Title: Quantum chaos and late-time dynamics

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Abstract: <p>From a quantum information perspective, we will study universal features of chaotic quantum systems. Recent progress has made evident that quantifying chaos is a useful way to gain insight into strongly-coupled field theories, quantum many-body systems, as well as the quantum nature of black holes. We will derive relations between different diagnostics of chaos and scrambling (OTOCs, spectral functions, and frame potentials) and define a quantity to capture the onset of a random matrix description. We will review and use tools from quantum information and random matrix theory, but our goal will be to understand strongly-interacting systems.</p>

Quantum chaos and late-time dynamics

A look through the (translucent) QI lens

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November 29, 2017 - Perimeter Institute

Based on:
J. Cotler, NHJ, J. Liu, B. Yoshida, "Chaos, Complexity, and Random Matrices," JHEP11(2017)048, 1706.05400
and "Symmetry, k-invariance, and late-time chaos," 1801.hopefully soon
(also related: NHJ, J. Liu; 1711.08184)

I'll be talking about quantum chaos and random matrix theory in quantum mechanical systems

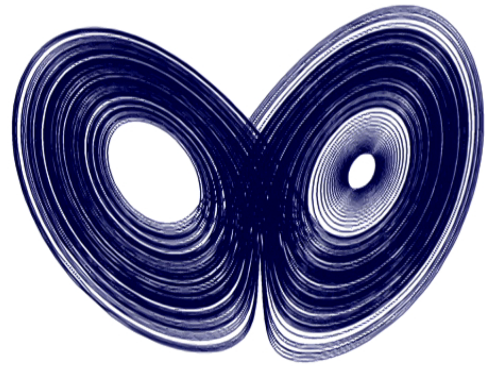
$\rightarrow$ finite dimensional $\mathcal{H}$ and discrete spectrum

Chaos

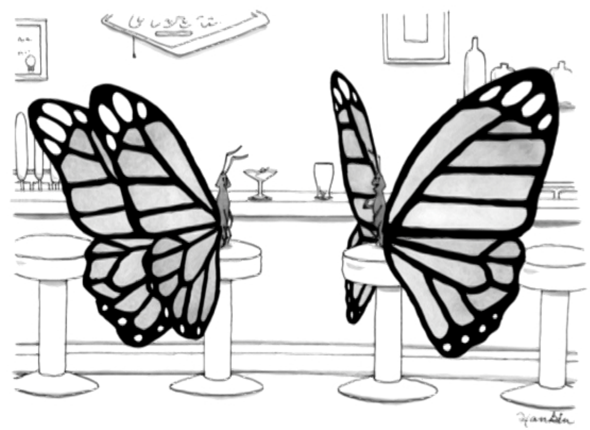
(ubiquitous but not boring)

we are familiar with classical chaos and sensitivity to initial conditions, but chaos in quantum systems is **different**

intuition: small perturbations grow to affect the system



[chaotic butterfly]



"Remember that hurricane a thousand miles away? That was me!"

[chaos in culture]

Quantum Chaos - a historical tour

The definition of quantum chaos seems to depend on the decade and subfield you work in:

70s-80s - QM chaos = quantization of a classically chaotic system

[Gutzwiller]

80s-90s - QM chaos = statement about universal properties of the spectrum [Bohigas, Giannoni, Schmit], [Berry] → Random matrix theory

([Berry], [Srednicki] → ETH)

2010s - QM chaos can be probed by correlation functions in thermal states → Out-of-time ordered correlators (OTOCs)

[Kitaev, Stanford, Shenker, Maldacena, Roberts, Yoshida, Susskind, . . .]

Quantum Chaos - an admission

some semantic guidance:

Michael Berry - Quantum Chaology, Not Quantum Chaos!

remember: classical chaos is **different** from quantum chaos

and furthermore, a precise definition of quantum chaos remains elusive

in this sense, we should be careful when discussing “probes” of chaos

OTOCs: Chaos in the modern era

quick review of out-of-time ordered correlation functions (OTOCs)
an old idea [Larkin, Ovchinnikov]

Consider the 4-point function of a pair of local(ish) operators in thermal states

$$\langle A(t) B A(t) B \rangle_\beta$$

OTOCs: Chaos in the modern era

Rough intuition: [Roberts, Stanford, Susskind] think about the time evolution of a Pauli operator in a chaotic spin system

$$Z_1(t) = e^{-iHt} Z_1 e^{iHt} = Z_1 - it[H, Z_1] - \frac{t^2}{2}[H, [H, Z_1]] + \dots$$

operator 'grows' in time, measured by commutator
consider $[Z_1(t), Z_8]$ (for a chaotic spin chain)

expand out $\langle [Z_1(t), Z_8]^2 \rangle_\beta$:

$$\begin{aligned}\langle [Z_1(t), Z_8]^2 \rangle_\beta = & \langle Z_1(t) Z_8 Z_8 Z_1(t) \rangle_\beta + \langle Z_8 Z_1(t) Z_1(t) Z_8 \rangle_\beta \\ & - \langle Z_1(t) Z_8 Z_1(t) Z_8 \rangle_\beta - \langle Z_8 Z_1(t) Z_8 Z_1(t) \rangle_\beta\end{aligned}$$

OTOCs: Chaos in the modern era

Rough intuition: [Roberts, Stanford, Susskind] think about the time evolution of a Pauli operator in a chaotic spin system

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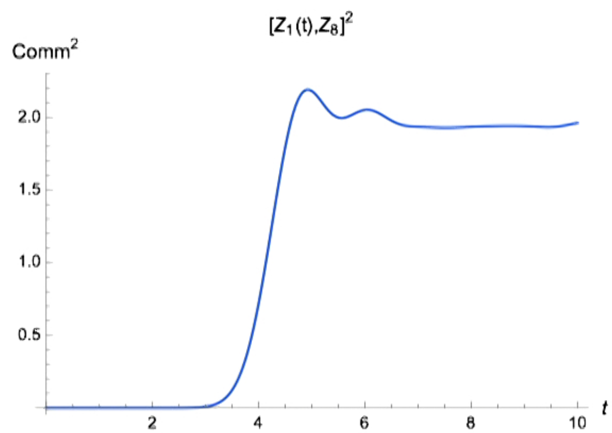
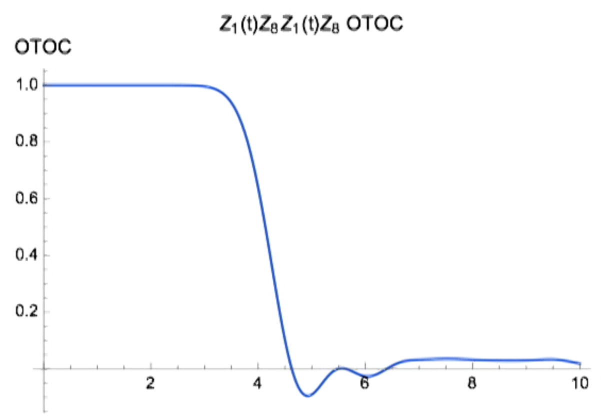


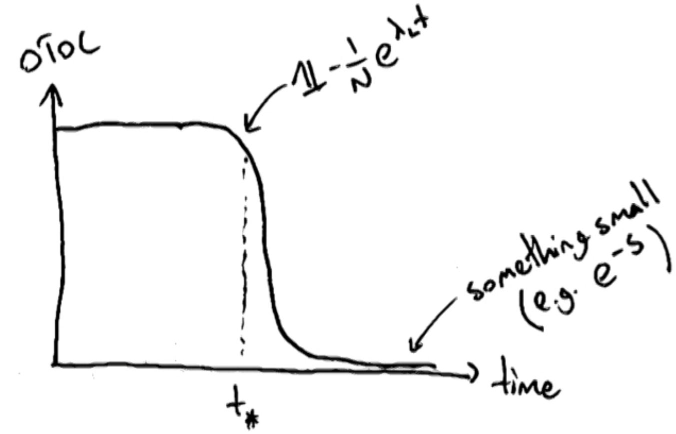


OTOCs: Chaos in the modern era

OTOC: 4-point function of a pair of local(ish) operators in thermal states

$$\langle A(t)BA(t)B\rangle_\beta$$

salient features of (chaotic) OTOCs:



