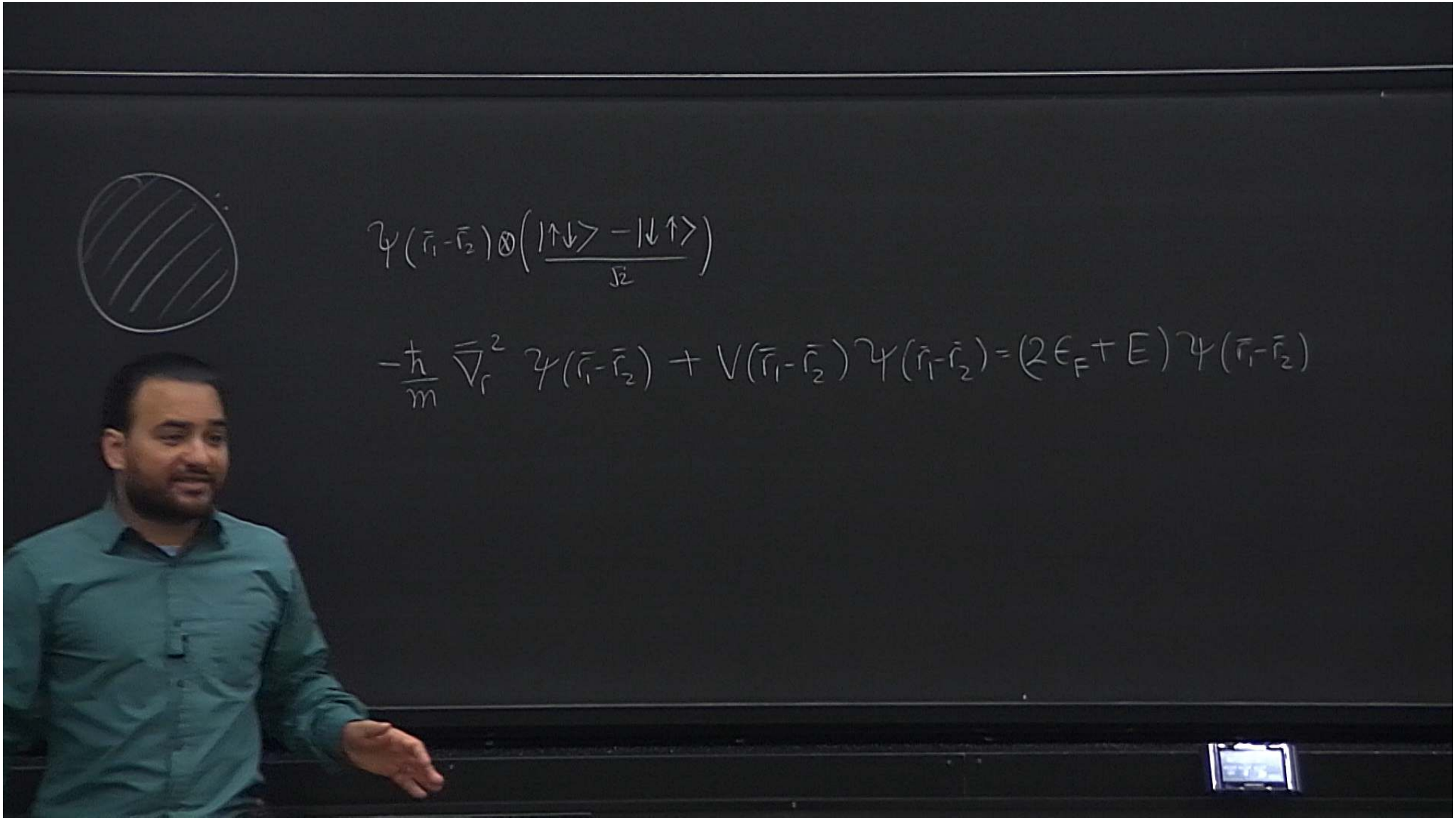
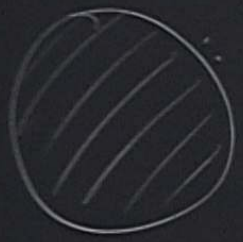
Title: PSI 17/18 - Condensed Matter - Lecture 11

Date: Nov 20, 2017 10:45 AM

URL: http://pirsa.org/17110036

Abstract:


$\psi(\vec{r}_1-\vec{r}_2)\otimes\left(\frac{|\uparrow\downarrow\rangle-|\downarrow\uparrow\rangle}{\sqrt{2}}\right)$
$-\frac{\hbar}{m}\vec{\nabla}_r^2\,\psi(\vec{r}_1-\vec{r}_2)+V(\vec{r}_1-\vec{r}_2)\psi(\vec{r}_1-\vec{r}_2)=(2\epsilon_F+E)\psi(\vec{r}_1-\vec{r}_2)$

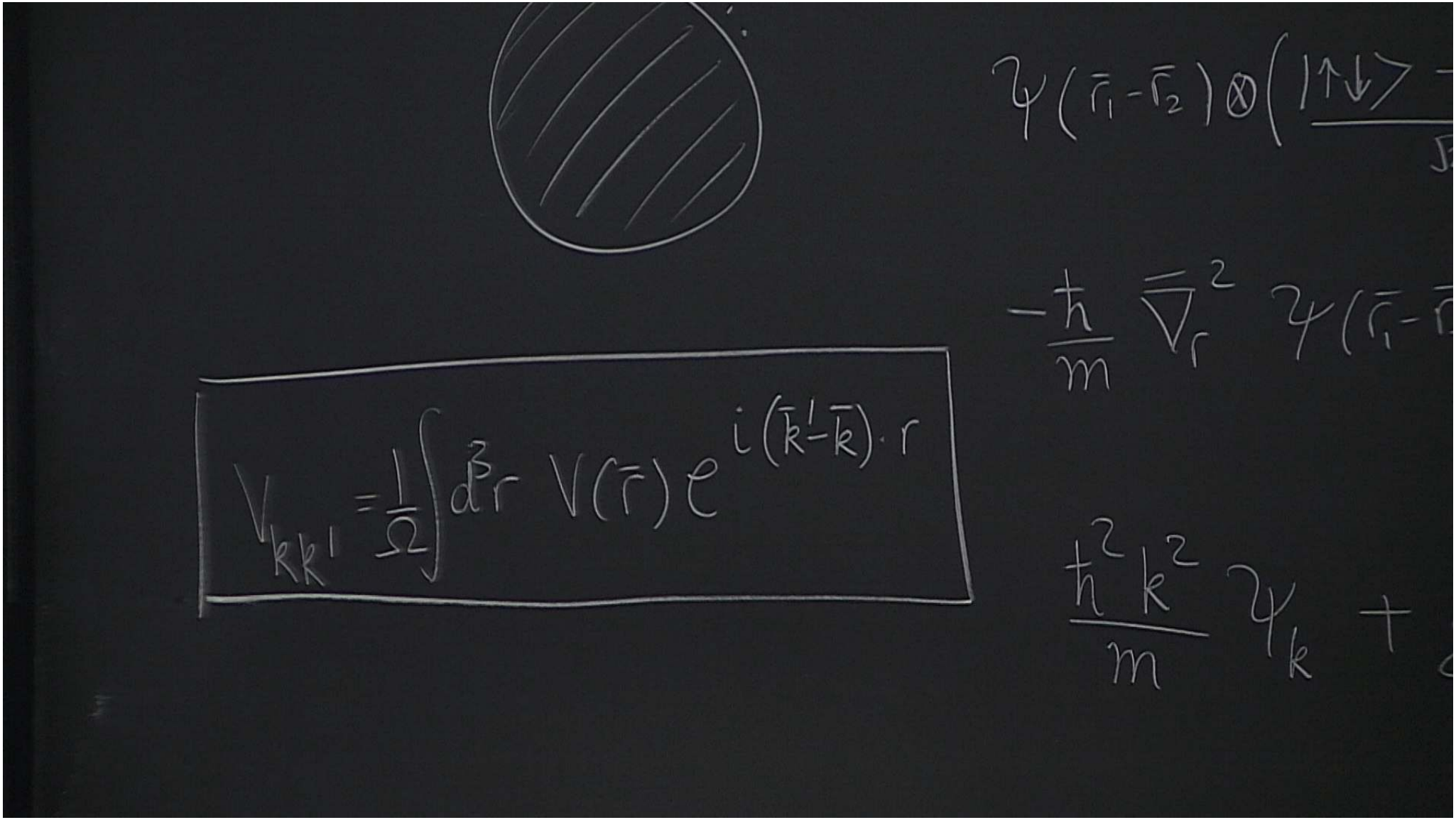


$$\psi(\vec{r}_1-\vec{r}_2)\otimes\left(\frac{|\uparrow\downarrow\rangle-|\downarrow\uparrow\rangle}{\sqrt{2}}\right)$$

$$-\frac{\hbar}{m}\vec{\nabla}_r^2\,\psi(\vec{r}_1-\vec{r}_2)+V(\vec{r}_1-\vec{r}_2)\,\psi(\vec{r}_1-\vec{r}_2)=(2\epsilon_F+E)\,\psi(\vec{r}_1-\vec{r}_2)$$

$$\downarrow \text{Fourier transform}$$

$$\frac{\hbar^2k^2}{m}\psi_k+\sum_{k'}V_{kk'}\psi_{k'}=(2\epsilon_F+E)\psi_k$$


$\psi(\bar{r}_1-\bar{r}_2)\otimes(|\uparrow\downarrow\rangle -$
$-\frac{\hbar}{m}\bar{\nabla}_r^2\,\psi(\bar{r}_1-\bar{r}$
$V_{kk'}=\frac{1}{\Omega}\int d^3r\, V(\bar{r})\,e^{i(\bar{k}'-\bar{k})\cdot r}$
$\frac{\hbar^2k^2}{m}\psi_k +$

$$\psi(\vec{r}_1-\vec{r}_2)\otimes\left(\frac{|\uparrow\downarrow\rangle-|\downarrow\uparrow\rangle}{\sqrt{2}}\right)$$

$$-\frac{\hbar}{m}\vec{\nabla}_r^2\,\psi(\vec{r}_1-\vec{r}_2)+V(\vec{r}_1-\vec{r}_2)\,\psi(\vec{r}_1-\vec{r}_2)=(2\epsilon_F+E)\,\psi(\vec{r}_1-\vec{r}_2)$$

↓ Fourier transform

$$\frac{\hbar^2k^2}{m}\psi_k+\sum_{k'}V_{kk'}\,\psi_{k'}=(2\epsilon_F+E)\,\psi_k \Rightarrow \boxed{(E-2\xi_{sk})\,\psi_k=\sum_{k'}V_{kk'}\,\psi_{k'}} \qquad \xi_{sk}=\frac{\hbar^2k^2}{2m}-\epsilon_F$$

