

Title: Quantum chaos and late-time dynamics

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Abstract: 

From a quantum information perspective, we will study universal features of chaotic quantum systems. Recent progress has made evident that quantifying chaos is a useful way to gain insight into strongly-coupled field theories, quantum many-body systems, as well as the quantum nature of black holes. We will derive relations between different diagnostics of chaos and scrambling (OTOCs, spectral functions, and frame potentials) and define a quantity to capture the onset of a random matrix description. We will review and use tools from quantum information and random matrix theory, but our goal will be to understand strongly-interacting systems.

# Quantum chaos and late-time dynamics

A look through the (translucent) QI lens

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Based on:

J. Cotler, NHJ, J. Liu, B. Yoshida, "Chaos, Complexity, and Random Matrices," [JHEP11\(2017\)048, 1706.05400](#)  
and "Symmetry,  $k$ -invariance, and late-time chaos," 1801.hopefully soon  
(also related: NHJ, J. Liu; 1711.08184)



I'll be talking about quantum chaos and random matrix theory in  
quantum mechanical systems

→ finite dimensional  $\mathcal{H}$  and discrete spectrum

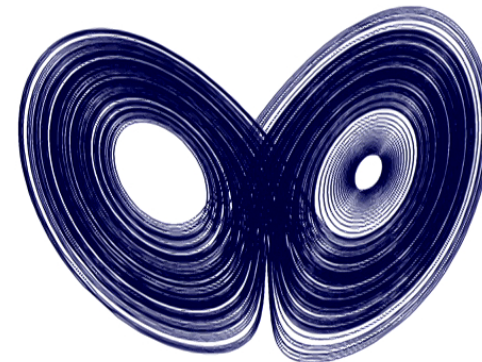


# Chaos

(ubiquitous but not boring)

we are familiar with classical chaos and sensitivity to initial conditions, but chaos in quantum systems is **different**

intuition: small perturbations grow to affect the system



[chaotic butterfly]



*"Remember that hurricane a thousand miles away? That was me!"*

[chaos in culture]



# Quantum Chaos - a historical tour

The definition of quantum chaos seems to depend on the decade and subfield you work in:

70s-80s - QM chaos = quantization of a classically chaotic system

[Gutzwiller]

80s-90s - QM chaos = statement about universal properties of the spectrum [Bohigas, Giannoni, Schmit], [Berry] → Random matrix theory

([Berry], [Srednicki] → ETH)

2010s - QM chaos can be probed by correlation functions in thermal states → Out-of-time ordered correlators (OTOCs)

[Kitaev, Stanford, Shenker, Maldacena, Roberts, Yoshida, Susskind, ...]

## Quantum Chaos - an admission

some semantic guidance:

Michael Berry - Quantum Chaology, Not Quantum Chaos!

remember: classical chaos is **different** from quantum chaos

and furthermore, a precise definition of quantum chaos **remains elusive**

in this sense, we should be careful when discussing “**probes**” of chaos

## OTOCs: Chaos in the modern era

quick review of out-of-time ordered correlation functions (OTOCs)  
an old idea [Larkin, Ovchinnikov]

Consider the 4-point function of a pair of local(ish) operators in thermal states

$$\langle A(t)BA(t)B \rangle_{\beta}$$

## OTOCs: Chaos in the modern era

Rough intuition: [Roberts, Stanford, Susskind] think about the time evolution of a Pauli operator in a chaotic spin system

$$Z_1(t) = e^{-iHt} Z_1 e^{iHt} = Z_1 - it[H, Z_1] - \frac{t^2}{2}[H, [H, Z_1]] + \dots$$

operator 'grows' in time, measured by commutator  
consider  $[Z_1(t), Z_8]$  (for a chaotic spin chain)

expand out  $\langle [Z_1(t), Z_8]^2 \rangle_\beta$ :

$$\begin{aligned} \langle [Z_1(t), Z_8]^2 \rangle_\beta &= \langle Z_1(t) Z_8 Z_8 Z_1(t) \rangle_\beta + \langle Z_8 Z_1(t) Z_1(t) Z_8 \rangle_\beta \\ &\quad - \langle Z_1(t) Z_8 Z_1(t) Z_8 \rangle_\beta - \langle Z_8 Z_1(t) Z_8 Z_1(t) \rangle_\beta \end{aligned}$$

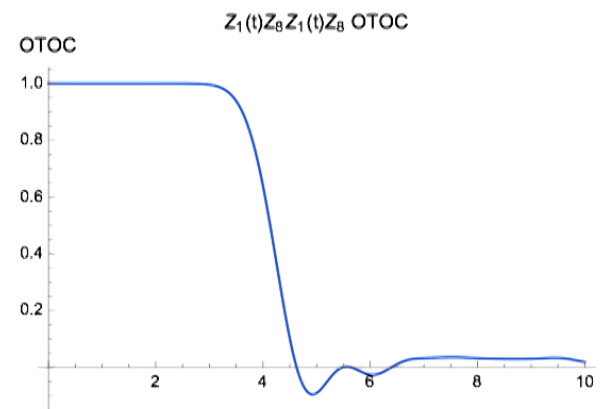
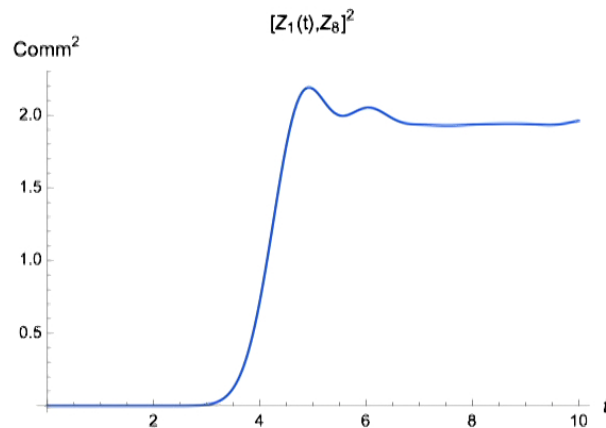


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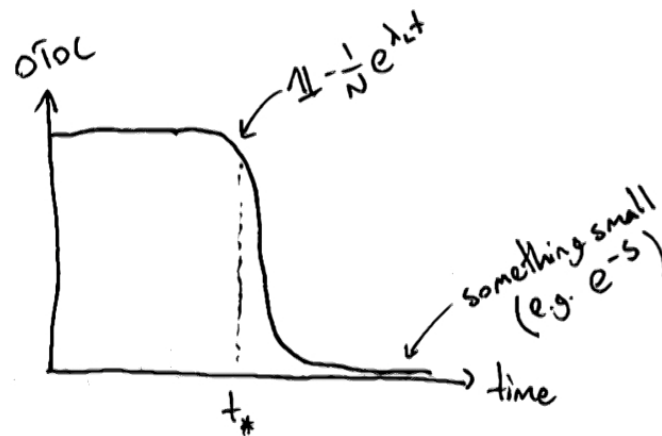


## OTOCs: Chaos in the modern era

OTOC: 4-point function of a pair of local(ish) operators in thermal states

$$\langle A(t)BA(t)B \rangle_{\beta}$$

salient features of (chaotic) OTOCs:



(exponentially) growing  
corrections at early times  
small value at late times

## A chaotic resurgence

In recent years, a revival in quantum chaos spurred by studying OTOCs

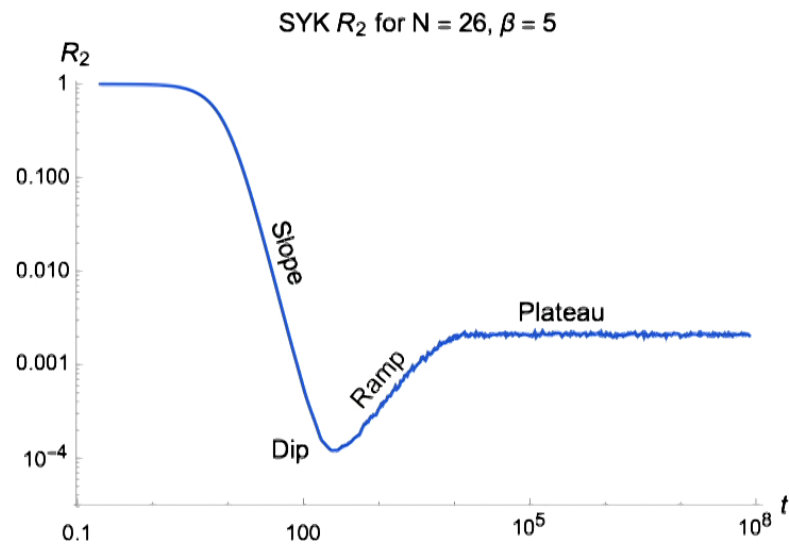
- ▶ Chaos in gravity: [Kitaev], [Shenker, Stanford] (also, Yoni's talk yesterday)
- ▶ Chaos in CFT: [Roberts, Stanford] (later: [Fitzpatrick, Kaplan], [Dyer, Gur-Ari])
- ▶ Chaos enfeathered: a chaos bound [Maldacena, Shenker, Stanford]
- ▶ Chaos in SYK: [Kitaev], [Stanford, Maldacena]
- ▶ RMT in SYK: [Cotler, Gur-Ari, Hanada, Polchinski, Saad, Shenker, Stanford, Streicher, Tezuka]