(exponentially) growing corrections at early times

small value at late times

A chaotic resurgence

In recent years, a revival in quantum chaos spurred by studying OTOCs

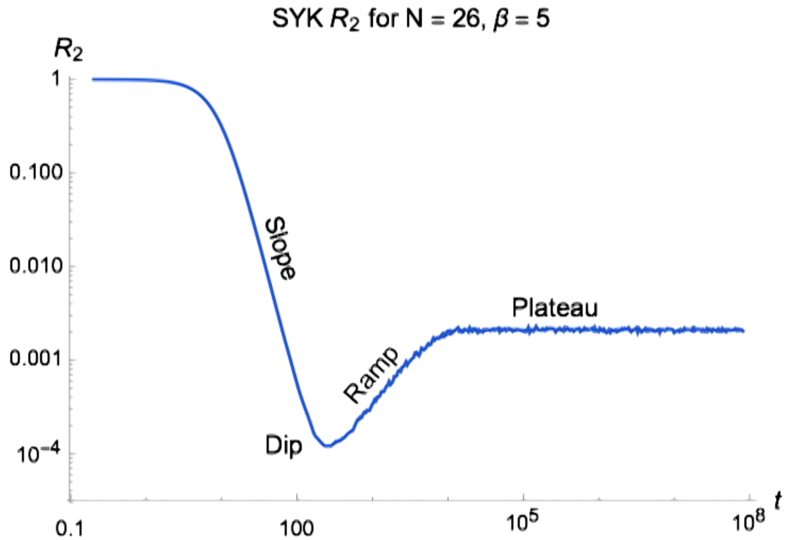
- Chaos in gravity: [Kitaev], [Shenker, Stanford] (also, Yoni's talk yesterday)
- Chaos in CFT: [Roberts, Stanford] (later: [Fitzpatrick, Kaplan], [Dyer, Gur-Ari])
- Chaos enfettered: a chaos bound [Maldacena, Shenker, Stanford]
- Chaos in SYK: [Kitaev], [Stanford, Maldacena]
- RMT in SYK: [Cotler, Gur-Ari, Hanada, Polchinski, Saad, Shenker, Stanford, Streicher, Tezuka]

Form factor in SYK

RMT in SYK: [Cotler, Gur-Ari, Hanada, Polchinski, Saad, Shenker, Stanford, Streicher, Tezuka]

Studied spectral form factor in SYK (to investigate BH info loss à la [Maldacena])

$$\mathcal{R}_2(\beta,t) \equiv \langle Z(\beta,t)Z^*(\beta,t)\rangle_{\rm SYK} = \langle {\rm Tr}\,(e^{-\beta H - iHt}){\rm Tr}\,(e^{-\beta H + iHt})\rangle_{\rm SYK}$$



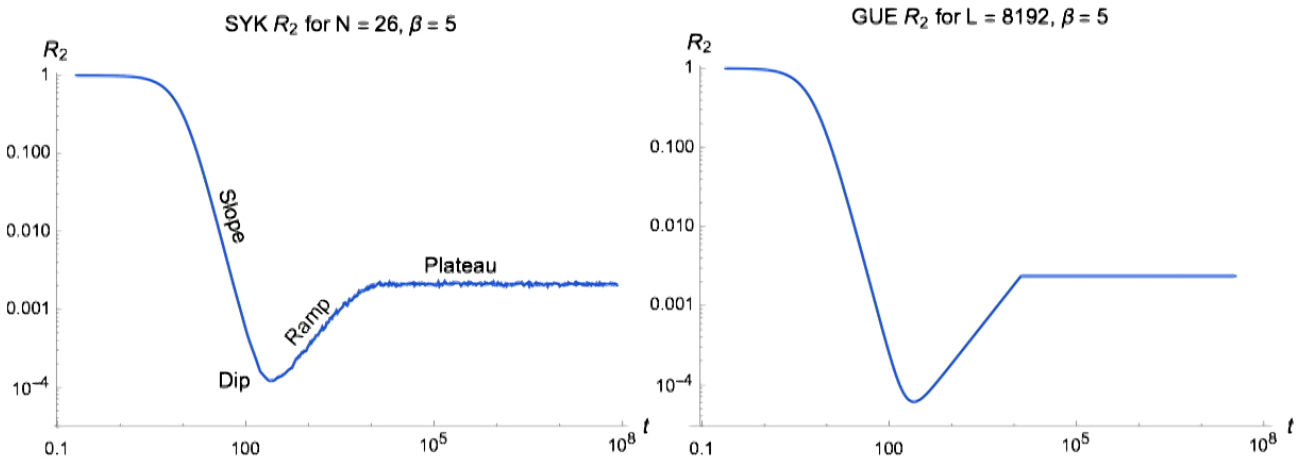
Form factor in SYK

RMT in SYK: [Cotler, Gur-Ari, Hanada, Polchinski, Saad, Shenker, Stanford, Streicher, Tezuka]

Studied spectral form factor in SYK (to investigate BH info loss à la [Maldacena])

$$\mathcal{R}_2(\beta, t) \equiv \langle Z(\beta,t) Z^*(\beta,t)\rangle_{\mathrm{SYK}} = \langle \mathrm{Tr}\,(e^{-\beta H - iHt}) \mathrm{Tr}\,(e^{-\beta H + iHt})\rangle_{\mathrm{SYK}}$$

found agreement with RMT



Goal: Chaos through the QI lens

a QI view can provide guiding intuition [Hayden, Preskill]

recently, OTOCs were cast in the QI light
[Hosur, Qi, Roberts, Yoshida] $\rightarrow$ relating OTOCs to MI, quantify scrambling
[Roberts, Yoshida] $\rightarrow$ relating OTOCs to the frame potential, quantify randomness

chaos is a ubiquitous feature of strongly-interacting systems $\rightarrow$ understand strongly-coupled systems/BHs
as 80s chaos $\approx$ RMT, and (some aspects of) the late time behavior of SYK $\approx$ RMT

in a quantum information theoretic way:
- understand the role RMT plays in describing chaotic dynamics
- want to relate symptoms of chaos (a chaotic first step)

distant motivation: understand BHs

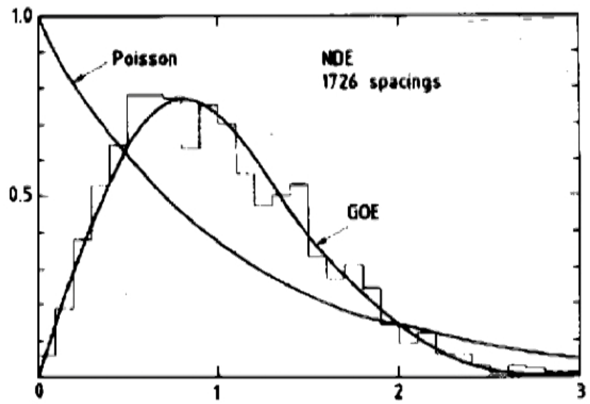
Outline

- Motivation ✓
- Quick overview of RMT
- Form factors and RMT
- Frame potentials and RMT
- OTOCs and RMT
- k-invariance

Overview of RMT

Early success in nuclear physics [Wigner], [Dyson]

famously reproducing the nearest-neighbor spacings of heavy nuclei resonances [Nuclear Data Ensemble]



taken from [Guhr, Müller-Groeling, Weidenmüller]

Overview of RMT

Early success in nuclear physics [Wigner], [Dyson]

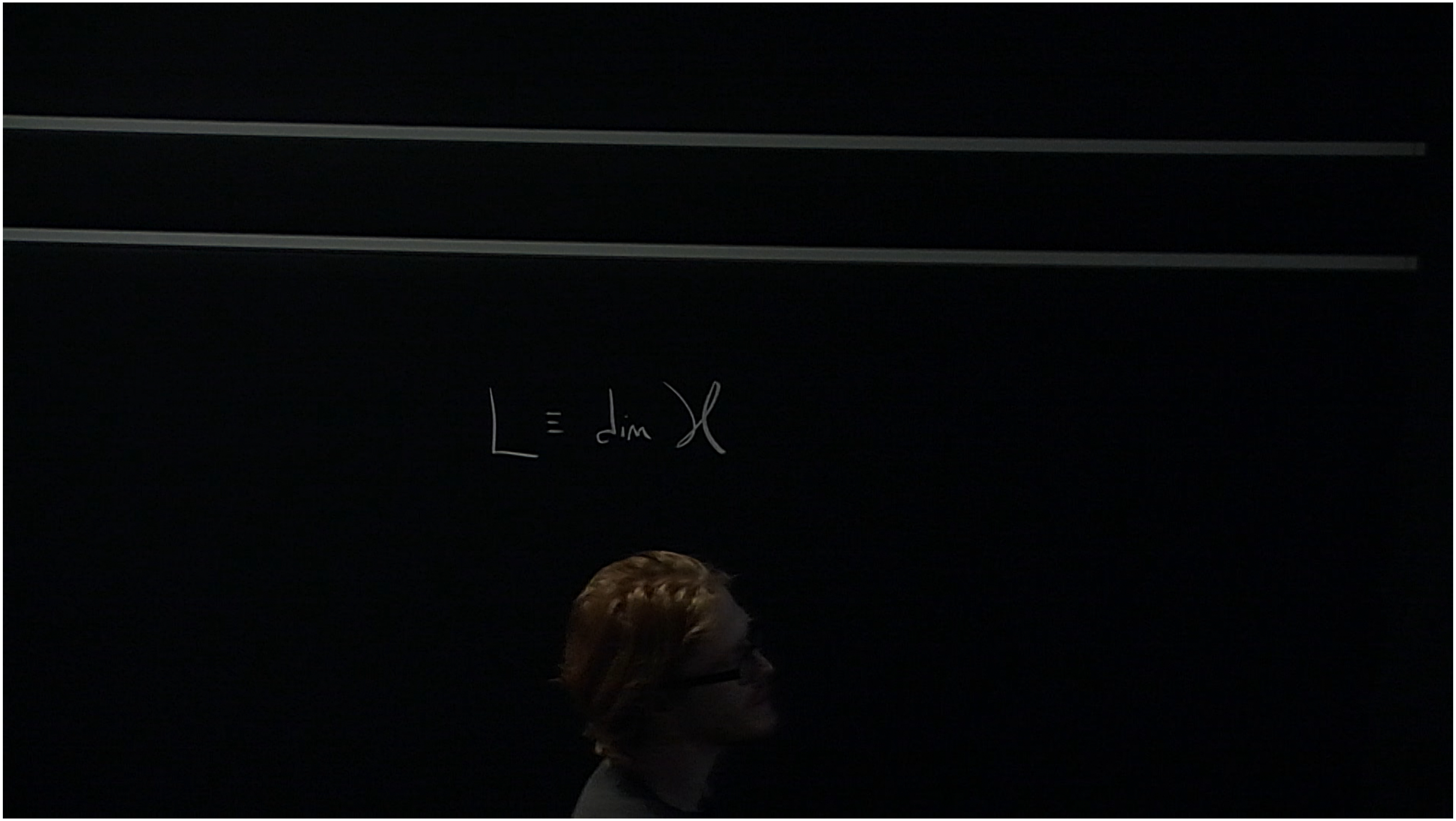
Since has pervaded many seemingly disparate subfields
large N QFT, string theory, transport in disordered quantum systems, ...

