Title: Higher-order quantum computations and causal structures

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Abstract: <p>Conventional quantum processes are described by quantum circuits, that represent evolutions of states of systems from input to output. In this seminar we consider transformations of an input circuit to an output circuit, which then represent the transformation of quantum evolutions. At this level, all the processes complying to admissibility conditions have in principle a physical realization scheme. The construction of a hierarchy of transformations of transformations, however, can proceed arbitrarily far, and in the higher orders one encounters admissible functions that have indefinite causal structures. These give rise to questions about possible realization schemes. Still, many of the maps in the hierarchy can be proved to have a realistic physical interpretation. In order to study the hierarchy, we introduce a simple rule for constructing new types of maps from known ones, and show how the tensor product can be rephrased in terms of the new rule. We use the hierarchy of types to introduce a partial order, which allows us to prove properties of maps by induction. We will then use induction proofs to discuss the characterisation of mathematically admissible maps at every level. We show an important structural result for a subclass of higher-order maps, and we conclude with the open question of their physical achievability.</p>

# Higher-order quantum computations and causal structures

Paolo Perinotti
Dipartimento di fisica - Università di Pavia

JOHN TEMPLETON FOUNDATION
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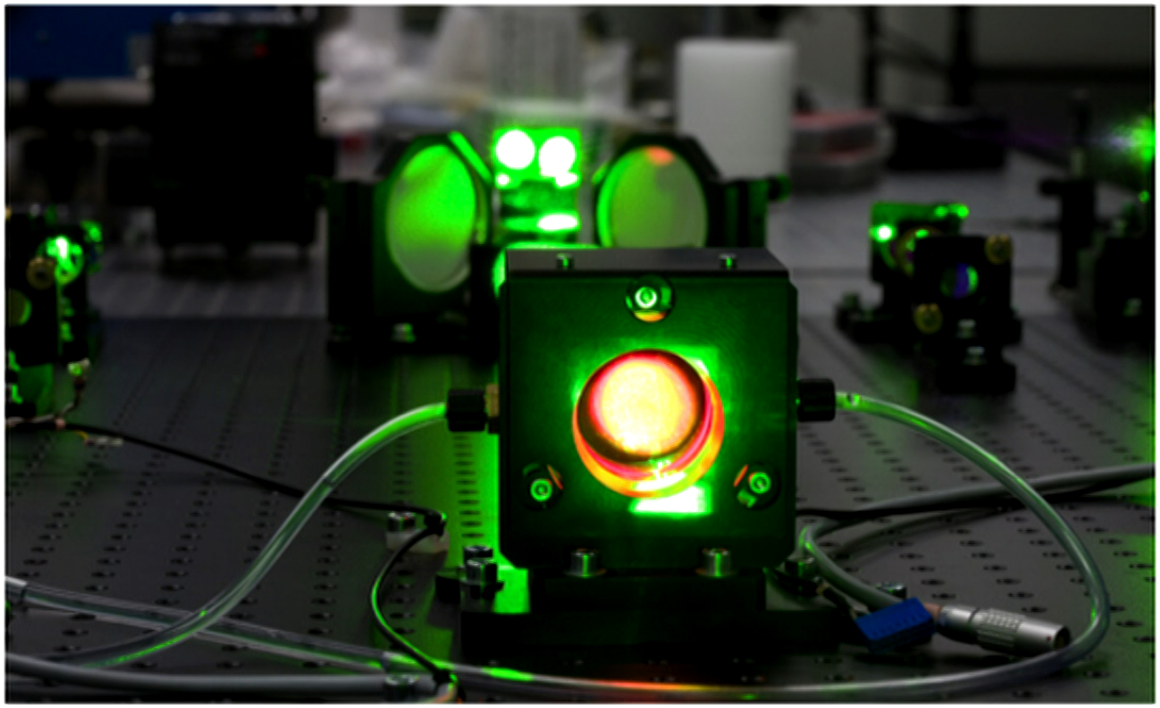
QUit quantum information theory group

Perimeter Institute - November 21 2017

# In collaboration with

A. Bisio

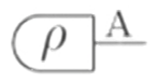


# Quantum channel

Sends input states into output states

# States, effects, transformations

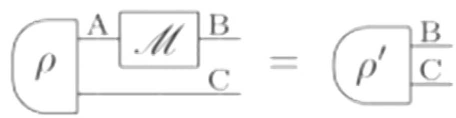
- State: density matrix



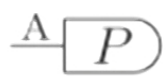

- Transformation: CP map



$$\mathscr{M}(\rho) = \sum_i K_i \rho K_i^\dagger$$




- Effect: positive operator

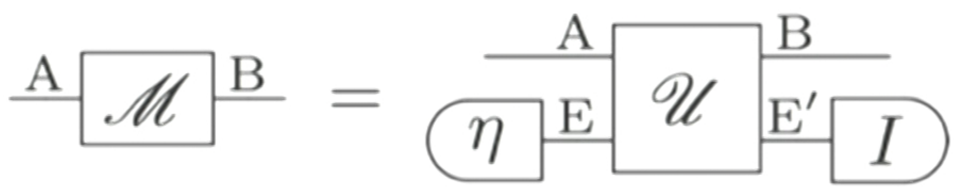




$= \mathrm{Tr}[\rho P]$

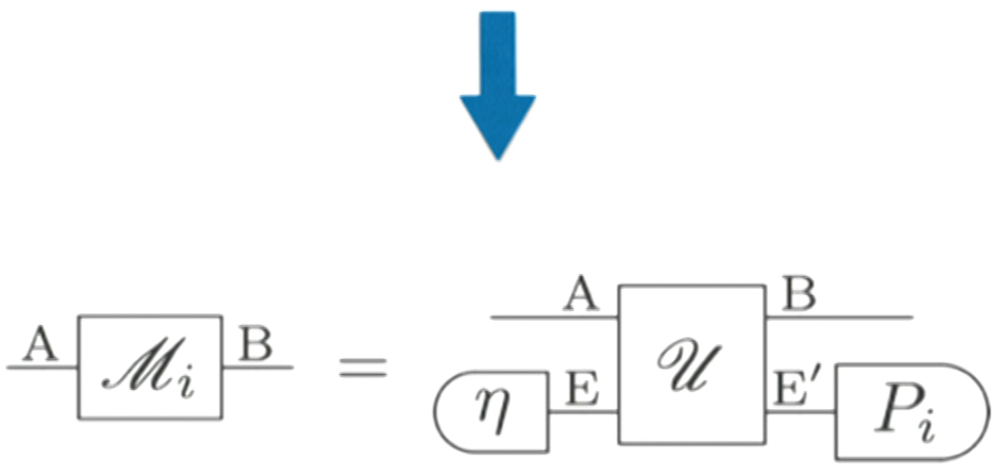
# Admissibility conditions

- Linear: preservation of convex combinations
- Completely Positive and trace non-increasing
    - Deterministic: trace preserving



# Non-deterministic maps

Quantum instrument: $\{\mathscr{M}_i\}, \quad \sum_i \mathscr{M}_i = \mathscr{M}$



# Causality and time

- Causality: no signalling from output to input

$$p_Q(\rho_i) := \sum_j \; \left( \rho_i \right] \xrightarrow{\mathrm{A}} \left[ Q_j \right) = p(\rho_i) = \mathrm{Tr}\,\rho_i$$

- In general Operational Probabilistic theories

$$p_Q(\rho_i) := \sum_j \; \left( \rho_i \right] \xrightarrow{\mathrm{A}} \left[ a_j \right) = p(\rho_i)$$

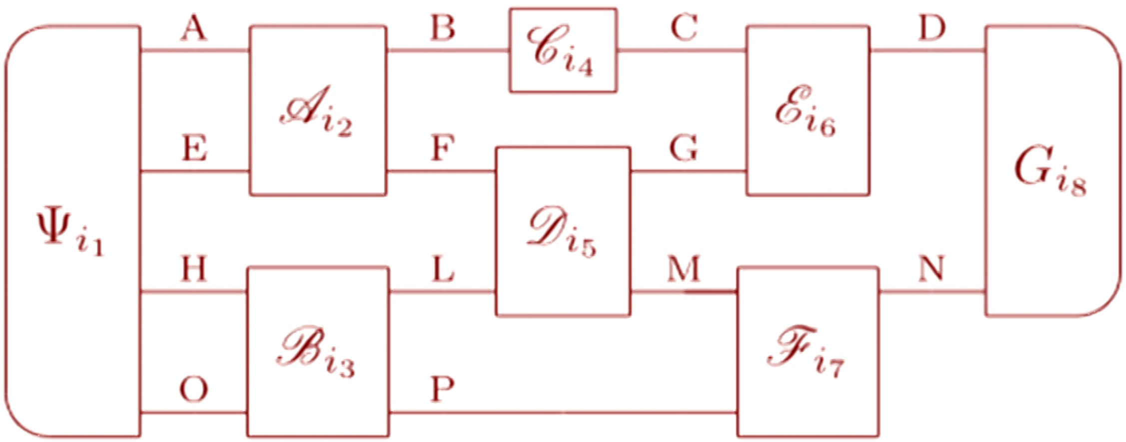
- Equivalently: unique deterministic effect

$$\sum_j \; \xrightarrow{\mathrm{A}} \left[ Q_j \right) = \xrightarrow{\mathrm{A}} \left[ e \right)$$

G. Chiribella, G. M. D'Ariano, PP, Phys. Rev. A **81**, 62348 (2010)

# Causality and time

- No signalling without interaction

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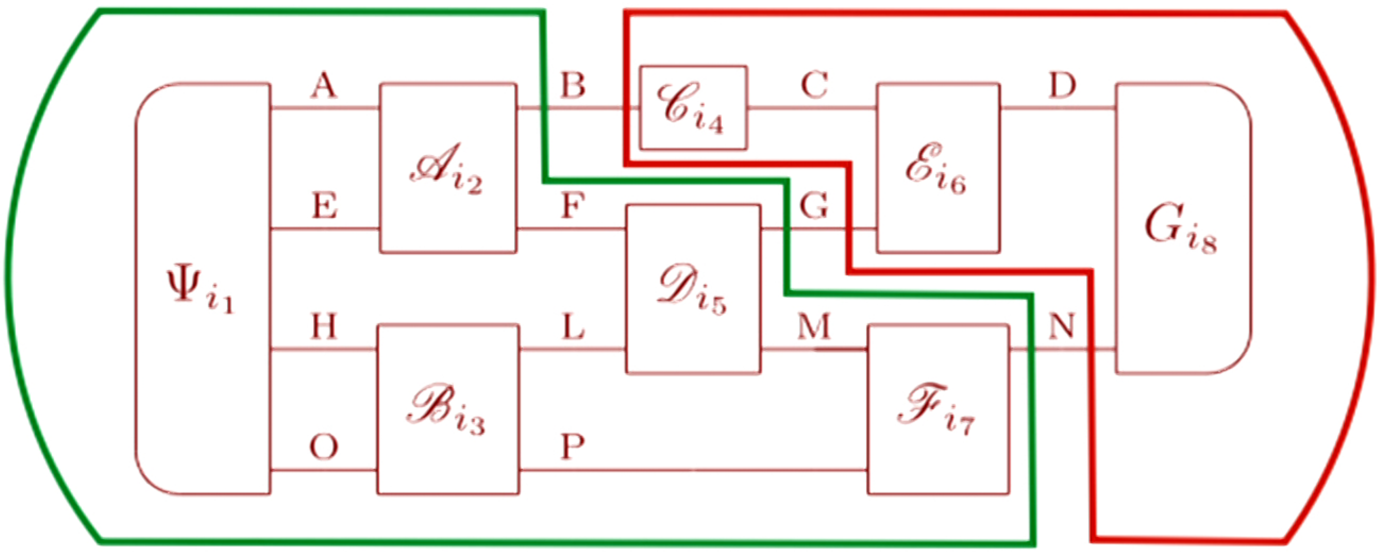