# Form factor in SYK

RMT in SYK: [Cotler, Gur-Ari, Hanada, Polchinski, Saad, Shenker, Stanford, Streicher, Tezuka]

Studied spectral form factor in SYK (to investigate BH info loss à la [Maldacena])

$$\mathcal{R}_2(\beta, t) \equiv \langle Z(\beta, t) Z^*(\beta, t) \rangle_{\text{SYK}} = \langle \text{Tr} (e^{-\beta H - iHt}) \text{Tr} (e^{-\beta H + iHt}) \rangle_{\text{SYK}}$$



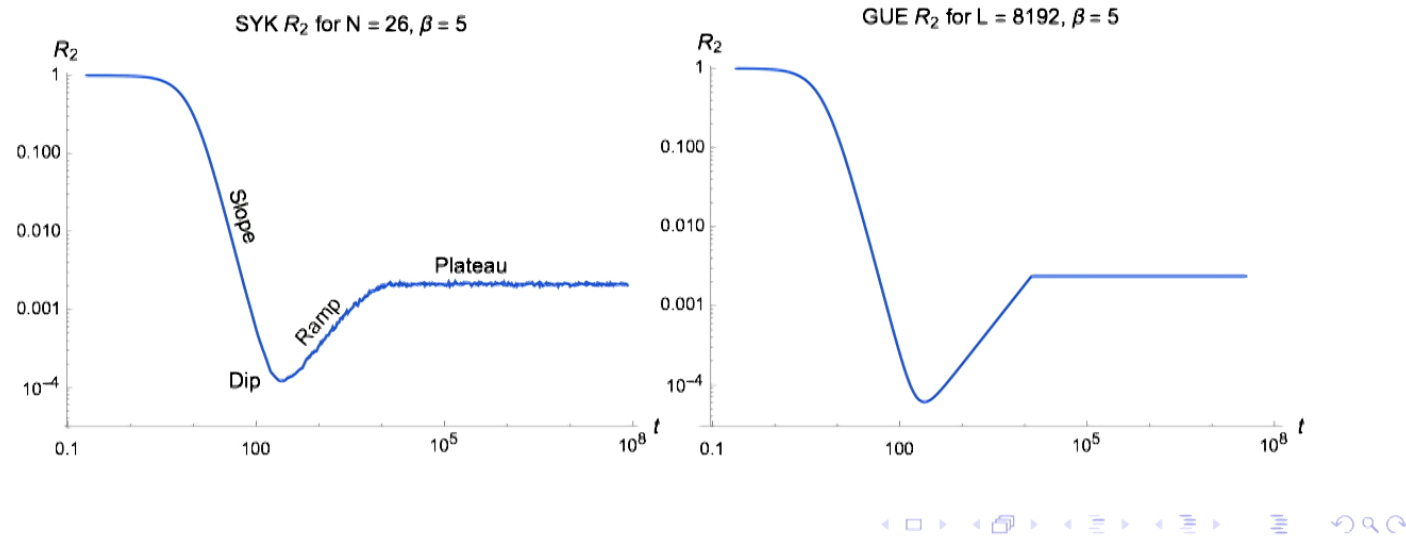
# Form factor in SYK

RMT in SYK: [Cotler, Gur-Ari, Hanada, Polchinski, Saad, Shenker, Stanford, Streicher, Tezuka]

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found agreement with RMT



## Goal: Chaos through the QI lens

a QI view can provide guiding intuition [Hayden, Preskill]

recently, OTOCs were cast in the QI light

[Hosur, Qi, Roberts, Yoshida] → relating OTOCs to MI, quantify scrambling

[Roberts, Yoshida] → relating OTOCs to the frame potential, quantify randomness

chaos is a ubiquitous feature of strongly-interacting systems → understand strongly-coupled systems/BHs

as 80s chaos  $\approx$  RMT, and (some aspects of) the late time behavior of SYK  $\approx$  RMT

in a **quantum information theoretic** way:

- understand the role RMT plays in describing chaotic dynamics
- want to relate symptoms of chaos (a chaotic first step)

distant motivation: understand BHs



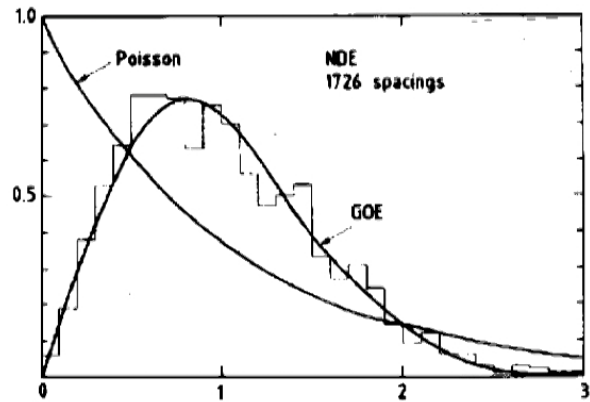
## Outline

- ▶ Motivation ✓
- ▶ Quick overview of RMT
- ▶ Form factors and RMT
- ▶ Frame potentials and RMT
- ▶ OTOCs and RMT
- ▶  $k$ -invariance

# Overview of RMT

Early success in nuclear physics [Wigner], [Dyson]

famously reproducing the nearest-neighbor spacings of heavy nuclei resonances [Nuclear Data Ensemble]



taken from [Guhr, Müller-Groeling, Weidenmüller]



# Overview of RMT

Early success in nuclear physics [Wigner], [Dyson]

Since has pervaded many seemingly disparate subfields  
large  $N$  QFT, string theory, transport in disordered quantum systems, ...

Classic matrix ensembles (GUE, GOE, GSE)

→ focus on the **Gaussian unitary ensemble** with  
 $\text{GUE}(L, 0, 1/\sqrt{L})$

**GUE** = ensemble of  $L \times L$  **Hermitian** matrices  
off-diagonal  $N(0, 1/\sqrt{L})_{\mathbb{C}}$  and diagonal  $N(0, 1/\sqrt{L})_{\mathbb{R}}$

note: use different norm than RMT  
also, sorry about  $L$



$$L \equiv \dim X$$



## Overview of GUE

GUE has a probability weight,  $P(H) \propto e^{-\frac{L}{2}\text{Tr}(H^2)}$   
and invariant measure  $dH = d(UHU^\dagger)$

In the eigenvalue basis:

$$P(\lambda_1, \dots, \lambda_L) = C e^{-\frac{L}{2} \sum_i \lambda_i^2} |\Delta(\lambda)|^2.$$

Average over GUE as

$$\langle O(\lambda) \rangle_{\text{GUE}} \equiv \int D\lambda O(\lambda), \quad \text{where} \quad \int D\lambda = \int \prod_i d\lambda_i P(\lambda) = 1.$$

The **density of states**:

$$\rho(\lambda) = \left\langle \sum_{i=1}^L \delta(\lambda - \lambda_i) \right\rangle_{\text{GUE}}.$$

The **spectral  $n$ -point correlation function**:

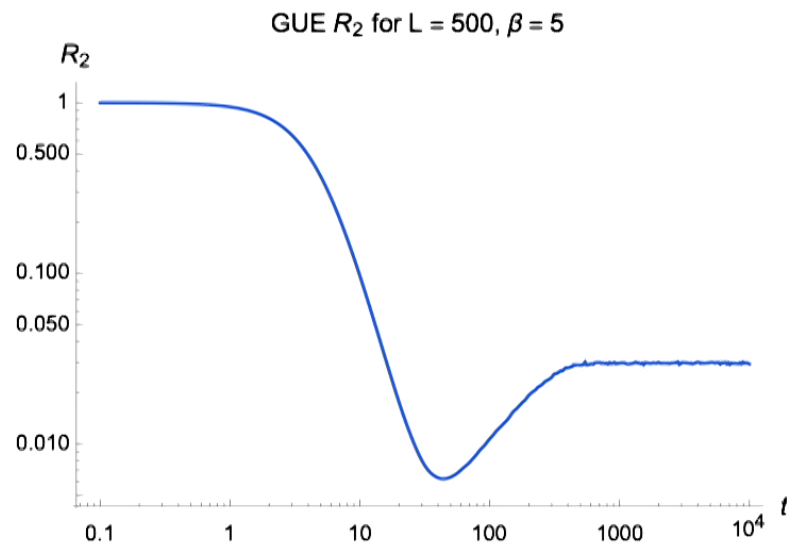
$$\rho^{(n)}(\lambda_1, \dots, \lambda_n) = \frac{L!}{(L-n)!} \int d\lambda_{n+1} \dots d\lambda_L P(\lambda_1, \dots, \lambda_L).$$