Classic matrix ensembles (GUE, GOE, GSE)

$\rightarrow$ focus on the **Gaussian unitary ensemble** with
GUE$(L, 0, 1/\sqrt{L})$

GUE $=$ ensemble of $L \times L$ **Hermitian** matrices
off-diagonal $N(0, 1/\sqrt{L})_{\mathbb{C}}$ and diagonal $N(0, 1/\sqrt{L})_{\mathbb{R}}$

note: use different norm than RMT
also, sorry about L


$L \equiv \dim \mathcal{H}$

Overview of GUE

GUE has a probability weight, $P(H) \propto e^{-\frac{L}{2}\mathrm{Tr}(H^2)}$
and invariant measure $dH = d(UHU^\dagger)$
In the eigenvalue basis:

$$P(\lambda_1, \ldots, \lambda_L) = Ce^{-\frac{L}{2}\sum_i \lambda_i^2}|\Delta(\lambda)|^2 .$$

Average over GUE as

$$\langle O(\lambda)\rangle_{\mathrm{GUE}} \equiv \int D\lambda\, O(\lambda) , \quad \text{where} \quad \int D\lambda = \int \prod_i d\lambda_i P(\lambda) = 1 .$$

The density of states:

$$\rho(\lambda) = \left\langle \sum_{i=1}^{L} \delta(\lambda - \lambda_i) \right\rangle_{\mathrm{GUE}} .$$

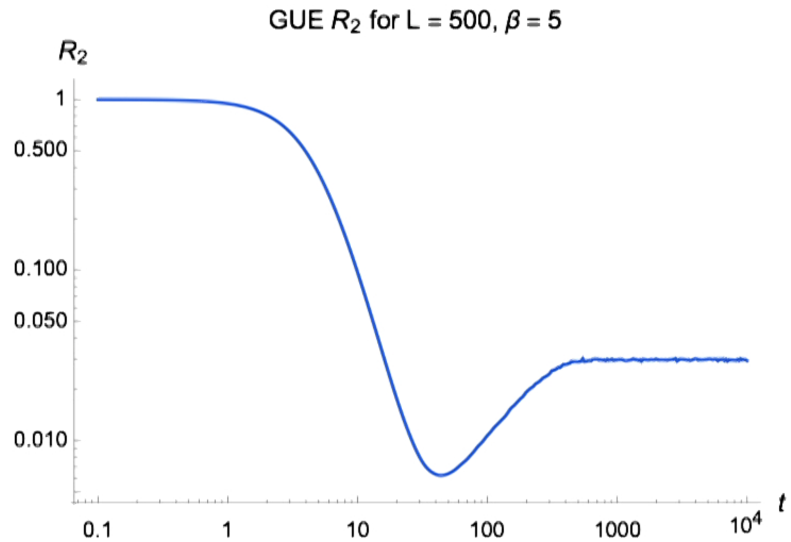
The spectral n-point correlation function:

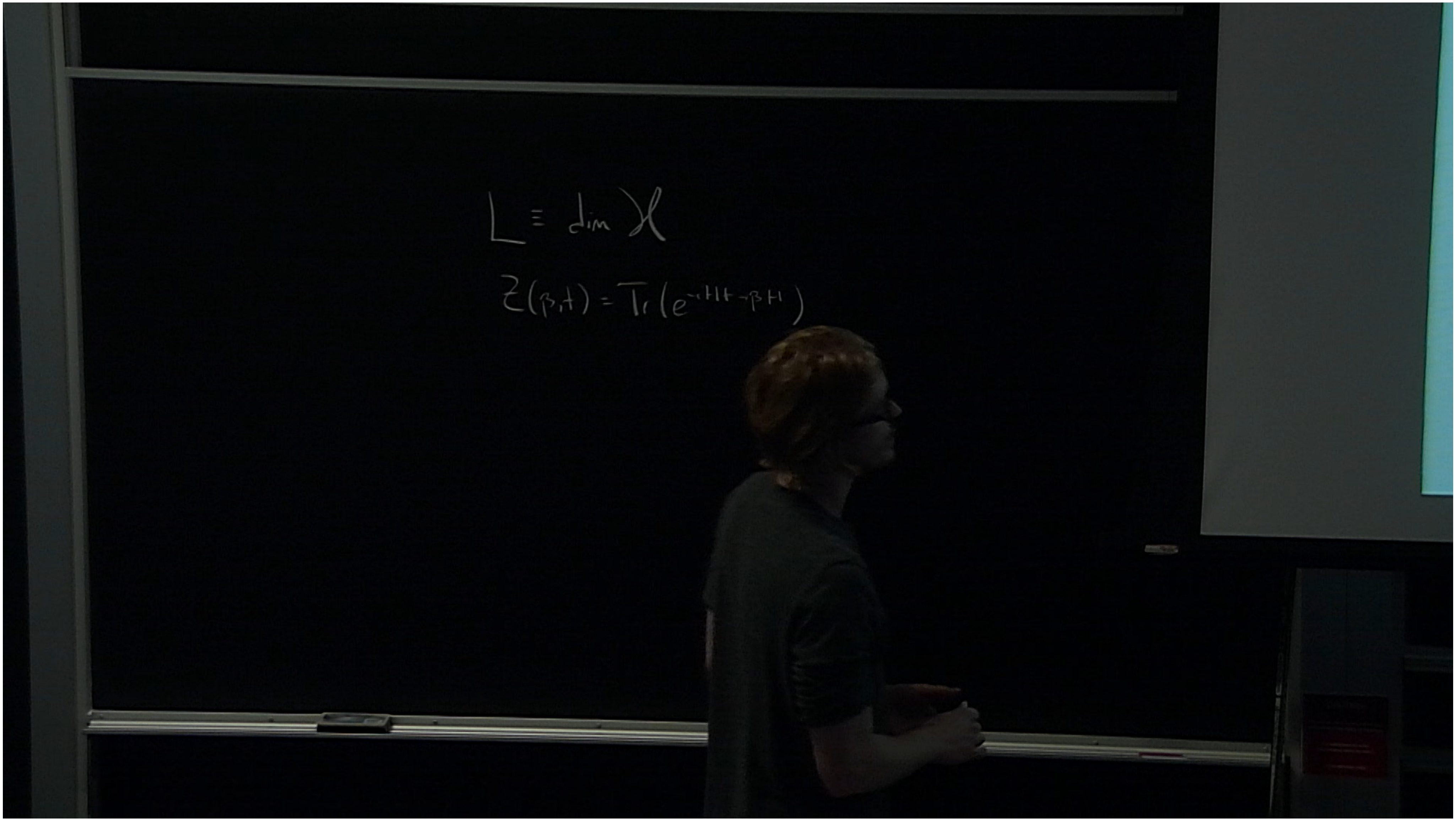
$$\rho^{(n)}(\lambda_1, \ldots, \lambda_n) = \frac{L!}{(L-n)!} \int d\lambda_{n+1} \ldots d\lambda_L P(\lambda_1, \ldots, \lambda_L) .$$