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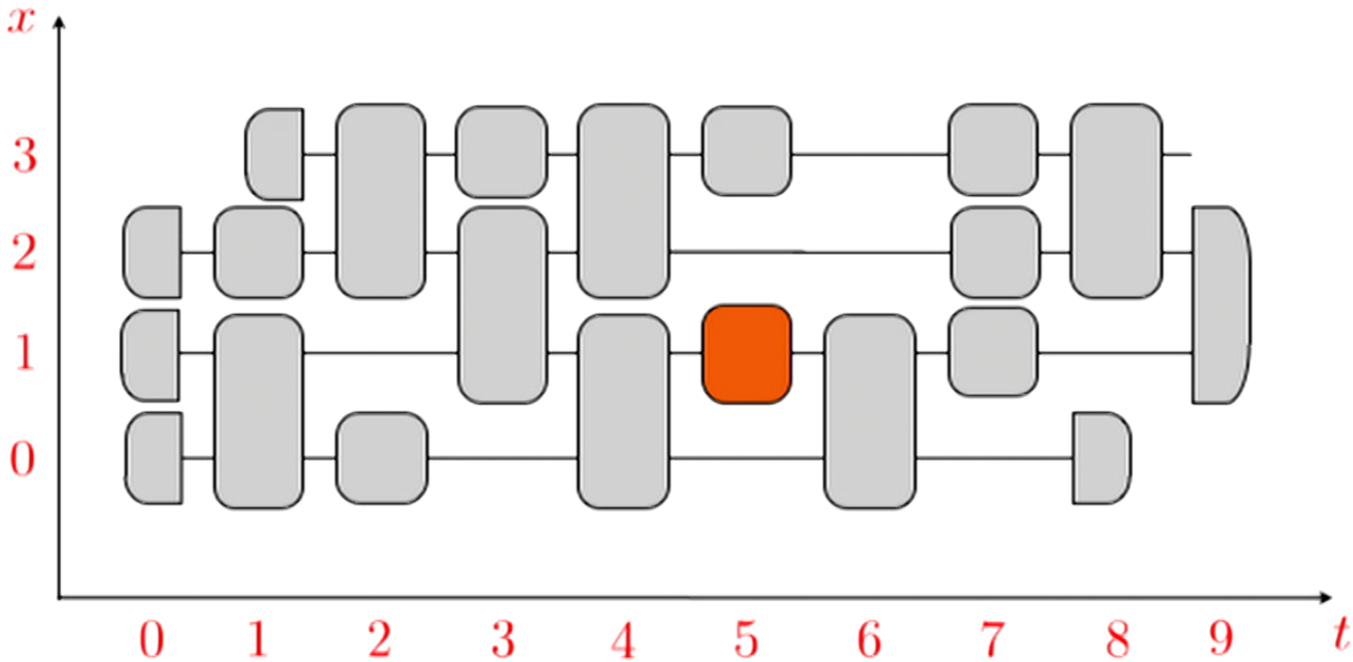
# Computational space-time

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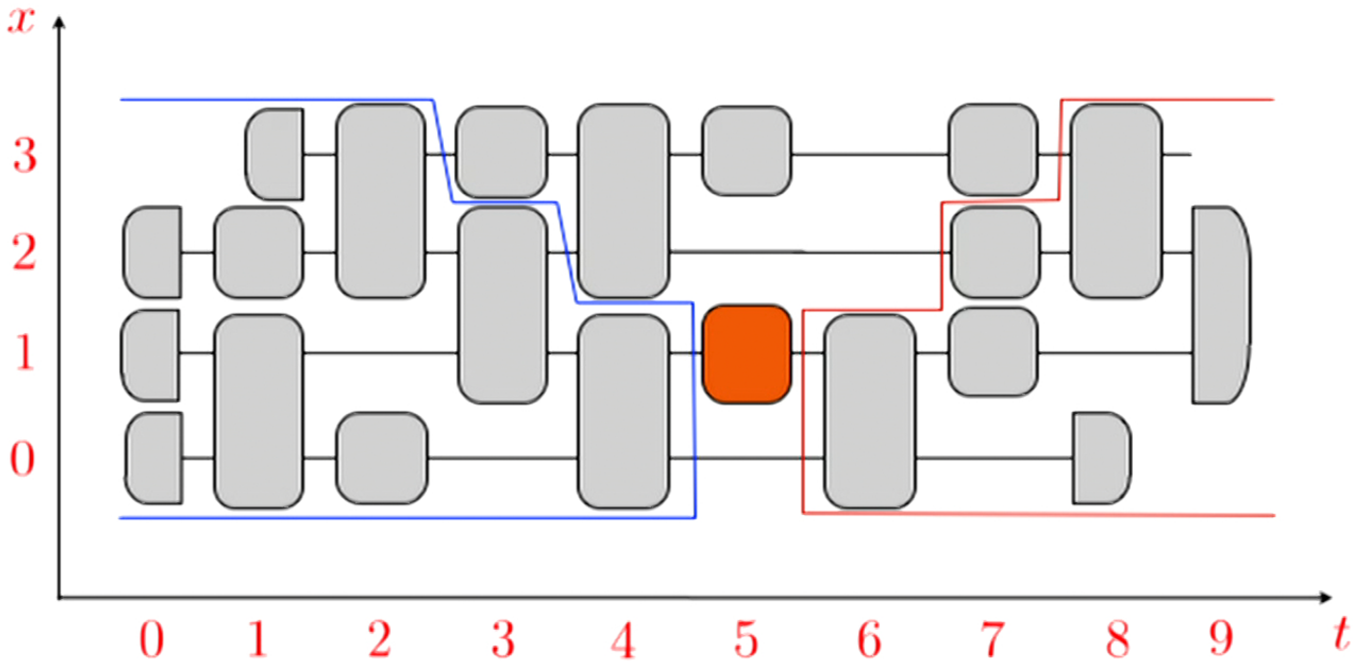
# Computational space-time

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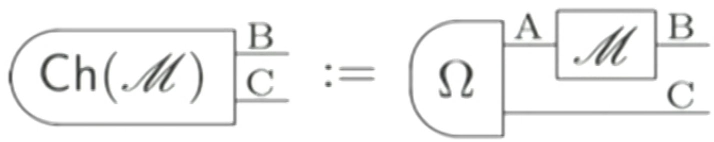
# Choi isomorphism

- Invertible map between linear maps and operators




$$\Omega := |\Psi\rangle\langle\Psi|, \quad |\Psi\rangle := \sum_{n=1}^{d_A} |n\rangle|n\rangle$$

- The map $\mathscr{M}$ is CP iff Ch($\mathscr{M}$) is *positive*
- We can treat maps as positive operators
    - Maps on maps are CP maps

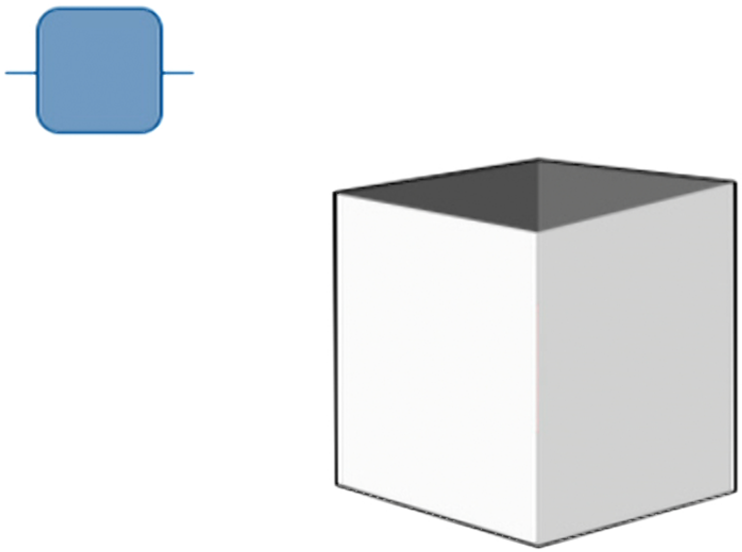
# Normalization

- Deterministic states: $\mathrm{Tr}\rho = 1$


- Channels (det. transf.): $\mathrm{Tr}_{\mathrm{B}}[R] = I_{\mathrm{A}}$


- Deterministic effect: $I_{\mathrm{A}}$

# Transformations of transformations



# Transformations of transformations

- Causal ordering between input and output is lost

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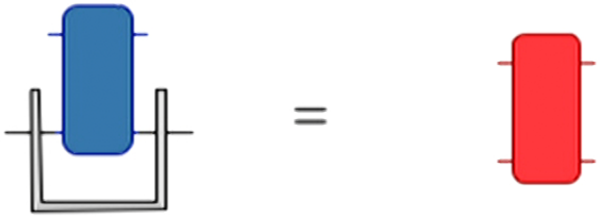
- Example: given a fixed (green) channel


- The red channel does not come *after* the blue one

# Admissibility conditions

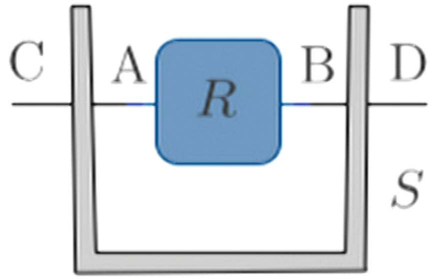
- Linearity: preservation of convex combinations
- Complete Positivity



- Deterministic maps: preservation of channels
  - Probabilistic map: has a complement to a det.

G. Chiribella, G. M. D'Ariano and PP, Europhys. Lett. **83**, 30004 (2008)

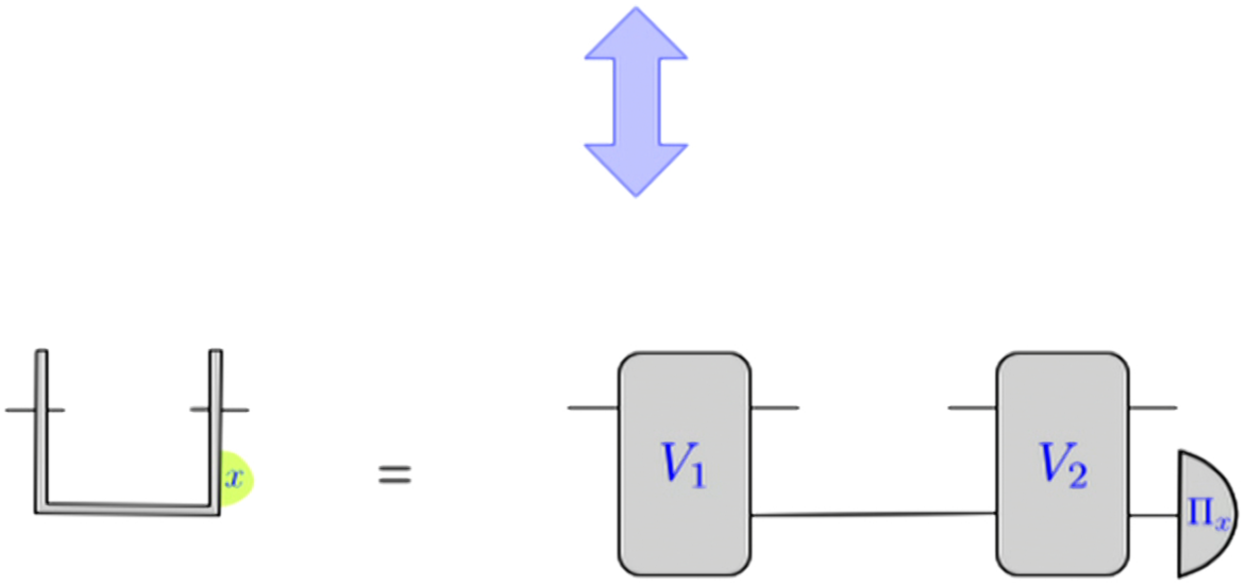
# Normalization




- Channels: $\mathrm{Tr}_{\mathrm{B}}[R] = I_{\mathrm{A}}$
- Deterministic maps: $\begin{cases} \mathrm{Tr}_{\mathrm{D}}[S] = I_{\mathrm{B}} \otimes S' \\ \mathrm{Tr}_{\mathrm{A}}[S'] = I_{\mathrm{C}} \end{cases}$
- Deterministic effect: $P = I_{\mathrm{B}} \otimes \rho_{\mathrm{A}}$

# General theorem

Admissibility conditions



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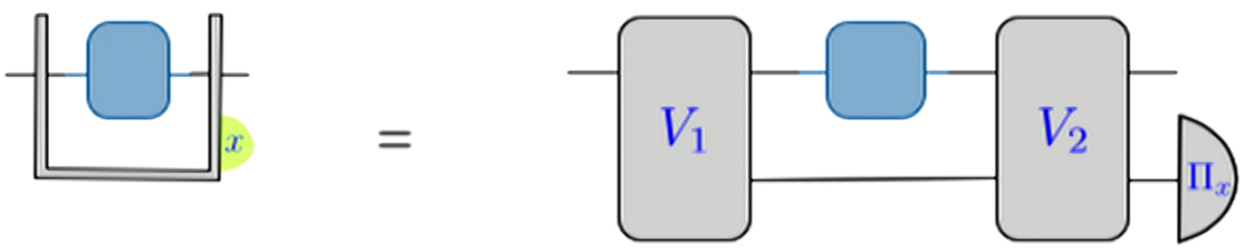
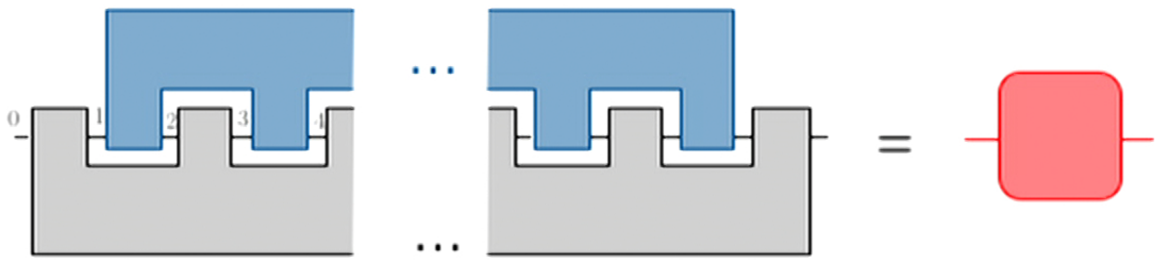