## Spectral form factor

$$\mathcal{R}_2(\beta, t) \equiv \langle Z(\beta, t) Z^*(\beta, t) \rangle = \int D\lambda \sum_{i,j} e^{i(\lambda_i - \lambda_j)t} e^{-\beta(\lambda_i + \lambda_j)}$$

Let's discuss some universal aspects of the form factor:

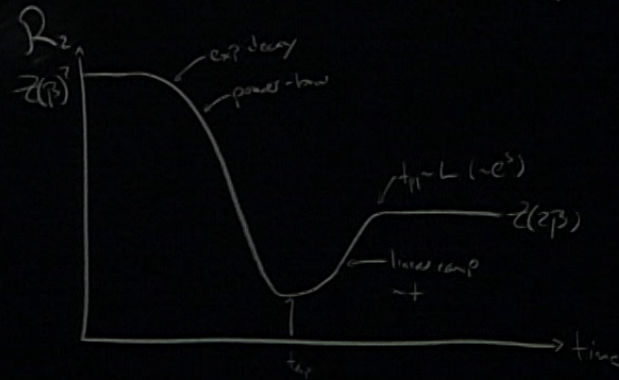


$$L \equiv \dim \mathcal{X}$$

$$Z(\beta, t) = \text{Tr}(e^{-tH - \beta H})$$

$$L \equiv \dim \mathcal{X}$$

$$\zeta(\beta, t) = \text{Tr}(e^{-tHt - \beta H})$$



## 2-pt form factor at $\beta = 0$

The infinite temperature 2-point form factor

$$\mathcal{R}_2(t) = \int D\lambda \sum_{i,j} e^{i(\lambda_i - \lambda_j)t}$$

can be computed as

$$\mathcal{R}_2(t) = L^2 r_1^2(t) - L r_2(t) + L,$$

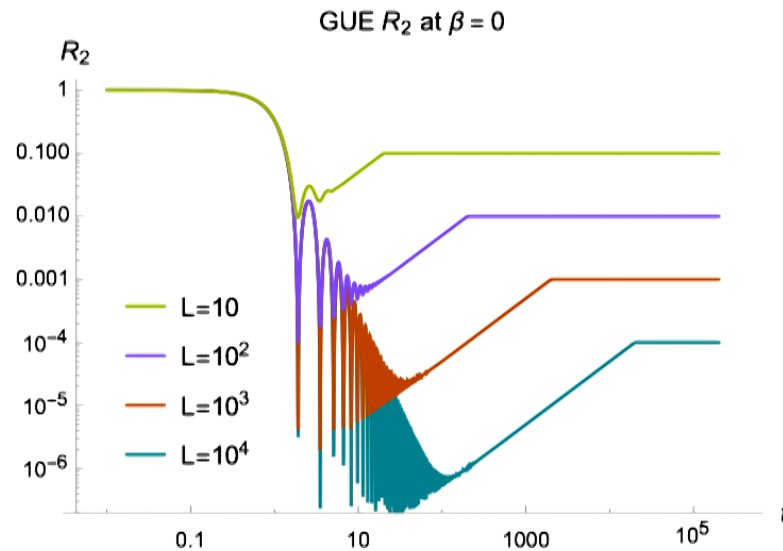
where we define the functions

$$r_1(t) = \frac{J_1(2t)}{t}, \quad \text{and} \quad r_2(t) = \begin{cases} 1 - \frac{t}{2L}, & \text{for } t < 2L \\ 0, & \text{for } t > 2L \end{cases}.$$

## 2-pt form factor at $\beta = 0$

The infinite temperature 2-point form factor

$$\mathcal{R}_2(t) = L^2 r_1^2(t) - L r_2(t) + L,$$



dip time:  $t_d = \sqrt{L}$  and dip value:  $\mathcal{R}_2 = \sqrt{L}$  → linear rise

plateau time:  $t_p = 2L$  and plateau value:  $\mathcal{R}_2 = L$



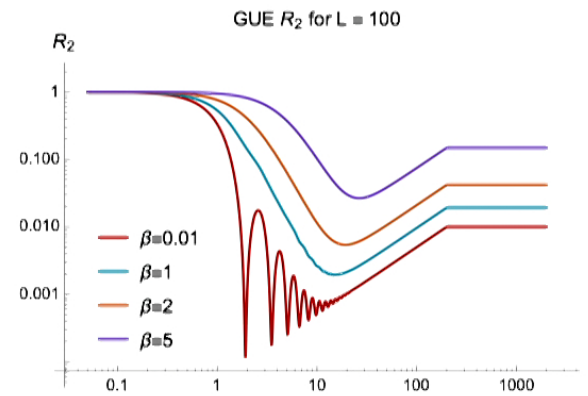
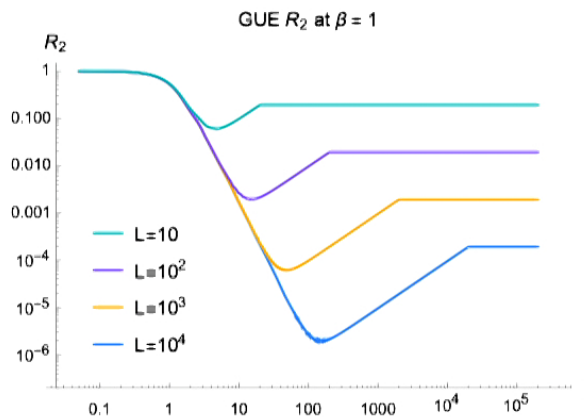
## 2-pt form factor at finite $\beta$

at finite temperature

$$\mathcal{R}_2(t, \beta) = \langle Z(t, \beta) Z^*(t, \beta) \rangle_{\text{GUE}} = \int D\lambda \sum_{i,j} e^{i(\lambda_i - \lambda_j)t} e^{-\beta(\lambda_i + \lambda_j)},$$

we insert the spectral  $\rho^{(2)}$  and use the short-distance kernel [Brézin, Hikami] to find

$$\mathcal{R}_2(t, \beta) = L^2 r_1(t + i\beta) r_1(-t + i\beta) + L r_1(2i\beta) - L r_1(2i\beta) r_2(t).$$



dip time:  $t_d = h_2(\beta)\sqrt{L}$  and dip value:  $\mathcal{R}_{2\beta} = h_3(\beta)\sqrt{L}$  → linear rise

plateau time:  $t_p = 2L$  and plateau value:  $\mathcal{R}_{2\beta} = h_1(2\beta)L$

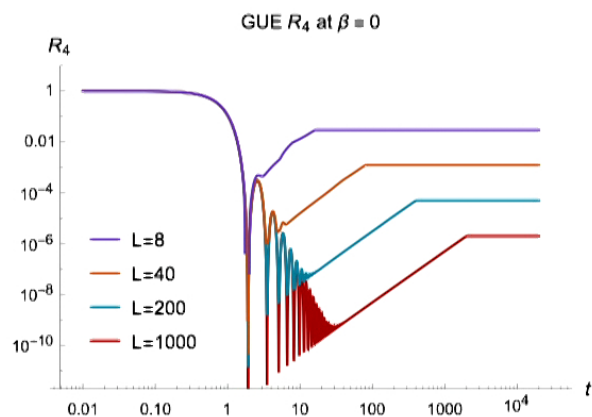
## 4-pt form factor at $\beta = 0$

we compute the 4-pt form factor at infinite temperature

$$\mathcal{R}_4(t) = \langle Z(t)Z(t)Z^*(t)Z^*(t) \rangle_{\text{GUE}} = \int D\lambda \sum_{i,j,k,\ell} e^{i(\lambda_i + \lambda_j - \lambda_k - \lambda_\ell)t}$$

insert  $\rho^{(4)}$  and carefully integrate each term, we find

$$\begin{aligned} \mathcal{R}_4(t) = & L^4 r_1^4(t) + 2L^2 r_2^2(t) - 4L^2 r_2(t) \\ & - 7L^2 r_2(2t) + 4Lr_2(3t) + 4Lr_2(t) + 2L^2 - L. \end{aligned}$$



dip time:  $t_d = \sqrt{L}$  and dip value:  $\mathcal{R}_2 = L \rightarrow$  quadratic rise

plateau time:  $t_p = 2L$  and plateau value:  $\mathcal{R}_2 = 2L^2$



## Overview of QI Machinery

**Haar:** (unique left/right invariant) measure on unitary group  $U(L)$

Consider an operator  $O$  acting on  $\mathcal{H}^{\otimes k}$ , the  **$k$ -fold channel** of  $O$  with respect to Haar:

$$\Phi_{\text{Haar}}^{(k)}(O) \equiv \int_{\text{Haar}} dU (U^{\otimes k})^\dagger O U^{\otimes k}.$$