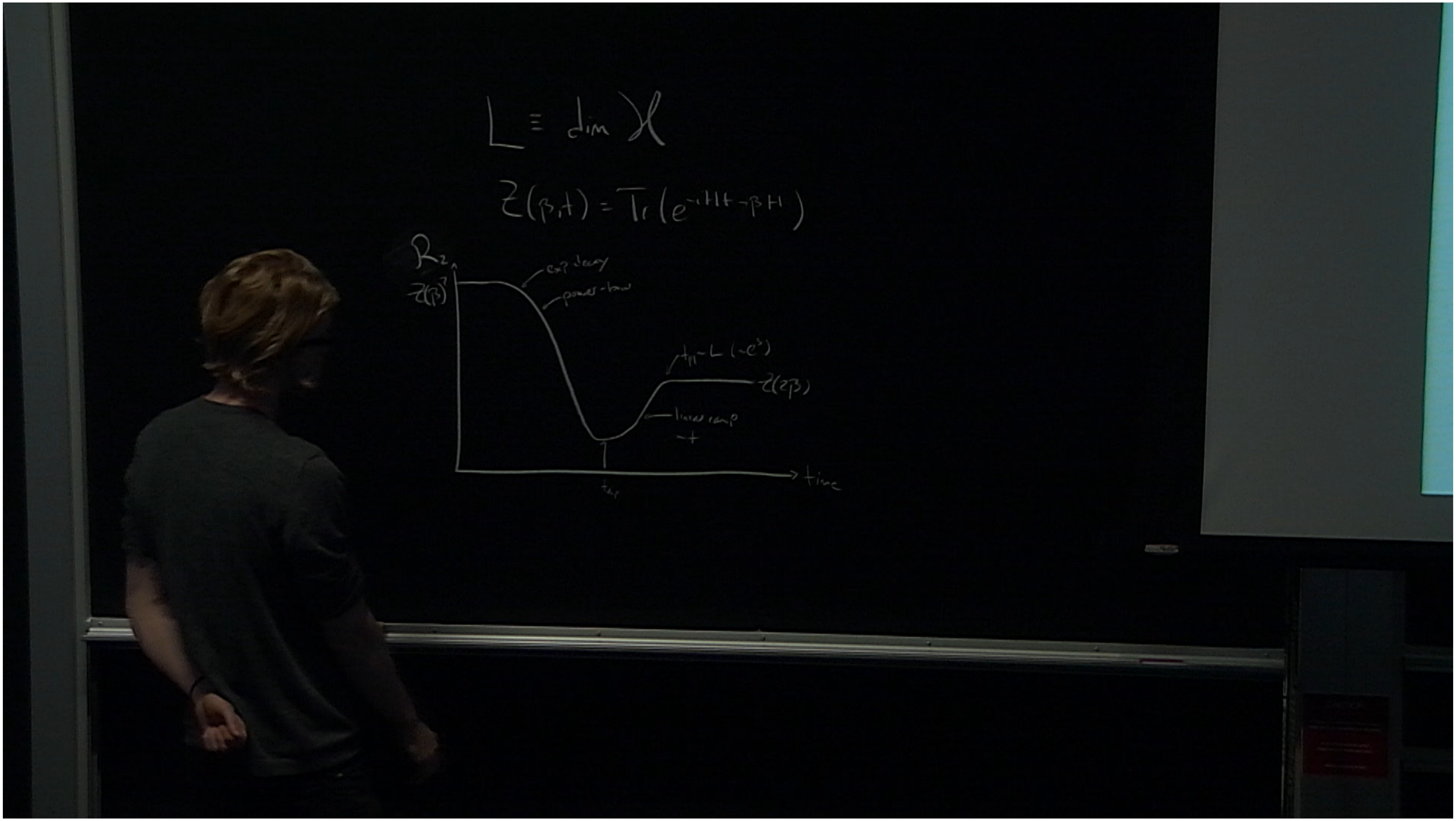
Spectral form factor

$$\mathcal{R}_2(\beta,t) \equiv \langle Z(\beta,t)Z^*(\beta,t)\rangle = \int D\lambda \sum_{i,j} e^{i(\lambda_i-\lambda_j)t} e^{-\beta(\lambda_i+\lambda_j)}$$

Let's discuss some universal aspects of the form factor:




L ≡ dim H
Z(β,t) = Tr(e^{-iHt-βH})


L ≡ dim H
Z(β,t) = Tr(e^{-iHt-βH})
R_2
Z(β)^2
exp decay
power-law
Z(2β)
linear ramp
~t
time

2-pt form factor at $\beta = 0$

The infinite temperature 2-point form factor

$$\mathcal{R}_2(t) = \int D\lambda \sum_{i,j} e^{i(\lambda_i - \lambda_j)t}$$

can be computed as

$$\mathcal{R}_2(t) = L^2 r_1^2(t) - L r_2(t) + L \,,$$

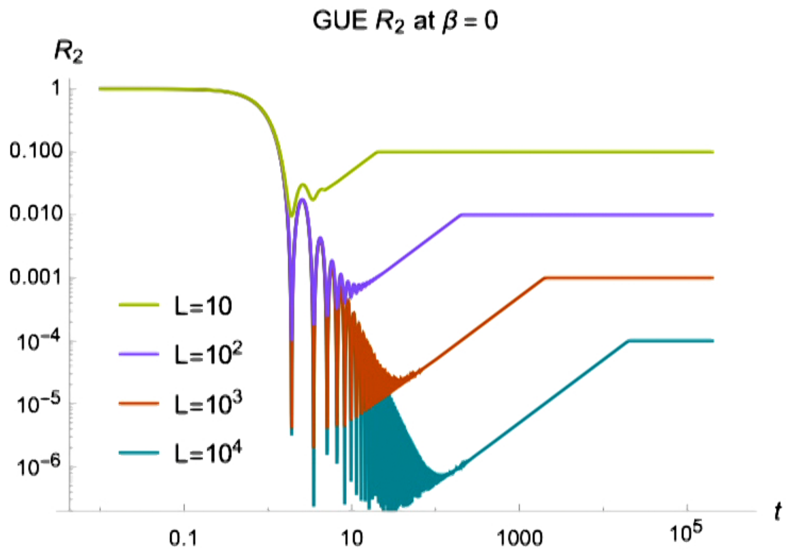
where we define the functions

$$r_1(t) = \frac{J_1(2t)}{t}, \quad \text{and} \quad r_2(t) = \begin{cases} 1 - \frac{t}{2L}\,, & \text{for} \quad t < 2L \\ 0\,, & \text{for} \quad t > 2L \end{cases} .$$

2-pt form factor at $\beta = 0$

The infinite temperature 2-point form factor

$$\mathcal{R}_2(t) = L^2 r_1^2(t) - L r_2(t) + L\,,$$



dip time: $t_d = \sqrt{L}$ and dip value: $\mathcal{R}_2 = \sqrt{L}$ $\quad \rightarrow$ linear rise

plateau time: $t_p = 2L$ and plateau value: $\mathcal{R}_2 = L$

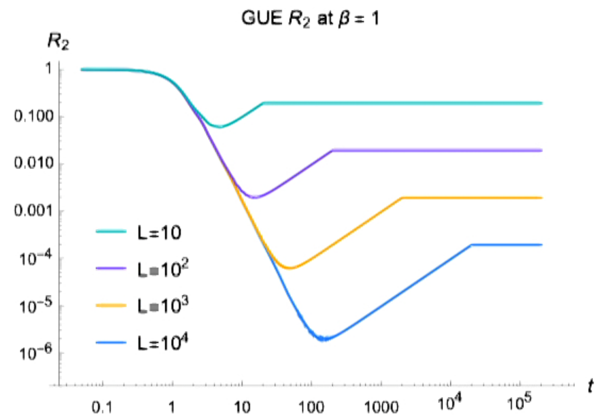
2-pt form factor at finite β

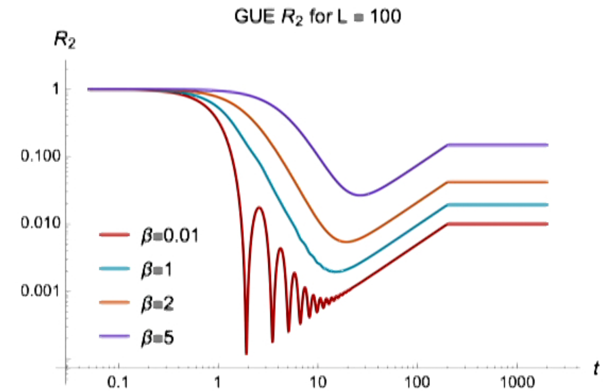
at finite temperature

$$\mathcal{R}_2(t,\beta) = \langle Z(t,\beta) Z^*(t,\beta) \rangle_{\mathrm{GUE}} = \int D\lambda \sum_{i,j} e^{i(\lambda_i - \lambda_j)t} e^{-\beta(\lambda_i + \lambda_j)} ,$$

we insert the spectral $\rho^{(2)}$ and use the short-distance kernel [Brézin, Hikami] to find

$$\mathcal{R}_2(t,\beta) = L^2 r_1(t+i\beta) r_1(-t+i\beta) + L r_1(2i\beta) - L r_1(2i\beta) r_2(t) .$$





dip time: $t_d = h_2(\beta)\sqrt{L}$ and dip value: $\mathcal{R}_{2\beta} = h_3(\beta)\sqrt{L}$ $\longrightarrow$ linear rise

plateau time: $t_p = 2L$ and plateau value: $\mathcal{R}_{2\beta} = h_1(2\beta)L$

