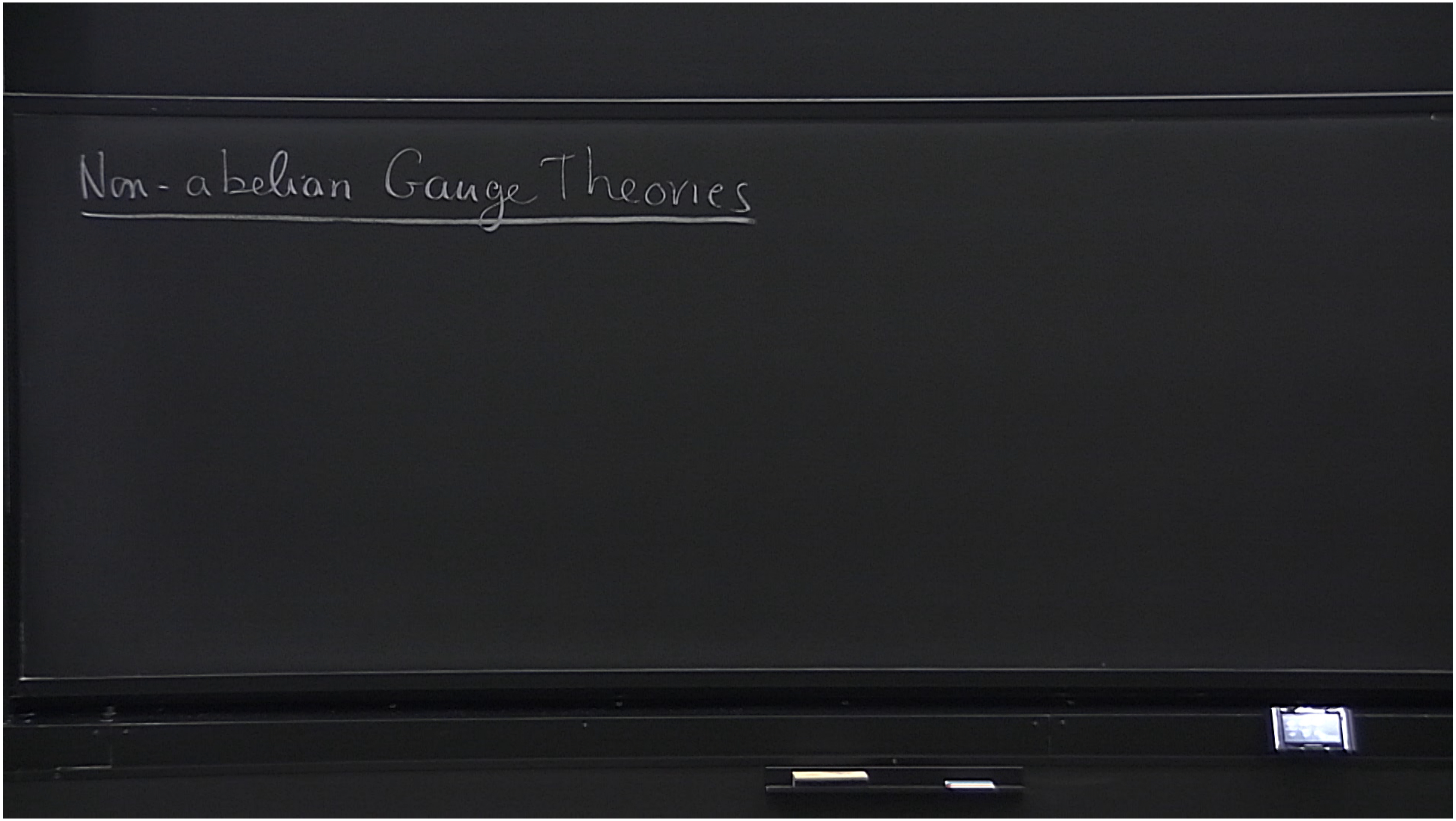
Title: PSI 17/18 - Quantum Field Theory II - Lecture 12

Date: Nov 21, 2017 09:00 AM

URL: http://pirsa.org/17110022

Abstract:



Non-abelian Gauge Theories

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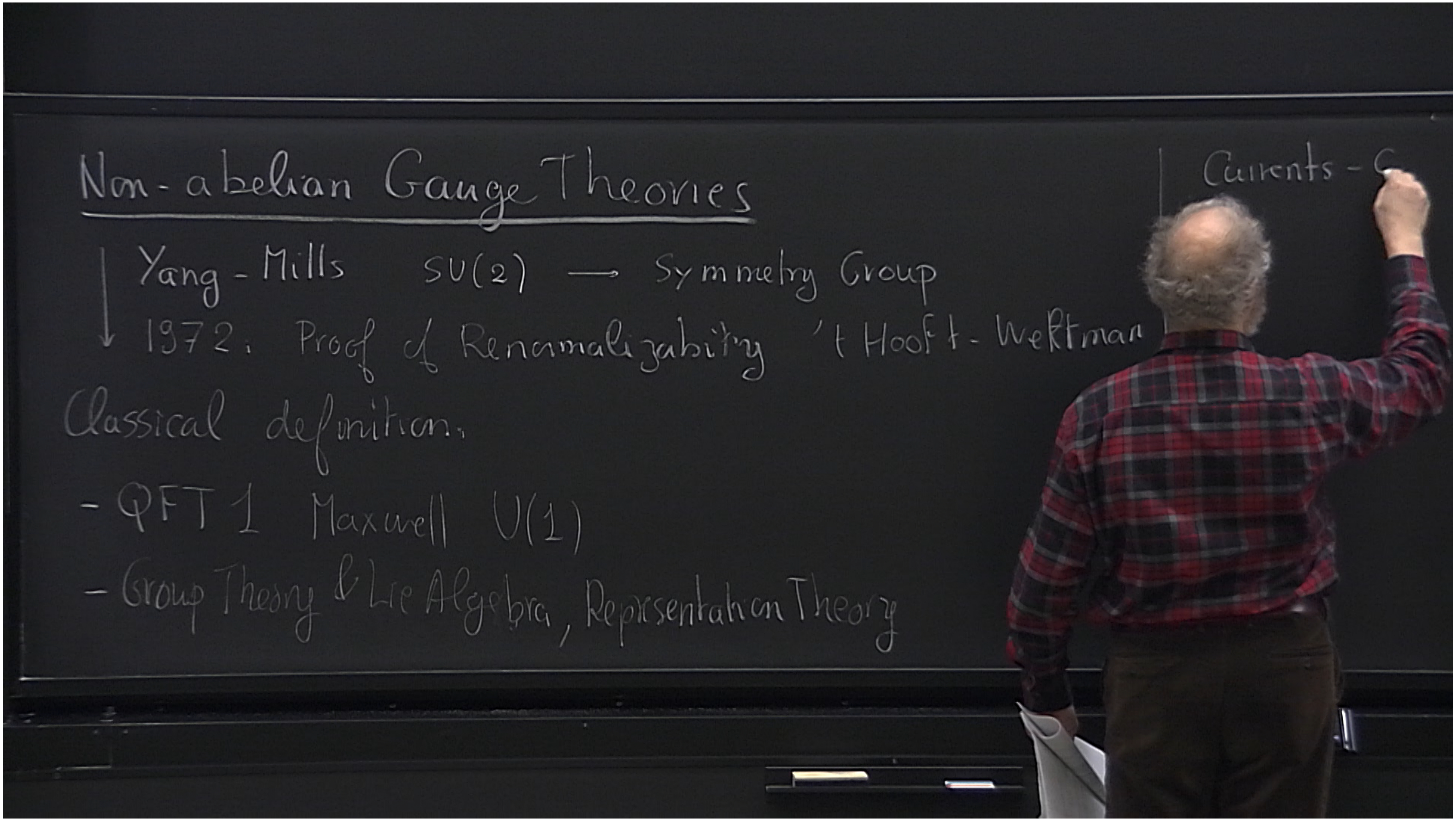
↓ Yang-Mills SU(2) ⟶ Symmetry Group