# Hierarchy of combs

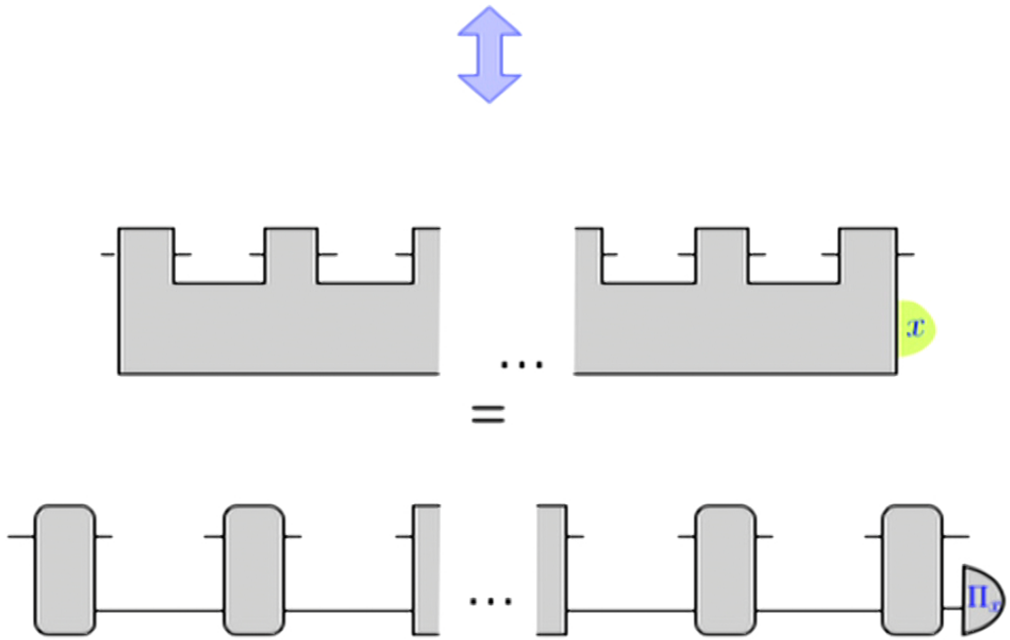
- 1-comb: quantum channel/operation
- N-comb: map from (N-1)-combs to 1-combs




$$\mathrm{Tr}_{2j-1}[R^{(j)}] = I_{2j-2} \otimes R^{(j-1)}, \quad 2 \leqslant j \leqslant N$$

# Realisation Theorem

Admissibility conditions




G. Chiribella, G. M. D'Ariano and PP, Phys. Rev. Lett. **101**, 060401 (2008)
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# Realisation Theorem

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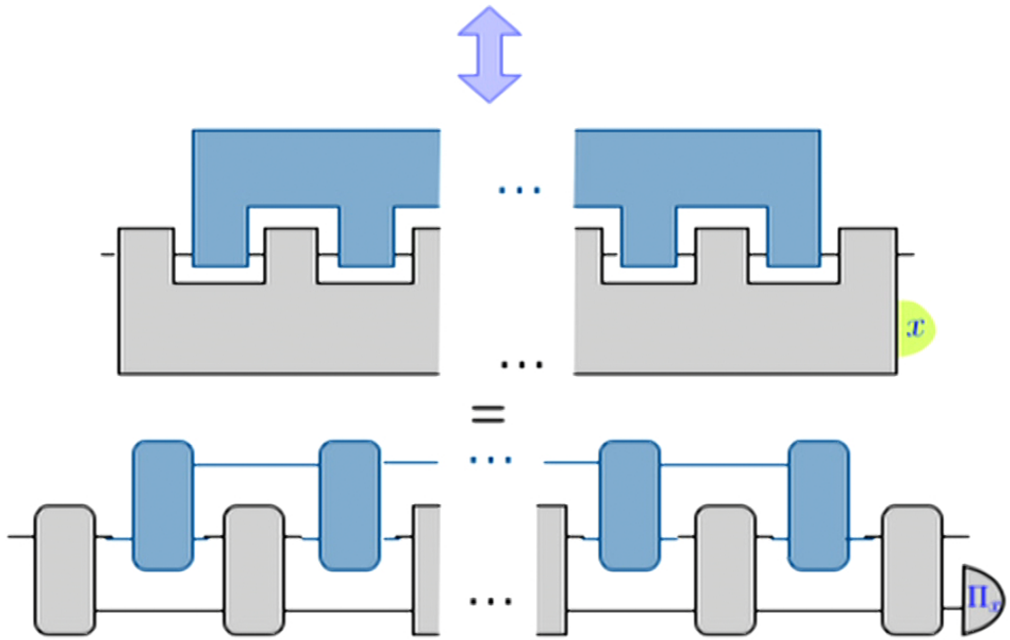



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# Why generalisations?

- Combs: one-to-one correspondence with circuits

G. Chiribella, G. M. D'Ariano and PP, Phys. Rev. A **80**, 022339 (2009)
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# Why generalisations?

- Combs: one-to-one correspondence with circuits
- Do we need more maps?
  - Higher order transformations
  - Proved to provide advantages
- Largely unexplored subject

G. Chiribella, G. M. D'Ariano and PP, Phys. Rev. A **80**, 022339 (2009)
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# Hierarchy of transformations

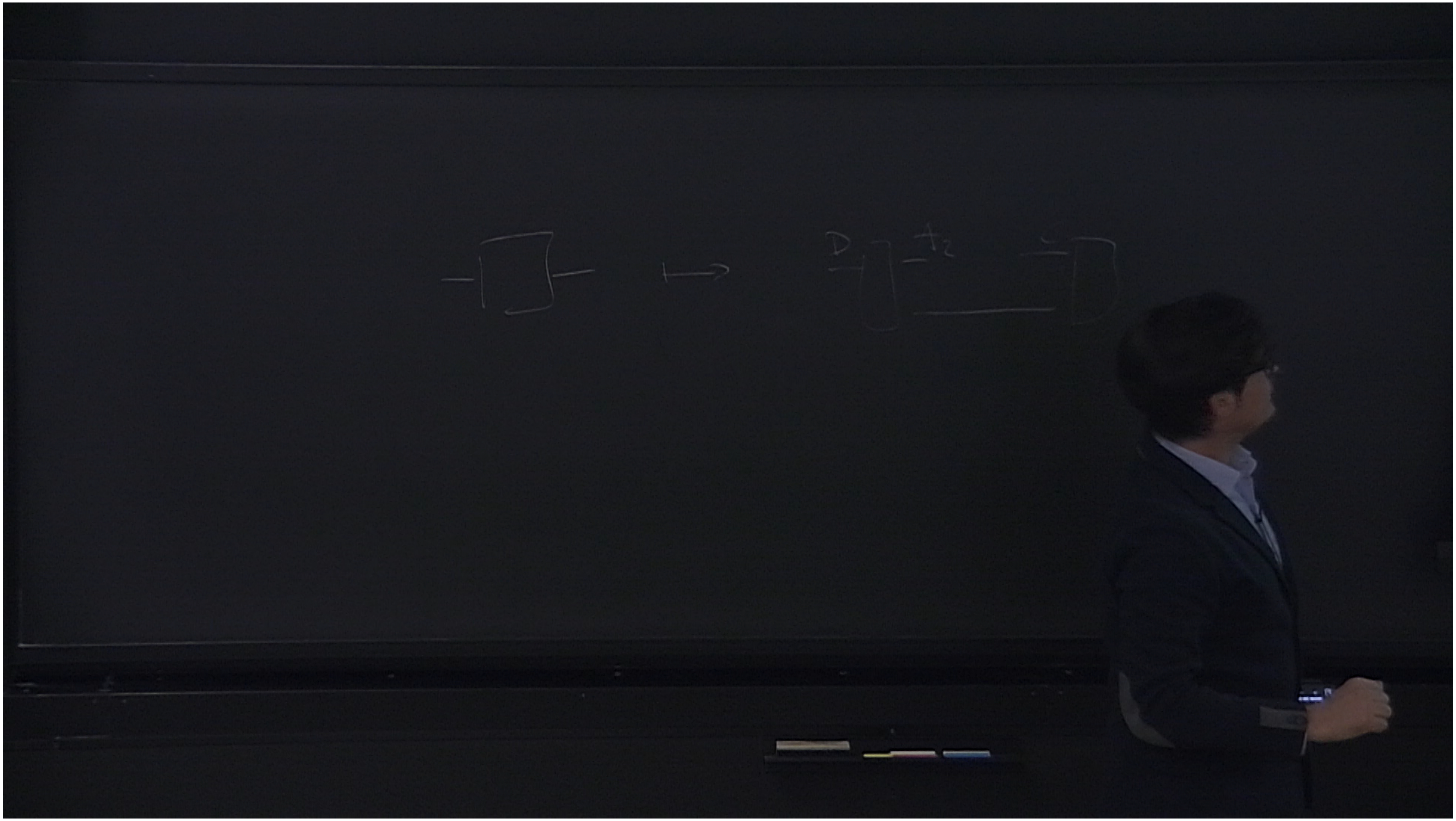
- Elementary system types $\mathrm{A}, \mathrm{B}, \ldots, \mathrm{Z}$ $\mathrm{I}$
- Sequential composition $\rightarrow: x, y \mapsto (x \rightarrow y)$
  - e.g. $\rightarrow: \mathrm{A}, \mathrm{B} \mapsto (\mathrm{A} \rightarrow \mathrm{B})$
  - Linear functionals $\bar{x} := (x \rightarrow \mathrm{I})$
    - e.g.: effects $\bar{\mathrm{A}}$

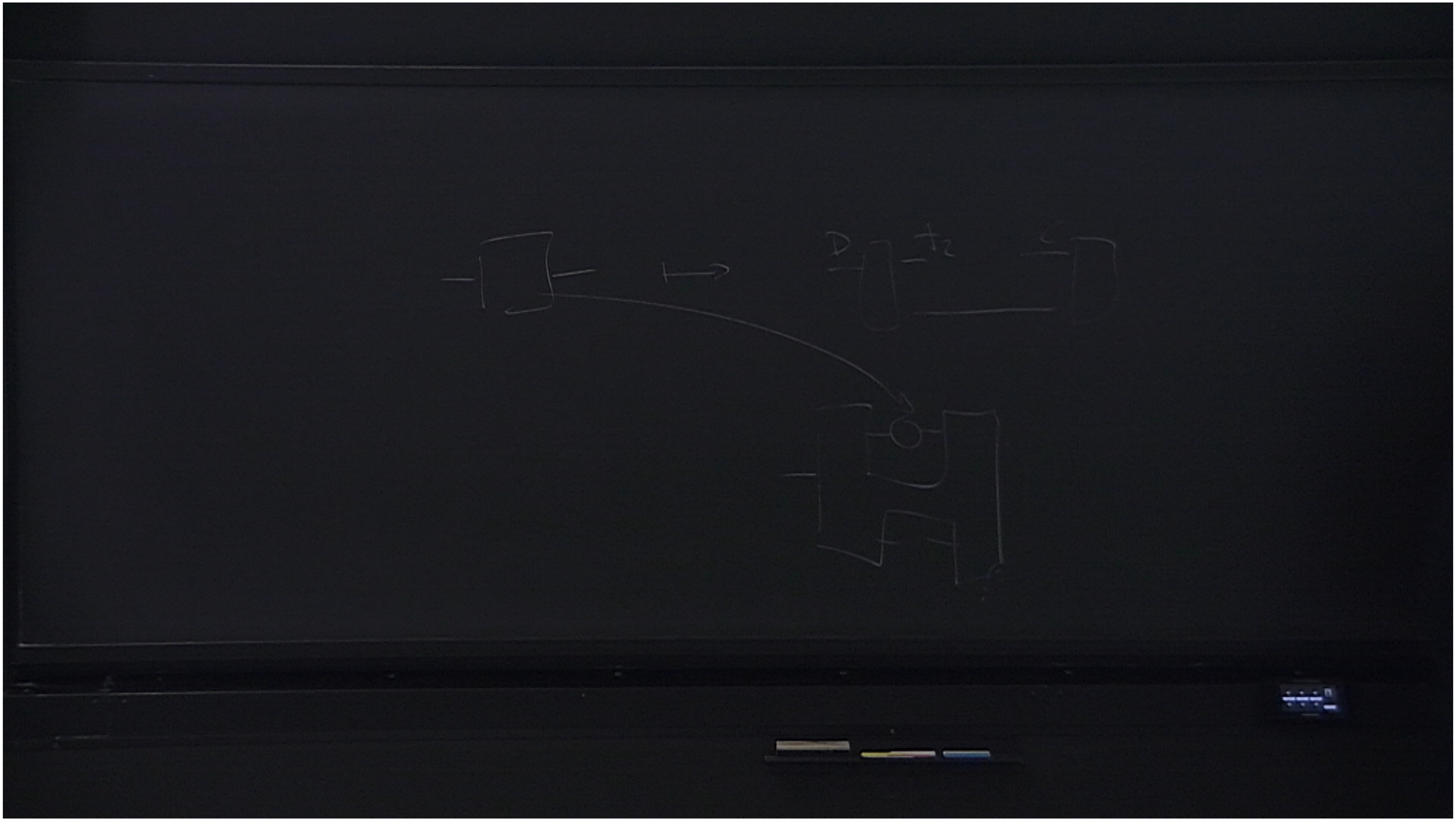
PP, in "Time in Physics", R. Renner and S. Stupar eds., Springer (2017)