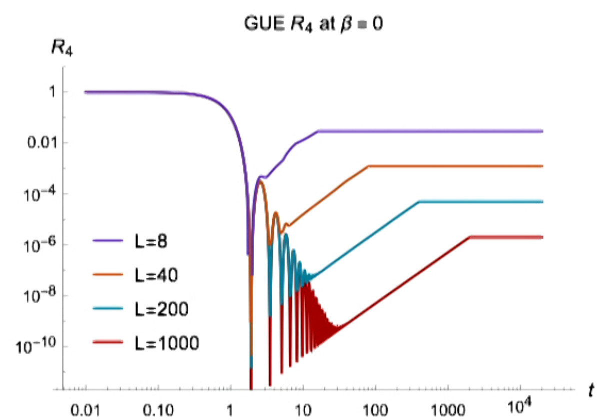
4-pt form factor at $\beta = 0$

we compute the 4-pt form factor at infinite temperature

$$\mathcal{R}_4(t) = \langle Z(t)Z(t)Z^*(t)Z^*(t)\rangle_{\text{GUE}} = \int D\lambda \sum_{i,j,k,\ell} e^{i(\lambda_i+\lambda_j-\lambda_k-\lambda_\ell)t}$$

insert $\rho^{(4)}$ and carefully integrate each term, we find

$$\mathcal{R}_4(t) = L^4 r_1^4(t) + 2L^2 r_2^2(t) - 4L^2 r_2(t) \\ - 7L^2 r_2(2t) + 4L r_2(3t) + 4L r_2(t) + 2L^2 - L\,.$$



dip time: $t_d = \sqrt{L}$ and dip value: $\mathcal{R}_2 = L$ $\rightarrow$ quadratic rise

plateau time: $t_p = 2L$ and plateau value: $\mathcal{R}_2 = 2L^2$

Frame potentials and random matrices

Overview of QI Machinery

Haar: (unique left/right invariant) measure on unitary group $U(L)$

Consider an operator O acting on $\mathcal{H}^{\otimes k}$, the k-fold channel of O with respect to Haar:

$$\Phi^{(k)}_{\mathrm{Haar}}(O) \equiv \int_{\mathrm{Haar}} dU \, (U^{\otimes k})^{\dagger} O U^{\otimes k} \, .$$

Overview of QI Machinery

Haar: (unique left/right invariant) measure on unitary group $U(L)$

Consider an operator O acting on $\mathcal{H}^{\otimes k}$, the k-fold channel of O with respect to Haar:

$$\Phi^{(k)}_{\mathrm{Haar}}(O) \equiv \int_{\mathrm{Haar}} dU\, (U^{\otimes k})^\dagger O U^{\otimes k} \,.$$

Instead, consider an ensemble of unitaries $\mathcal{E} = \{p_i, U_i\}$, where the k-fold channel of $\mathcal{E}$ is

$$\Phi^{(k)}_{\mathcal{E}}(O) \equiv \int_{U\in\mathcal{E}} dU (U^{\otimes k})^\dagger O U^{\otimes k} \,.$$

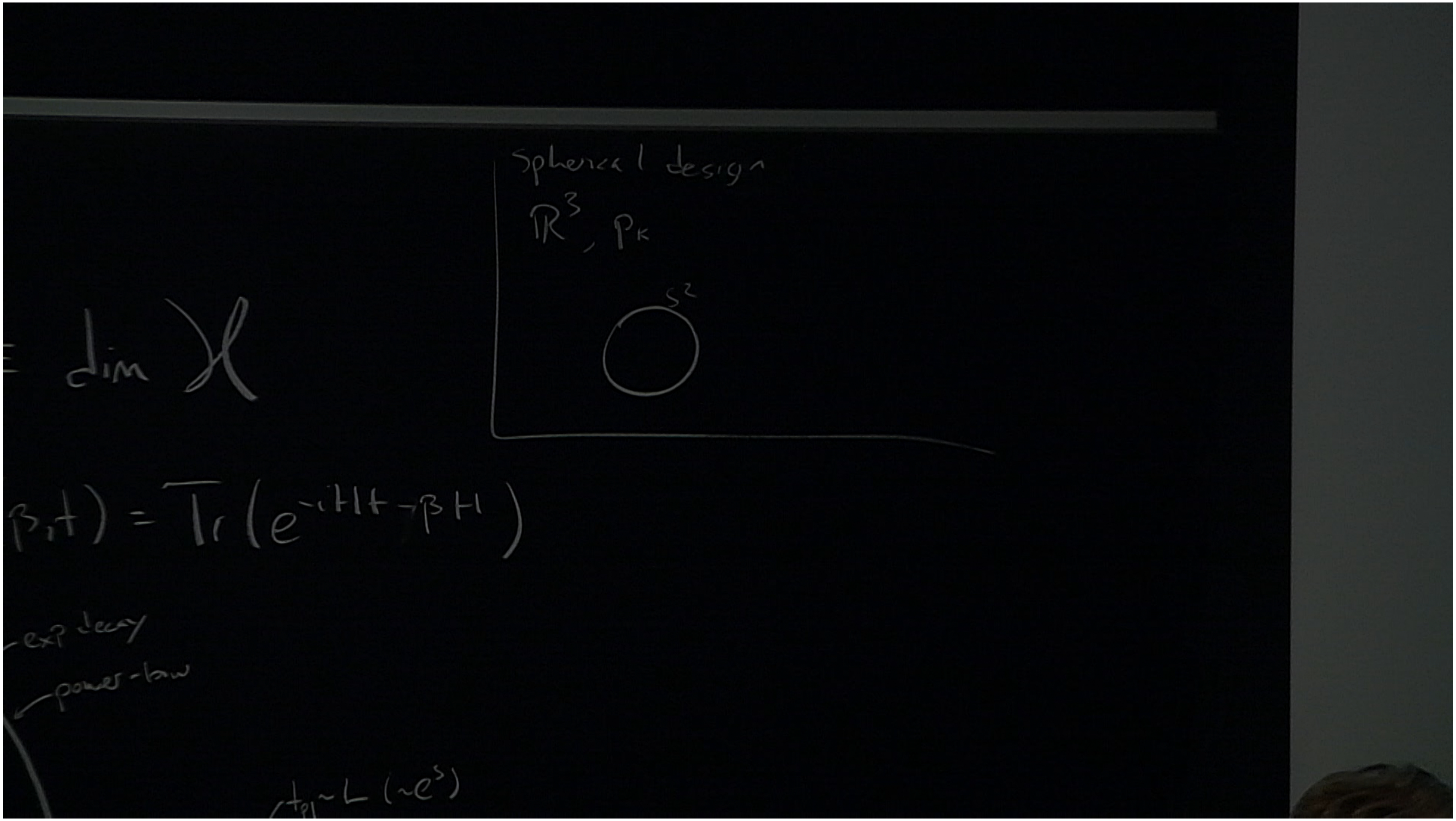
Ensemble $\mathcal{E}$ is a unitary k-design if and only if

$$\Phi^{(k)}_{\mathcal{E}}(O) = \Phi^{(k)}_{\mathrm{Haar}}(O) \,,$$

meaning we reproduce the first k moments of Haar.

$\mathbb{R}^3 \qquad p_k$

$$\frac{1}{|X|} \sum_{x \in X} p_k(x) = \int_{S^2} d\mu \, p_k$$


Spherical design
$\mathbb{R}^3$, p_K
S^2
= dim $\mathcal{H}$
$(\beta, t) = \mathrm{Tr}(e^{-iHt-\beta H})$
exp decay
power-law
$t_{pl} \sim L \; (\sim e^S)$

Overview of QI Machinery

Ensemble $\mathcal{E}$ is a unitary k-design if and only if

$$\Phi_{\mathcal{E}}^{(k)}(O) = \Phi_{\text{Haar}}^{(k)}(O)\,,$$

Measure of this $\to$ **Frame potential** [Scott]:

$$\boxed{\mathcal{F}_{\mathcal{E}}^{(k)} = \int_{U,V\in\mathcal{E}} dU\,dV\,\left|\text{Tr}(U^\dagger V)\right|^{2k}},$$

(distance to Haar)

k-th frame potential for Haar: $\mathcal{F}_{\text{Haar}}^{(k)} = k! \quad \text{for} \quad k \le L\,.$

For ensemble $\mathcal{E}$, the frame potential is lower bounded by

$$\mathcal{F}_{\mathcal{E}}^{(k)} \ge \mathcal{F}_{\text{Haar}}^{(k)}\,,$$

($=$ iff $\mathcal{E}$ is a k-design)

Why frame potentials?

in BH physics we often approximate by Haar unitaries [Page], [Hayden, Preskill], $\rightarrow$ k-design behavior sufficient

[Roberts, Yoshida] made progress quantifying chaos by relating $2k$-OTOCs to the k-th frame potential

$$\frac{1}{L^{4k}} \sum_{A's,B's} \left| \langle A_1 B_1(t) \ldots A_k B_k(t) \rangle_{\mathcal{E}} \right|^2 = \frac{\mathcal{F}_{\mathcal{E}}^{(k)}}{L^{2(k+1)}}$$

where "$B(t)$" $= UBU^\dagger$, and for any ensemble $\mathcal{E}$.

makes precise an approach to *randomness*

(also complexity. . .)