1972: Proof of Renormalizabity 't Hooft-Veltman

Classical definition:

- QFT 1 Maxwell U(1)
- Group Theory & Lie Algebra



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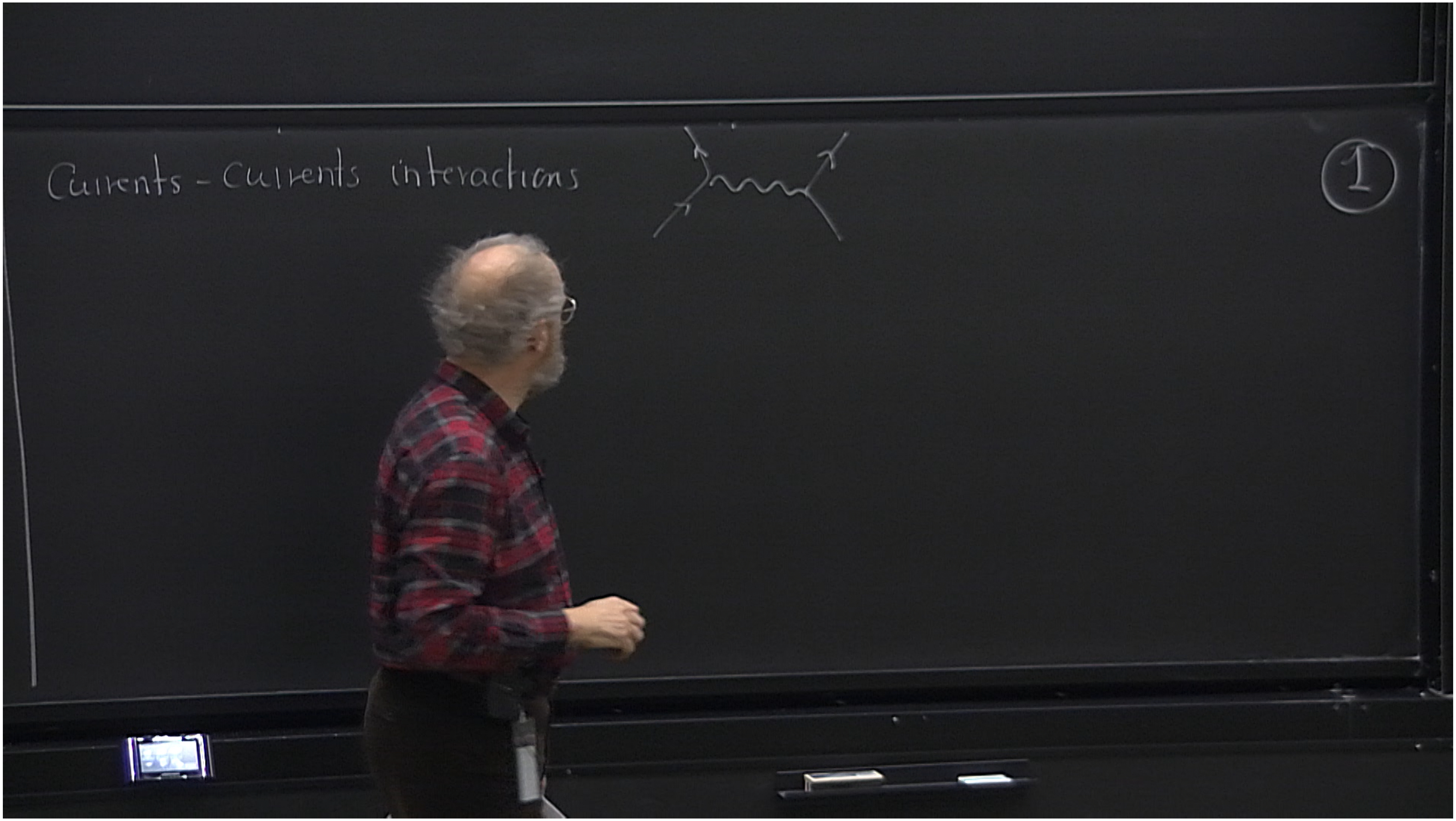
Currents - G



Currents - currents interactions

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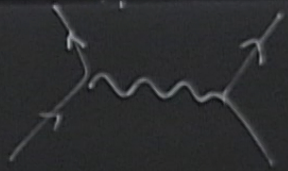
Currents - Currents interactions



Spin 1 vector particles

massless (symmetry & quantum consistency) ⇒ long range interactions
(classically at least)

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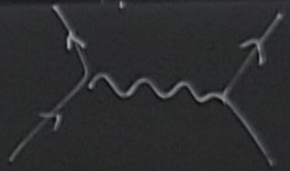
Currents - Currents interactions