# Types of transformations

- General type $x = (\mathrm{A}_1 \to \mathrm{B}) \to ((\mathrm{A}_2 \to \mathrm{C}) \to \bar{\mathrm{D}})$
  - Basic rule $\mathrm{I} \to x = x$
- Ordering of types $x \to y \succ x, y$
  - Well founded: it allows for proofs by induction

PP, in "Time in Physics", R. Renner and S. Stupar eds., Springer (2017)





# Extension of types

- Extension by an elementary system

$$(x \to y) \parallel \mathrm{A} := x \to (y \parallel \mathrm{A})$$

$$\mathrm{B} \parallel \mathrm{A} := \mathrm{BA}$$

A. Bisio and PP, in preparation

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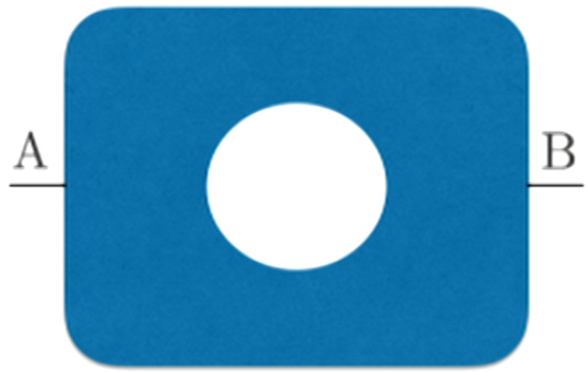


$$y = x \to (\mathrm{A} \to \mathrm{B})$$

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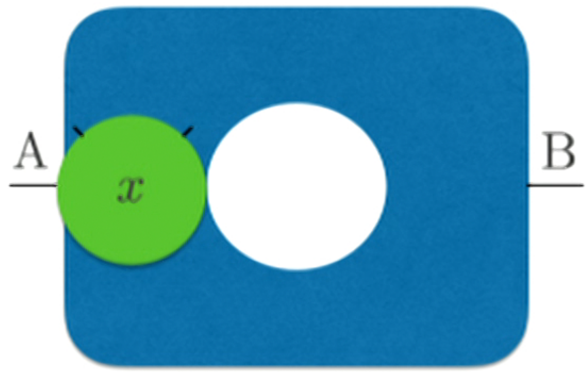


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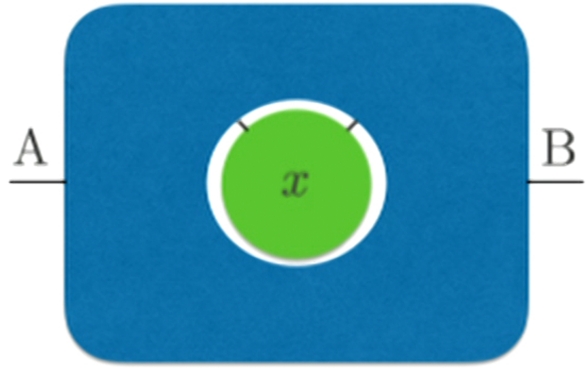


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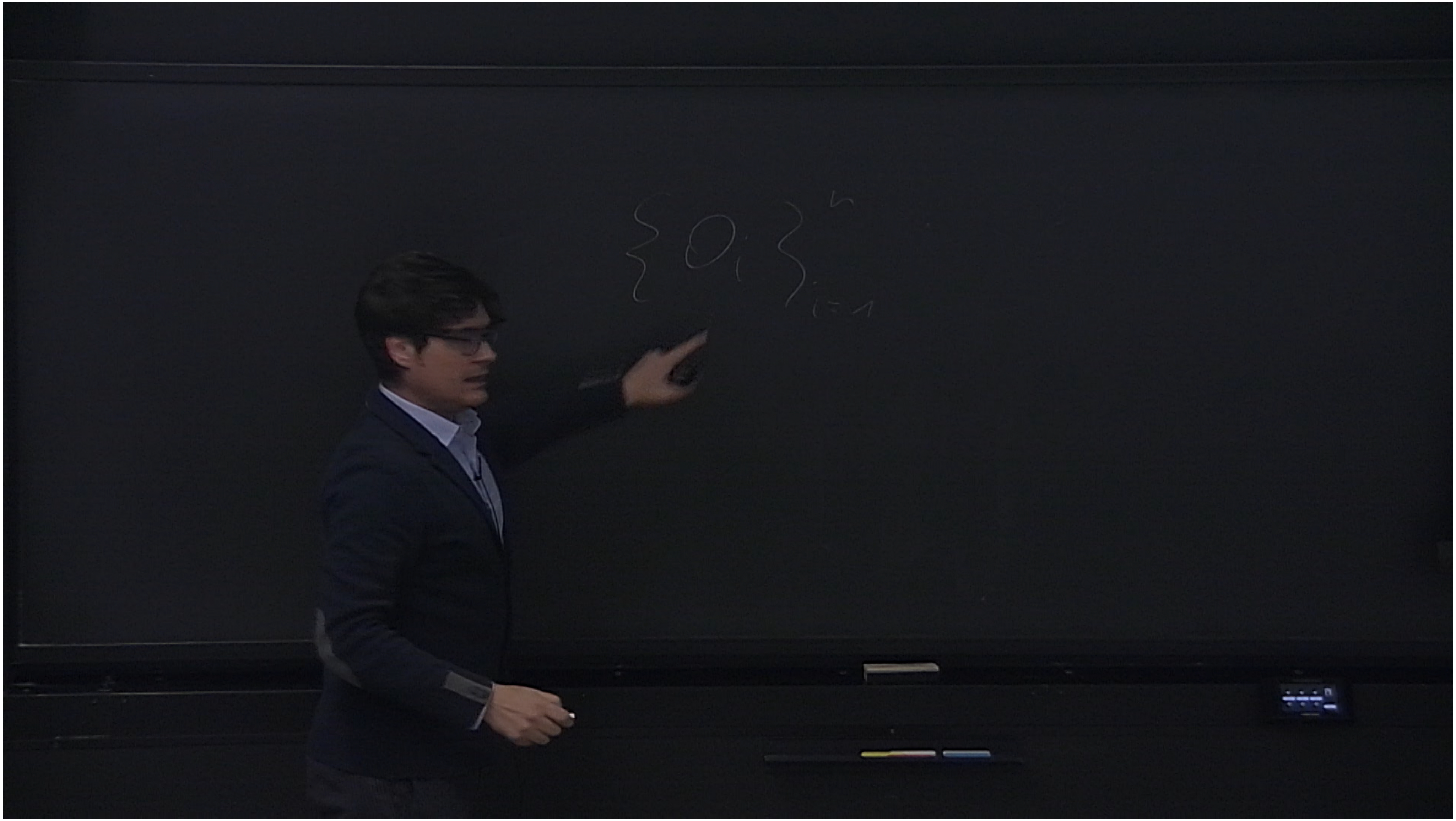
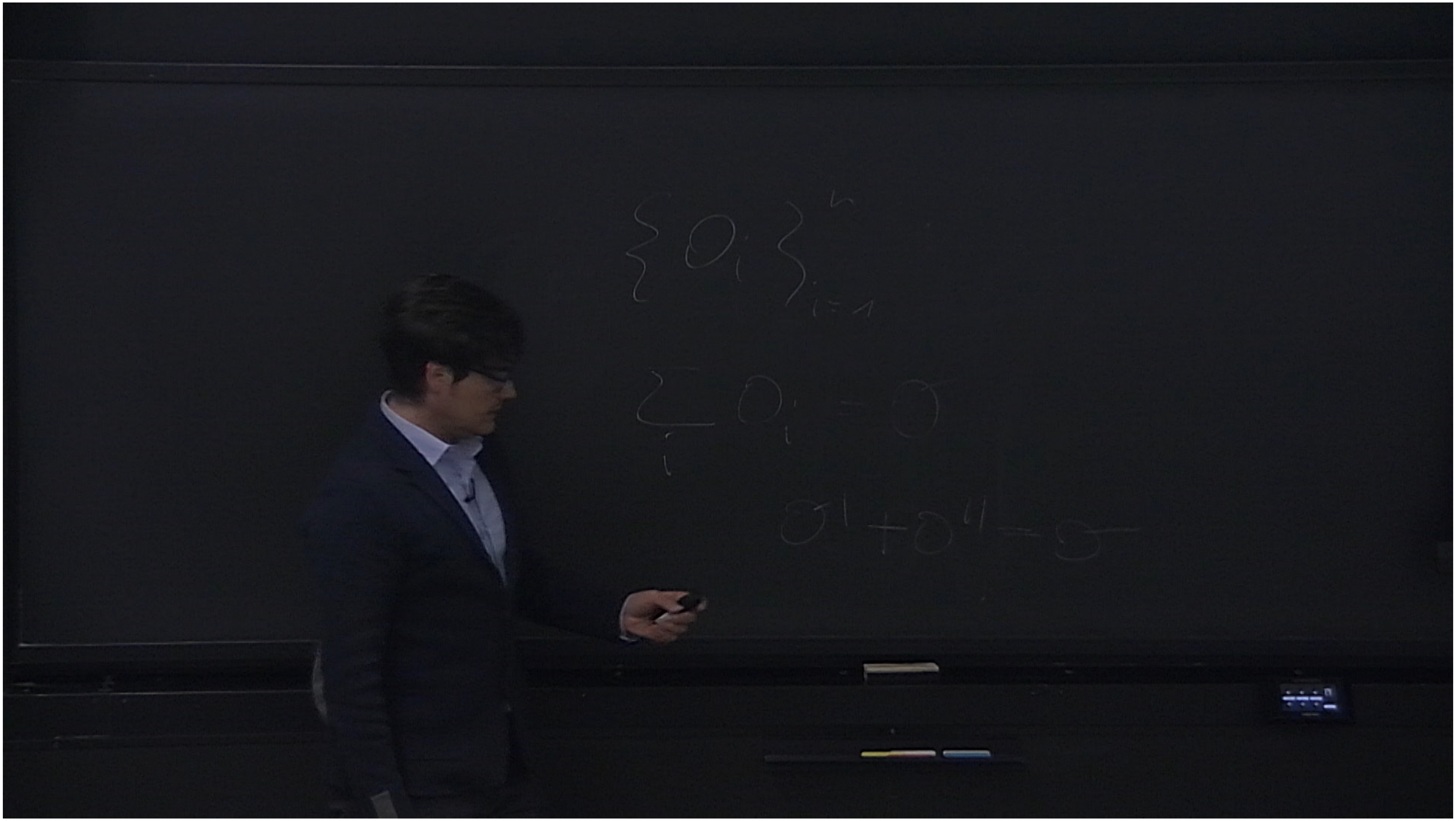
A. Bisio and PP, in preparation