$k = 1$ frame potential for GUE

Consider the ensemble of unitary time evolutions at a fixed time t with GUE Hamiltonians

$$\mathcal{E}_t^{\rm GUE} = \left\{ e^{-iHt} \,, \;\text{with}\;\; H \in {\rm GUE} \right\}.$$

$k = 1$ frame potential for GUE

Consider the ensemble: $\mathcal{E}_t^{\mathrm{GUE}} = \{e^{-iHt}, \; H \in \mathrm{GUE}\}$

The first frame potential for GUE:

$$\mathcal{F}^{(1)}_{\mathrm{GUE}} = \int dH_1 dH_2 \, e^{-\frac{L}{2}\mathrm{Tr}H_1^2} e^{-\frac{L}{2}\mathrm{Tr}H_2^2} \left| \mathrm{Tr}\left(e^{iH_1 t} e^{-iH_2 t}\right) \right|^2$$

insert Haar unitaries, use L/R invariance of Haar, and integrate using the second moment

$$\boxed{\mathcal{F}^{(1)}_{\mathrm{GUE}} = \frac{1}{L^2-1}\left(\mathcal{R}_2^2 + L^2 - 2\mathcal{R}_2\right),}$$

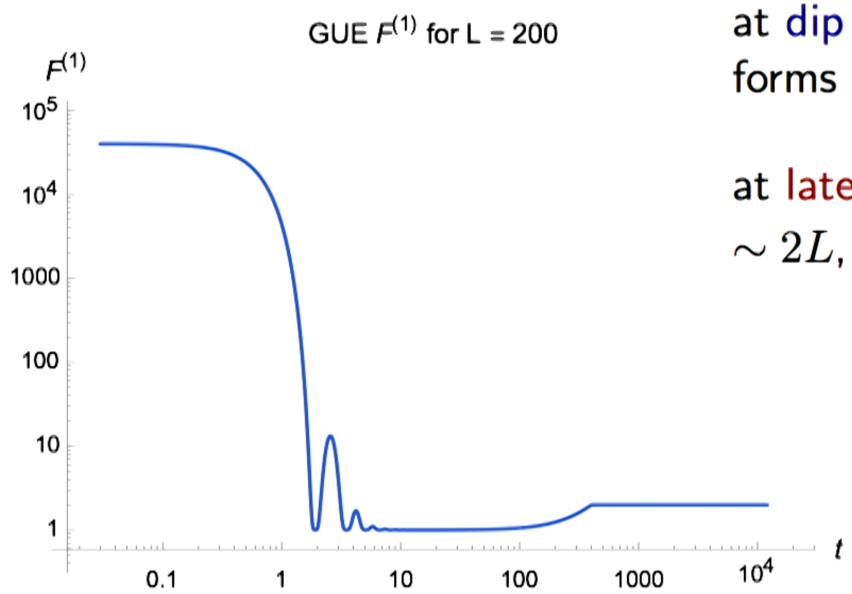
written in terms of the 2-point form factor

$$\mathcal{R}_2 = \big\langle Z(t) Z^*(t) \big\rangle_{\mathrm{GUE}} = \int D\lambda \sum_{i,j} e^{i(\lambda_i - \lambda_j)t} \, .$$

$k = 1$ frame potential for GUE

use explicit expressions for $\mathcal{R}_2$ for GUE

$$\mathcal{F}^{(1)}_{\text{GUE}} \approx \frac{\mathcal{R}_2^2}{L^2} - \frac{2\mathcal{R}_2}{L^2} + 1\,,$$

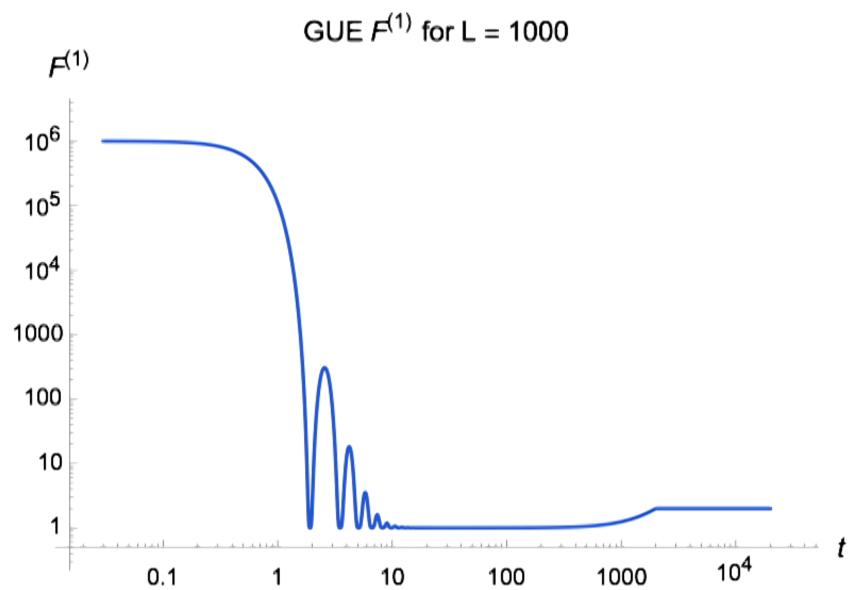


at dip time $\sim \sqrt{L}$, $\mathcal{F}^{(1)}_{\text{GUE}} = 1$, forms a 1-design

at late times, after plateau time $\sim 2L$, $\mathcal{F}^{(1)}_{\text{GUE}} = 2$, no longer Haar

$k = 1$ frame potential for GUE

late time behavior not necessarily too surprising...



Higher k frame potentials

We computed the second frame potential for GUE to be

$$
\begin{aligned}
\mathcal{F}^{(2)}_{\rm GUE} = \Big(& \left(L^4-8L^2+6\right)\mathcal{R}_4^2 + 4L^2\left(L^2-9\right)\mathcal{R}_4 + 4\left(L^6-9L^4+4L^2+24\right)\mathcal{R}_2^2 \\
& - 8L^2\left(L^4-11L^2+18\right)\mathcal{R}_2 + 2\left(L^4-7L^2+12\right)\mathcal{R}_{4,1}^2 - 4L^2\left(L^2-9\right)\mathcal{R}_{4,2} \\
& + \left(L^4-8L^2+6\right)\mathcal{R}_{4,2}^2 - 8\left(L^4-8L^2+6\right)\mathcal{R}_2\mathcal{R}_4 - 4L\left(L^2-4\right)\mathcal{R}_4\mathcal{R}_{4,1} \\
& + 16L\left(L^2-4\right)\mathcal{R}_2\mathcal{R}_{4,1} - 8\left(L^2+6\right)\mathcal{R}_2\mathcal{R}_{4,2} + 2\left(L^2+6\right)\mathcal{R}_4\mathcal{R}_{4,2} \\
& - 4L\left(L^2-4\right)\mathcal{R}_{4,1}\mathcal{R}_{4,2} + 2L^4\left(L^4-12L^2+27\right) \Big) \\
& \Big/ \Big((L-3)(L-2)(L-1)L^2(L+1)(L+2)(L+3)\Big).
\end{aligned}
$$

and computed the third frame potential for GUE to be

$$\mathcal{F}^{(3)}_{\rm GUE} = \Big(6L^{12} + 18\mathcal{R}_2^2 L^{12} - 36\mathcal{R}_4 L^{12} - 318L^{12} - \ldots$$

[illegible]

$$\Big/\Big((L-5)(L-4)(L-3)(L-2)(L-1)L^2(L+1)(L+2)(L+3)(L+4)(L+5)\Big)$$

Higher k frame potentials

We computed the second frame potential for GUE to be

$$\begin{aligned}\mathcal{F}^{(2)}_{\text{GUE}} =\Big(& \left(L^4-8L^2+6\right)\mathcal{R}_4^2+4L^2\left(L^2-9\right)\mathcal{R}_4+4\left(L^6-9L^4+4L^2+24\right)\mathcal{R}_2^2 \\ & -8L^2\left(L^4-11L^2+18\right)\mathcal{R}_2+2\left(L^4-7L^2+12\right)\mathcal{R}_{4,1}^2-4L^2\left(L^2-9\right)\mathcal{R}_{4,2} \\ & +\left(L^4-8L^2+6\right)\mathcal{R}_{4,2}^2-8\left(L^4-8L^2+6\right)\mathcal{R}_2\mathcal{R}_4-4L\left(L^2-4\right)\mathcal{R}_4\mathcal{R}_{4,1} \\ & +16L\left(L^2-4\right)\mathcal{R}_2\mathcal{R}_{4,1}-8\left(L^2+6\right)\mathcal{R}_2\mathcal{R}_{4,2}+2\left(L^2+6\right)\mathcal{R}_4\mathcal{R}_{4,2} \\ & -4L\left(L^2-4\right)\mathcal{R}_{4,1}\mathcal{R}_{4,2}+2L^4\left(L^4-12L^2+27\right)\Big) \\ & \Big/\Big((L-3)(L-2)(L-1)L^2(L+1)(L+2)(L+3)\Big).\end{aligned}$$

and computed the third frame potential for GUE to be

$$\mathcal{F}^{(3)}_{\text{GUE}} = \Big(\text{[illegible]} \Big) \Big/\Big((L-5)(L-4)(L-3)(L-2)(L-1)L^2(L+1)(L+2)(L+3)(L+4)(L+5)\Big)$$