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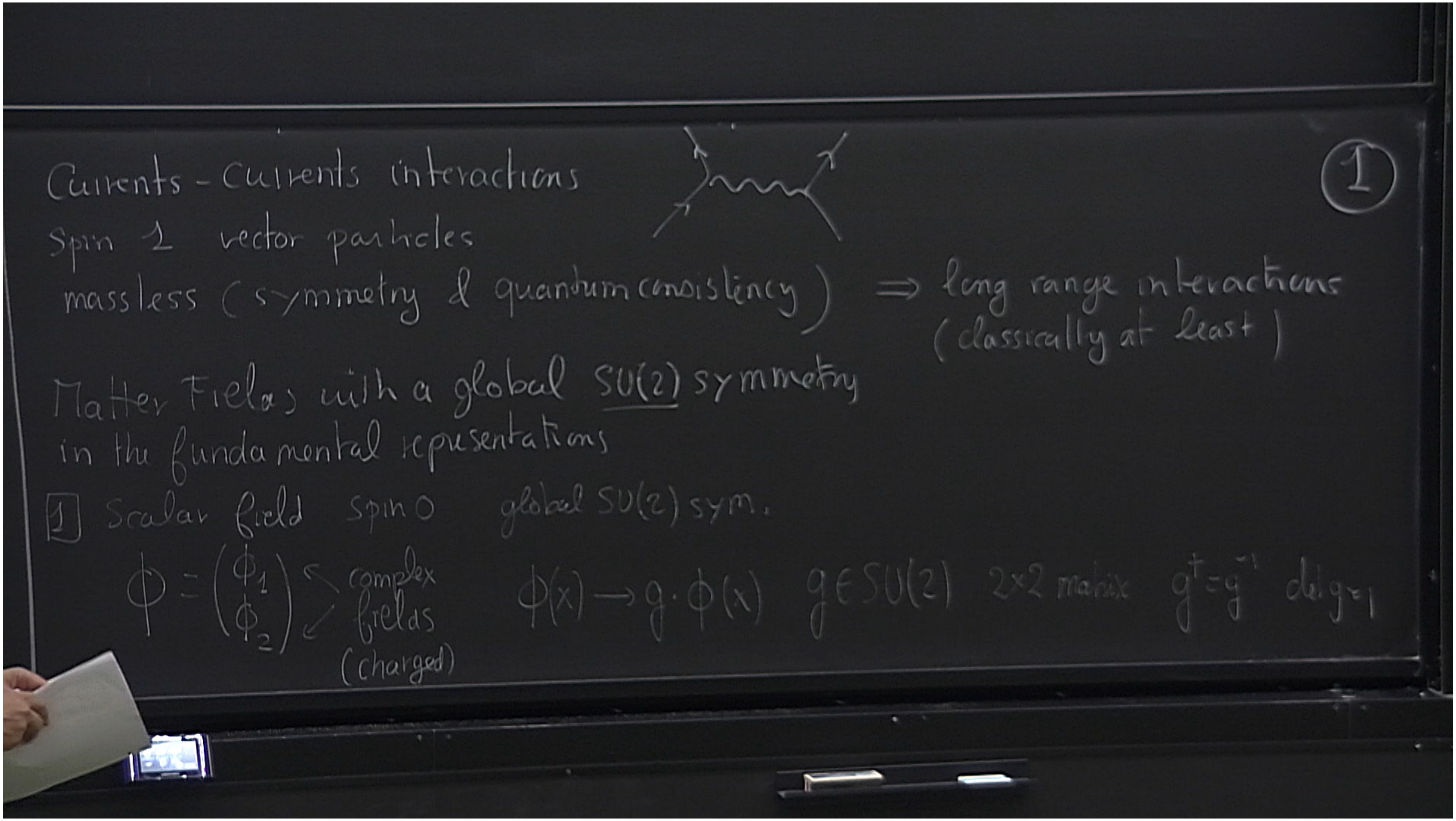
Matter Fields with a global SU(2) symmetry



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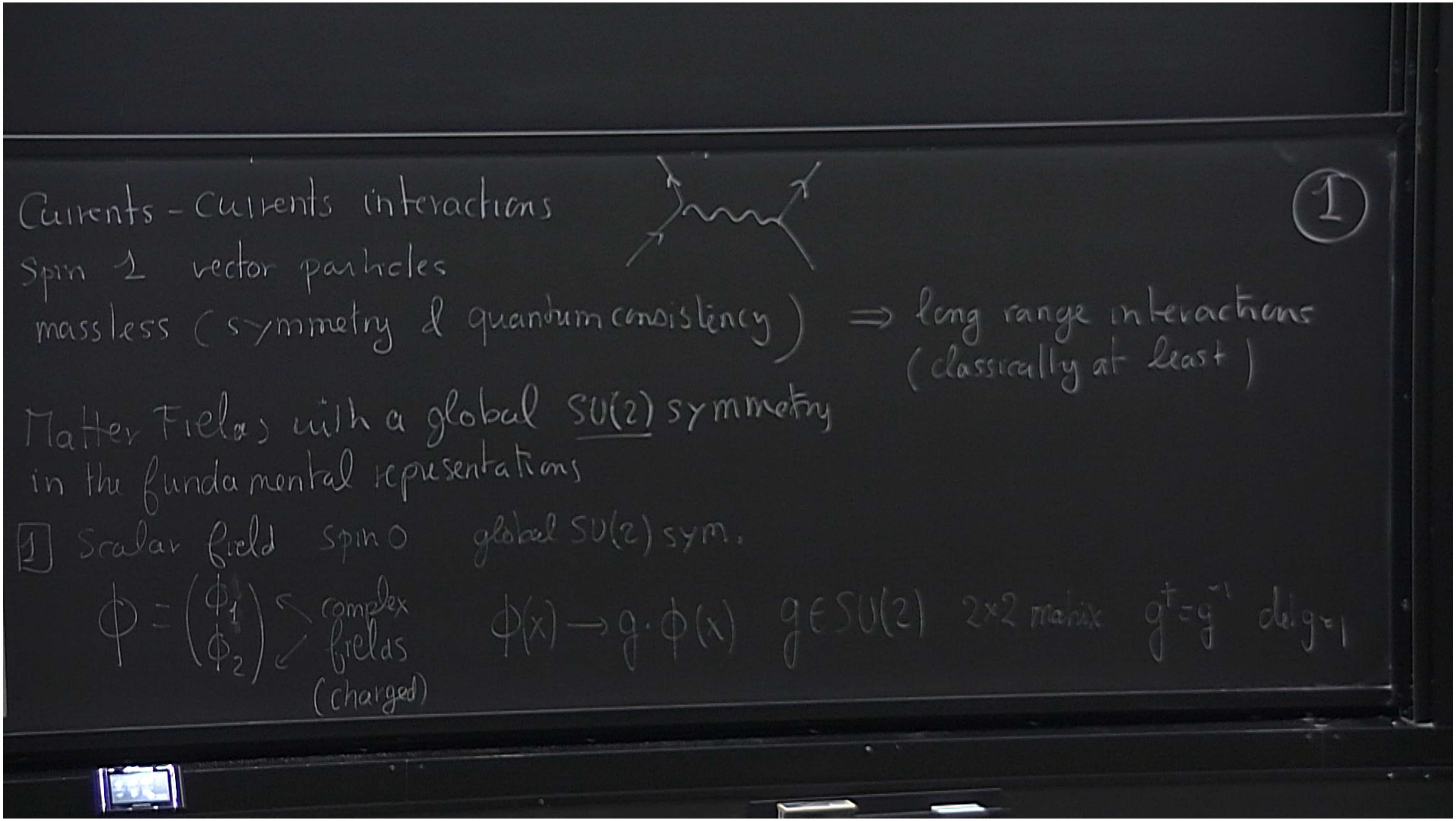
Matter Fields with a global $\underline{SU(2)}$ symmetry in the fundamental representations

$\boxed{1}$ Scalar field spin 0 global SU(2) sym.