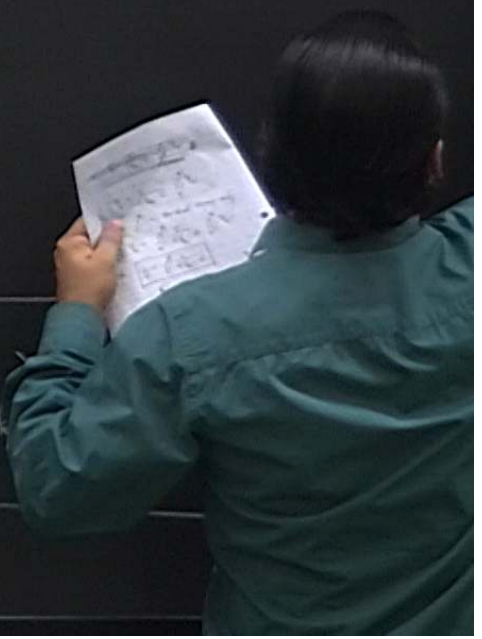
$$\psi_k = \sum_{k'}$$

$$\psi(\vec{r}_1 - \vec{r}_2) \otimes \left(\frac{|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle}{\sqrt{2}} \right)$$

$$-\frac{\hbar}{m} \vec{\nabla}_r^2 \psi(\vec{r}_1 - \vec{r}_2) + V(\vec{r}_1 - \vec{r}_2)\psi(\vec{r}_1 - \vec{r}_2) = (2\epsilon_F + E)\psi(\vec{r}_1 - \vec{r}_2)$$

$$\downarrow \text{Fourier transform}$$

$$\frac{\hbar^2 k^2}{m} \psi_k + \sum_{k'} V_{kk'} \psi_{k'} = (2\epsilon_f + E)\psi_k \Rightarrow \boxed{(E - 2\xi$$



$$\psi_k = \frac{\sum_{k'} V_{kk'} \psi_{k'}}{E - 2\xi_k}$$

$$(E - 2\xi_k)\psi_k = \sum_{k'} V_{kk'} \psi_{k'}$$

$$\xi_k = \frac{\hbar^2 k^2}{2m} - \epsilon_F$$

$$\psi_k = \frac{\sum_{k'} V_{kk'} \psi_{k'}}{E - 2\xi_k}$$

Onshell Approximation

$$V_{kk'} = \begin{cases} -V & \text{If } |\epsilon_k - \epsilon_F| \leq \hbar\omega_c \text{ and } |\epsilon_{k'} - \epsilon_F| \leq \hbar\omega_c \\ 0 & \text{otherwise} \end{cases}$$

$$\psi_k = \sum_{k'} V_{kk'} \psi_{k'}$$

$$\xi_k = \frac{\hbar^2 k^2}{2m} - \epsilon_F$$

$$= \epsilon_k - \epsilon_F$$

$$\psi_k = \frac{\sum'_{k'} V_{kk'} \psi_{k'}}{E - 2\xi_k}$$

$$\psi_k = \frac{V}{2\xi_k - E} \sum'^{*}_{k'} \psi_{k'}$$

$$\psi_k = \sum'_{k'} V_{kk'} \psi_{k'}$$

$$\xi_k = \frac{\hbar^2 k^2}{2m} - \epsilon_F$$

$$= \epsilon_k - \epsilon_F$$

Onshell Approximation

$$V_{kk'} = \begin{cases} -V & \text{If } |\epsilon_k - \epsilon_F| \le \hbar\omega_c \text{ and } |\epsilon_{k'} - \epsilon_F| \le \hbar\omega_c \\ 0 & \text{otherwise} \end{cases}$$

$$\psi_k = \frac{\sum_{k'}^{'} V_{kk'} \psi_{k'}}{E - 2\xi_k}$$

$$\psi_k = \frac{V}{2\xi_k - E} \sum_{k'}^{'*} \psi_{k'}$$

$$\sum_{k}^{'*} \psi_k = \sum_{k}^{'*} \frac{V}{2\xi_k - E} \sum_{k'}^{'*} \psi_{k'}$$

$$\psi_k = \sum_{k'}^{'} V_{kk'} \psi_{k'}$$

$$\xi_k = \frac{\hbar^2 k^2}{2m} - \epsilon_F$$

$$= \epsilon_k - \epsilon_F$$

Onshell Approximation

$$V_{kk'} = \begin{cases} -V & \text{If } |\epsilon_k - \epsilon_F| \leq \hbar\omega_c \text{ and } |\epsilon_{k'} - \epsilon_F| \leq \hbar\omega_c \\ 0 & \text{otherwise} \end{cases}$$

$$\psi_k = \frac{\sum_{k'} V_{kk'} \psi_{k'}}{E - 2\xi_k}$$

$$\psi_k = \frac{V}{2\xi_k - E} \sum_{k'}^{*} \psi_{k'}$$

$$\sum_{k}^{*} \psi_k = \sum_{k}^{*} \frac{V}{2\xi_k - E} \sum_{k'}^{*} \psi_{k'}$$