## Overview of QI Machinery

**Haar:** (unique left/right invariant) measure on unitary group  $U(L)$

Consider an operator  $O$  acting on  $\mathcal{H}^{\otimes k}$ , the  **$k$ -fold channel** of  $O$  with respect to Haar:

$$\Phi_{\text{Haar}}^{(k)}(O) \equiv \int_{\text{Haar}} dU (U^{\otimes k})^\dagger O U^{\otimes k}.$$

Instead, consider an ensemble of unitaries  $\mathcal{E} = \{p_i, U_i\}$ , where the  $k$ -fold channel of  $\mathcal{E}$  is

$$\Phi_{\mathcal{E}}^{(k)}(O) \equiv \int_{U \in \mathcal{E}} dU (U^{\otimes k})^\dagger O U^{\otimes k}.$$

Ensemble  $\mathcal{E}$  is a **unitary  $k$ -design** if and only if

$$\Phi_{\mathcal{E}}^{(k)}(O) = \Phi_{\text{Haar}}^{(k)}(O),$$

meaning we reproduce the first  $k$  moments of Haar.

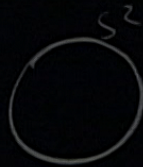


$$\mathbb{R}^3 \quad P_K$$
$$\frac{1}{|X|} \sum_{x \in X} P_K(x) = \int_{S^2} d\mu P_K$$

$\dim \mathcal{H}$

Spherical design

$\mathbb{R}^3, P_K$



$$\beta(t) = \text{Tr}(e^{-iHt - \beta H})$$

exp decay  
power-law

$$|t| \sim L(\sim e^S)$$

## Overview of QI Machinery

Ensemble  $\mathcal{E}$  is a **unitary  $k$ -design** if and only if

$$\Phi_{\mathcal{E}}^{(k)}(O) = \Phi_{\text{Haar}}^{(k)}(O),$$

Measure of this  $\rightarrow$  **Frame potential** [Scott]:

$$\mathcal{F}_{\mathcal{E}}^{(k)} = \int_{U, V \in \mathcal{E}} dU dV |\text{Tr}(U^\dagger V)|^{2k},$$

(distance to Haar)

$k$ -th frame potential **for Haar**:  $\mathcal{F}_{\text{Haar}}^{(k)} = k!$  for  $k \leq L$ .

For ensemble  $\mathcal{E}$ , the frame potential is **lower bounded** by

$$\mathcal{F}_{\mathcal{E}}^{(k)} \geq \mathcal{F}_{\text{Haar}}^{(k)},$$

( = iff  $\mathcal{E}$  is a  $k$ -design)





## Why frame potentials?

in BH physics we often approximate by Haar unitaries [Page], [Hayden, Preskill],  $\rightarrow$   $k$ -design behavior sufficient

[Roberts, Yoshida] made progress quantifying chaos by relating  $2k$ -OTOCs to the  $k$ -th frame potential

$$\frac{1}{L^{4k}} \sum_{A's, B's} \left| \langle A_1 B_1(t) \dots A_k B_k(t) \rangle_{\mathcal{E}} \right|^2 = \frac{\mathcal{F}_{\mathcal{E}}^{(k)}}{L^{2(k+1)}}$$

where " $B(t)$ " =  $UBU^\dagger$ , and for any ensemble  $\mathcal{E}$ .

makes precise an approach to *randomness*

(also complexity...)

## $k = 1$ frame potential for GUE

Consider the ensemble of unitary time evolutions at a fixed time  $t$  with GUE Hamiltonians

$$\mathcal{E}_t^{\text{GUE}} = \{e^{-iHt}, \text{ with } H \in \text{GUE}\}.$$

## $k = 1$ frame potential for GUE

Consider the ensemble:  $\mathcal{E}_t^{\text{GUE}} = \{e^{-iHt}, H \in \text{GUE}\}$

The **first frame potential** for GUE:

$$\mathcal{F}_{\text{GUE}}^{(1)} = \int dH_1 dH_2 e^{-\frac{L}{2} \text{Tr} H_1^2} e^{-\frac{L}{2} \text{Tr} H_2^2} \left| \text{Tr} (e^{iH_1 t} e^{-iH_2 t}) \right|^2$$

insert Haar unitaries, use L/R invariance of Haar, and integrate using the second moment

$$\mathcal{F}_{\text{GUE}}^{(1)} = \frac{1}{L^2 - 1} \left( \mathcal{R}_2^2 + L^2 - 2\mathcal{R}_2 \right),$$

written in terms of the 2-point form factor

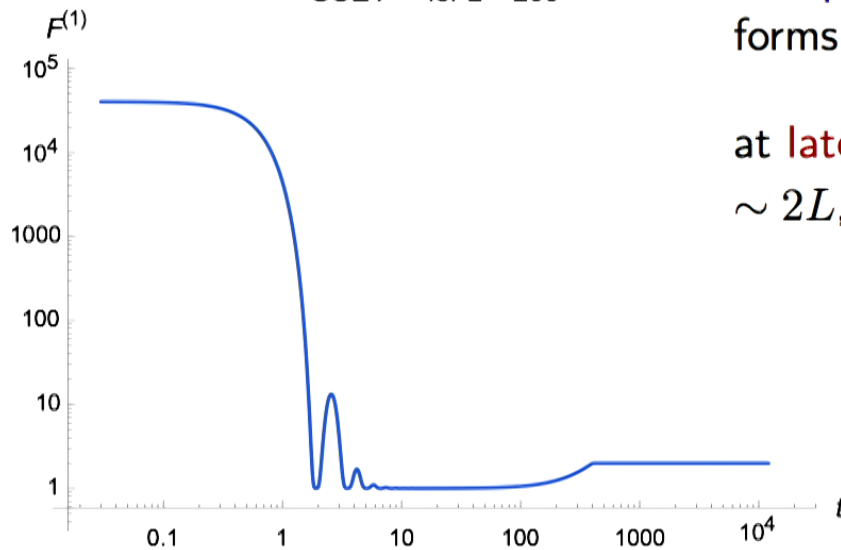
$$\mathcal{R}_2 = \langle Z(t) Z^*(t) \rangle_{\text{GUE}} = \int D\lambda \sum_{i,j} e^{i(\lambda_i - \lambda_j)t}.$$

## $k = 1$ frame potential for GUE

use explicit expressions for  $\mathcal{R}_2$  for GUE

$$\mathcal{F}_{\text{GUE}}^{(1)} \approx \frac{\mathcal{R}_2^2}{L^2} - \frac{2\mathcal{R}_2}{L^2} + 1,$$

GUE  $F^{(1)}$  for  $L = 200$

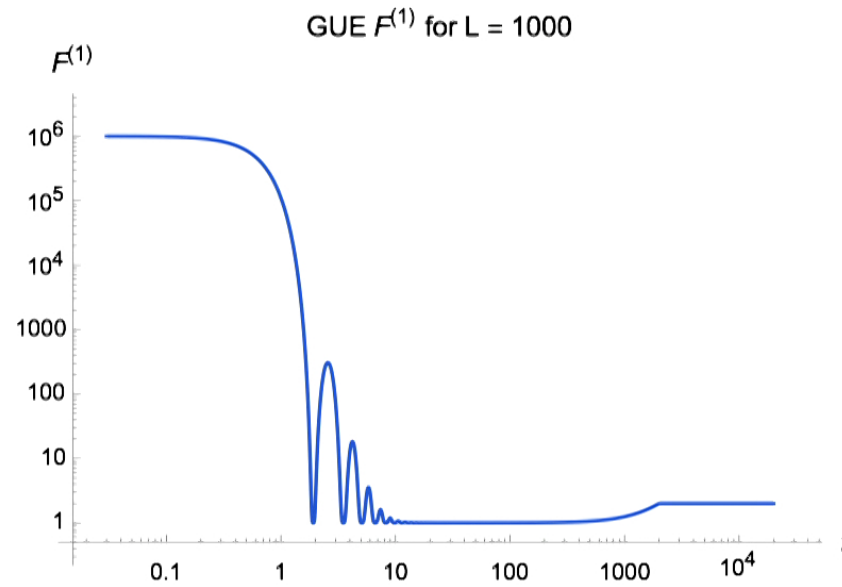


at dip time  $\sim \sqrt{L}$ ,  $\mathcal{F}_{\text{GUE}}^{(1)} = 1$ ,  
forms a 1-design

at late times, after plateau time  
 $\sim 2L$ ,  $\mathcal{F}_{\text{GUE}}^{(1)} = 2$ , no longer Haar

# $k = 1$ frame potential for GUE

late time behavior not necessarily too surprising...