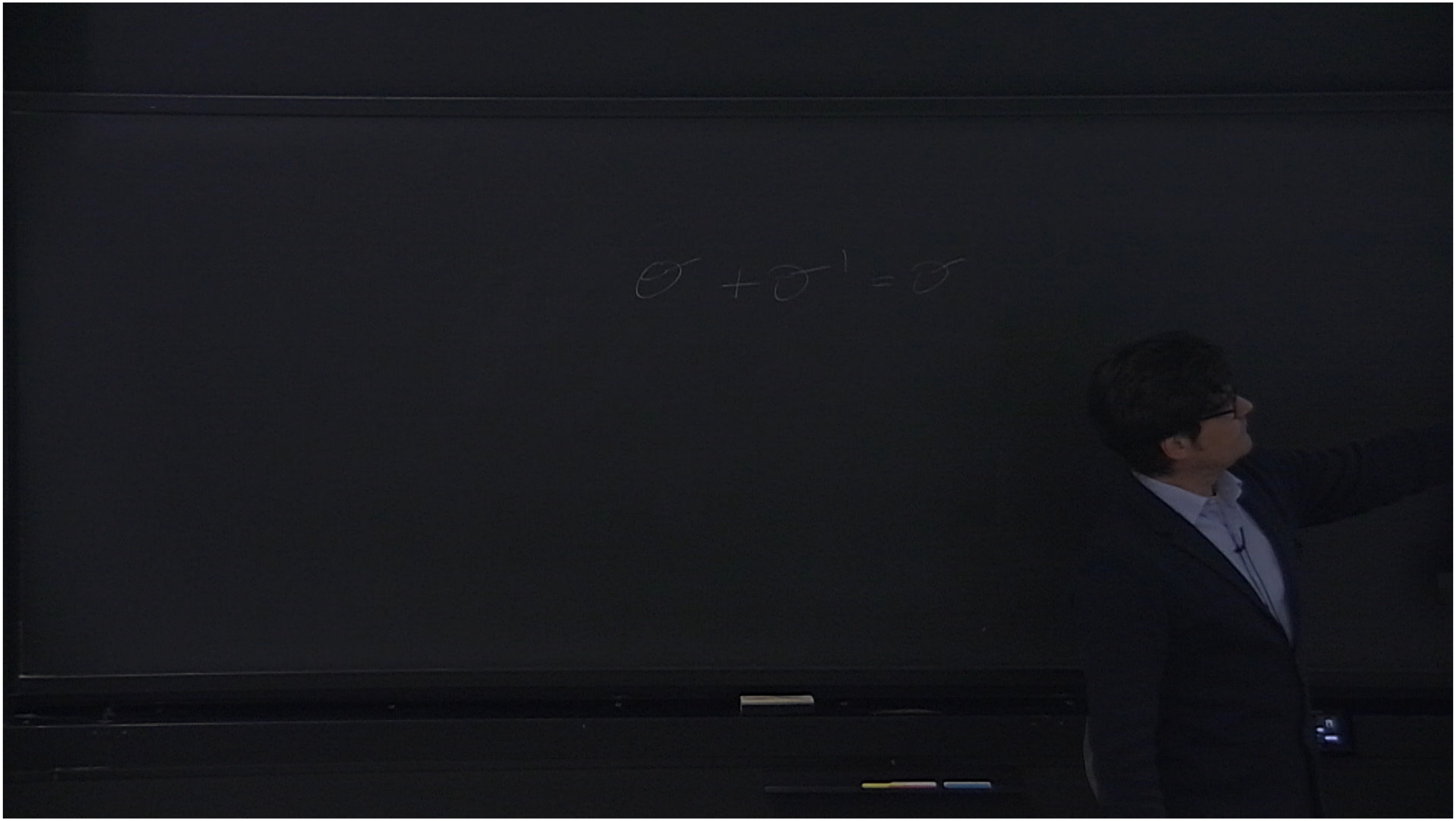
# Admissibility axioms

- Admissibility: $\mathscr{T}$ of type $x \to y$ is admissible if
  - $\mathscr{T} \otimes \mathscr{I}_{\mathrm{A}}$ maps $x \parallel A$ to $y \parallel A$ preserving admissibility
  - There is $\{\mathscr{S}_i\}_{i=1}^{n}$ admissible s.t. $\mathscr{T} + \sum_{i=1}^{n} \mathscr{S}_i$ is admissible and preserves deterministic property
- By induction: $\mathscr{T}$ admissible* iff $\mathsf{Ch}(\mathscr{T}) \geq 0$
- Ready for general OPTs

PP, in "Time in Physics", R. Renner and S. Stupar eds. Springer (2017)
A. Bisio and PP, in preparation


$\{O_i\}_{i=1}^{n}$


$\{O_i\}_{i=1}^{n}$
$\sum_i O_i = O$



# Types and cones

- Type $x$ → cone of positive operators $\mathsf{K}(x)$
  - Deterministic maps: $\mathsf{K}(x) \cap \mathsf{N}(x)$
  - Probabilistic map: $S \leq T, \quad T$ deterministic
- The correspondence is not one-to-one
  - e.g. $\mathsf{K}(\mathrm{AB}) = \mathsf{K}(\mathrm{A} \to \mathrm{B})$
- Type structures of combs: number of teeth $\mathrm{m}$

# Normalisation constraints

- Types with same cone: different normalization
  - e.g. $\mathsf{K}(\mathrm{AB}) = \mathsf{K}(\mathrm{A} \to \mathrm{B})$

$$\mathsf{N}(\mathrm{AB}) = \{\rho \geq 0 | \mathrm{Tr}\rho = 1\}$$
$$\mathsf{N}(\mathrm{A} \to \mathrm{B}) = \{R \geq 0 | \mathrm{Tr}_{\mathrm{B}} R = I_{\mathrm{A}}\}$$

- We define $x = y \;\Leftrightarrow\; \mathsf{K}(x) = \mathsf{K}(y),\; \mathsf{N}(x) = \mathsf{N}(y)$
- Normalisation: crucial for type equivalence

# Parallel composition

$$x \otimes y := \overline{x \to \bar{y}}$$

$$\mathsf{K}(x \otimes y) = \{R \in \mathcal{L}(\mathsf{H}_x \otimes \mathsf{H}_y) \mid R \geq 0\}$$
$$\mathsf{N}(x \otimes y) = \mathsf{Aff}\{X \otimes Y | X \in \mathsf{N}(x), Y \in \mathsf{N}(y)\}$$

- Maps normalized on the convex hull are normalized also on the affine span

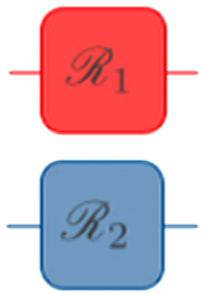
- Parallel composition is associative

$$(x \otimes y) \otimes z = x \otimes (y \otimes z)$$

PP, in "Time in Physics", R. Renner and S. Stupar eds., Springer (2017)

# Parallel composition

$x \otimes y$ ?

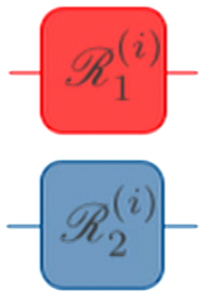



$R_1 \otimes R_2$

# Parallel composition

$x \otimes y$ ?

$$\sum_i p_i$$



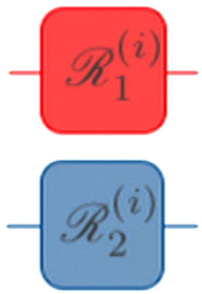

$$\sum_i p_i R_1^{(i)} \otimes R_2^{(i)}$$

LOSR*

# Parallel composition

$x \otimes y$ ?

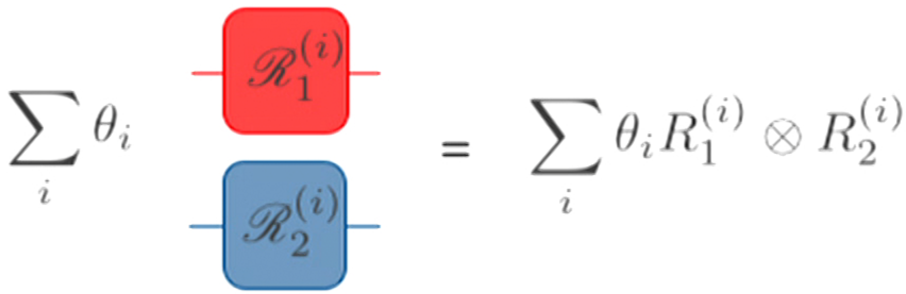
$\sum_i \theta_i$

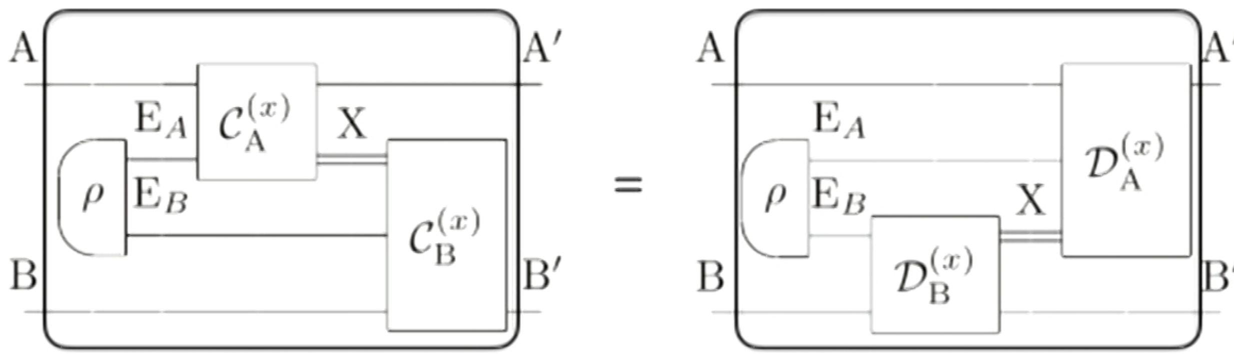



$\sum_i \theta_i R_1^{(i)} \otimes R_2^{(i)}$

No-signalling

# Non-signalling channels




G. M. D'Ariano, S. Facchini and P.P., PRL **106**, 010501 (2011)

# Parallel composition

$$x \otimes y = \overline{x \to \overline{y}}$$

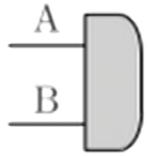
Why not $x \to y = \overline{x \otimes \overline{y}}$ ?  $\overline{z} := z \to I$

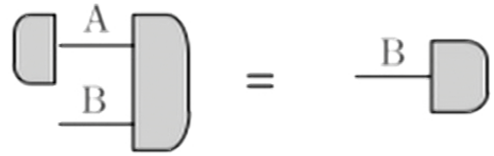
# Parallel composition

$$x \otimes y = \overline{x \rightarrow \overline{y}}$$

Why not $x \rightarrow y = \overline{x \otimes \overline{y}}$ ? $\quad \overline{z} := z \rightarrow I$

Example: $\mathrm{AB} = \overline{\mathrm{A} \rightarrow \overline{\mathrm{B}}}$

$\mathrm{A} \rightarrow \overline{\mathrm{B}} \ni$ 


$\overline{\mathrm{A} \rightarrow \overline{\mathrm{B}}} \ni$ 

# Uncurrying

- Associativity of ⊗ is equivalent to

$$x \to (y \to z) = (x \otimes y) \to z$$

PP, in "Time in Physics", R. Renner and S. Stupar eds., Springer (2017)

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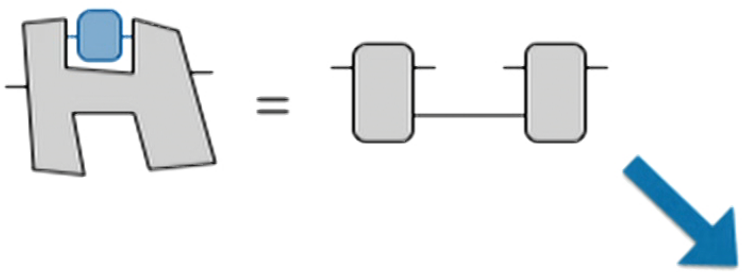

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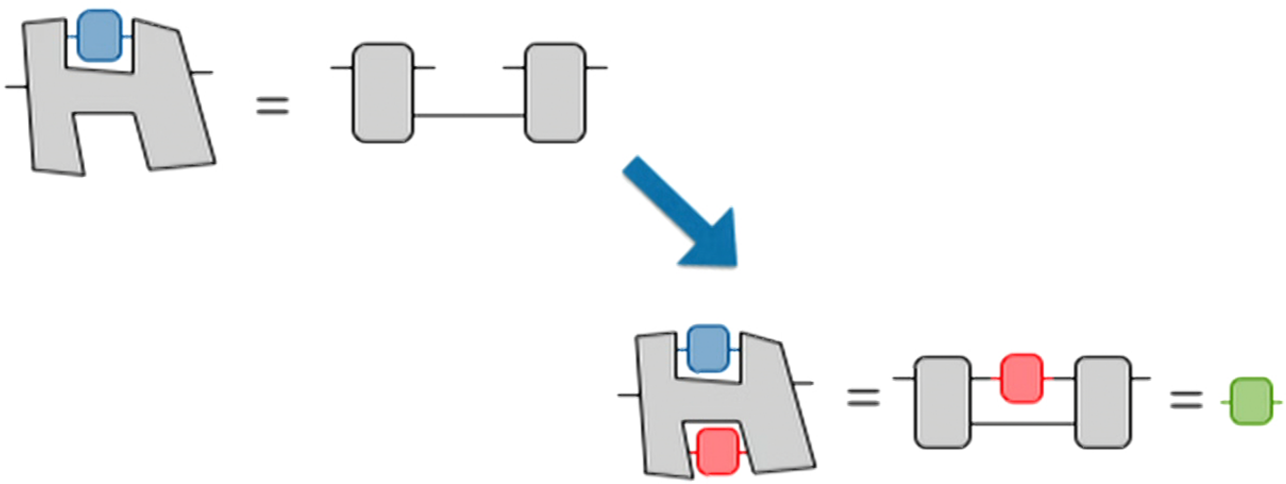