$$\Rightarrow \boxed{\frac{1}{V} = \sum_{k}^{*} \frac{1}{2\xi_k - E}}$$

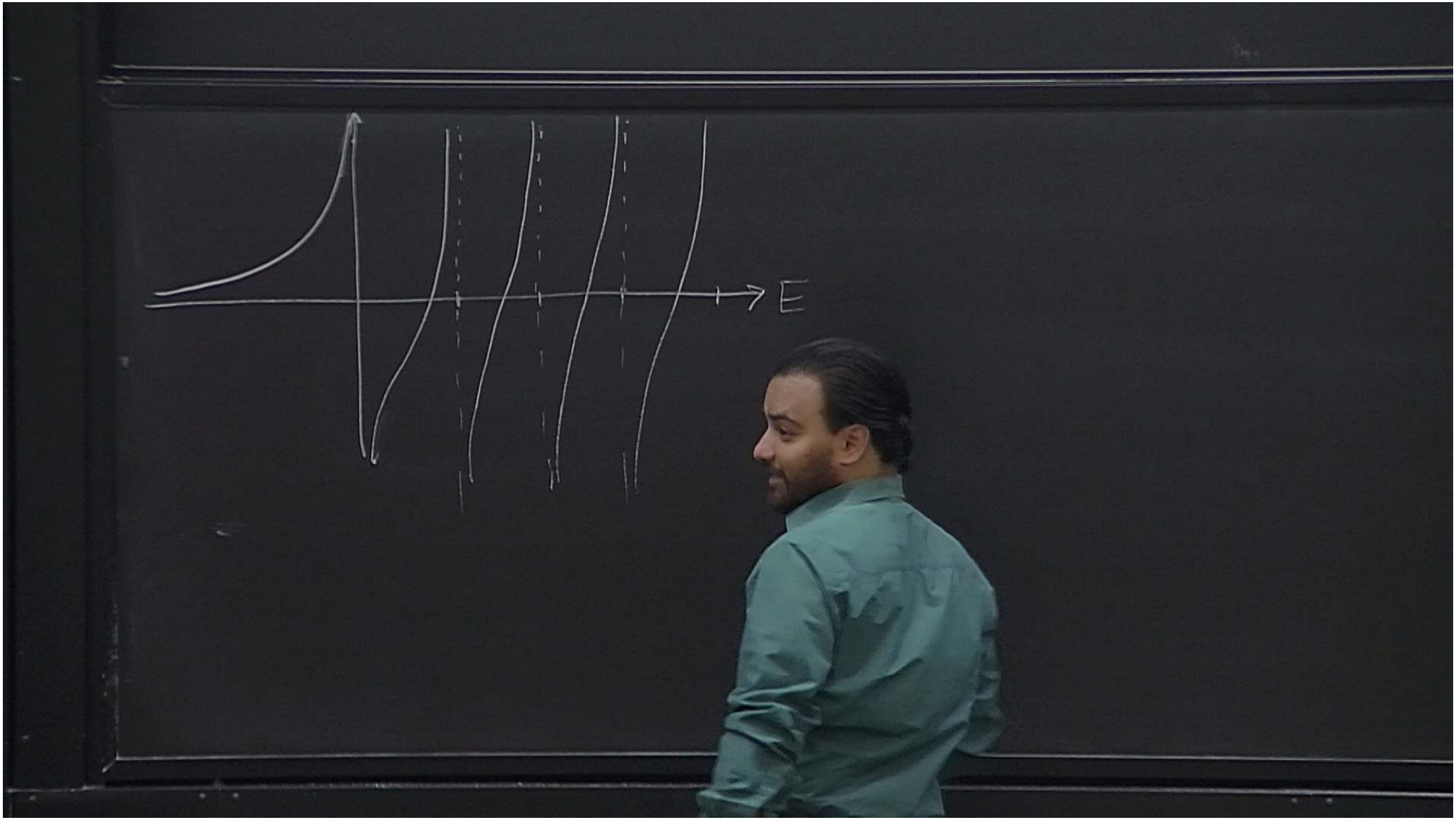
Onshell Approximation

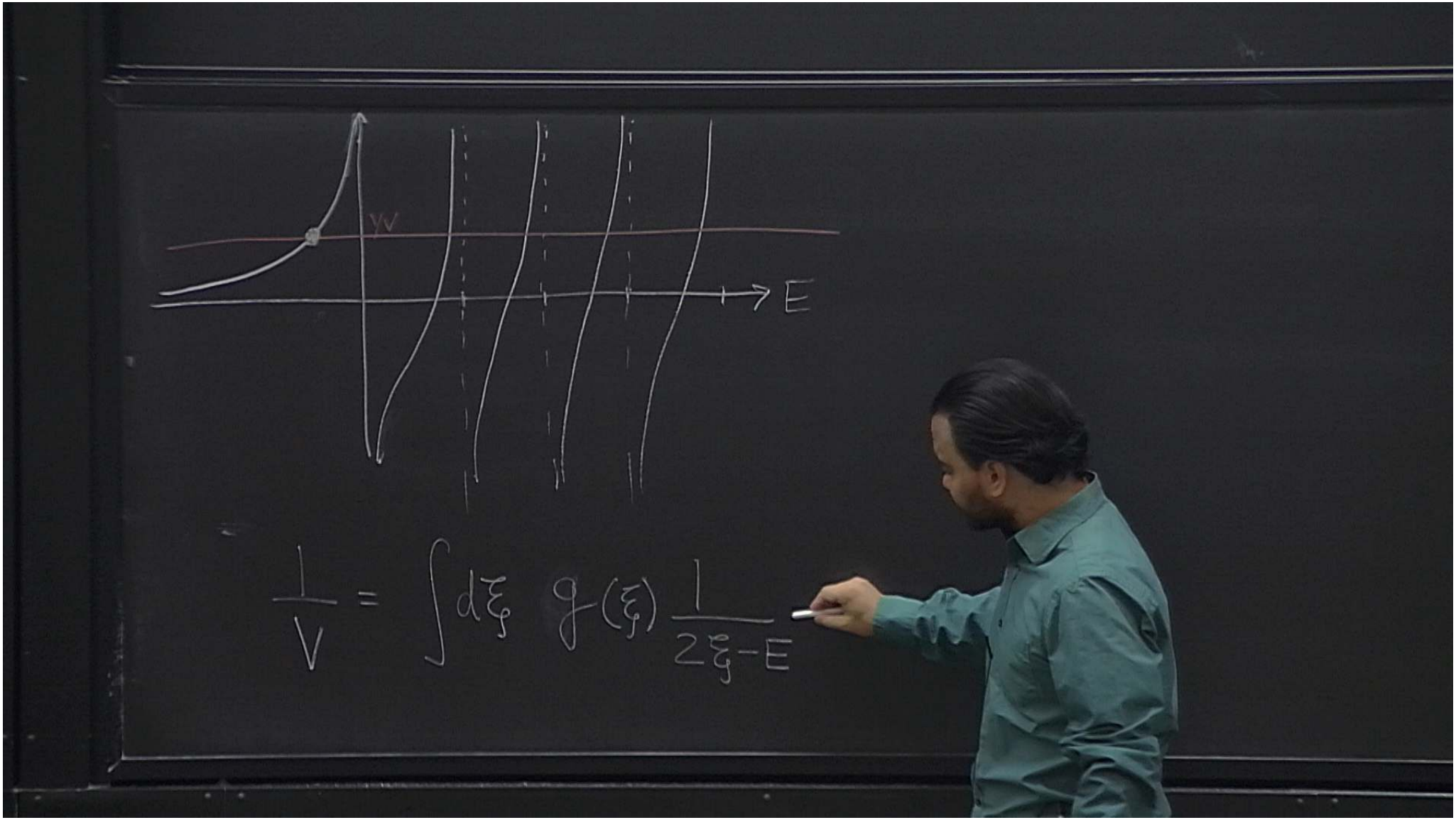
$$V_{kk'} = \begin{cases} -V & \text{If } |\epsilon_k - \epsilon_F| \le \hbar\omega_c \text{ and } |\epsilon_{k'} - \epsilon_F| \le \hbar\omega_c \\ 0 & \text{otherwise} \end{cases}$$

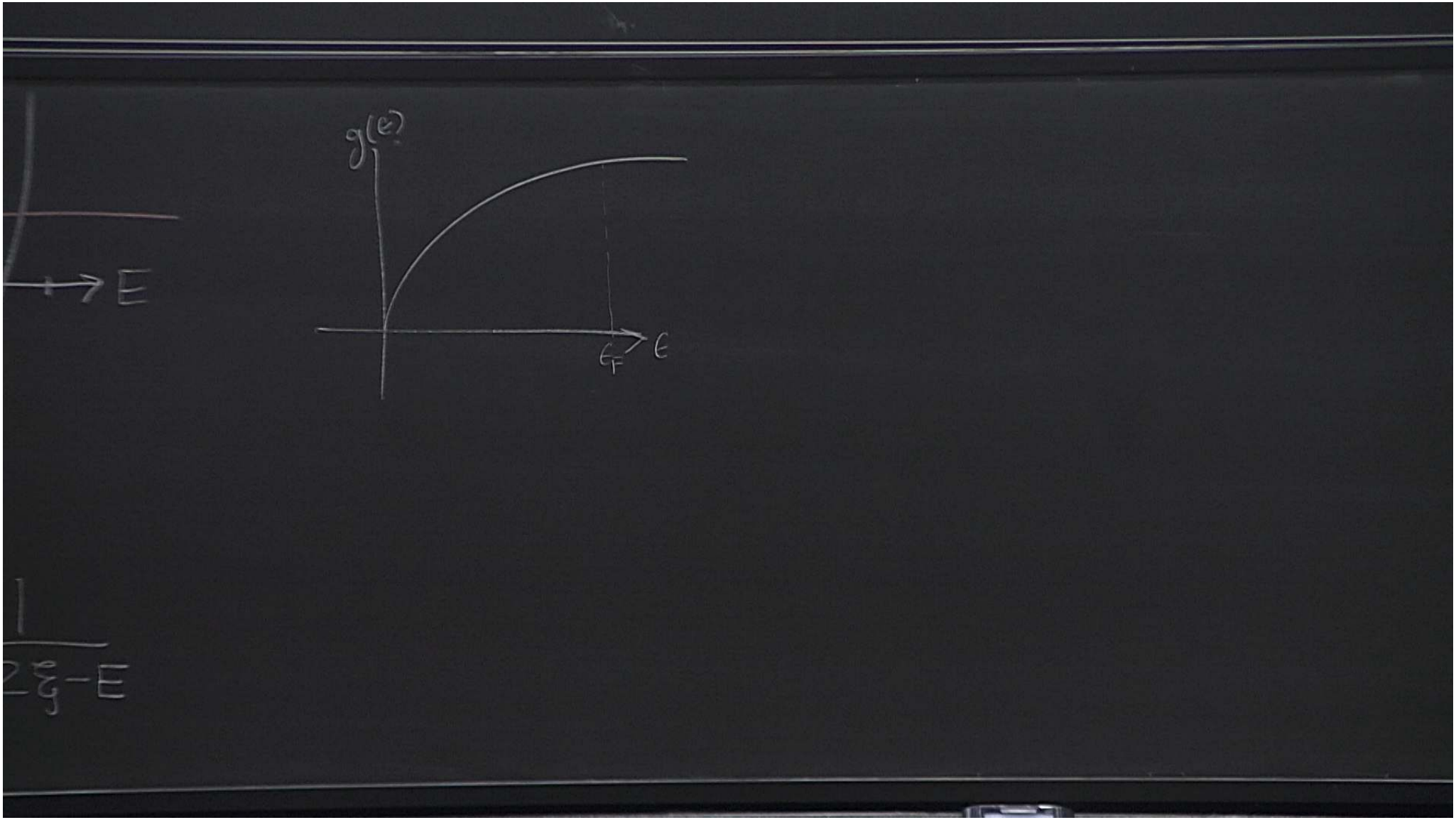
$$\boxed{\psi_k = \sum_{k'} V_{kk'} \psi_{k'}}$$

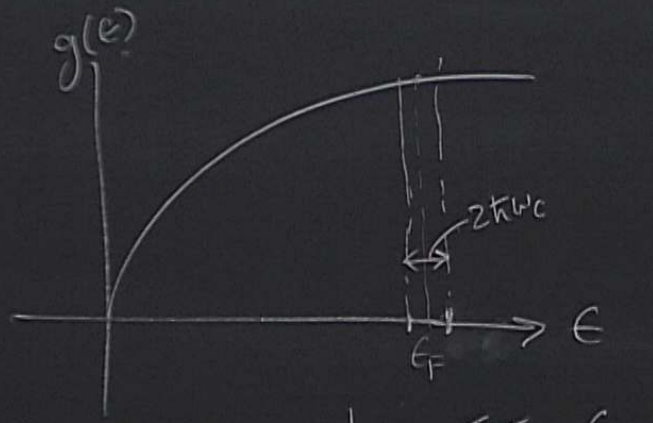
$$\xi_k = \frac{\hbar^2 k^2}{2m} - \epsilon_F$$

$$= \epsilon_k - \epsilon_F$$


E

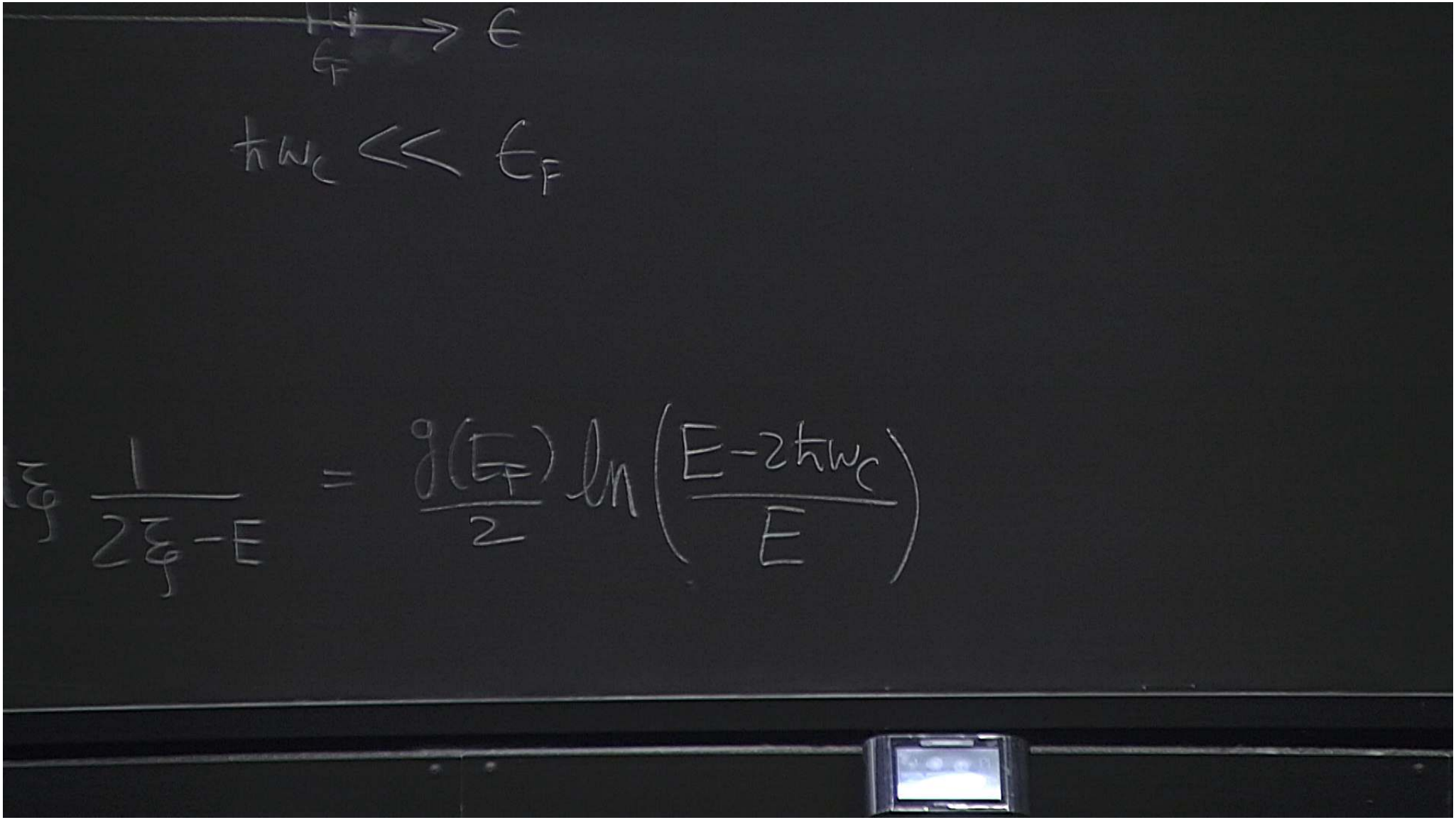

V
E
$\frac{1}{V} = \int d\xi \; g(\xi) \frac{1}{2\xi - E}$


g(ε)
E
ϵ_F
E



$\hbar\omega_c \ll \epsilon_F$

$\frac{1}{2\xi - E} \approx g(E_F) \int d\xi \frac{1}{2\xi - E}$


ϵ_F
ϵ
$\hbar\omega_c \ll \epsilon_F$
$\int_{\xi} \frac{1}{2\xi - E} = \frac{g(\epsilon_F)}{2} \ln\left(\frac{E - 2\hbar\omega_c}{E}\right)$

$$E = 2\hbar\omega_c \left(1 - e^{\frac{2}{g(E_F)V}}\right)^{-1} = 2\hbar\omega_c \Big/ \left(1 - e^{\frac{2}{g(E_F)V}}\right) \qquad g(E_F)V \ll 1$$