# Higher $k$ frame potentials

We computed the second frame potential for GUE to be

$$\begin{aligned} \mathcal{F}_{\text{GUE}}^{(2)} = & \left( (L^4 - 8L^2 + 6) \mathcal{R}_4^2 + 4L^2 (L^2 - 9) \mathcal{R}_4 + 4(L^6 - 9L^4 + 4L^2 + 24) \mathcal{R}_2^2 \right. \\ & - 8L^2 (L^4 - 11L^2 + 18) \mathcal{R}_2 + 2(L^4 - 7L^2 + 12) \mathcal{R}_{4,1}^2 - 4L^2 (L^2 - 9) \mathcal{R}_{4,2} \\ & + (L^4 - 8L^2 + 6) \mathcal{R}_{4,2}^2 - 8(L^4 - 8L^2 + 6) \mathcal{R}_2 \mathcal{R}_4 - 4L(L^2 - 4) \mathcal{R}_4 \mathcal{R}_{4,1} \\ & + 16L(L^2 - 4) \mathcal{R}_2 \mathcal{R}_{4,1} - 8(L^2 + 6) \mathcal{R}_2 \mathcal{R}_{4,2} + 2(L^2 + 6) \mathcal{R}_4 \mathcal{R}_{4,2} \\ & \left. - 4L(L^2 - 4) \mathcal{R}_{4,1} \mathcal{R}_{4,2} + 2L^4 (L^4 - 12L^2 + 27) \right) \\ & / \left( (L-3)(L-2)(L-1)L^2(L+1)(L+2)(L+3) \right). \end{aligned}$$

and computed the third frame potential for GUE to be

$$\begin{aligned} \mathcal{F}_{\text{GUE}}^{(3)} = & \left( 6L^{13} + 18R_2^2 L^{12} - 36R_4 L^{11} - 318L^{10} - 846R_2^2 L^9 + 9R_4^2 L^9 + 18R_2^4 L^{10} + 9R_4^2 L^{10} + 1836R_2 L^{10} - 72R_4 R_2 L^{10} + 86R_4 L^{10} + 5550L^{10} \right. \\ & + 144R_4 R_2 L^{10} - 36R_4 R_4 L^{10} + 11574R_2^2 L^9 - 360R_4^2 L^9 + R_4^2 L^9 - 828R_2^2 L^9 + 9R_4^2 R_2 L^9 - 18R_4 R_2^2 L^9 - 441R_4^2 L^9 + 6R_4^2 L^9 \\ & + 48R_4^2 L^9 + 192R_4^2 L^9 + 4R_4^2 L^9 - 39772R_2^2 L^8 + 3976R_4 R_2 L^8 - 1728R_4^2 L^8 + 36R_4 R_2 L^8 - 18R_4 R_2 L^8 - 12R_4 L^8 - 36R_4^2 R_2 L^8 + 18R_4 R_2 L^8 \\ & + 1140R_4 R_2 L^8 - 36R_4 R_4 L^8 + 24R_4 R_2 L^8 - 8718L^8 - 9192R_2 R_4 L^8 + 1332R_4 R_2 L^8 + 36R_4 R_2 L^8 + 108R_4 R_2 L^8 + 15144R_4 R_2 L^8 \\ & - 144R_4 R_2 L^8 + 108R_4 R_2 L^8 - 12R_4 R_2 L^8 - 30R_4 R_2 L^8 + 36R_4 R_2 L^8 + 72R_4 R_2 L^8 - 24R_4 R_2 L^8 + 144R_4 R_2 L^8 - 72R_4 R_2 L^8 \\ & + 72R_4 R_2 L^8 - 24R_4 R_2 L^8 - 48R_4 R_2 L^8 - 30976R_2^2 L^7 + 3726R_4 R_2 L^7 - 41R_4^2 L^7 + 11610R_4^2 L^7 - 207R_4 R_2 L^7 + 504R_4 R_2 L^7 + 6750R_4^2 L^7 \\ & - 208R_4^2 L^7 - 150R_4^2 L^7 - 1848R_4^2 L^7 - 1848R_4^2 L^7 + 109912R_2 R_4 L^7 - 42768R_4 R_2 L^7 + 24792R_4^2 L^7 - 15132R_4 R_2 L^7 + 788R_4 R_2 L^7 + 328R_4 L^7 \\ & + 11512R_4 R_2 L^7 - 432R_4 R_2 L^7 - 102R_4 R_2 L^7 - 496R_4 R_2 L^7 + 18R_4 R_2 L^7 - 18R_4 R_2 L^7 - 27072R_4 R_2 L^7 + 1224R_4 R_2 L^7 + 144R_4 R_2 L^7 \\ & - 144R_4 R_2 L^7 + 102R_4 R_2 L^7 + 72R_4 R_2 L^7 - 72R_4 R_2 L^7 - 48R_4 R_2 L^7 - 360R_4 R_2 L^7 + 120R_4 R_2 L^7 - 144R_4 R_2 L^7 + 72R_4 R_2 L^7 \\ & - 72R_4 R_2 L^7 + 92R_4 R_2 L^7 + 1032R_4 R_2 L^7 + 8000L^7 + 72576R_2 R_4 L^7 - 11292R_4 R_2 L^7 - 4188R_4 R_2 L^7 - 3132R_4 R_2 L^7 - 18792R_4 R_2 L^7 \\ & + 4040R_4 R_2 L^7 - 3584R_4 R_2 L^7 + 309R_4 R_2 L^7 + 1044R_4 R_2 L^7 - 1844R_4 R_2 L^7 - 2232R_4 R_2 L^7 + 744R_4 R_2 L^7 - 3040R_4 R_2 L^7 \\ & + 1432R_4 R_2 L^7 - 48R_4 R_2 L^7 + 2088R_4 R_2 L^7 - 2088R_4 R_2 L^7 + 648R_4 R_2 L^7 + 288R_4 R_2 L^7 - 90R_4 R_2 L^7 + 1488R_4 R_2 L^7 - 522R_4 R_2 L^7 \\ & - 52128R_4^2 L^7 + 4584R_4^2 L^7 - 5502R_4^2 L^7 + 2430R_4^2 L^7 - 4860R_4^2 L^7 - 35100R_4^2 L^7 + 1704R_4^2 L^7 + 1660R_4^2 L^7 + 2388R_4^2 L^7 + 1440R_4^2 L^7 \\ & - 27432R_4^2 L^7 + 146412R_2 R_4 L^7 + 17172R_4 R_2 L^7 - 8244R_4 R_2 L^7 - 6976R_4 R_2 L^7 - 10876R_4 R_2 L^7 + 18144R_4 R_2 L^7 + 3078R_4 R_2 L^7 + 324R_4 R_2 L^7 \\ & - 342R_4 R_2 L^7 + 342R_4 R_2 L^7 + 14100R_4 R_2 L^7 - 10704R_4 R_2 L^7 - 4608R_4 R_2 L^7 + 3072R_4 R_2 L^7 - 608R_4 R_2 L^7 - 1200R_4 R_2 L^7 \\ & + 1136R_4 R_2 L^7 + 1868R_4 R_2 L^7 + 7200R_4 R_2 L^7 - 2400R_4 R_2 L^7 + 3312R_4 R_2 L^7 - 288R_4 R_2 L^7 + 32R_4 R_2 L^7 - 1368R_4 R_2 L^7 \\ & + 1368R_4 R_2 L^7 - 752R_4 R_2 L^7 - 1156R_4 R_2 L^7 - 96000L^6 - 109728R_2 R_4 L^6 - 4392R_4 R_2 L^6 + 9144R_4 R_2 L^6 + 26352R_4 R_2 L^6 \\ & + 151824R_4 R_2 L^6 - 37368R_4 R_2 L^6 + 27432R_4 R_2 L^6 - 2048R_4 R_2 L^6 - 8784R_4 R_2 L^6 + 8784R_4 R_2 L^6 + 17088R_4 R_2 L^6 - 19792R_4 R_2 L^6 \\ & + 87296R_4 R_2 L^6 - 1080R_4 R_2 L^6 + 1205R_4 R_2 L^6 - 17568R_4 R_2 L^6 + 17368R_4 R_2 L^6 - 196512R_4 R_2 L^6 - 100800R_4 R_2 L^6 - 6736R_4 R_2 L^6 \\ & - 720R_4 R_2 L^6 + 240R_4 R_2 L^6 - 11052R_4 R_2 L^6 + 141840R_4 R_2 L^6 - 49284R_4 R_2 L^6 - 1258R_4 R_2 L^6 + 11182R_4 R_2 L^6 + 1098R_4 R_2 L^6 - 2196R_4 R_2 L^6 \\ & + 88712R_4 R_2 L^6 - 3756R_4 R_2 L^6 - 8188R_4 R_2 L^6 + 108R_4 R_2 L^6 - 2796R_4 R_2 L^6 + 98800R_4 R_2 L^6 + 8479R_4 R_2 L^6 - 47376R_4 R_2 L^6 + 79044R_4 R_2 L^6 + 14400R_4 R_2 L^6 \\ & + 14100R_4 R_2 L^6 - 9396R_4 R_2 L^6 + 4092R_4 R_2 L^6 + 1044R_4 R_2 L^6 - 1044R_4 R_2 L^6 - 11800R_4 R_2 L^6 + 9336R_4 R_2 L^6 + 84624R_4 R_2 L^6 \\ & - 10488R_4 R_2 L^6 + 1832R_4 R_2 L^6 + 4176R_4 R_2 L^6 - 4176R_4 R_2 L^6 - 19200R_4 R_2 L^6 - 45720R_4 R_2 L^6 + 15240R_4 R_2 L^6 + 8352R_4 R_2 L^6 \\ & - 8352R_4 R_2 L^6 + 928R_4 R_2 L^6 + 4176R_4 R_2 L^6 + 4176R_4 R_2 L^6 + 5220R_4 R_2 L^6 + 19200R_4 R_2 L^6 + 134208R_4 R_2 L^6 + 53208R_4 R_2 L^6 \\ & - 12012R_4 R_2 L^6 - 62208R_4 R_2 L^6 + 4608R_4 R_2 L^6 + 92400R_4 R_2 L^6 - 30036R_4 R_2 L^6 + 4104R_4 R_2 L^6 + 97760R_4 R_2 L^6 - 20736R_4 R_2 L^6 \\ & - 32048R_4 R_2 L^6 + 11016R_4 R_2 L^6 - 32400R_4 R_2 L^6 - 22772R_4 R_2 L^6 - 2808R_4 R_2 L^6 + 41472R_4 R_2 L^6 - 41472R_4 R_2 L^6 - 16032R_4 R_2 L^6 \\ & - 16848R_4 R_2 L^6 + 5616R_4 R_2 L^6 + 22632R_4 R_2 L^6 - 21600R_4 R_2 L^6 - 2160R_4 R_2 L^6 - 105840R_4 R_2 L^6 - 12600R_4 R_2 L^6 + 2592R_4 R_2 L^6 - 34560R_4 R_2 L^6 \\ & - 2160R_4 R_2 L^6 - 19440R_4 R_2 L^6 - 960R_4 R_2 L^6 + 43200R_4 R_2 L^6 + 14400R_4 R_2 L^6 - 4320R_4 R_2 L^6 + 172800R_4 R_2 L^6 + 20920R_4 R_2 L^6 - 69120R_4 R_2 L^6 \\ & - 2880R_4 R_2 L^6 + 2880R_4 R_2 L^6 + 12000R_4 R_2 L^6 + 14400R_4 R_2 L^6 - 4320R_4 R_2 L^6 - 480R_4 R_2 L^6 - 11520R_4 R_2 L^6 + 11520R_4 R_2 L^6 + 90720R_4 R_2 L^6 \\ & - 30240R_4 R_2 L^6 - 2880R_4 R_2 L^6 - 9840R_4 R_2 L^6 + 25200R_4 R_2 L^6 - 11520R_4 R_2 L^6 - 11520R_4 R_2 L^6 - 6720R_4 R_2 L^6 \\ & \left. / \left( (L-5)(L-4)(L-3)(L-2)(L-1)L^2(L+1)(L+2)(L+3) \right) \right) \end{aligned}$$