$\phi = \begin{pmatrix}\phi_1 \\ \phi_2\end{pmatrix}$ complex fields (charged)

$\phi(x) \to g \cdot \phi(x)$ $g \in SU(2)$ 2×2 matrix $g^\dagger = g^{-1}$ $\det g = 1$



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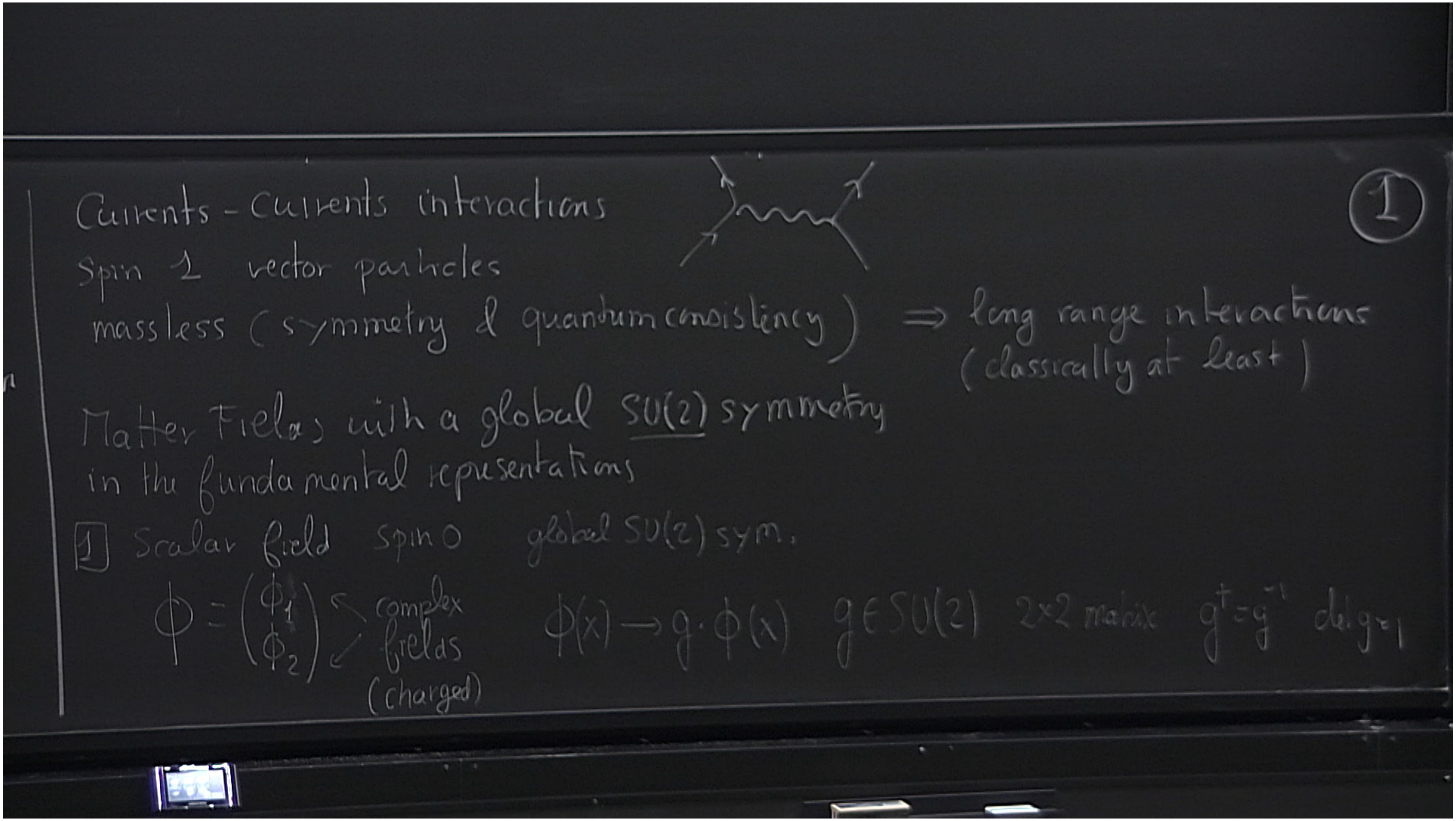
$$\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \quad \text{complex fields (charged)} \qquad \phi(x) \rightarrow g \cdot \phi(x) \quad g \in SU(2) \quad 2\times 2 \text{ matrix} \quad g^\dagger = g^{-1} \quad \det g = 1$$



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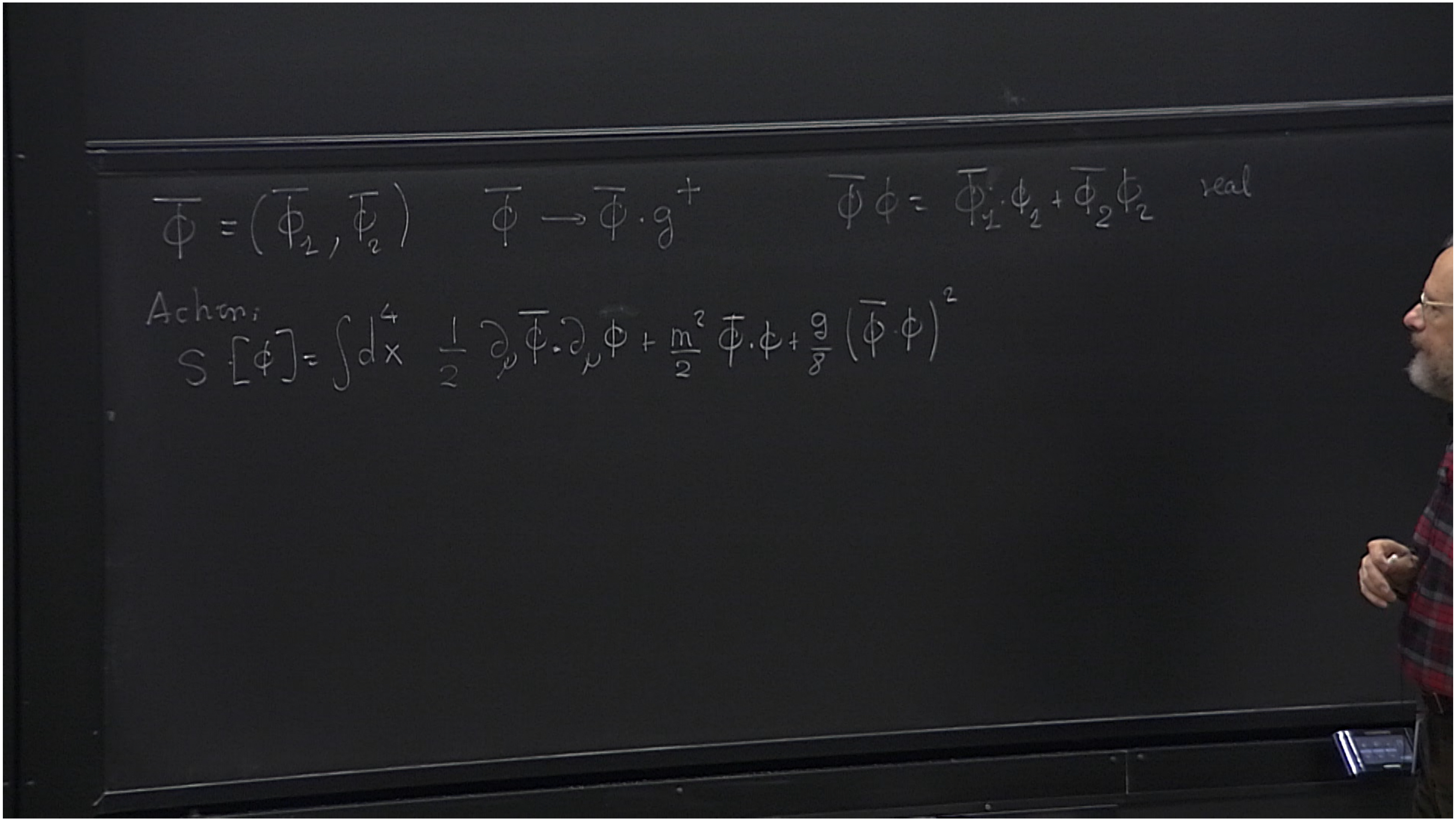
$$\bar{\phi} = (\bar{\phi}_1, \bar{\phi}_2) \qquad \bar{\phi} \longrightarrow \bar{\phi} \cdot g^{+}$$