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# Uncurrying

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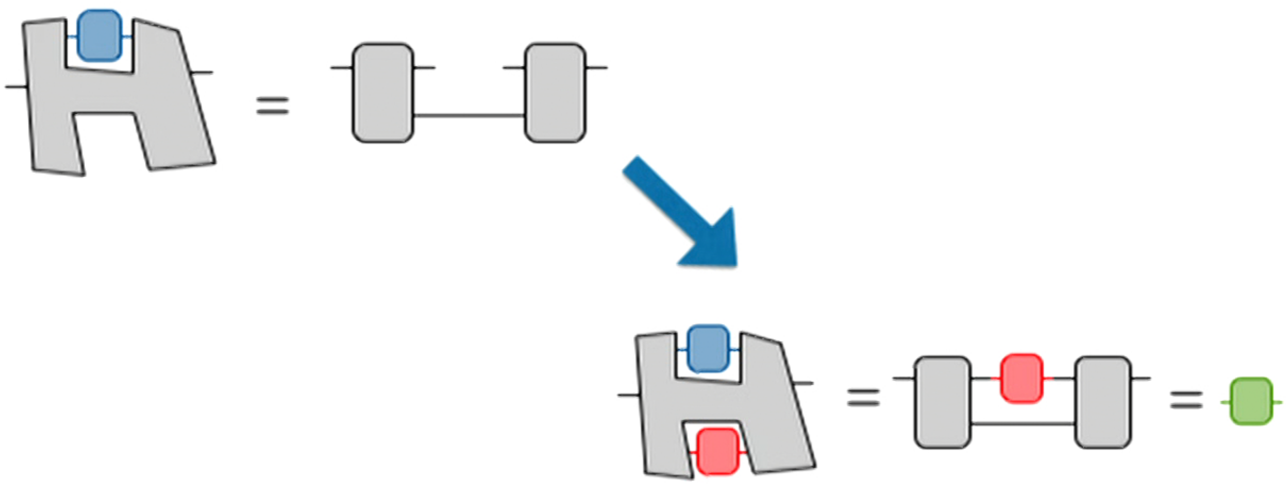
$$h(x, y) := g_x(y)$$



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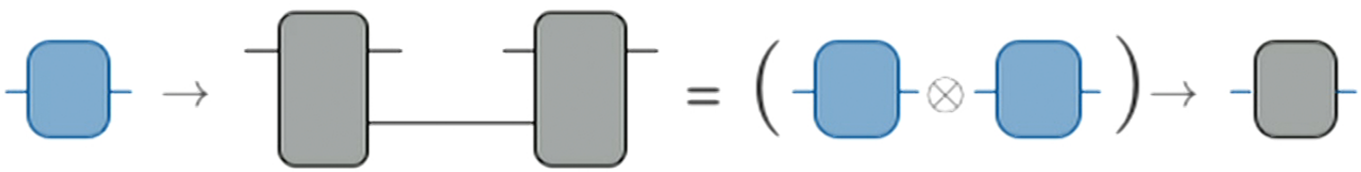
# Maps on tensor products

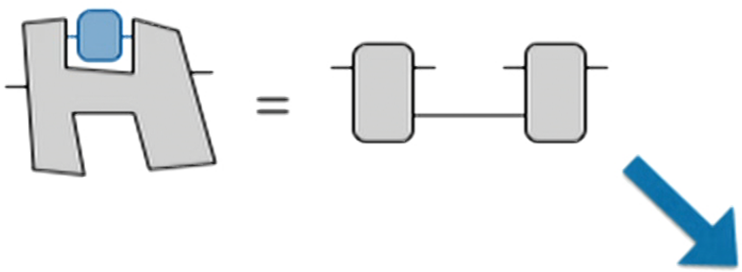
- How shall we define the type $\mathrm{m} \to \mathrm{n}$ ?
- Remind $\mathrm{n} = (\mathrm{n}-1) \to 1$
- Uncurrying $\mathrm{m} \to ((\mathrm{n}-1) \to 1) = (\mathrm{m} \otimes (\mathrm{n}-1)) \to 1$
  - e.g. $1 \to 2 = (1 \otimes 1) \to 1$



# Uncurrying

- Associativity of ⊗ is equivalent to

$$x \to (y \to z) = (x \otimes y) \to z$$



PP, in "Time in Physics", R. Renner and S. Stupar eds., Springer (2017)

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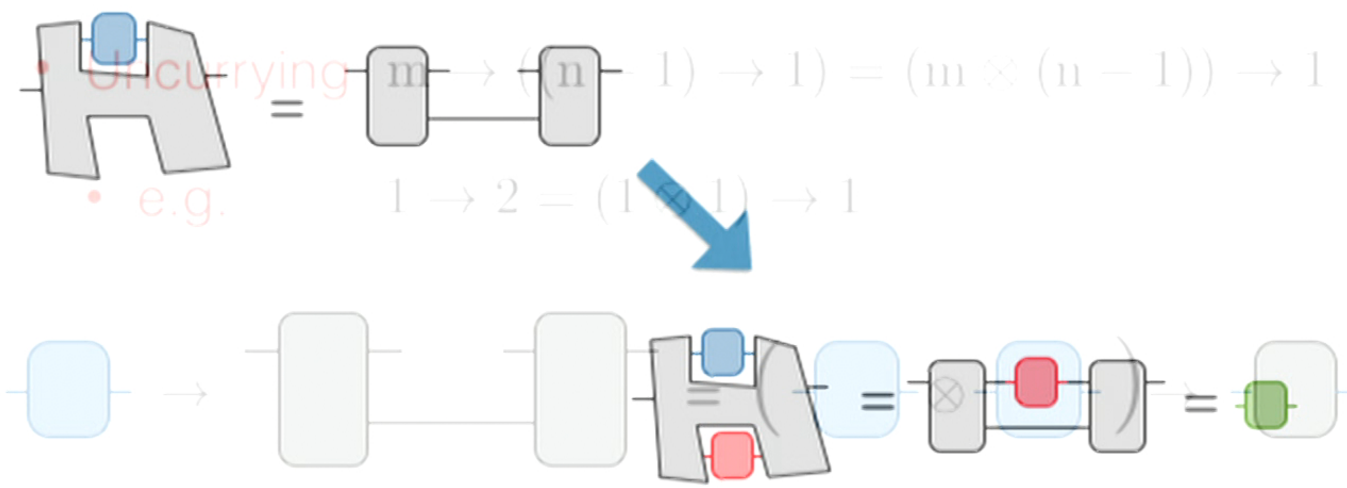
$$h(x, y) := g_x(y)$$



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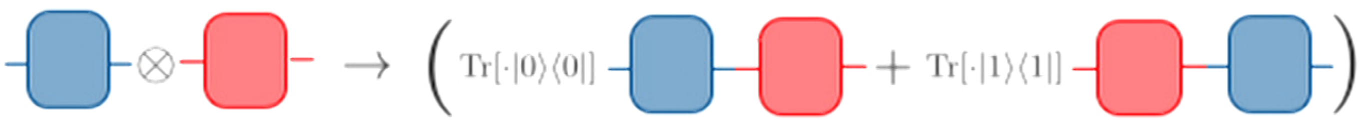
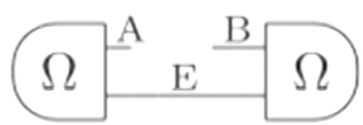
# The SWITCH

- Example of map $(1 \otimes 1) \to 1$




No-switch theorem:
a **quantum circuit** implementing the switch map
would be equivalent to a "time loop"




G. Chiribella, G. M. D'Ariano, PP, and B. Valiron, PRA **88**, 022318 (2013)

# Mathematical structure

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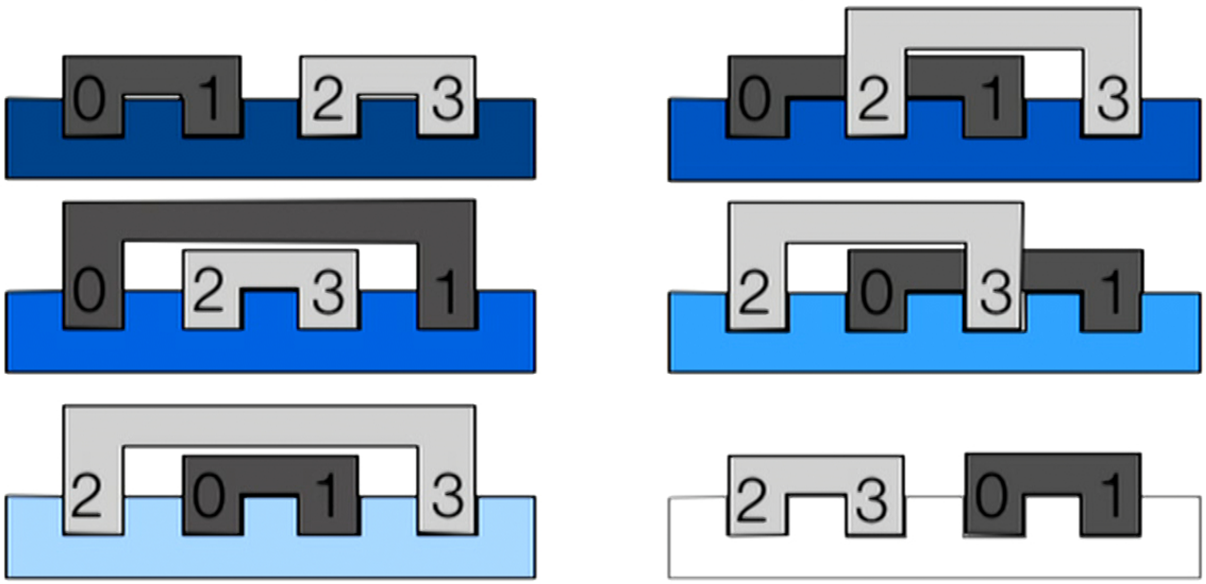

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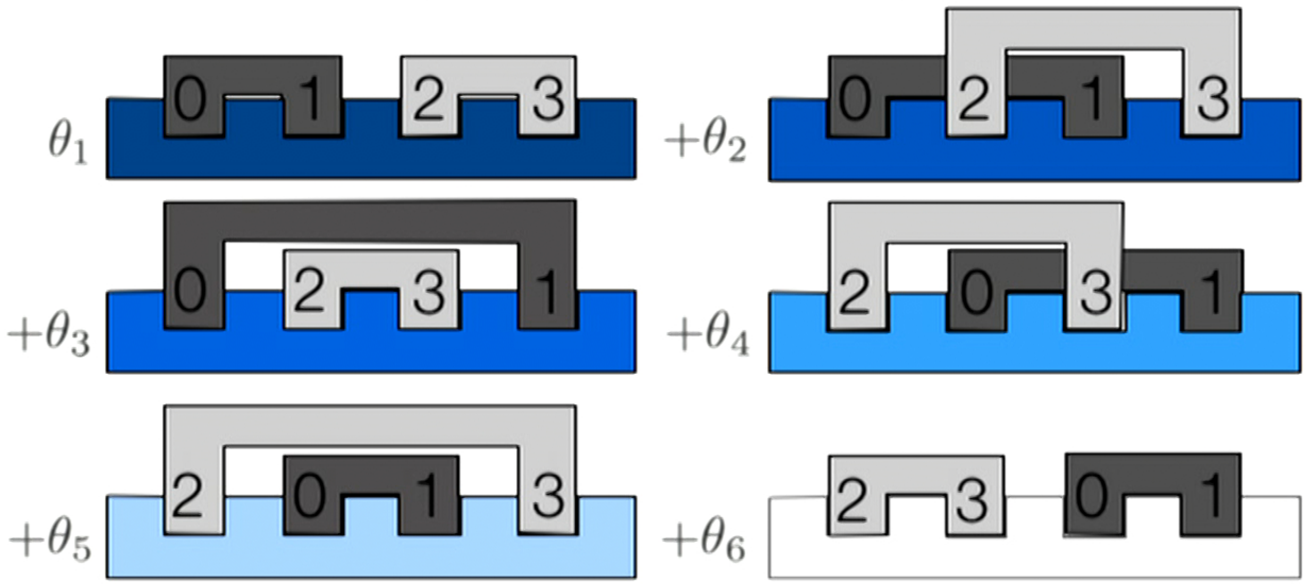


Or what else?