Weak Coupling limit

$$\approx -2\hbar\omega_c \exp\left(-\frac{2}{g(E_F)V}\right)$$

$= 2\hbar\omega_c \left(1 - e^{\frac{2}{g(E_F)V}}\right)^{-1} = 2\hbar\omega_c \Big/ \left(1 - e^{\frac{2}{g(E_F)V}}\right)$ $\quad g(E_F)V \ll 1$ Weak Coupling limit

$\approx -2\hbar\omega_c \exp\left(-\frac{2}{g(E_F)V}\right)$

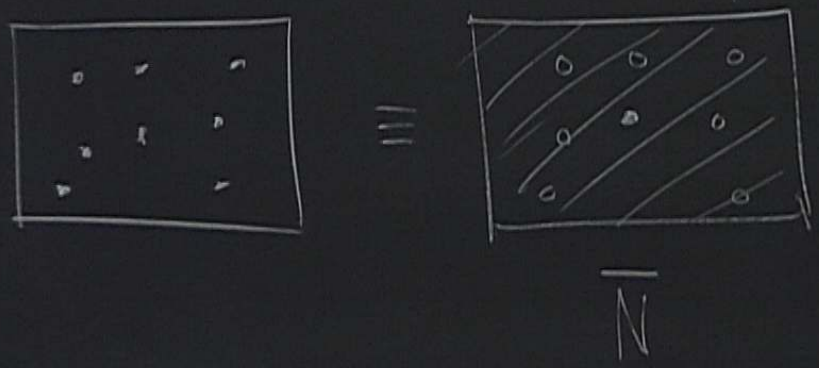
$$|\psi_0\rangle = \sum_{k>k_F} \psi_k \, c^+_{k\uparrow} \, c^+_{-k\downarrow} \, |F\rangle$$

$$|\psi_0\rangle = \sum_{k>k_F} \psi_k \, c^+_{k\uparrow} \, c^+_{-k\downarrow} \, |F\rangle$$

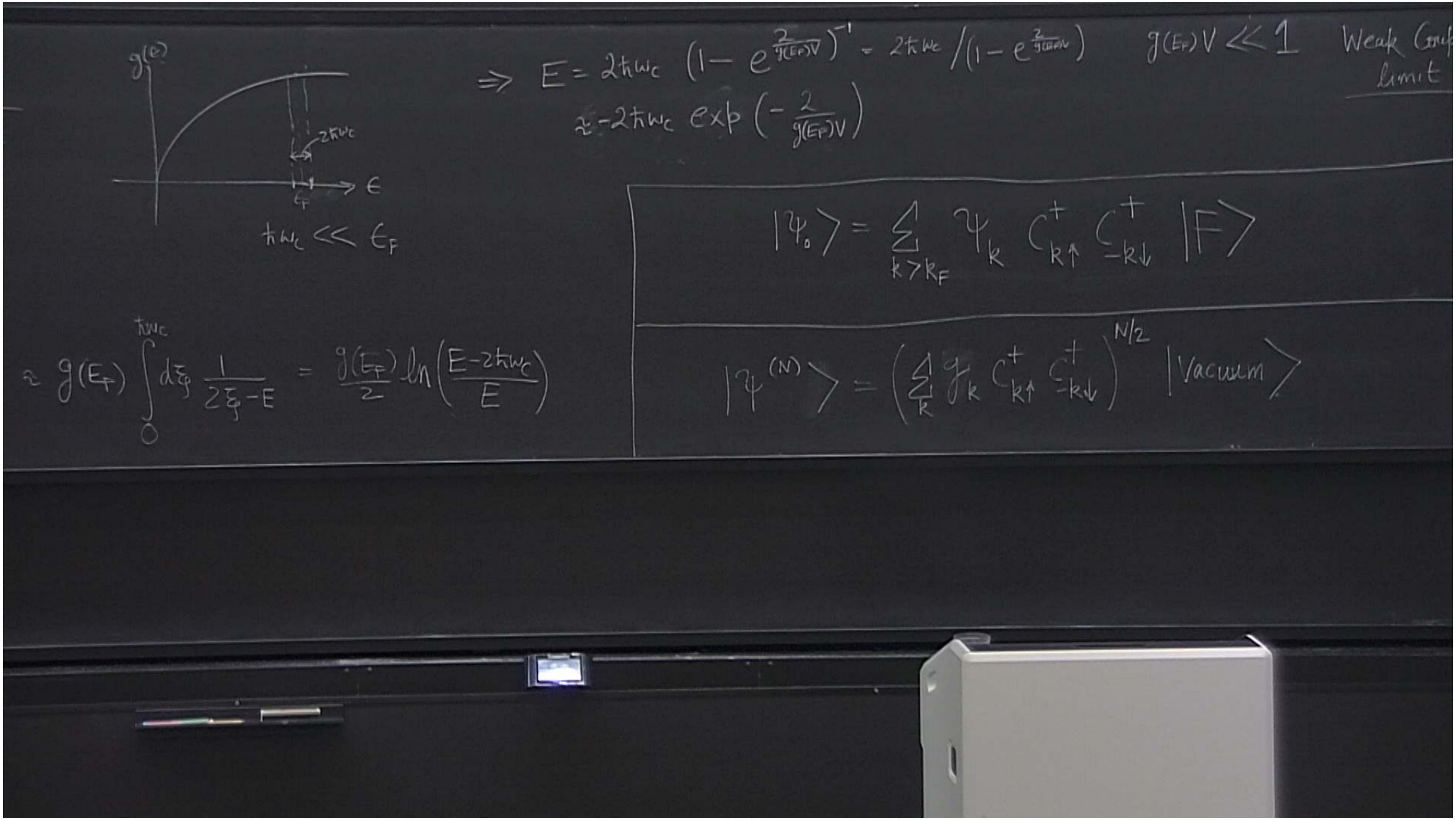
$$|\psi^{(N)}\rangle = \left(\sum_k \psi_k \, c^+_{k\uparrow} \, c^+_{-k\downarrow} \right)^{N/2} |\text{Vacuum}\rangle$$

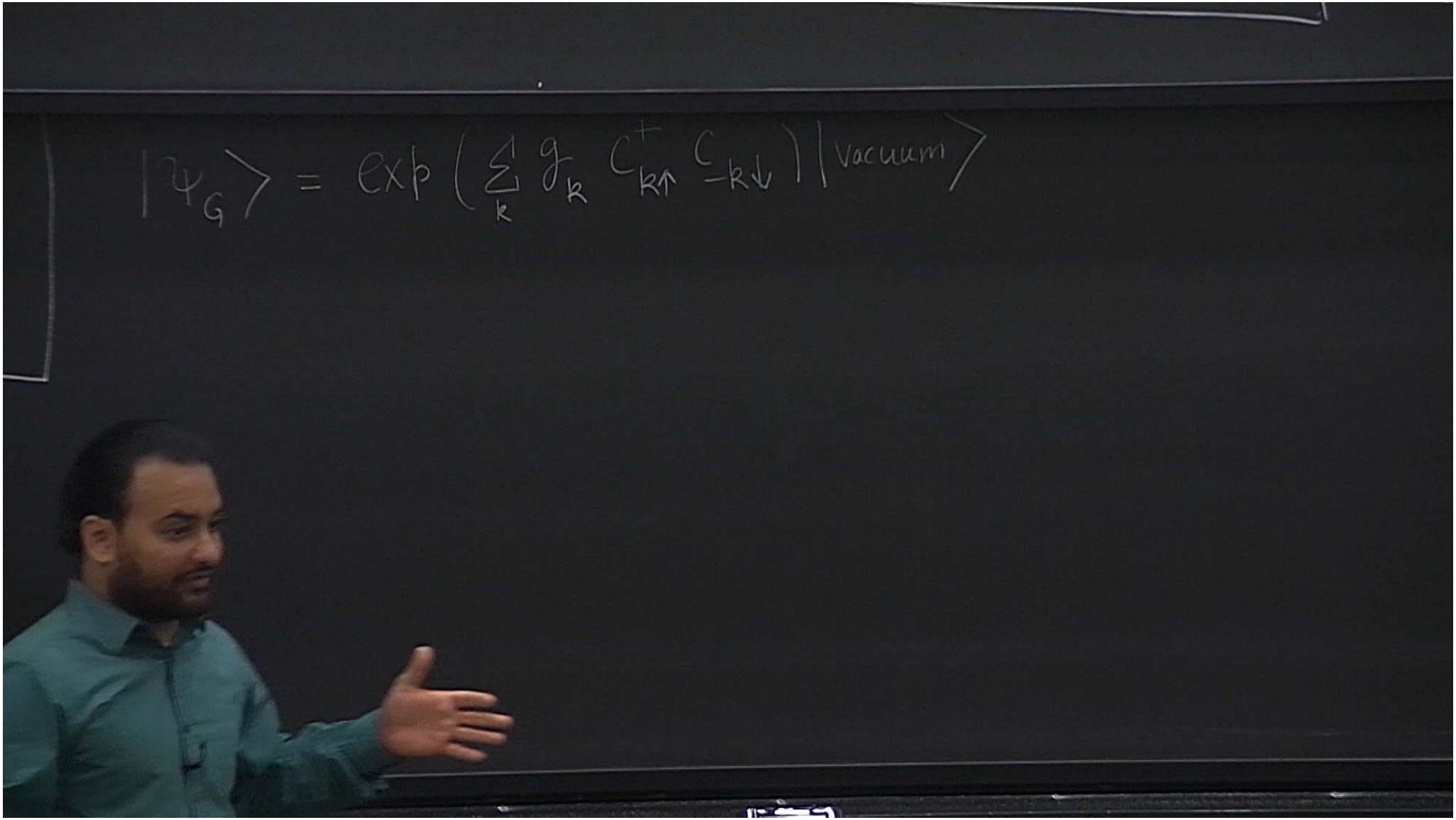
k can take M values

$$\frac{M!}{\frac{N}{2}!\left(M-\frac{N}{2}\right)!} \sim 10^{10^{20}}$$



N


g(ε)
2ħω_c
ε
ε_F
ħω_c ≪ ε_F
⇒ E = 2ħω_c (1 − e^{2/g(E_F)V})^{-1} ≈ 2ħω_c / (1 − e^{2/g(E_F)V})
≈ −2ħω_c exp(−2/g(E_F)V)
g(E_F)V ≪ 1
Weak Coup
limit
|ψ_0⟩ = Σ_{k>k_F} φ_k c†_{k↑} c†_{−k↓} |F⟩
|ψ^(N)⟩ = (Σ_k φ_k c†_{k↑} c†_{−k↓})^{N/2} |vacuum⟩
≈ g(E_F) ∫_0^{ħω_c} dξ 1/(2ξ − E) = g(E_F)/2 ln((E − 2ħω_c)/E)