Action:

$$S[\phi] = \int d^4x \;\; \frac{1}{2} \partial_\mu \bar{\phi} \cdot \partial_\mu \phi + \frac{m^2}{2} \bar{\phi} \cdot \phi + \frac{g}{8} (\bar{\phi} \cdot \phi)^2$$



$\bar\Phi = (\bar\Phi_1, \bar\Phi_2)$ $\quad \bar\Phi \longrightarrow \bar\Phi \cdot g^\dagger$ $\quad \bar\Phi\phi = \bar\Phi_1 \cdot \phi_2 + \bar\Phi_2 \phi_2$ real

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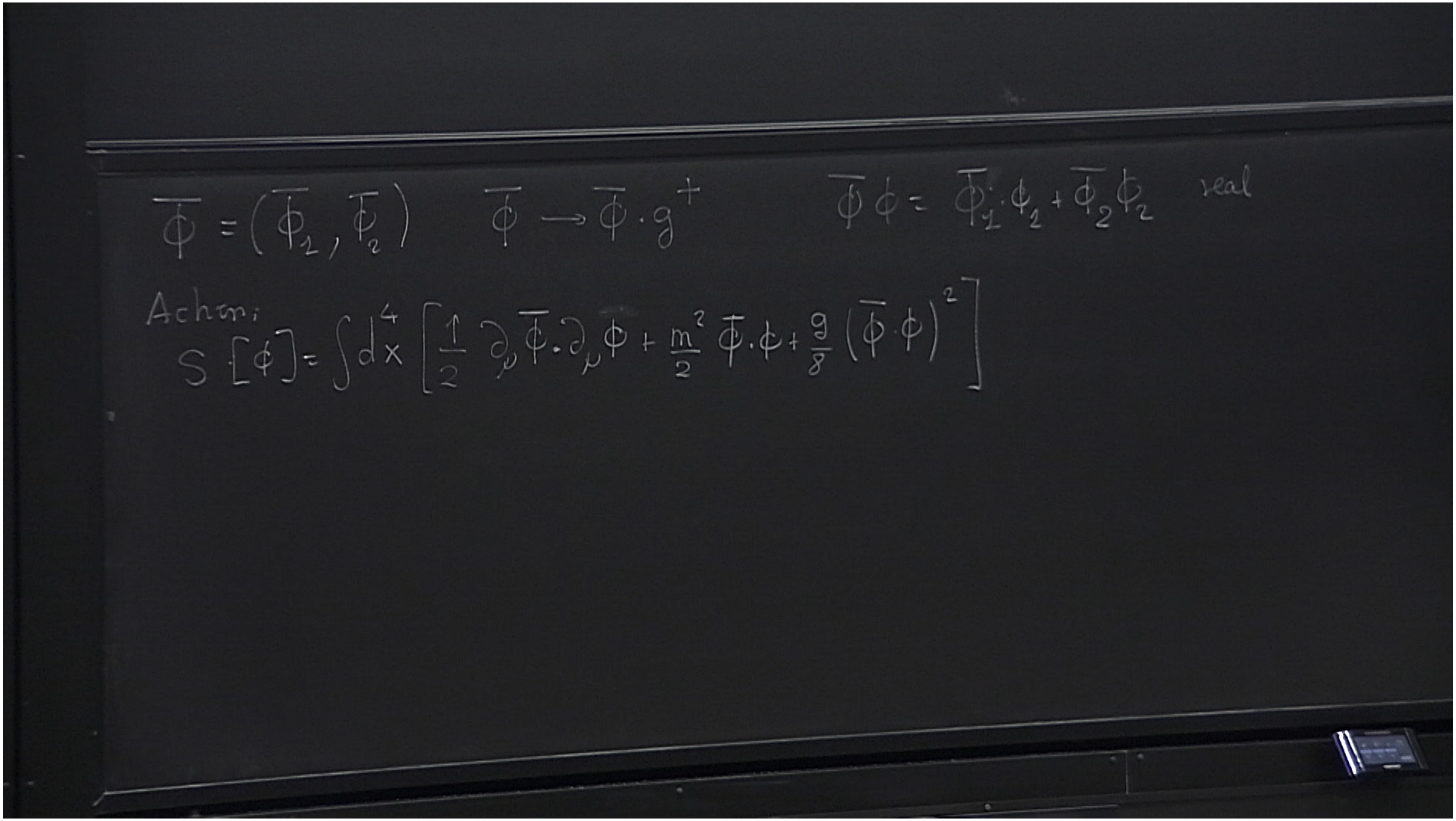

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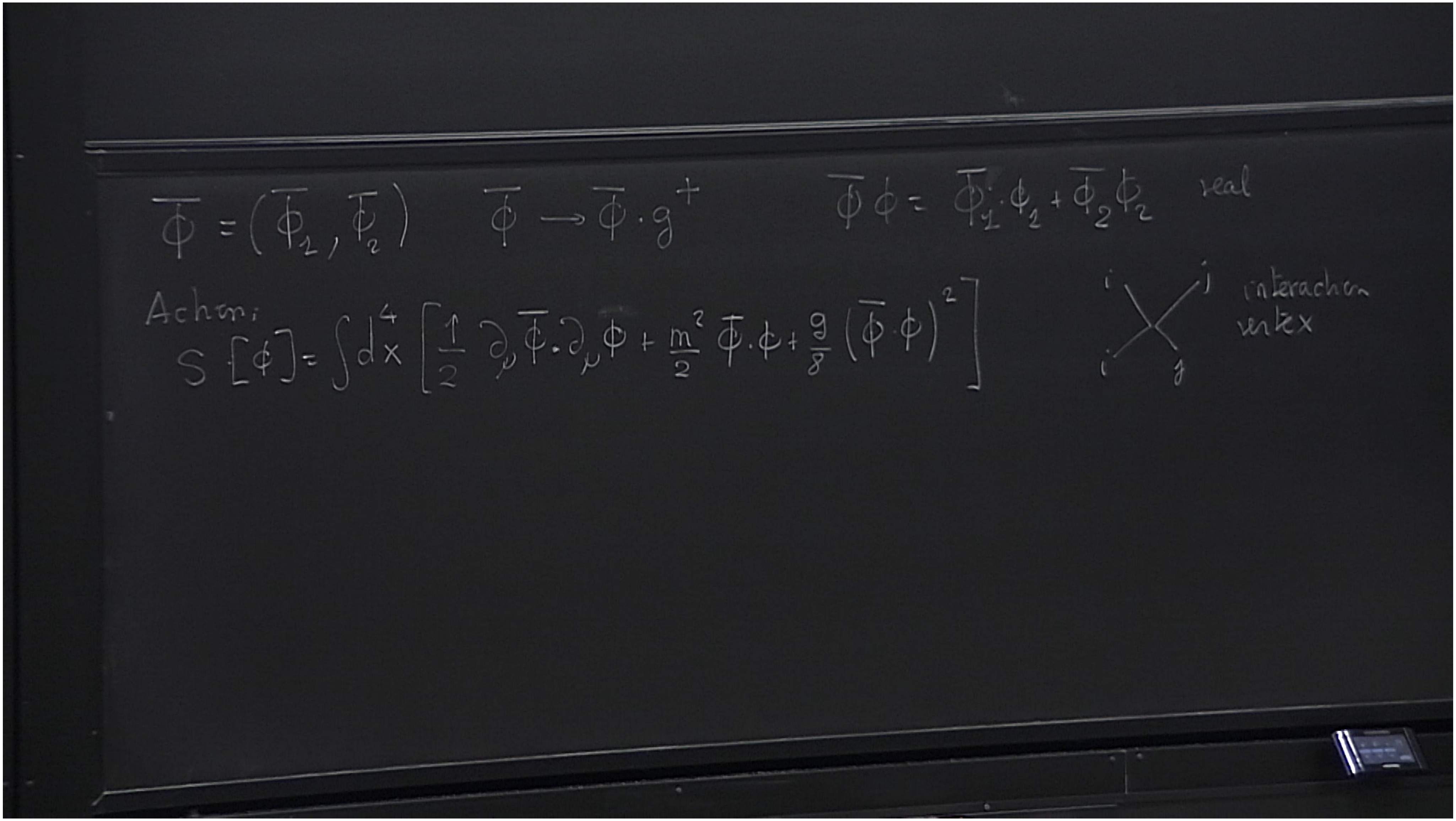