# Theorem

$$\mathrm{m} \to \mathrm{n} = (\mathrm{m} \otimes (\mathrm{n} - 1)) \to 1$$

maps m→n are *affine combinations* of combs of **two types**

PP, in "Time in Physics", R. Renner and S. Stupar eds., Springer (2017)

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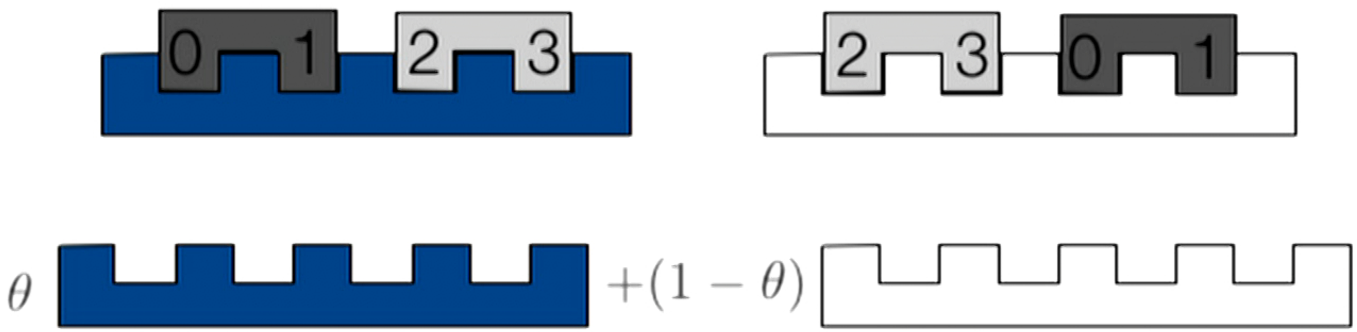


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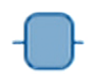
# Example: process matrices

$$(\mathrm{A}_1 \to \mathrm{B}_1) \to [(\mathrm{A}_2 \to \mathrm{B}_2)] \to \mathrm{I}]$$

Equivalently by uncurrying

$$[(\mathrm{A}_1 \to \mathrm{B}_1) \otimes (\mathrm{A}_2 \to \mathrm{B}_2)] \to \mathrm{I}$$



O. Oreshkov F. Costa, C. Brukner, Nat. Comm. **3**, 1092 (2012)

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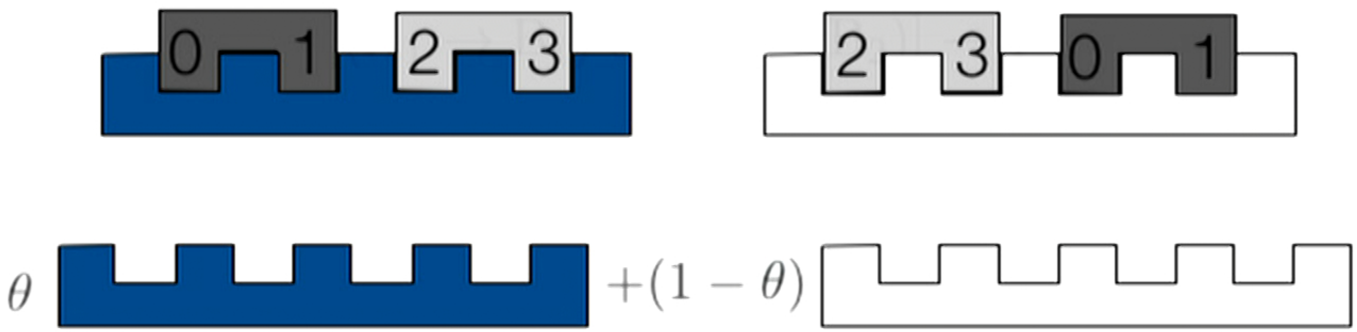


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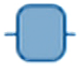

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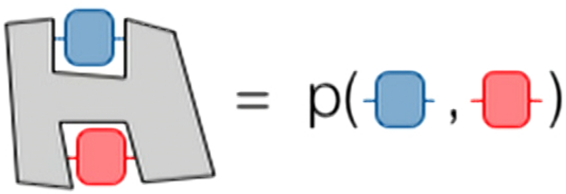


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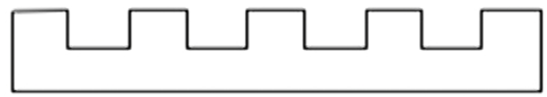
# Feasibility

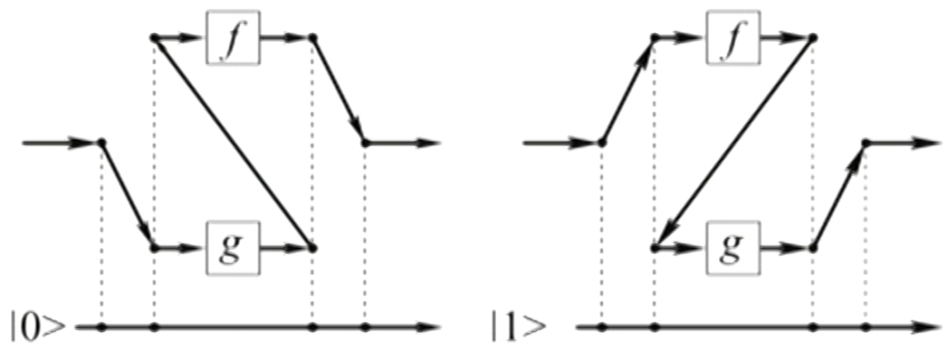
Implementation scheme for *convex combinations*:
random choice between different orders

$p$  $+(1-p)$ 

$p \geq 0$

# Purifications

Implementation schemes for *purifications* of convex combinations via *quantum control*




G. Chiribella, G. M. D'Ariano, PP, and B. Valiron, PRA **88**, 022318 (2013)
arXiv:0912.0195 (2009)

# Other maps

Feasibility of other affine combinations

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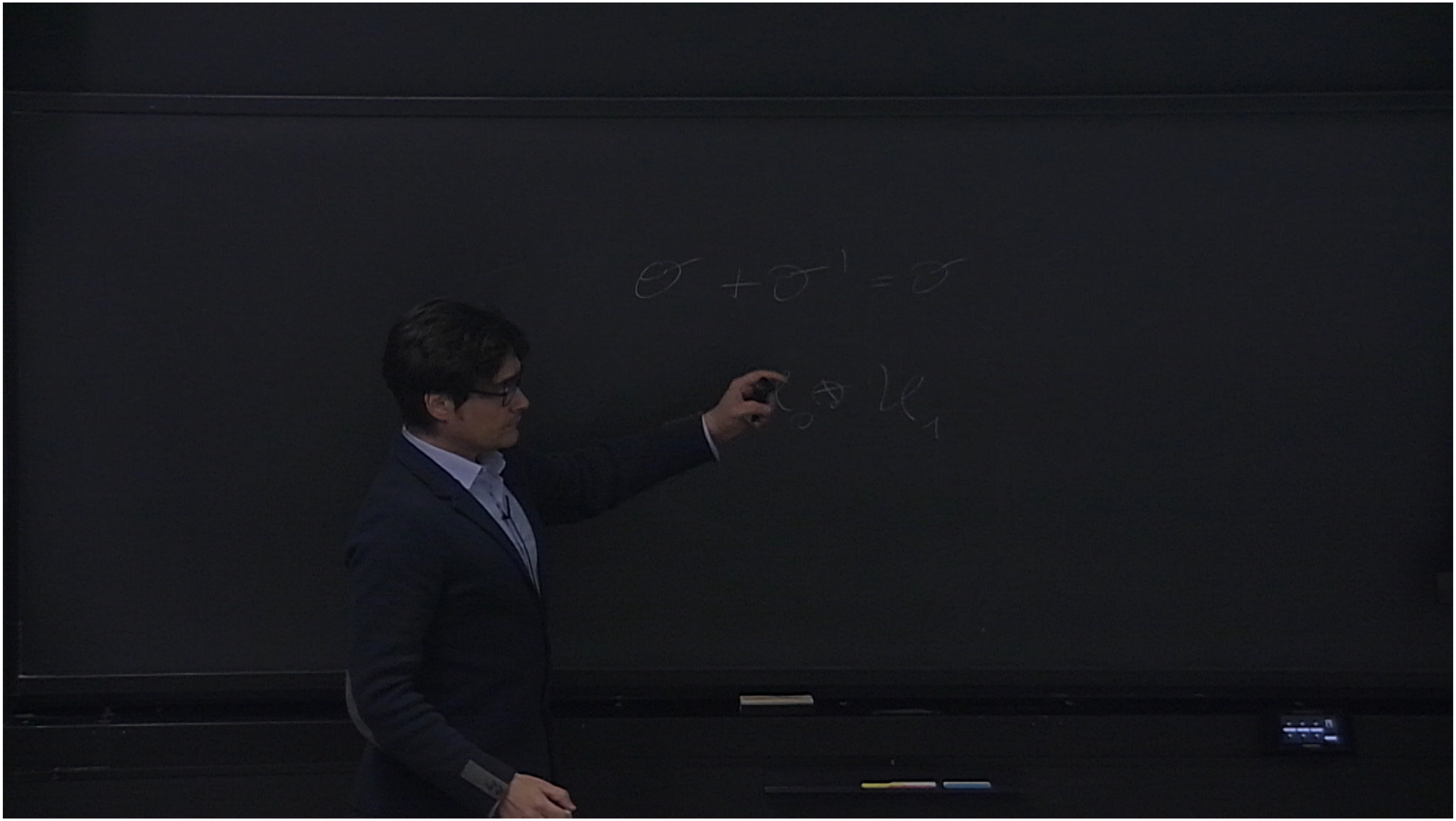
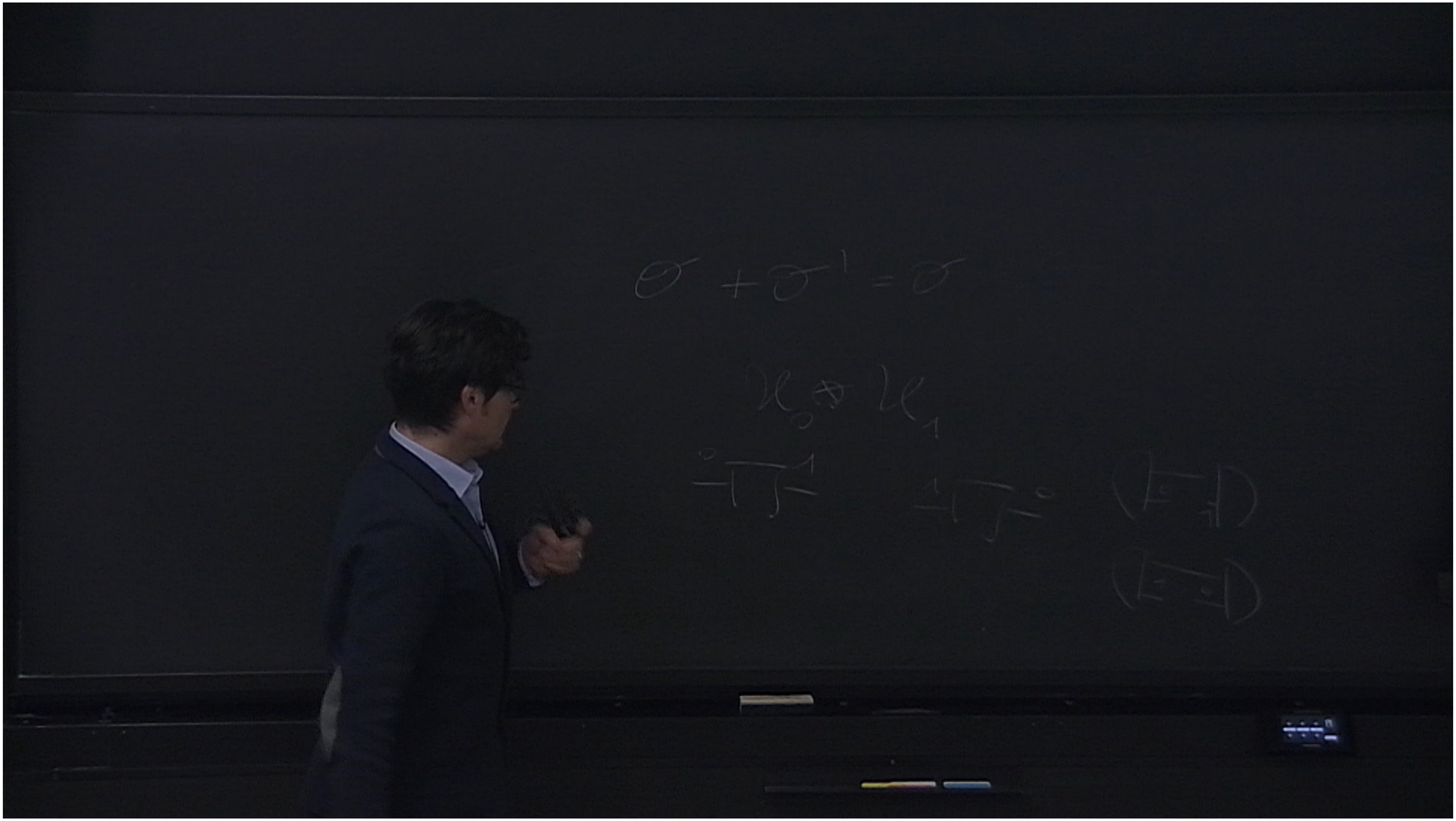
Any relevant constraint besides admissibility?