$|\psi_G\rangle = \exp\left(\sum_k g_k c^+_{k\uparrow} c_{-k\downarrow}\right)|\text{vacuum}\rangle$

$$|\psi_G\rangle = \exp\left(\sum_k g_k\, c^+_{k\uparrow}\, c^+_{-k\downarrow}\right)|\text{vacuum}\rangle$$

$$= \prod_k \exp\left(g_k\, c^+_{k\uparrow}\, c^+_{-k\downarrow}\right)|\text{vacuum}\rangle$$

$$= \prod_k \left(1 + g_k\, c^+_{k\uparrow}\, c^+_{-k\downarrow}\right)|\text{vacuum}\rangle \qquad c^{+2}_{k\sigma} = 0$$

$$= \prod_k \left(u_k + v_k\, c^+_{k\uparrow}\, c^+_{-k\downarrow}\right)|\text{vacuum}\rangle$$

$|u_k|^2$ $(k\uparrow, -k\downarrow)$ empty

$|v_k|^2$ $(k\uparrow, -k\downarrow)$ occupied

$$|\psi_G\rangle = \exp\left(\sum_k g_k\, c^+_{k\uparrow}\, c^+_{-k\downarrow}\right)|\text{vacuum}\rangle$$

$$= \prod_k \exp\left(g_k\, c^+_{k\uparrow}\, c^+_{-k\downarrow}\right)|\text{vacuum}\rangle$$

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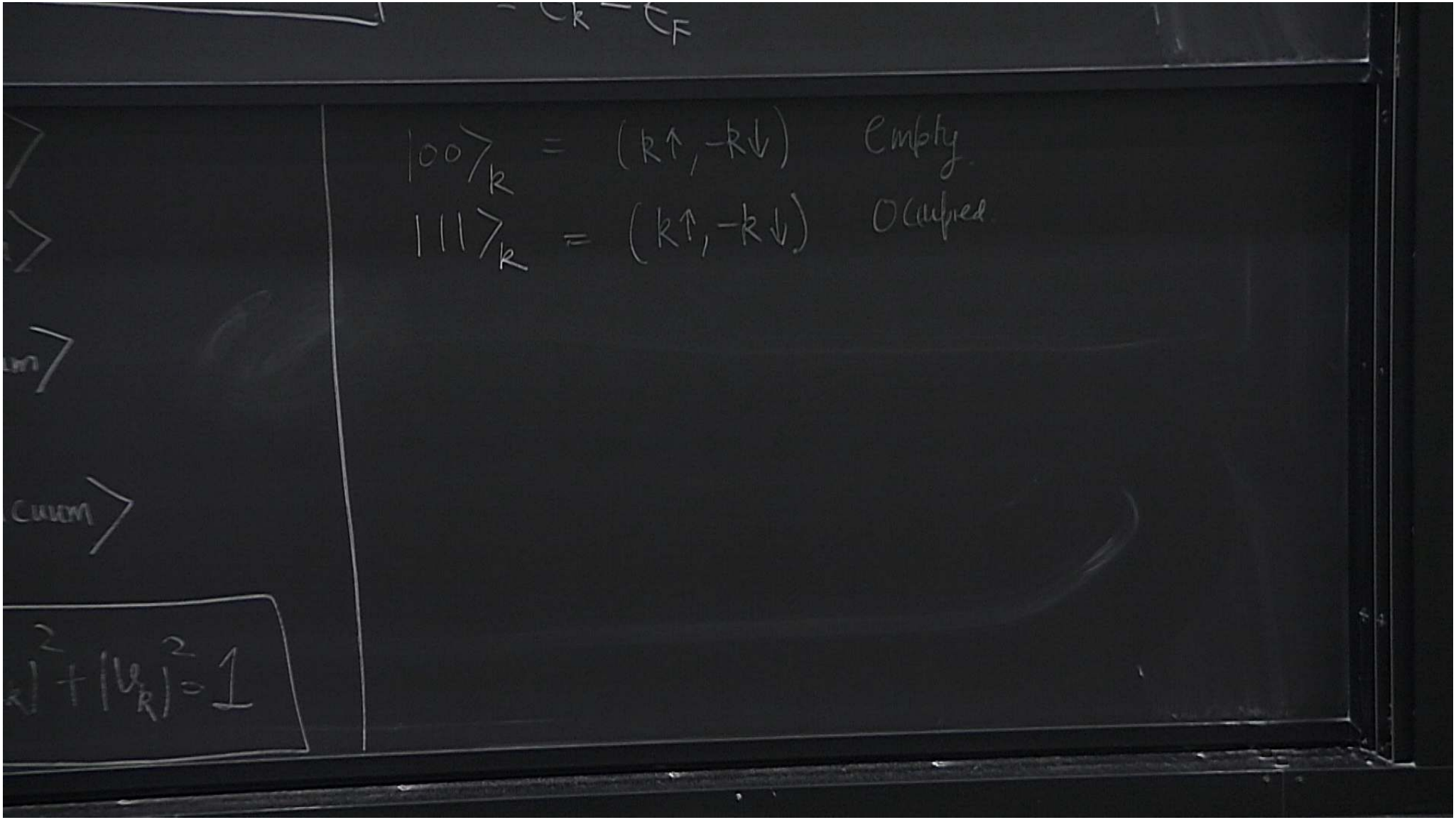
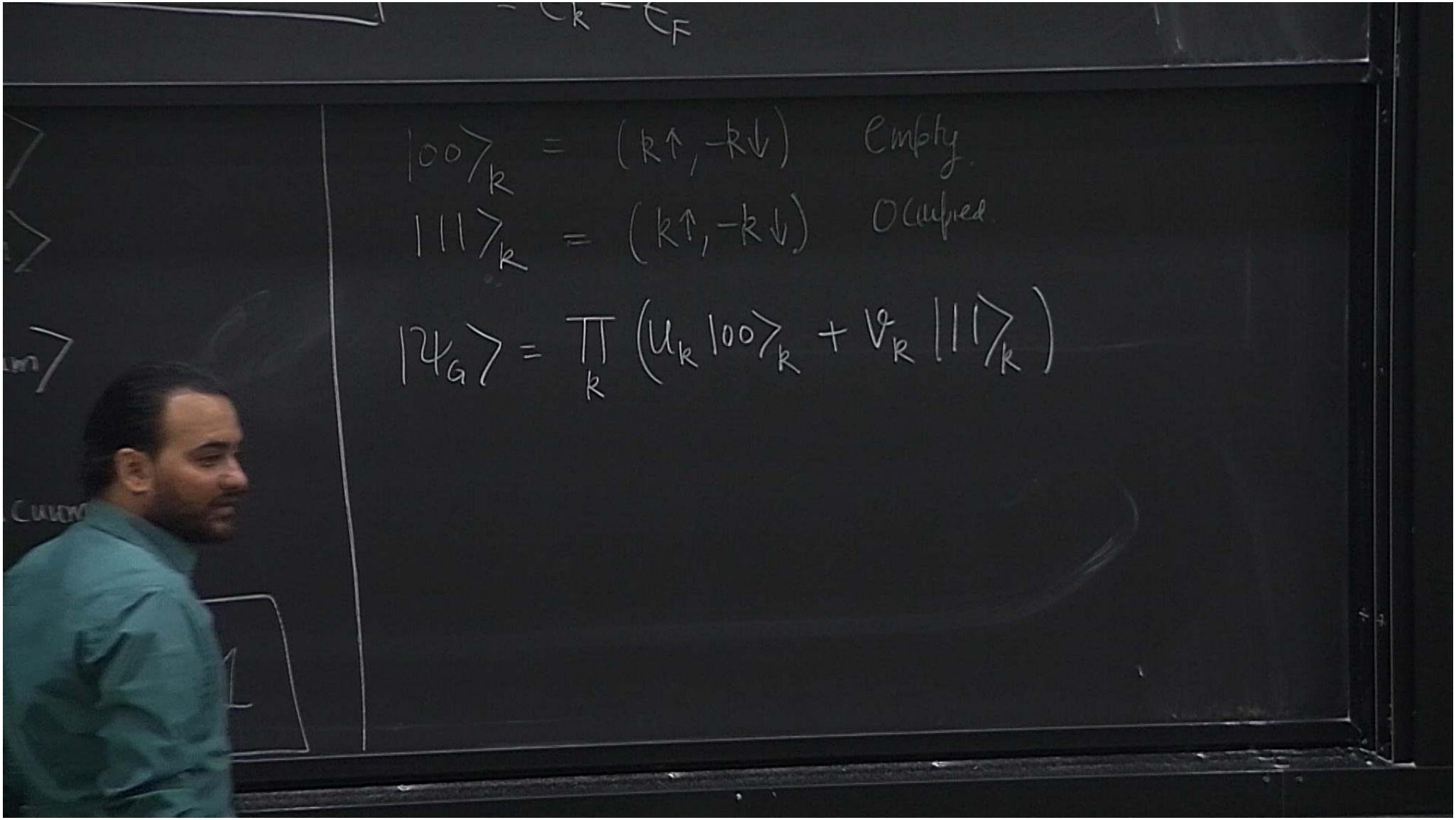
$|u_k|^2$ $(k\uparrow, -k\downarrow)$ empty

$|v_k|^2$ $(k\uparrow, -k\downarrow)$ occupied

$$\boxed{|u_k|^2 + |v_k|^2 = 1}$$


$|00\rangle_k$ = $(k\uparrow, -k\downarrow)$ Empty
$|11\rangle_k$ = $(k\uparrow, -k\downarrow)$ Occupied
$|u_k|^2 + |v_k|^2 = 1$


$|00\rangle_k$ = $(k\uparrow, -k\downarrow)$ Empty
$|11\rangle_k$ = $(k\uparrow, -k\downarrow)$ Occupied
$|\psi_G\rangle = \prod_k (u_k |00\rangle_k + v_k |11\rangle_k)$