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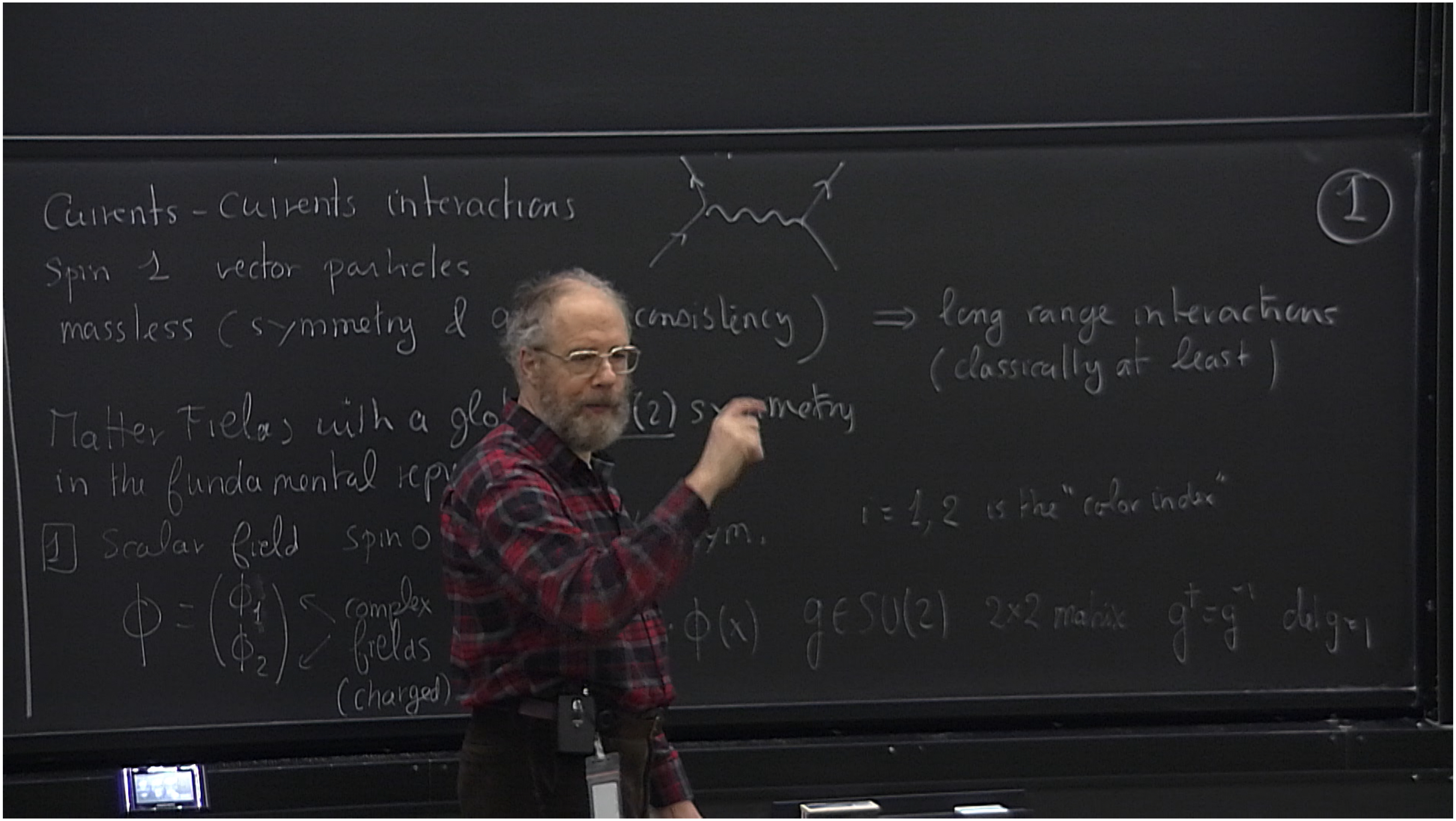
interaction vertex



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Currents - Currents interactions

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Matter Fields with a glo $\underline{(2)}$ s metry

in the fundamental repr

$i = 1, 2$ is the "color index"

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$$\phi = \begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}$$

complex fields (charged)