# Higher $k$ frame potentials

We computed the second frame potential for GUE to be

$$\begin{aligned} \mathcal{F}_{\text{GUE}}^{(2)} = & \left( (L^4 - 8L^2 + 6) \mathcal{R}_4^2 + 4L^2 (L^2 - 9) \mathcal{R}_4 + 4(L^6 - 9L^4 + 4L^2 + 24) \mathcal{R}_2^2 \right. \\ & - 8L^2 (L^4 - 11L^2 + 18) \mathcal{R}_2 + 2(L^4 - 7L^2 + 12) \mathcal{R}_{4,1}^2 - 4L^2 (L^2 - 9) \mathcal{R}_{4,2} \\ & + (L^4 - 8L^2 + 6) \mathcal{R}_{4,2}^2 - 8(L^4 - 8L^2 + 6) \mathcal{R}_2 \mathcal{R}_4 - 4L(L^2 - 4) \mathcal{R}_4 \mathcal{R}_{4,1} \\ & + 16L(L^2 - 4) \mathcal{R}_2 \mathcal{R}_{4,1} - 8(L^2 + 6) \mathcal{R}_2 \mathcal{R}_{4,2} + 2(L^2 + 6) \mathcal{R}_4 \mathcal{R}_{4,2} \\ & \left. - 4L(L^2 - 4) \mathcal{R}_{4,1} \mathcal{R}_{4,2} + 2L^4 (L^4 - 12L^2 + 27) \right) \\ & / \left( (L-3)(L-2)(L-1)L^2(L+1)(L+2)(L+3) \right). \end{aligned}$$

and computed the third frame potential for GUE to be

$$\begin{aligned} \mathcal{F}_{\text{GUE}}^{(3)} = & \left( 6L^{13} + 18R_1^2 L^{12} - 36R_1 L^{11} - 318L^{10} - 846R_1^2 L^9 + 9R_1^3 L^8 + 18R_1^2 L^{10} + 9R_1^3 L^9 + 1836R_1 L^8 - 72R_1^2 L^7 + 36R_1^3 L^6 + 5550L^{10} \right. \\ & + 144R_1 R_2 L^9 - 36R_1^2 R_2 L^8 - 36R_1 R_3 L^7 + 11574R_1^2 L^6 - 360R_1^3 L^5 + R_2^2 L^4 - 828R_1^2 L^3 + 9R_1^3 R_2 L^2 - 18R_1 R_3 L^2 - 441R_1^2 L^2 + 6R_1^3 L^2 \\ & + 48R_1 L^2 + 192R_1^2 L^2 + 8R_1^3 L^2 - 39772R_1 L^2 + 3976R_1^2 L^2 - 1728R_1^3 L^2 + 36R_1 R_2 L^2 - 18R_1 R_3 L^2 - 12R_1 L^2 - 36R_1^2 R_2 L^2 + 18R_1 R_3 L^2 \\ & + 1140R_1 R_2 L^2 - 36R_1 R_3 L^2 + 24R_1 L^2 - 8718L^8 - 4192R_1 R_2 L^7 + 1332R_1 R_3 L^7 + 36R_1 R_2 L^7 + 108R_1 R_3 L^7 + 5148R_1 R_2 L^7 \\ & - 144R_1 R_3 L^7 + 108R_1 R_2 L^7 - 12R_1 R_3 L^7 - 36R_1 R_2 L^7 + 36R_1 R_3 L^7 + 72R_1 R_2 L^7 + 24R_1 R_3 L^7 + 144R_1 R_2 L^7 - 72R_1 R_3 L^7 \\ & + 72R_1 R_2 L^7 - 24R_1 R_3 L^7 - 48R_1 R_2 L^7 - 30078R_1 L^6 + 3726R_1^2 L^5 - 41R_1^3 L^5 + 11610R_1^2 L^4 - 207R_1^3 L^4 + 504R_1 R_2 L^4 + 6750R_1^2 L^4 \\ & - 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## Higher $k$ frame potentials

$$k = 1 \quad \text{Early: } \mathcal{F}_{\text{GUE}}^{(1)} \sim \frac{\mathcal{R}_2^2}{L^2}, \quad \text{Dip: } \mathcal{F}_{\text{GUE}}^{(1)} = 1, \quad \text{Late: } \mathcal{F}_{\text{GUE}}^{(1)} = 2.$$

$$k = 2 \quad \text{Early: } \mathcal{F}_{\text{GUE}}^{(2)} \sim \frac{\mathcal{R}_4^2}{L^4}, \quad \text{Dip: } \mathcal{F}_{\text{GUE}}^{(2)} = 2, \quad \text{Late: } \mathcal{F}_{\text{GUE}}^{(2)} = 10.$$

$$k = 3 \quad \text{Early: } \mathcal{F}_{\text{GUE}}^{(3)} \sim \frac{\mathcal{R}_6^2}{L^6}, \quad \text{Dip: } \mathcal{F}_{\text{GUE}}^{(3)} = 6, \quad \text{Late: } \mathcal{F}_{\text{GUE}}^{(3)} = 96.$$