$|00\rangle_k = (k\uparrow, -k\downarrow)$ Empty

$|11\rangle_k = (k\uparrow, -k\downarrow)$ Occupied

$$|\Psi_G\rangle = \prod_k \left(u_k |00\rangle_k + v_k |11\rangle_k\right)$$

$$|\text{vacuum}\rangle = \prod_k |00\rangle_k$$

$$\langle \Psi_G | \Psi_G \rangle = \prod_k \left(|u_k|^2 + |v_k|^2\right) = 1$$

$|u_k|^2 + |v_k|^2 = 1$

$$\bar{N} = \langle N \rangle = \langle \Psi_G | \sum_k \left(c^\dagger_{k\uparrow} c_{k\uparrow} + c^\dagger_{-k\downarrow} c_{-k\downarrow} \right) | \Psi_G \rangle$$

$$= 2 \sum_k \langle \Psi_G | c^\dagger_{k\uparrow} c_{k\uparrow} | \Psi_G \rangle = 2 \sum_k |v_k|^2$$

$$\bar{N} = \langle N \rangle = \langle \Psi_G | \sum_k \left(c^\dagger_{k\uparrow} c_{k\uparrow} + c^\dagger_{-k\downarrow} c_{-k\downarrow} \right) | \Psi_G \rangle$$

$$= 2 \sum_k \langle \Psi_G | c^\dagger_{k\uparrow} c_{k\uparrow} | \Psi_G \rangle = 2 \sum_k |v_k|^2$$

$$\langle N^2 \rangle = 4 \sum_k |v_k|^2 + 4 \sum_{k, k' \neq k} |v_k|^2 |v_{k'}|^2$$

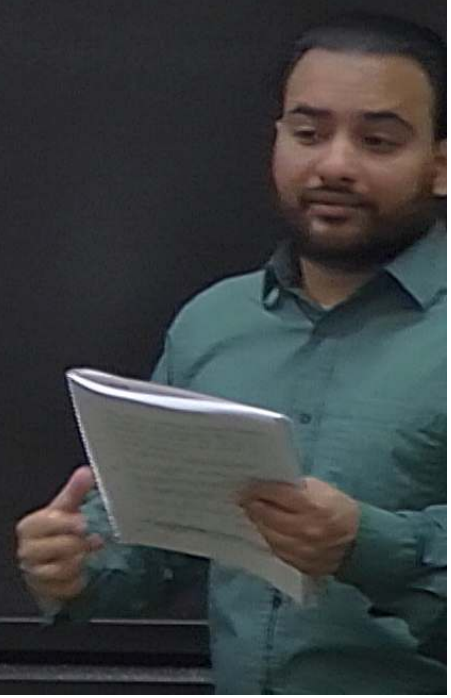
$$\bar{N} = \langle N \rangle = \langle \Psi_G | \sum_k \left(c^+_{k\uparrow} c_{k\uparrow} + c^+_{-k\downarrow} c_{-k\downarrow} \right) | \Psi_G \rangle$$

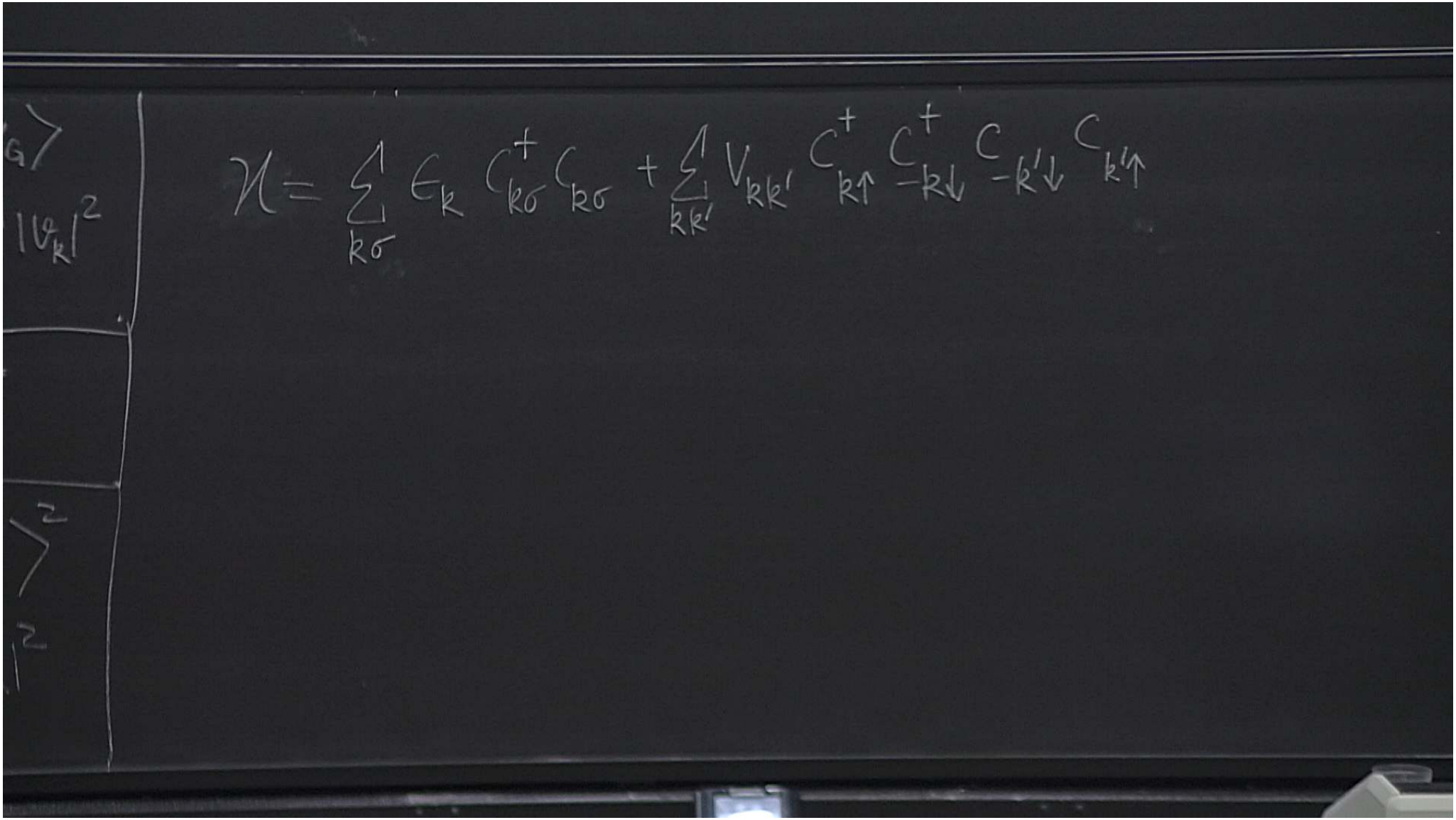
$$= 2 \sum_k \langle \Psi_G | c^+_{k\uparrow} c_{k\uparrow} | \Psi_G \rangle = 2 \sum_k |v_k|^2$$

$$\langle N^2 \rangle = 4 \sum_k |v_k|^2 + 4 \sum_{k, k' \neq k} |v_k|^2 |v_{k'}|^2$$

$$\langle (\delta N)^2 \rangle = \langle (N - \bar{N})^2 \rangle = \langle N^2 \rangle - \langle \bar{N} \rangle^2$$

$$= 4 \sum_k |v_k|^2 - |v_k|^4 = 4 \sum_k |v_k|^2 |u_k|^2$$

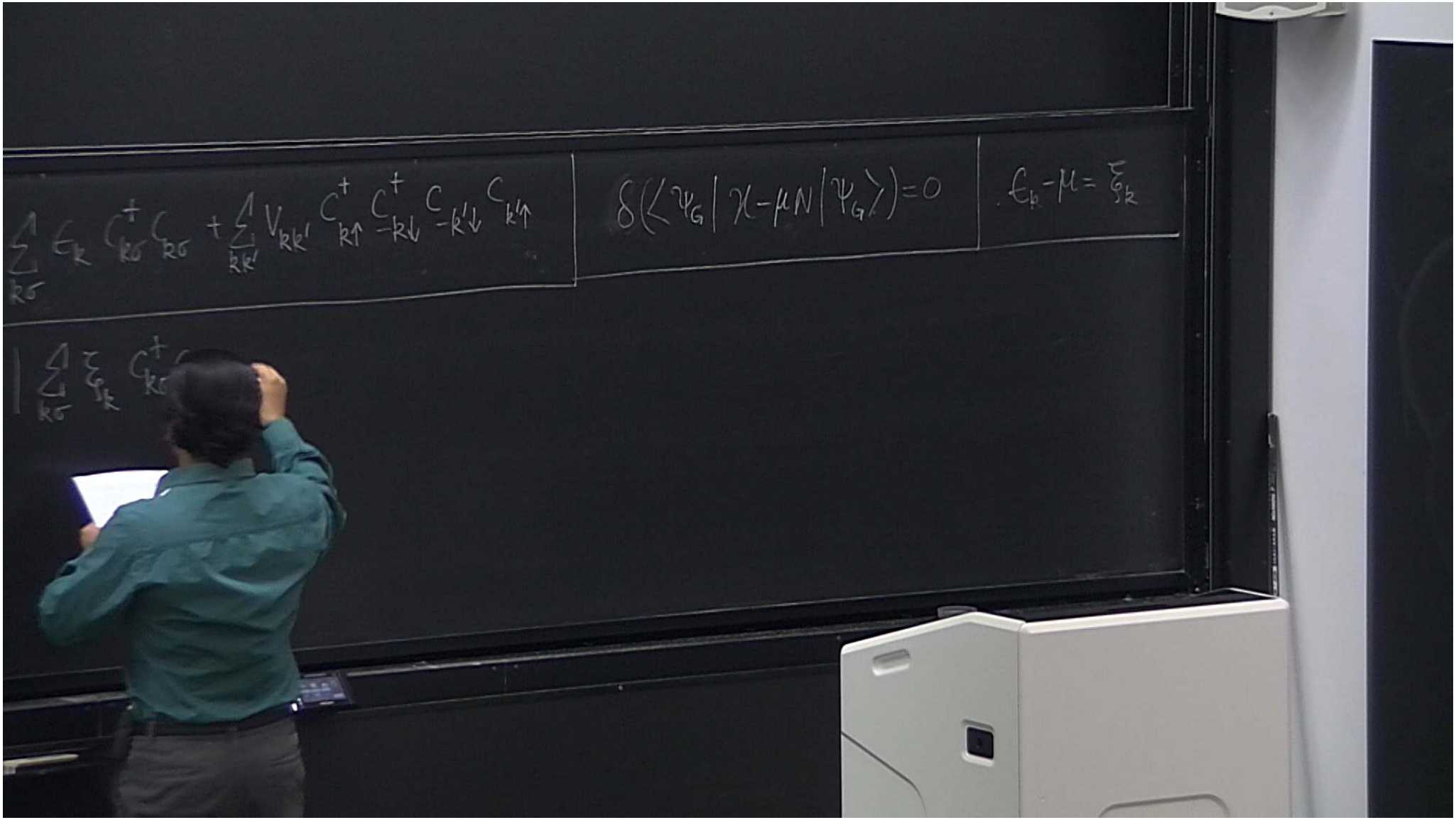



$\mathcal{H} = \sum_{k\sigma} \epsilon_k c^\dagger_{k\sigma} c_{k\sigma} + \sum_{kk'} V_{kk'} c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow} c_{-k'\downarrow} c_{k'\uparrow}$

$$\mathcal{H} = \sum_{k\sigma} \epsilon_k c^\dagger_{k\sigma} c_{k\sigma} + \sum_{kk'} V_{kk'} c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow} c_{-k'\downarrow} c_{k'\uparrow}$$