Higher k frame potentials

$$k=1 \quad \text{Early: } \mathcal{F}^{(1)}_{\rm GUE} \sim \frac{\mathcal{R}_2^2}{L^2}, \text{ Dip: } \mathcal{F}^{(1)}_{\rm GUE} = 1, \text{ Late: } \mathcal{F}^{(1)}_{\rm GUE} = 2.$$

$$k=2 \quad \text{Early: } \mathcal{F}^{(2)}_{\rm GUE} \sim \frac{\mathcal{R}_4^2}{L^4}, \text{ Dip: } \mathcal{F}^{(2)}_{\rm GUE} = 2, \text{ Late: } \mathcal{F}^{(2)}_{\rm GUE} = 10.$$

$$k=3 \quad \text{Early: } \mathcal{F}^{(3)}_{\rm GUE} \sim \frac{\mathcal{R}_6^2}{L^6}, \text{ Dip: } \mathcal{F}^{(3)}_{\rm GUE} = 6, \text{ Late: } \mathcal{F}^{(3)}_{\rm GUE} = 96.$$

k-th frame potential: form a k-design at the dip,
late times no longer Haar

- related symptoms of chaos
- GUE forms a k-design, late times not Haar random
- if late time physics is GUE then Haar/k-designs might miss important (global) aspects of chaotic systems at late times
- but what is GUE capturing about chaotic systems?

let's look more explicitly at why GUE only captures global physics

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let's look more explicitly at why GUE only captures global physics

Correlators and form factors

Let's look at correlation functions in spin systems

Consider a Hamiltonian H on $|\mathcal{H}| = L = 2^n$, and consider the averaged 2-point function (at $\beta = 0$)

$$\int dA \, \langle A(0) A^\dagger(t) \rangle_\beta = \frac{1}{L} \int dA \, \mathrm{Tr}(A e^{-iHt} A^\dagger e^{iHt}) \, ,$$

where A is a unitary integrated over the Haar measure.

$$\int dA \, \langle A(0) A^\dagger(t) \rangle_\beta = \frac{\mathcal{R}_2^H(t)}{L^2}$$

($\mathcal{R}_2^H$ for a single Hamiltonian)

Correlators and form factors

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or more generally, $2k$-OTOCs $\langle A_1 B_1(t) \dots A_k B_k(t) \rangle_\beta$ and average

$$\int dA_1 dB_1 \dots dA_k \langle A_1 B_1(t) \dots A_k B_k(t) \rangle_\beta = \frac{\mathcal{R}_{2k}^H(t)}{L^{2k}}$$

i.e. $2k$-OTOCs $\leftrightarrow$ $2k$-form factors

(can also understand universal $1/t^6$ OTOC behavior [Bagrets, Altland, Kamenev])

→ connects spectral statistics and physical observables
→ gives a practical way to measure the form factors

2-pt and form factors: a check

Pauli operators form a 1-design, so we average as

$$\int dA \langle A(0) A^\dagger(t) \rangle = \frac{1}{4^N} \sum_{A \in \mathrm{Pauli}} \langle A(0) A^\dagger(t) \rangle = \frac{\mathcal{R}_2^H(t)}{L^2},$$

if we pick a few random Pauli operators, we should be able to approximate $\mathcal{R}_2$
we can also check that $\Delta \langle A(0) A^\dagger(t) \rangle^2_{\mathrm{avg}} \sim \mathcal{O}(1/L^2)$

Let's check this!
consider a random non-local spin system, sum over all 2-body operators with random Gaussian couplings J

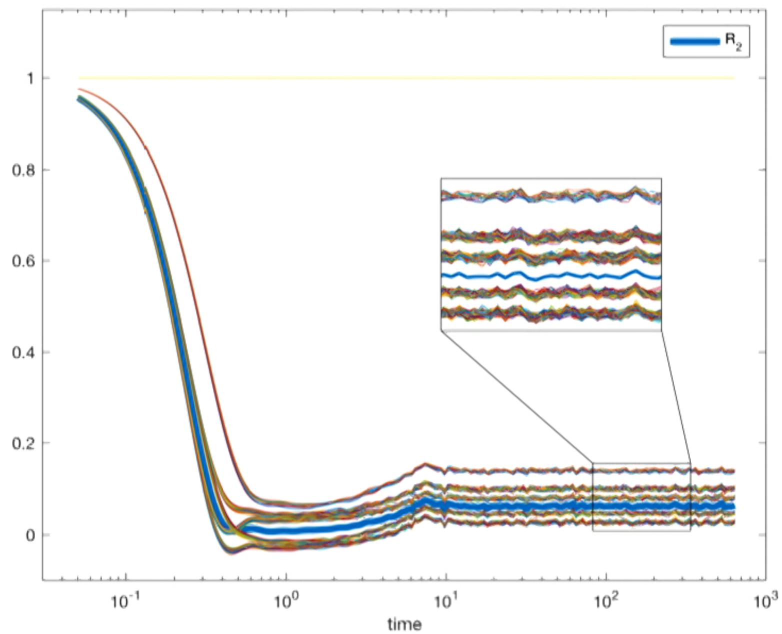
$$H_{\mathrm{RNL}} = \sum_{i,j,\alpha,\beta} J_{ij\alpha\beta} S_i^\alpha S_j^\beta,$$

2-pt and form factors: a check

Check: $\frac{1}{4^N}\sum_{A\in\text{Pauli}}\langle A(0)A^\dagger(t)\rangle = \frac{\mathcal{R}_2^H(t)}{L^2}$

consider a random non-local spin system:

$$H_{\text{RNL}} = \sum_{i,j,\alpha,\beta} J_{ij\alpha\beta} S_i^\alpha S_j^\beta \,,$$



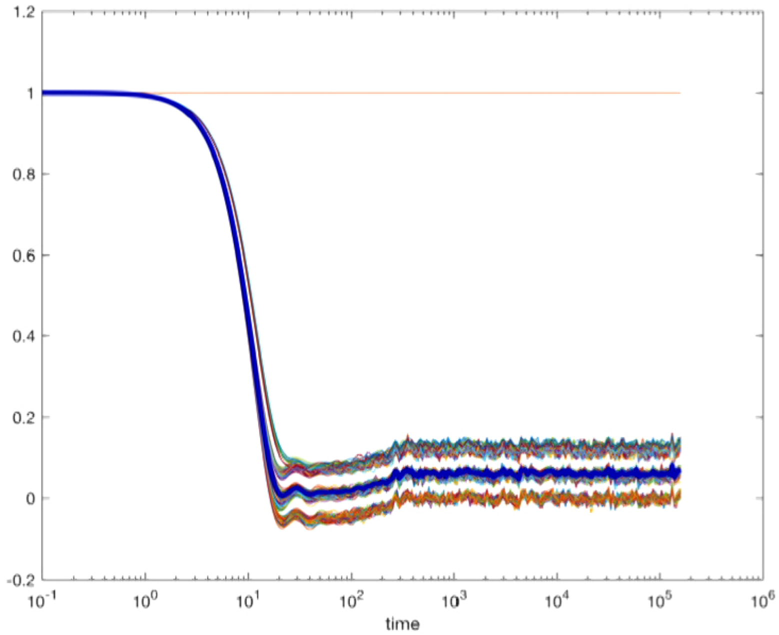
2-pt and form factors: an SYK check

just for fun, can also check this for SYK: $\langle \chi_i \chi_j(t) \rangle$

i.e. 2-pt correlators of Majoranas

where SYK:

$$H_{\rm SYK} = \sum_{i,j,k,\ell} J_{ijk\ell} \chi_i \chi_j \chi_k \chi_\ell$$



for SYK for N=10 Majoranas, $|\mathcal{E}| = 200$, $\beta = 0$

OTOCs and random matrices

consider a 2-pt function averaged over GUE (fix ops and average over H's, i.e. $A(t) = e^{-iHt} A e^{iHt}$ with GUE H's)

$$\text{GUE avg}: \quad \int dH \langle A(0) A^\dagger(t) \rangle_\beta = \frac{\mathcal{R}_2(t) - 1}{L^2 - 1} \approx \frac{\mathcal{R}_2(t)}{L^2},$$

recall from before that the operator average (for any H)

$$\text{Op avg}: \quad \int dA \, \langle A(0) A^\dagger(t) \rangle_\beta = \frac{\mathcal{R}_2^H(t)}{L^2}$$

GUE average is the same as operator average, but in taking GUE average we make no assumption about locality of operators
$\rightarrow$ GUE suited to capture global properties

OTOCs and random matrices

2-pt functions averaged over GUE:

$$\langle A(0)A^\dagger(t)\rangle_{\text{GUE}} \approx \frac{\mathcal{R}_2(t)}{L^2}\,.$$

furthermore, we can use the 4-th moment of Haar and compute the OTOC averaged over GUE:

$$\langle A(0)B(t)C(0)D(t)\rangle_{\text{GUE}} \approx \langle ABCD\rangle \frac{\mathcal{R}_4(t)}{L^4}\,.$$

what have we learned?