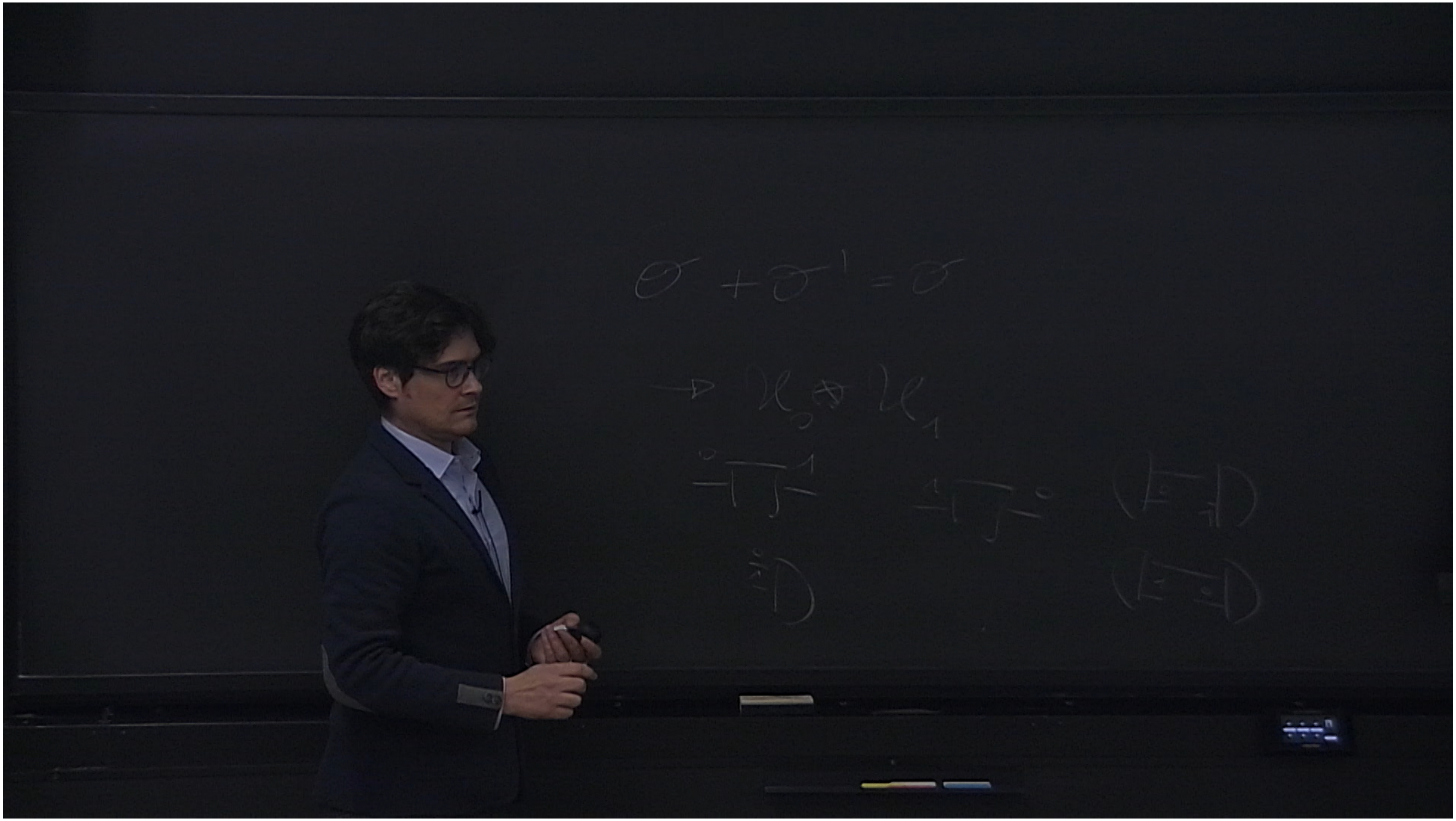
It cannot be “our ability to imagine a realisation”.

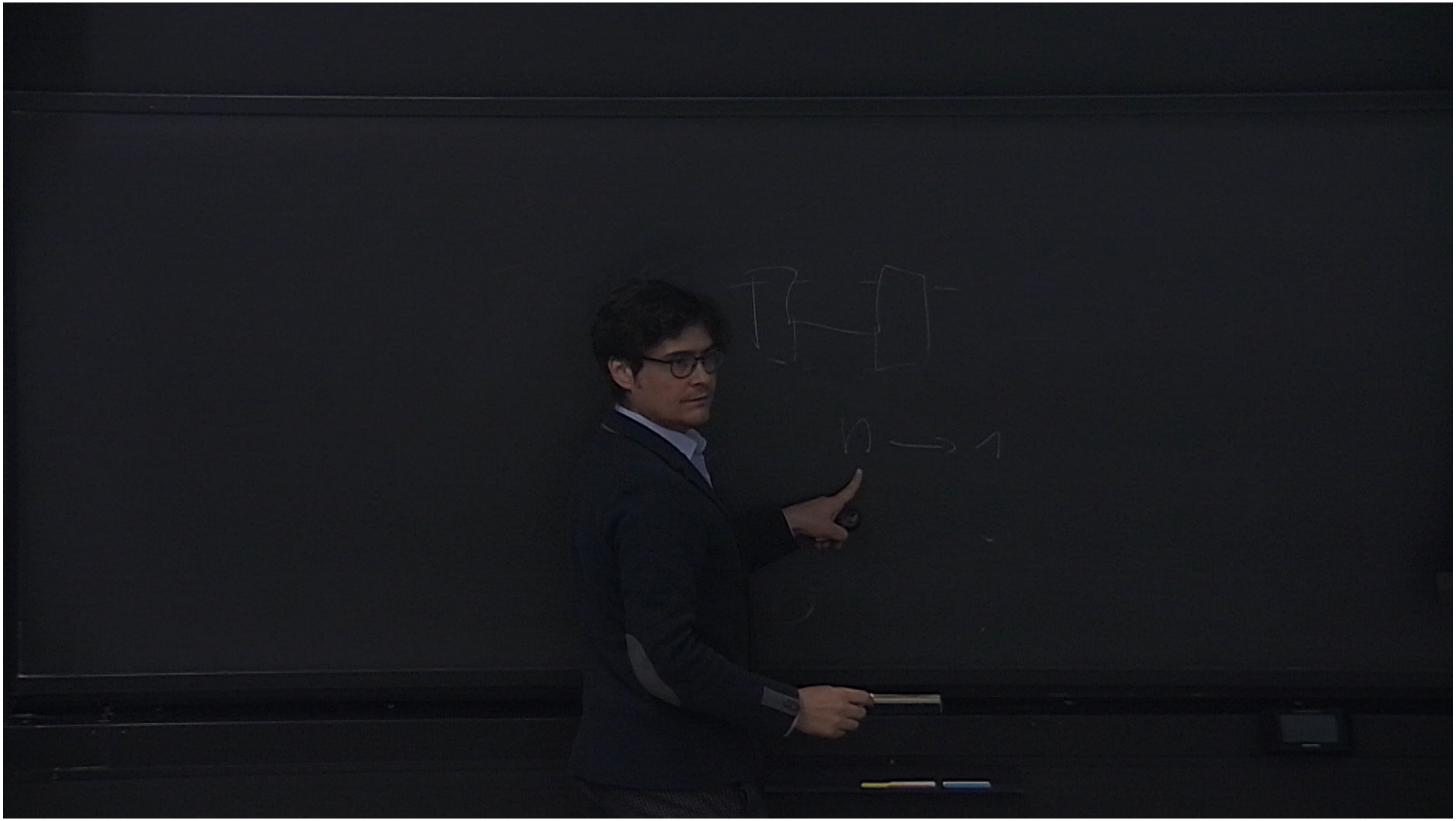
# Other problems

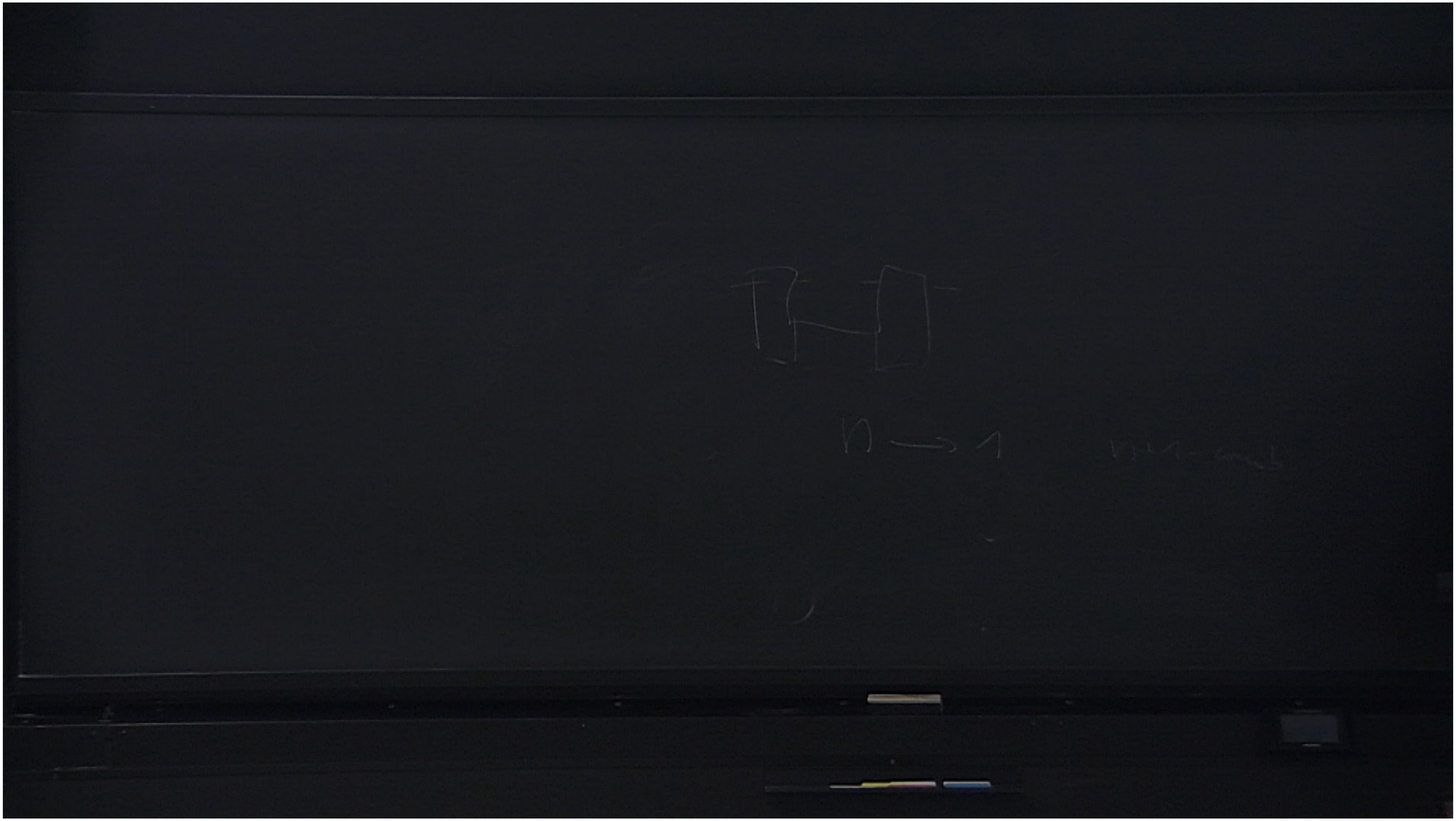
- Is there a simple rule to identify $x \mapsto \mathsf{N}(x)$ ?
- Is there a universal higher order set?
- Can there be a "speedup"?








n → 1


n ⟶ 1