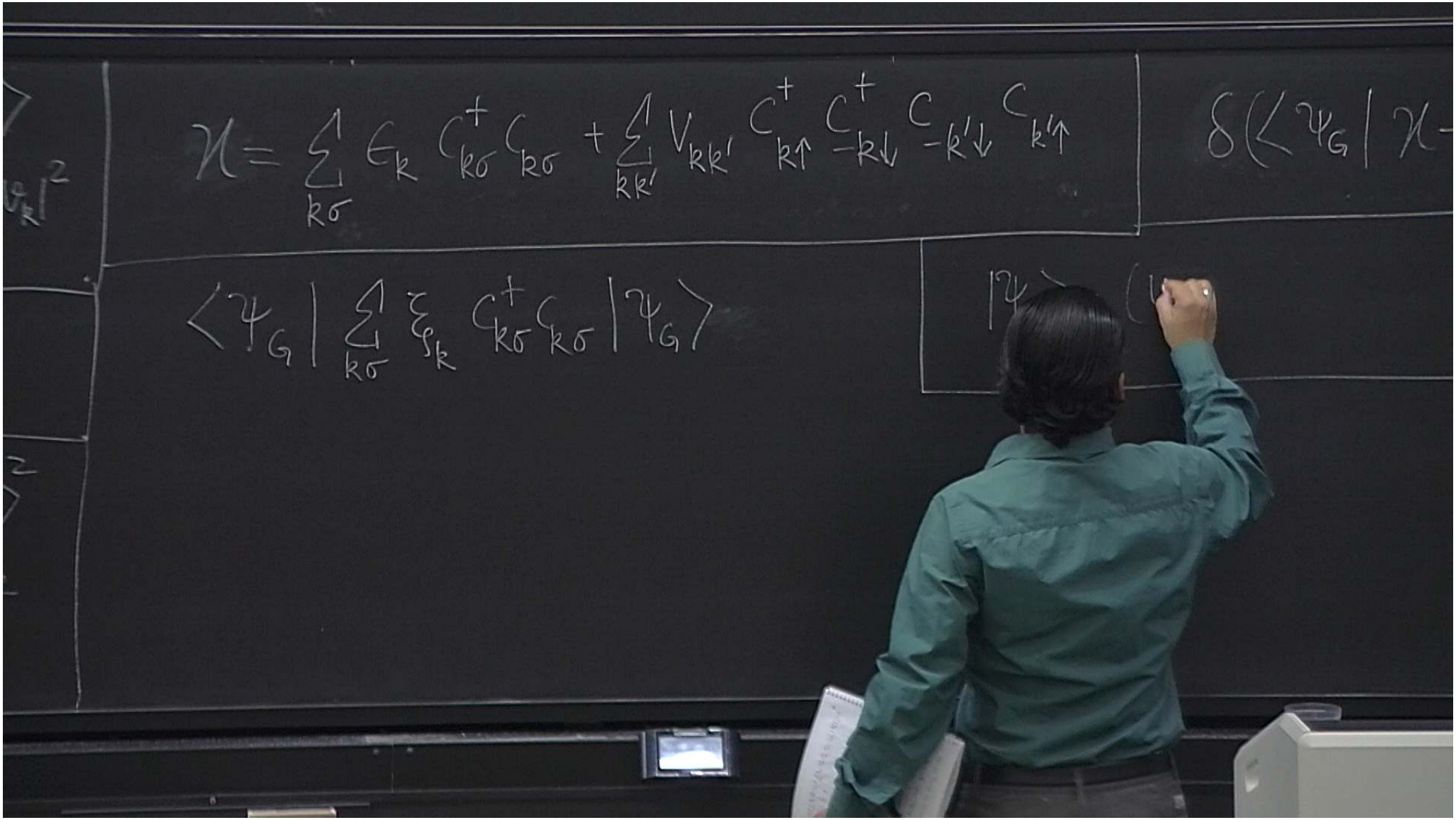
$$\delta(\langle \Psi_G | \mathcal{H} - \mu N | \Psi_G \rangle) = 0$$