$k$ -th frame potential: form a  $k$ -design at the dip,  
late times no longer Haar



- ▶ related symptoms of chaos
- ▶ GUE forms a  $k$ -design, late times **not Haar random**
- ▶ if late time physics is GUE then Haar/ $k$ -designs might miss important (global) aspects of chaotic systems at late times
- ▶ but what is GUE capturing about chaotic systems?

let's look more explicitly at why GUE only captures global physics

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let's look more explicitly at why GUE only captures global physics

## Correlators and form factors

Let's look at correlation functions in spin systems

Consider a Hamiltonian  $H$  on  $|\mathcal{H}| = L = 2^n$ , and consider the averaged 2-point function (at  $\beta = 0$ )

$$\int dA \langle A(0)A^\dagger(t) \rangle_\beta = \frac{1}{L} \int dA \text{Tr}(Ae^{-iHt}A^\dagger e^{iHt}),$$

where  $A$  is a unitary integrated over the Haar measure.

$$\int dA \langle A(0)A^\dagger(t) \rangle_\beta = \frac{\mathcal{R}_2^H(t)}{L^2}$$

( $\mathcal{R}_2^H$  for a single Hamiltonian)

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$$\int dA \langle A(0)A^\dagger(t) \rangle_\beta = \frac{\mathcal{R}_2^H(t)}{L^2}$$

or more generally,  $2k$ -OTOCs  $\langle A_1 B_1(t) \dots A_k B_k(t) \rangle_\beta$  and average

$$\int dA_1 dB_1 \dots dA_k \langle A_1 B_1(t) \dots A_k B_k(t) \rangle_\beta = \frac{\mathcal{R}_{2k}^H(t)}{L^{2k}}$$

i.e.  $2k$ -OTOCs  $\leftrightarrow$   $2k$ -form factors

(can also understand universal  $1/t^6$  OTOC behavior [Bagrets, Altland, Kamenev])

→ connects **spectral statistics** and **physical observables**

→ gives a practical way to **measure** the form factors



## 2-pt and form factors: a check

Pauli operators form a 1-design, so we average as

$$\int dA \langle A(0)A^\dagger(t) \rangle = \frac{1}{4^N} \sum_{A \in \text{Pauli}} \langle A(0)A^\dagger(t) \rangle = \frac{\mathcal{R}_2^H(t)}{L^2},$$

if we pick a few random Pauli operators, we should be able to approximate  $\mathcal{R}_2$   
we can also check that  $\Delta \langle A(0)A^\dagger(t) \rangle_{\text{avg}}^2 \sim \mathcal{O}(1/L^2)$

Let's check this!

consider a **random non-local spin system**, sum over all 2-body operators with random Gaussian couplings  $J$

$$H_{\text{RNL}} = \sum_{i,j,\alpha,\beta} J_{ij\alpha\beta} S_i^\alpha S_j^\beta,$$

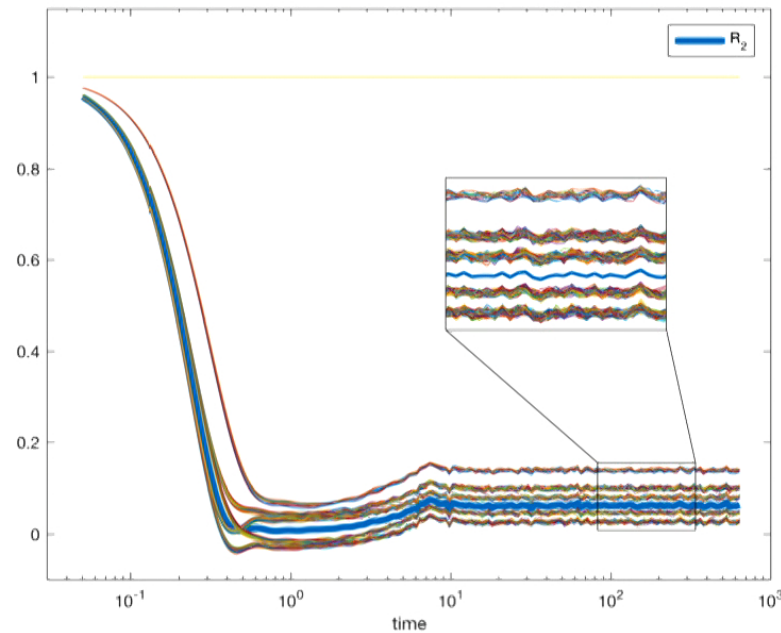


## 2-pt and form factors: a check

$$\text{Check: } \frac{1}{4^N} \sum_{A \in \text{Pauli}} \langle A(0) A^\dagger(t) \rangle = \frac{\mathcal{R}_2^H(t)}{L^2}$$

consider a **random non-local spin system**:

$$H_{\text{RNL}} = \sum_{i,j,\alpha,\beta} J_{ij\alpha\beta} S_i^\alpha S_j^\beta,$$



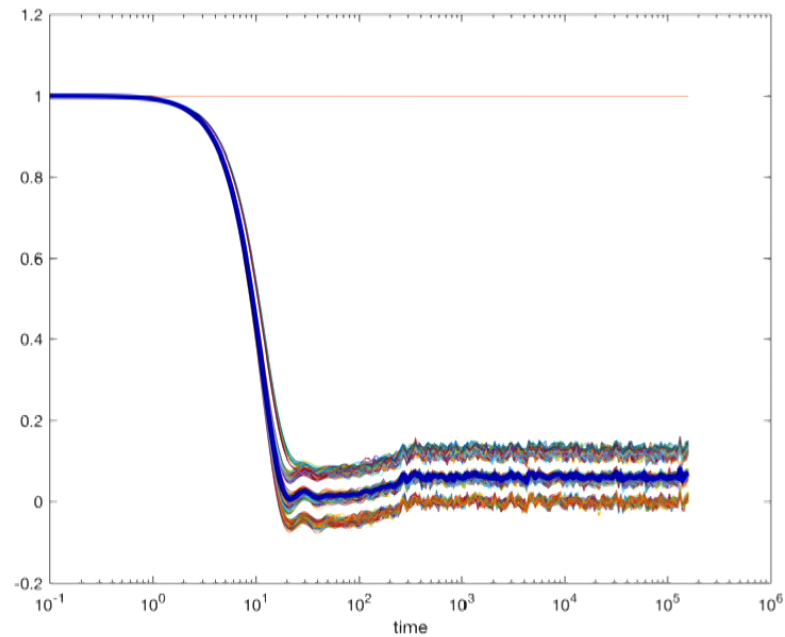
## 2-pt and form factors: an SYK check

just for fun, can also check this for SYK:  $\langle \chi_i \chi_j(t) \rangle$

i.e. 2-pt correlators of Majoranas

where SYK:

$$H_{\text{SYK}} = \sum_{i,j,k,l} J_{ijkl} \chi_i \chi_j \chi_k \chi_l$$



for SYK for N=10 Majoranas,  $|\mathcal{E}| = 200$ ,  $\beta = 0$



## OTOCs and random matrices

consider a 2-pt function **averaged over GUE** (fix ops and average over  $H$ 's, i.e.  $A(t) = e^{-iHt} A e^{iHt}$  with GUE  $H$ 's)

$$\text{GUE avg : } \int dH \langle A(0) A^\dagger(t) \rangle_\beta = \frac{\mathcal{R}_2(t) - 1}{L^2 - 1} \approx \frac{\mathcal{R}_2(t)}{L^2},$$

recall from before that the operator average (for any  $H$ )

$$\text{Op avg : } \int dA \langle A(0) A^\dagger(t) \rangle_\beta = \frac{\mathcal{R}_2^H(t)}{L^2}$$

GUE average is the same as operator average, but in taking GUE average we make no assumption about **locality** of operators

→ GUE suited to capture **global properties**



## OTOCs and random matrices

2-pt functions averaged over GUE:

$$\langle A(0)A^\dagger(t) \rangle_{\text{GUE}} \approx \frac{\mathcal{R}_2(t)}{L^2}.$$

furthermore, we can use the 4-th moment of Haar and compute the OTOC averaged over GUE:

$$\langle A(0)B(t)C(0)D(t) \rangle_{\text{GUE}} \approx \langle ABCD \rangle \frac{\mathcal{R}_4(t)}{L^4}.$$

what have we learned?