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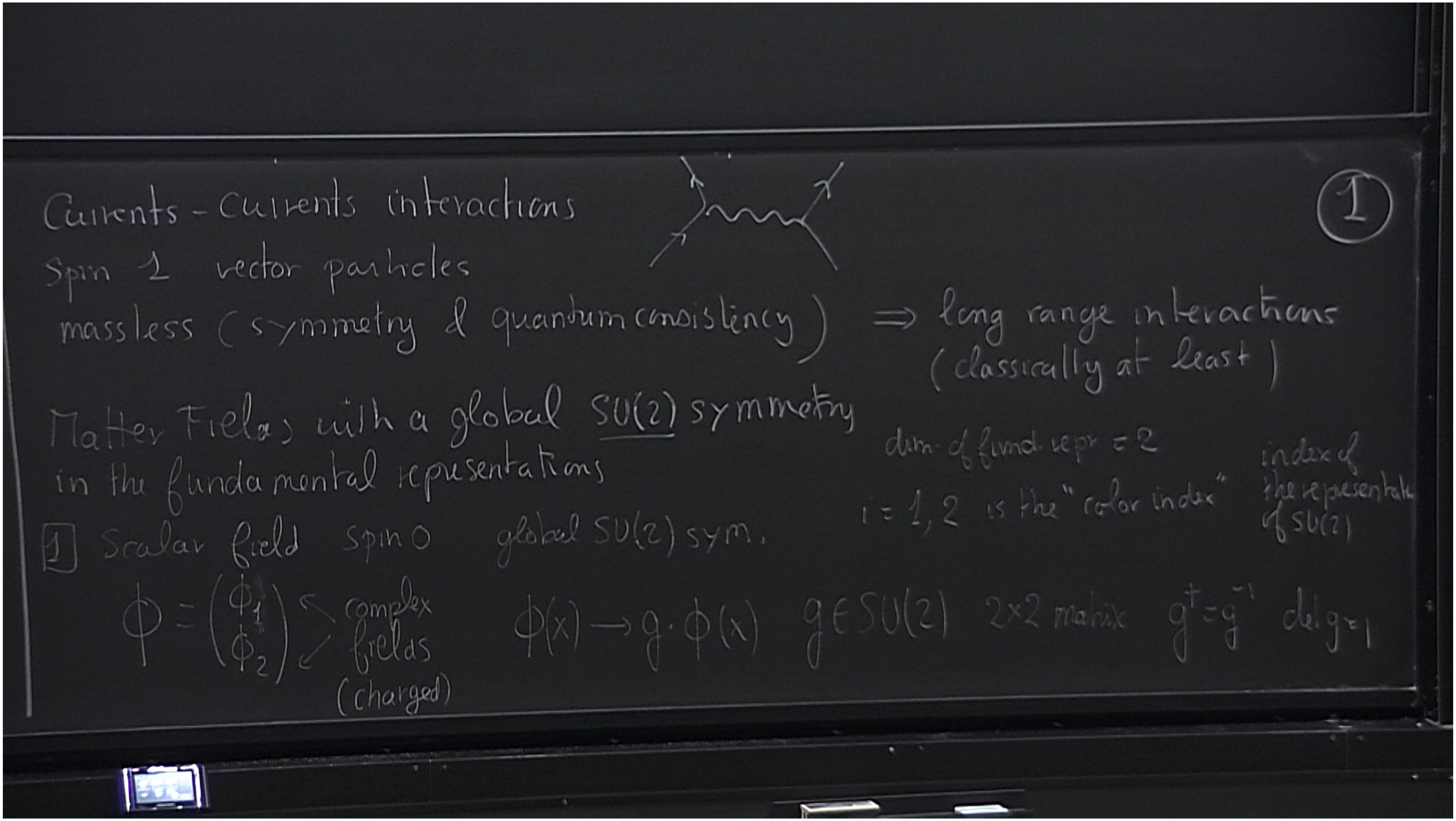

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Matter Fields with a global $\underline{SU(2)}$ symmetry in the fundamental representations

dim. of fund. repr = 2

$i = 1, 2$ is the "color index"

index of the representation of SU(2)

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$$\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$$

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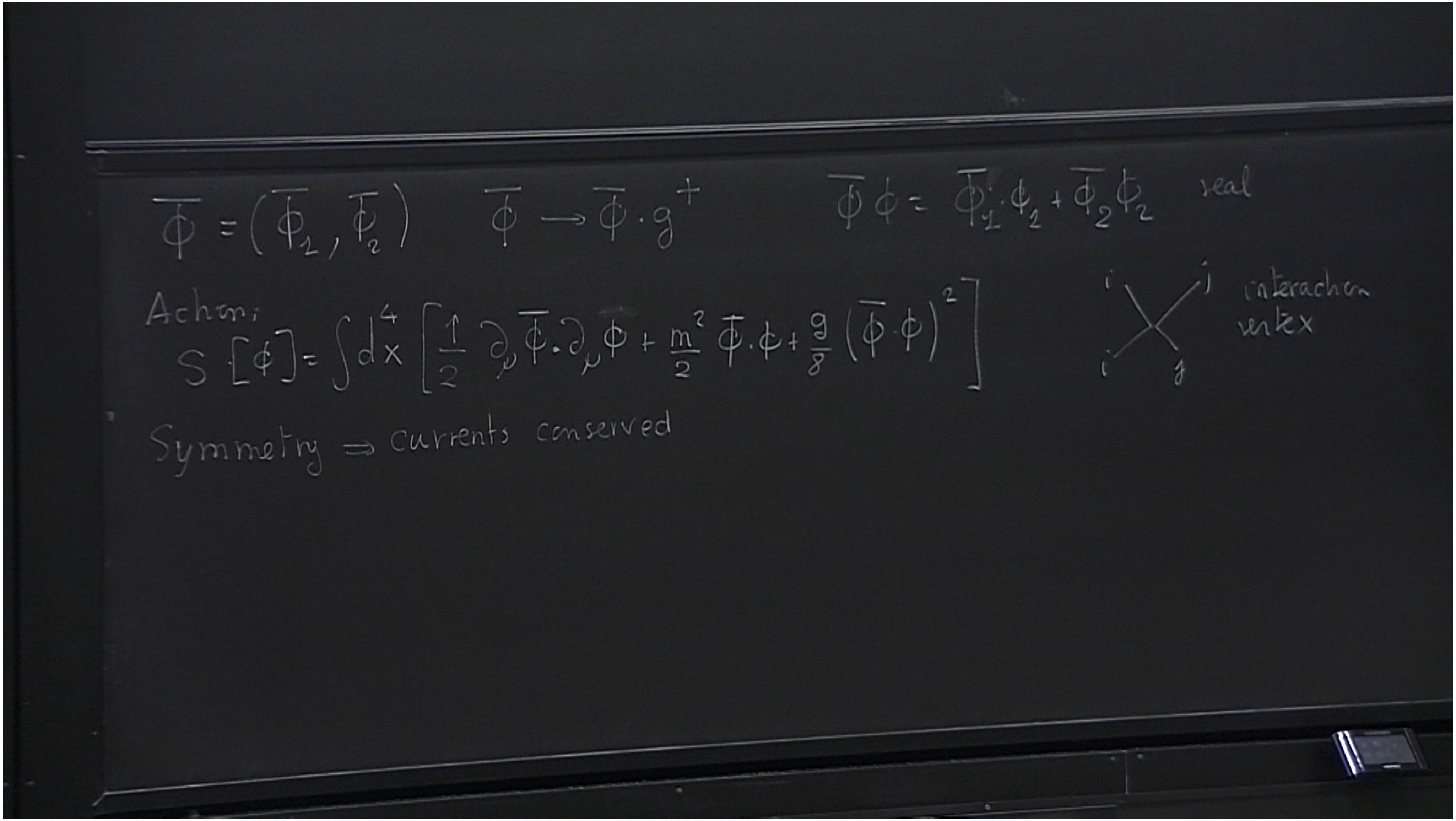