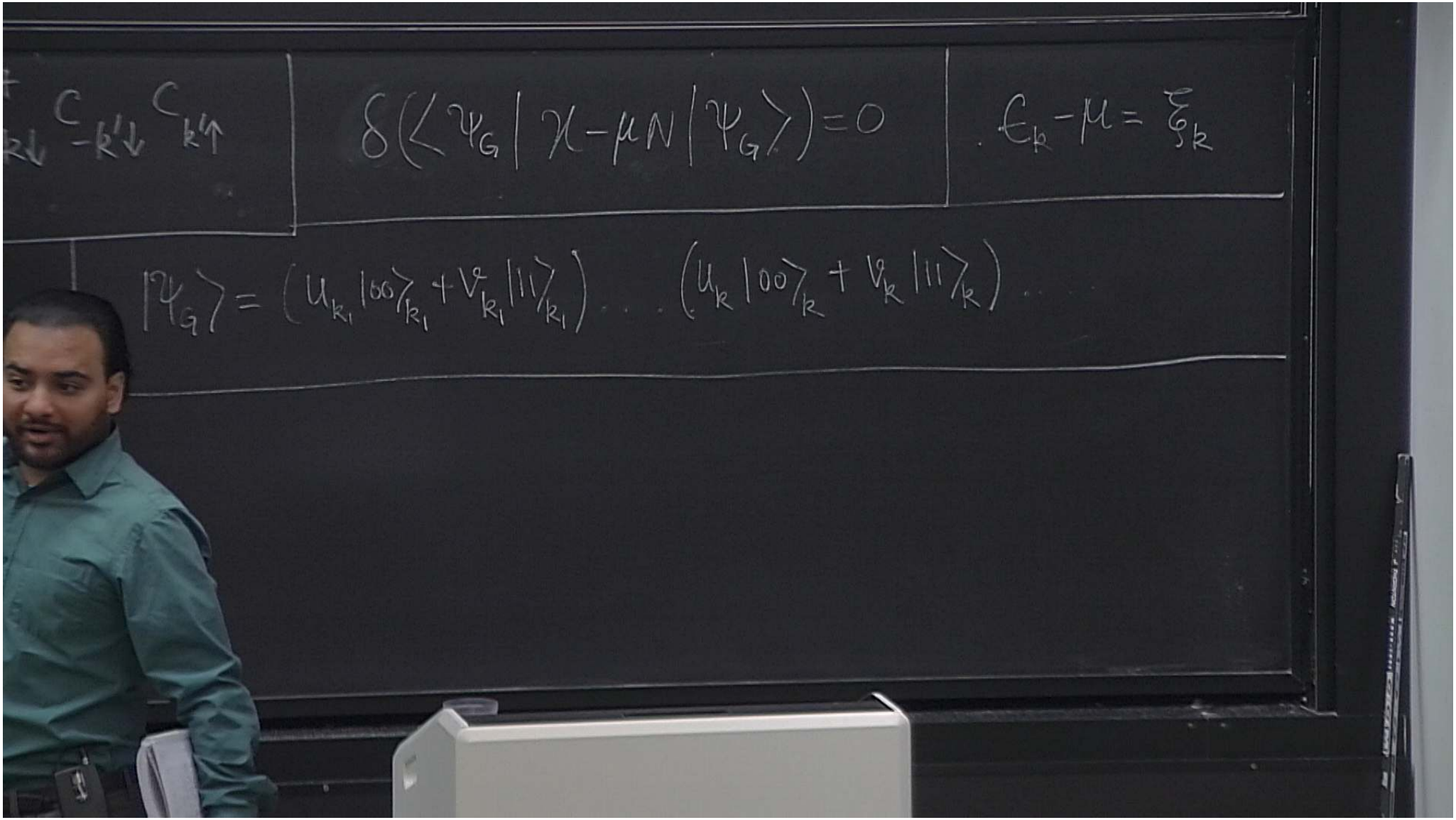
$$\langle \Psi_G | \sum_{k\sigma}$$

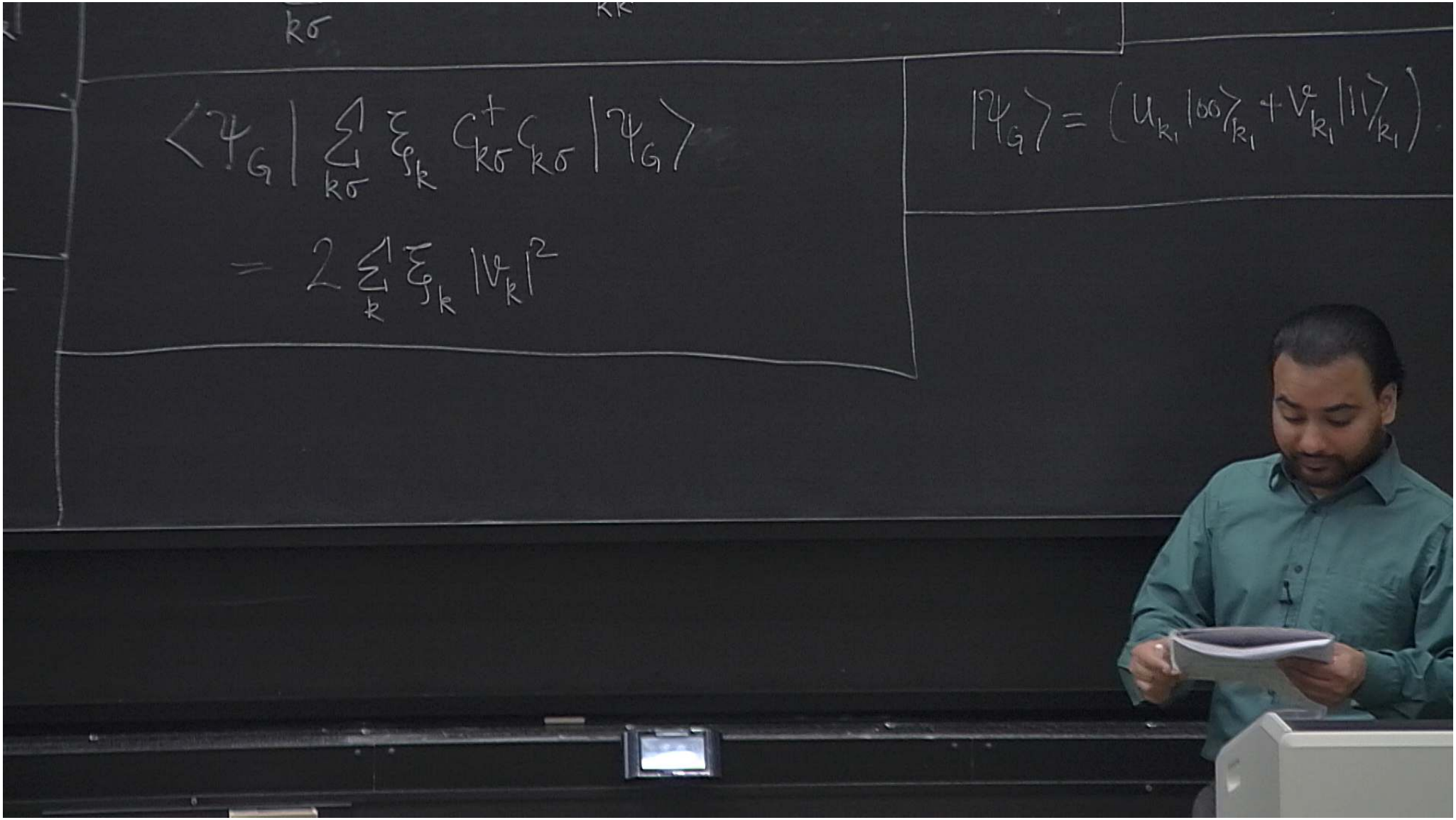

$\sum_{k\sigma} \epsilon_k c^\dagger_{k\sigma} c_{k\sigma} + \sum_{kk'} V_{kk'} c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow} c_{-k'\downarrow} c_{k'\uparrow}$
$\delta(\langle \Psi_G | \mathcal{H} - \mu N | \Psi_G \rangle) = 0$
$\epsilon_k - \mu = \xi_k$
$\sum_{k\sigma} \xi_k c^\dagger_{k\sigma}$

$\delta(\langle \psi_G | \mathcal{H} - \mu N | \psi_G \rangle) = 0$

$\epsilon_k - \mu = \xi_k$


$\mathcal{H} = \sum_{k\sigma} \epsilon_k c^\dagger_{k\sigma} c_{k\sigma} + \sum_{kk'} V_{kk'} c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow} c_{-k'\downarrow} c_{k'\uparrow}$
$\delta(\langle \Psi_G | \mathcal{H}$
$\langle \Psi_G | \sum_{k\sigma} \xi_k c^\dagger_{k\sigma} c_{k\sigma} | \Psi_G \rangle$
$|\Psi\rangle$


$C_{-k'\downarrow} C_{k'\uparrow}$
$\delta(\langle \psi_G | \mathcal{H} - \mu N | \psi_G \rangle) = 0$
$\epsilon_k - \mu = \xi_k$
$|\psi_G\rangle = (u_{k_1}|00\rangle_{k_1} + v_{k_1}|11\rangle_{k_1}) \ldots (u_k|00\rangle_k + v_k|11\rangle_k) \ldots$


$\langle \psi_G | \sum_{k\sigma} \xi_k c^\dagger_{k\sigma} c_{k\sigma} | \psi_G \rangle$
$= 2 \sum_k \xi_k |v_k|^2$
$|\psi_G\rangle = (u_{k_1} |00\rangle_{k_1} + v_{k_1} |11\rangle_{k_1})$

$c^\dagger_{-k\downarrow} c_{-k'\downarrow} c_{k'\uparrow}$

$\delta(\langle \Psi_G | \mathcal{H} - \mu N | \Psi_G \rangle) = 0$

$\epsilon_k - \mu = \xi_k$

$|\Psi_G\rangle = (u_{k_1}|00\rangle_{k_1} + v_{k_1}|11\rangle_{k_1}) \ldots (u_k|00\rangle_k + v_k|11\rangle_k) \ldots$

$c_{-k'\downarrow} c_{k'\uparrow} |\Psi_G\rangle = (u_{k_1}|00\rangle_{k_1} + v_{k_1}|11\rangle_k) \ldots (v_{k'}|00\rangle_{k'}) \ldots$

$c^\dagger_{-k\downarrow} c^\dagger_{k\uparrow} c_{-k'\downarrow} c_{k'\uparrow}$

$\delta(\langle \Psi_G | \mathcal{H} - \mu N | \Psi_G \rangle) = 0$

$\epsilon_k - \mu = \xi_k$

$$|\Psi_G\rangle = (u_{k_1}|00\rangle_{k_1} + v_{k_1}|11\rangle_{k_1}) \cdots (u_k|00\rangle_k + v_k|11\rangle_k) \cdots$$

$$c_{-k'\downarrow} c_{k'\uparrow} |\Psi_G\rangle = (u_{k_1}|00\rangle_{k_1} + v_{k_1}|11\rangle_{k_1}) \cdots (v_{k'}|00\rangle_{k'}) \cdots$$

$$c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow} c_{-k'\downarrow} c_{k'\downarrow} |\Psi_G\rangle = (u_{k_1}|00\rangle_{k_1} + v_{k_1}|11\rangle_{k_1}) \cdots (u_k|11\rangle_k) \cdots (v_{k'}|00\rangle_{k'}) \cdots$$

$$\mathcal{H} = \sum_{k\sigma} \epsilon_k c^\dagger_{k\sigma} c_{k\sigma} + \sum_{kk'} V_{kk'} c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow} c_{-k'\downarrow} c_{k'\uparrow}$$

$$\delta(\langle \Psi_G | \mathcal{H} - \mu N | \Psi_G$$

$$\langle \Psi_G | \sum_{k\sigma} \xi_k c^\dagger_{k\sigma} c_{k\sigma} | \Psi_G \rangle = 2 \sum_k \xi_k |v_k|^2$$

$$|\Psi_G\rangle = (u_{k_1} |00\rangle_{k_1} + v_{k_1} |11\rangle_{k_1}) \quad (u_k |$$

$$c_{-k'\downarrow} c_{k'\uparrow} |\Psi_G\rangle = (u_{k_1} |00\rangle_{k_1} + v_{k_1} |11\rangle_k)$$

$$c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow} c_{-k'\downarrow} c_{k'\downarrow} |\Psi_G\rangle = (u_{k_1} |00\rangle_{k_1} + v_{k_1} |11\rangle_k)$$

$$\delta(\langle \Psi_G | \mathcal{H} - \mu N | \Psi_G \rangle) = \delta\left(2 \sum_k \xi_k |v_k|^2 + \sum_{kk'} V_{kk'} u_k v_k^* u_{k'}^* v_{k'}\right) = 0$$

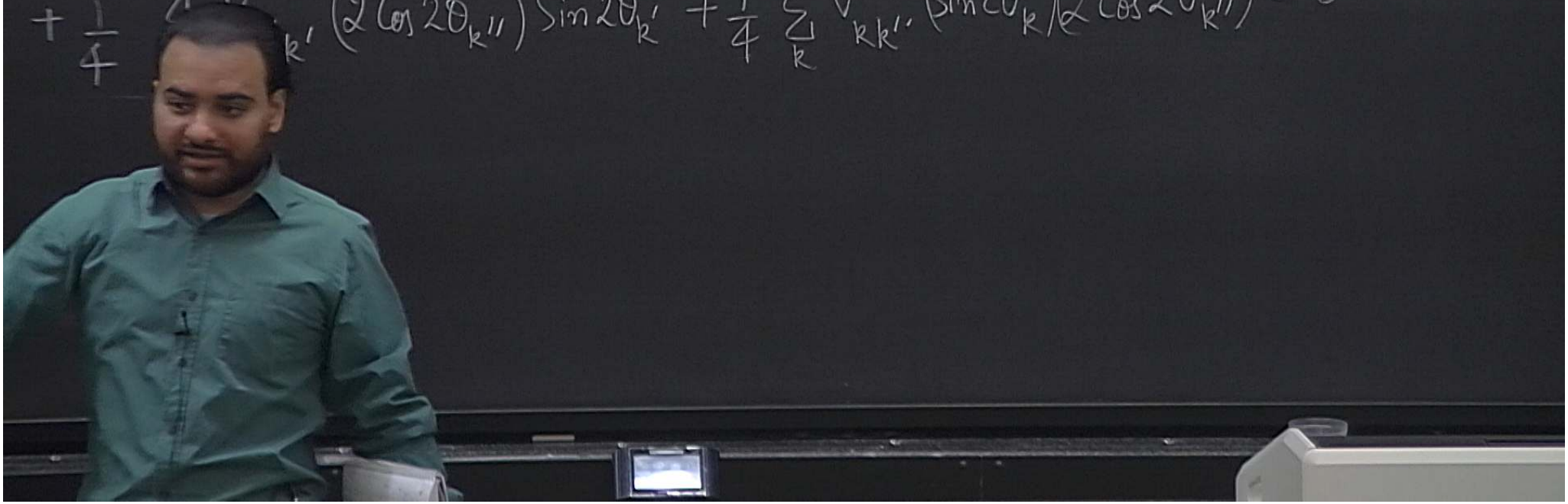
$$u_k = \sin\theta_k \qquad v_k = \cos\theta_k$$

$$\langle \psi_G | \mathcal{H} - \mu N | \psi_G \rangle = \sum_k \xi_k \left(1 + \cos 2\theta_k\right) + \frac{1}{4} \sum_{kk'} V_{kk'} \sin 2\theta_k \sin 2\theta_{k'}$$

$$\frac{\partial}{\partial \theta_{k''}} \langle \psi_G | \mathcal{H} - \mu N | \psi_G \rangle = 0 \Rightarrow -2\xi_{k''} \sin 2\theta_{k''} + \frac{1}{4} \sum_{k'} V_{k''k'} \left(2\cos 2\theta_{k''}\right) \sin 2\theta_{k'}$$

$$\tan 2\theta_k = \frac{\sum_{k'} V_{kk'} \sin 2\theta_{k'}}{2\xi_k}$$

$$\delta\left(\langle \Psi_G | \mathcal{H} - \mu N | \Psi_G \rangle\right) = \delta\left(2\sum_k \varepsilon_k |v_k|^2 + \sum_{kk'} V_{kk'} u_k v_k^* u_{k'}^* v_{k'}\right) = 0$$

$$V_{kk'} \sin 2\theta_k \sin 2\theta_{k'}$$

$$+\frac{1}{4} \sum_{k'} V_{kk'} (2\cos 2\theta_{k''}) \sin 2\theta_{k'} + \frac{1}{4} \sum_{k} V_{kk''} (\sin 2\theta_k)(2\cos 2\theta_{k''}) = 0$$