OTOCs and random matrices

2-pt functions averaged over GUE:

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what have we learned?

GUE as a chaotic pastiche

- GUE captures thermalization (apparent loss of QI)
- GUE does not capture scrambling ($t_4 \sim t_2/2$)
- GUE does not care about spatial/temporal locality

(BH viewpoint: GUE BHs are bad)

None of this is too surprising, but it at least formalizes why we fail to capture physics at early times

More interesting is why GUE captures chaotic physics at late times $\rightarrow$ Haar invariance

GUE is a good description of a system which has lost its notion of locality

k-invariance

In many of the expressions we derived (frame potential, OTOCs $\leftrightarrow$ form factors), the key ingredient was invariance of the measure

Consider an ensemble of unitary time evolutions by some (ensemble of) physical Hamiltonians:

$$\mathcal{E}_t = \{e^{-iHt}, \quad H \in \mathcal{E}_H\}$$

i.e. SYK, spin system, (QFT on a random lattice)

k-invariance

Consider an ensemble of unitary time evolutions by some (ensemble of) physical Hamiltonians: $\mathcal{E}_t = \{e^{-iHt}, H \in \mathcal{E}_H\}$

Define the Haar-invariant ensemble: $\widetilde{\mathcal{E}}_t = U\mathcal{E}_t U^\dagger$

ensemble is k-invariant iff $\mathcal{F}^{(k)}_{\mathcal{E}_t} = \mathcal{F}^{(k)}_{\widetilde{\mathcal{E}}_t}$ (reproduces the first k moments)

equivalently,

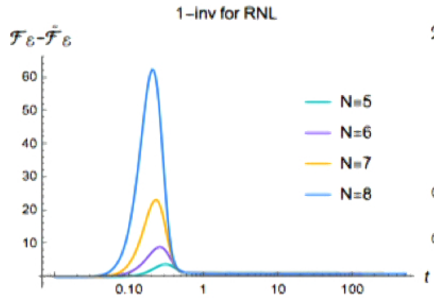
$$\boxed{\mathcal{F}^{(k)}_{\mathcal{E}_t}(t) - \mathcal{F}^{(k)}_{\widetilde{\mathcal{E}}_t}(t) \geq 0}$$

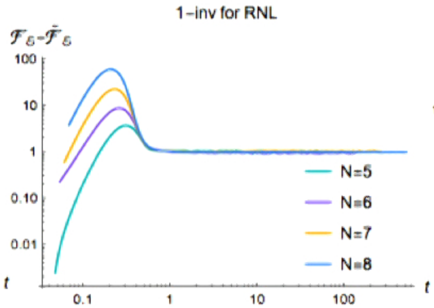
defines a distance to k-invariance

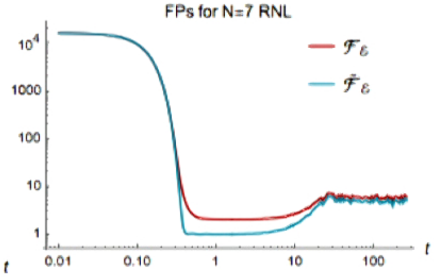
this quantity becoming small signifies the onset of a RMT description

k-invariance: a check

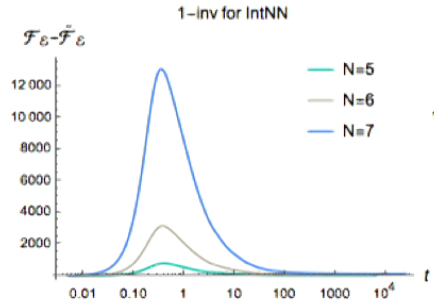
random non-local spin system: $H = \sum_{i,j,\alpha,\beta} J_{ij\alpha\beta} S_i^\alpha S_j^\beta$

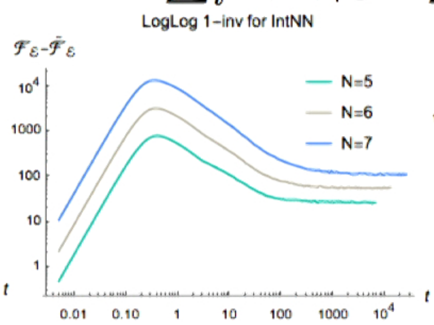


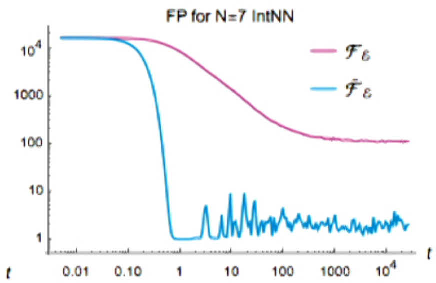




integrable spin chain: $H = -\sum_i Z_i Z_{i+1} - \sum_i h_i X_i$

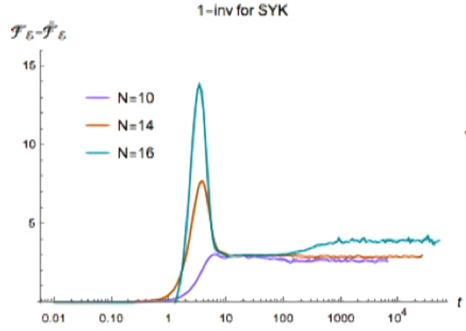


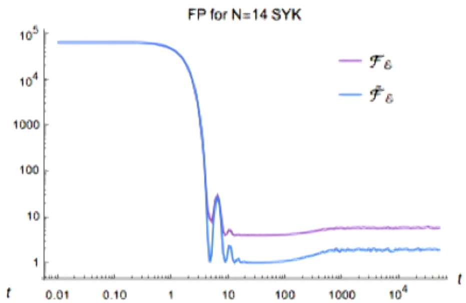




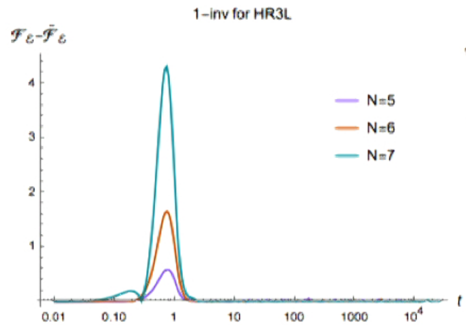
k-invariance: a check

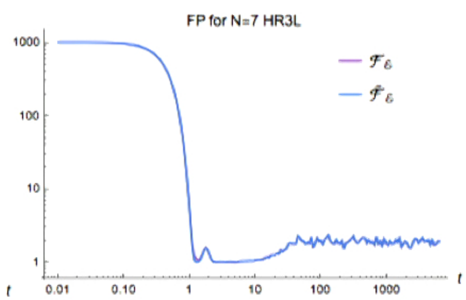
SYK: $H = \sum_{ijk\ell} J_{ijk\ell}\chi_i\chi_j\chi_k\chi_\ell$





random 3-local spin system: $H = \sum_{ijk\alpha\beta\gamma} J_{\alpha\beta\gamma ijk} S_i^\alpha S_j^\beta S_k^\gamma$





Summary

- computed form factors in RMT
- related symptoms of chaos (OTOCs, frame potentials, form factors)
- understood time scales at which GUE is Haar random
- and why GUE fails to capture early times
- related observables to spectral statistics

Late time behavior of chaotic systems understood as k-invariance

- Extensions of this work:

  i) Consider chaotic systems with symmetry
  generalize to GOE, GSE and to all extended ensembles in 10-fold symmetry classification (Altland-Zirnbauer) (as in classification of top. phases)
  → quantify randomness/chaos for symmetry classes using Weingarten calculus for the associated compact symmetric space $U(L)/H$

  ii) Apply these ideas to study chaos in SYK models
  (e.g. consider Wishart ensembles for supersymmetric SYK models [NHJ, Liu; 1711.08184])

  iii) Investigate k-invariance in random spin systems and SYK as a characterization of late-time chaos

Work in progress/future work

- Things to think about:
  - understand the precise role ETH plays in k-invariance, use these tools to study thermalization in quantum systems
  - understand late-time OTOCs
  - transport and hydrodynamics
  - complexity?
  - shockwaves on AdS-KN black holes
  - chaos in $d > 2$ CFTs
  - sparse RMT?