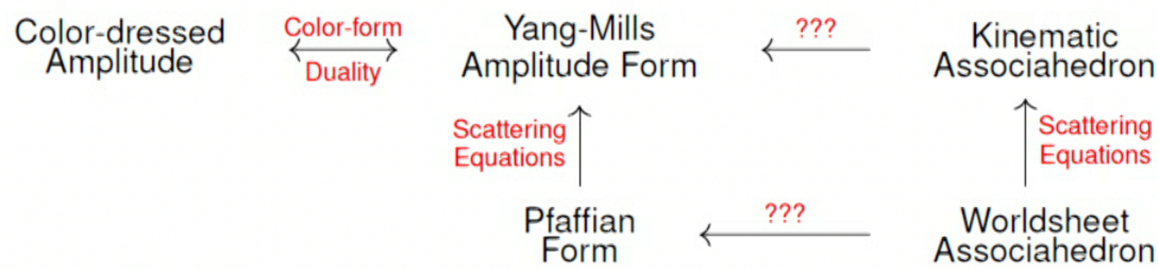


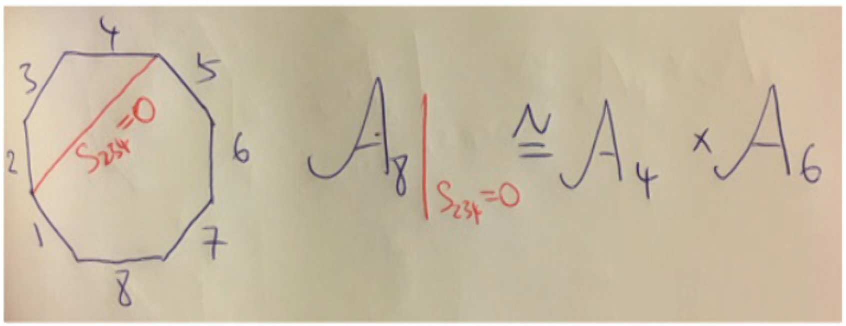
Locality and Unitarity

- Recall two basic properties of amplitudes: locality and unitarity.
- Both properties follow from the geometry of the associahedron.

Locality and Unitarity

- The associahedron "factorizes" combinatorially. That is, each facet $s_I = 0$ of the associahedron is the product of two lower associahedra:

$$\mathcal{A}_n|_{s_I=0} \cong \mathcal{A}_L \times \mathcal{A}_R$$



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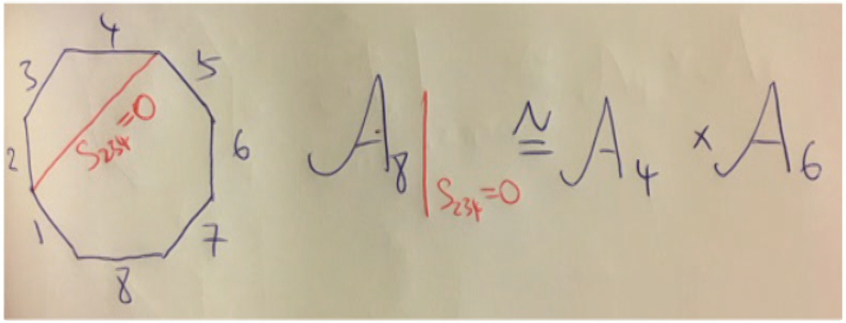

- It follows that

$$\text{Res}_{s_I=0}\Omega(\mathcal{A}_n) = \Omega(\mathcal{A}|_{s_I=0}) = \Omega(\mathcal{A}_L \times \mathcal{A}_R) = \Omega(\mathcal{A}_L) \wedge \Omega(\mathcal{A}_R)$$