## OTOCs and random matrices

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what have we learned?



## GUE as a chaotic pastiche

- ▶ GUE captures **thermalization** (apparent loss of QI)
- ▶ GUE does not capture **scrambling** ( $t_4 \sim t_2/2$ )
- ▶ GUE does not care about spatial/temporal **locality**

(BH viewpoint: GUE BHs are bad)

None of this is too surprising, but it at least formalizes why we fail to capture physics at early times

More interesting is why GUE captures chaotic physics at late times  
→ Haar invariance

GUE is a good description of a system which has **lost its notion of locality**

## $k$ -invariance

In many of the expressions we derived (frame potential, OTOCs  $\leftrightarrow$  form factors), the key ingredient was **invariance of the measure**

Consider an ensemble of unitary time evolutions by some (ensemble of) **physical Hamiltonians**:

$$\mathcal{E}_t = \{e^{-iHt}, H \in \mathcal{E}_H\}$$

i.e. SYK, spin system, (QFT on a random lattice)

## $k$ -invariance

Consider an ensemble of unitary time evolutions by some (ensemble of) **physical Hamiltonians**:  $\mathcal{E}_t = \{e^{-iHt}, H \in \mathcal{E}_H\}$

Define the **Haar-invariant ensemble**:  $\tilde{\mathcal{E}}_t = U\mathcal{E}_tU^\dagger$

ensemble is  **$k$ -invariant** iff  $\mathcal{F}_{\mathcal{E}_t}^{(k)} = \mathcal{F}_{\tilde{\mathcal{E}}_t}^{(k)}$  (reproduces the first  $k$  moments)

equivalently,

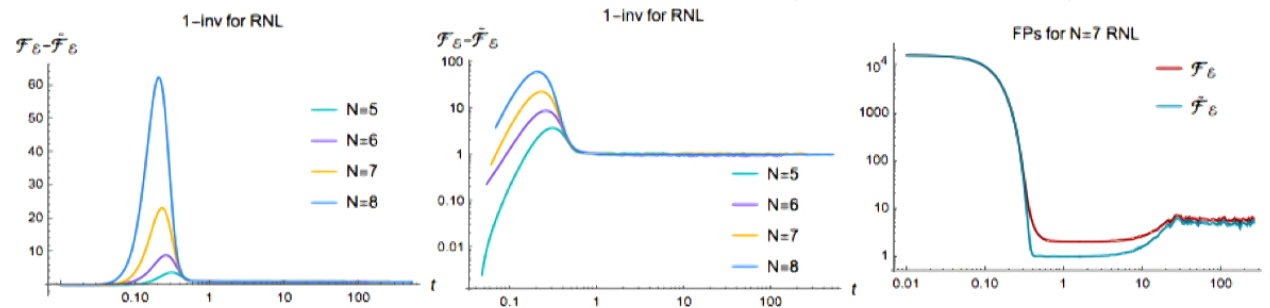
$$\mathcal{F}_{\mathcal{E}_t}^{(k)}(t) - \mathcal{F}_{\tilde{\mathcal{E}}_t}^{(k)}(t) \geq 0$$

defines a **distance to  $k$ -invariance**

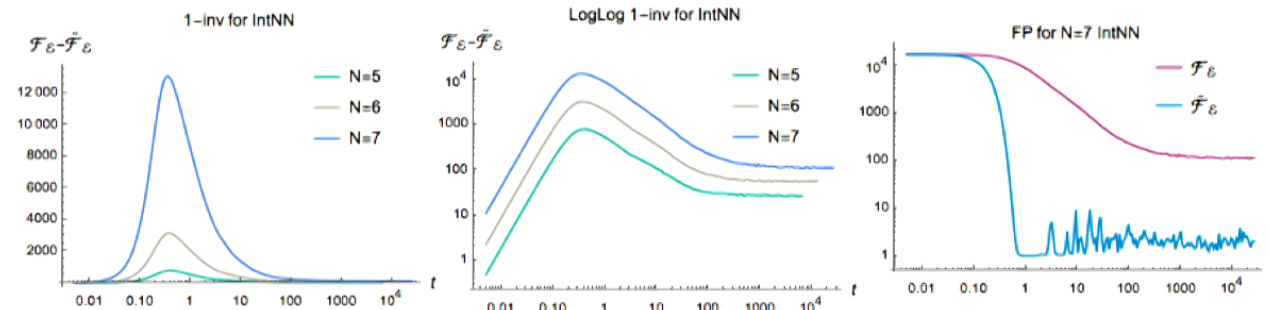
this quantity becoming small signifies the onset of a **RMT description**

# $k$ -invariance: a check

random non-local spin system:  $H = \sum_{i,j,\alpha,\beta} J_{ij\alpha\beta} S_i^\alpha S_j^\beta$

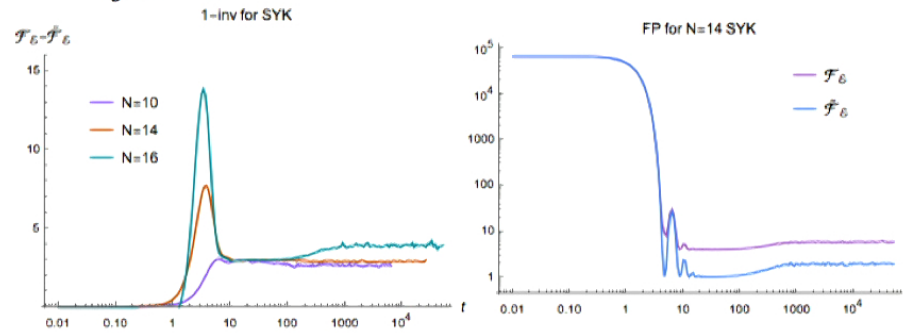


integrable spin chain:  $H = -\sum_i Z_i Z_{i+1} - \sum_i h_i X_i$

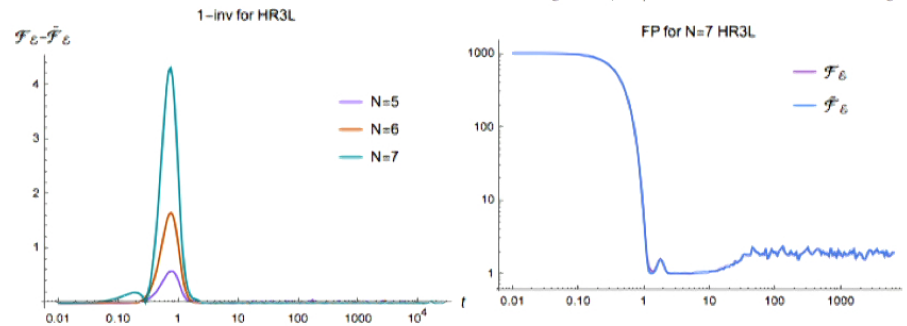


## $k$ -invariance: a check

$$\text{SYK: } H = \sum_{ijkl} J_{ijkl} \chi_i \chi_j \chi_k \chi_l$$



$$\text{random 3-local spin system: } H = \sum_{ijk\alpha\beta\gamma} J_{\alpha\beta\gammaijk} S_i^\alpha S_j^\beta S_k^\gamma$$



## Summary

- ▶ computed form factors in RMT
- ▶ related **symptoms of chaos** (OTOCs, frame potentials, form factors)
- ▶ understood time scales at which GUE is **Haar random**
- ▶ and why GUE **fails** to capture early times
- ▶ related **observables** to **spectral statistics**

Late time behavior of chaotic systems understood as  $k$ -invariance



► Extensions of this work:

i) Consider chaotic systems with **symmetry**

generalize to **GOE**, **GSE** and to all **extended ensembles** in 10-fold symmetry classification (Altland-Zirnbauer) (as in classification of top. phases)

→ quantify randomness/chaos for symmetry classes using Weingarten calculus for the associated compact symmetric space  $U(L)/H$

ii) Apply these ideas to study chaos in **SYK models**

(e.g. consider **Wishart ensembles** for supersymmetric SYK models [NHJ, Liu; 1711.08184] )

iii) Investigate  **$k$ -invariance** in random spin systems and SYK as a characterization of **late-time chaos**

## Work in progress/future work

► Things to think about:

- understand the precise role ETH plays in  $k$ -invariance, use these tools to study thermalization in quantum systems
- understand late-time OTOCs
- transport and hydrodynamics
- complexity?
- shockwaves on AdS-KN black holes
- chaos in  $d > 2$  CFTs
- sparse RMT?