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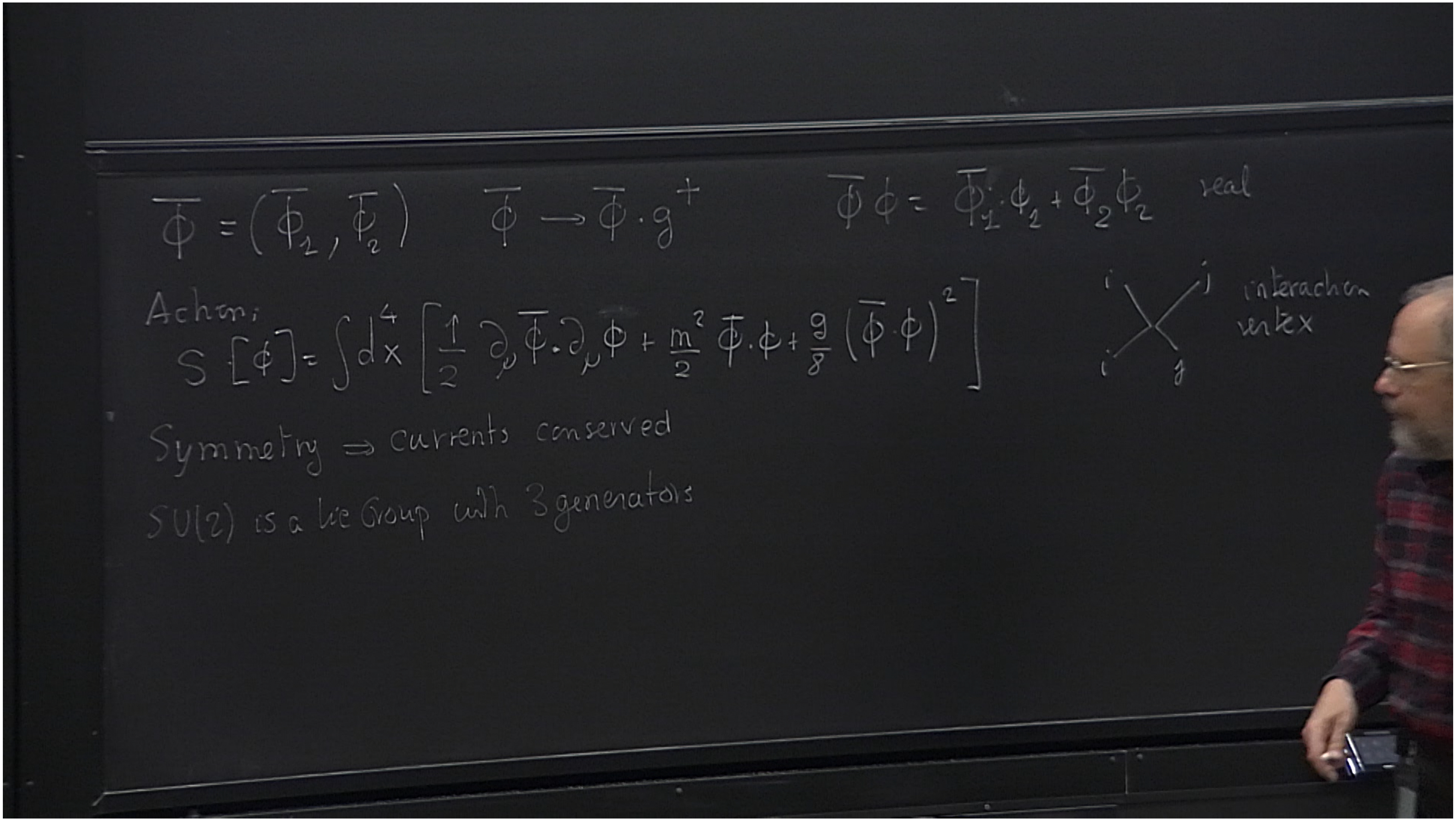
interaction vertex

Symmetry ⇒ currents conserved



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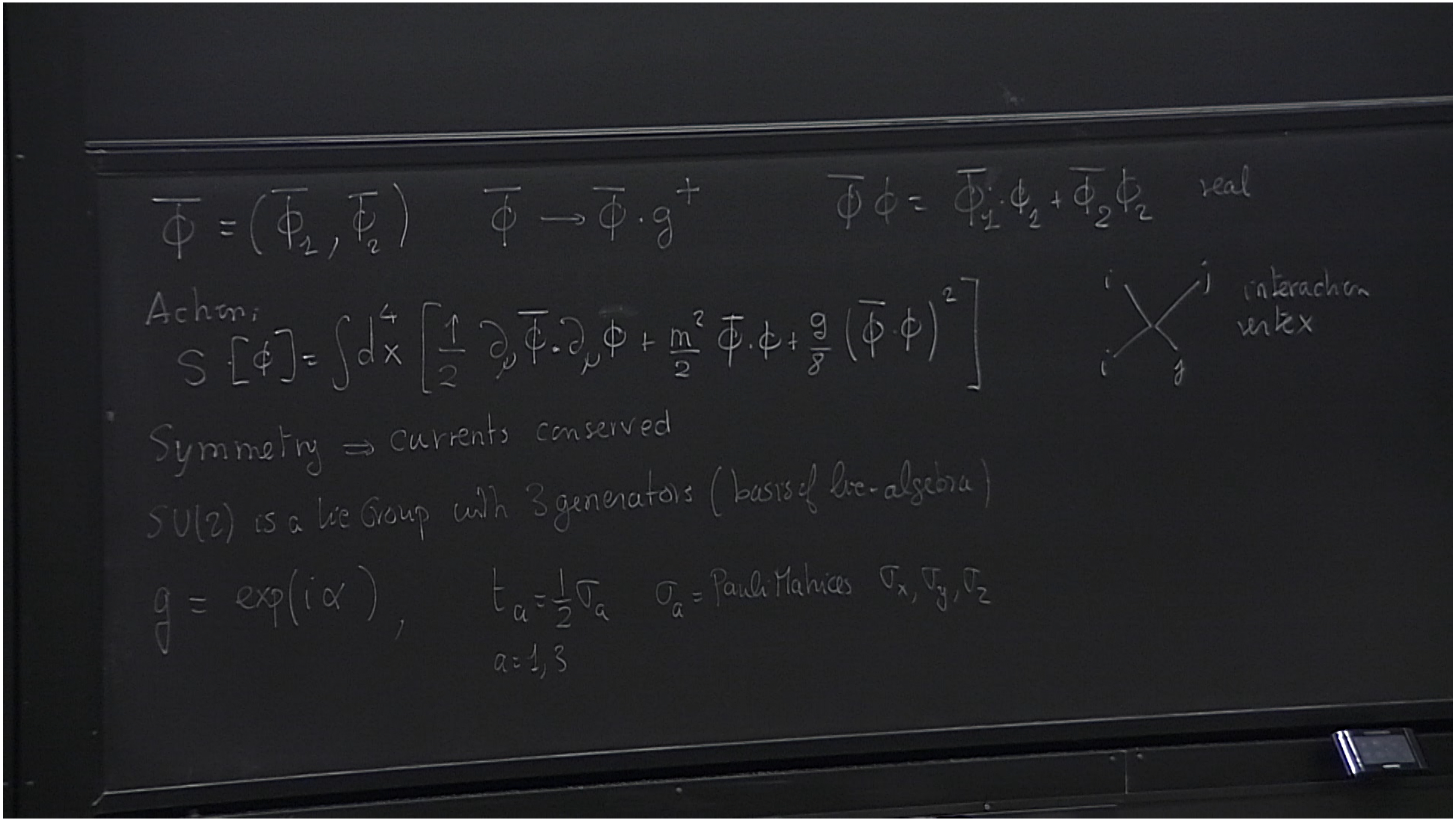
Symmetry $\Rightarrow$ currents conserved

SU(2) is a Lie Group with 3 generators



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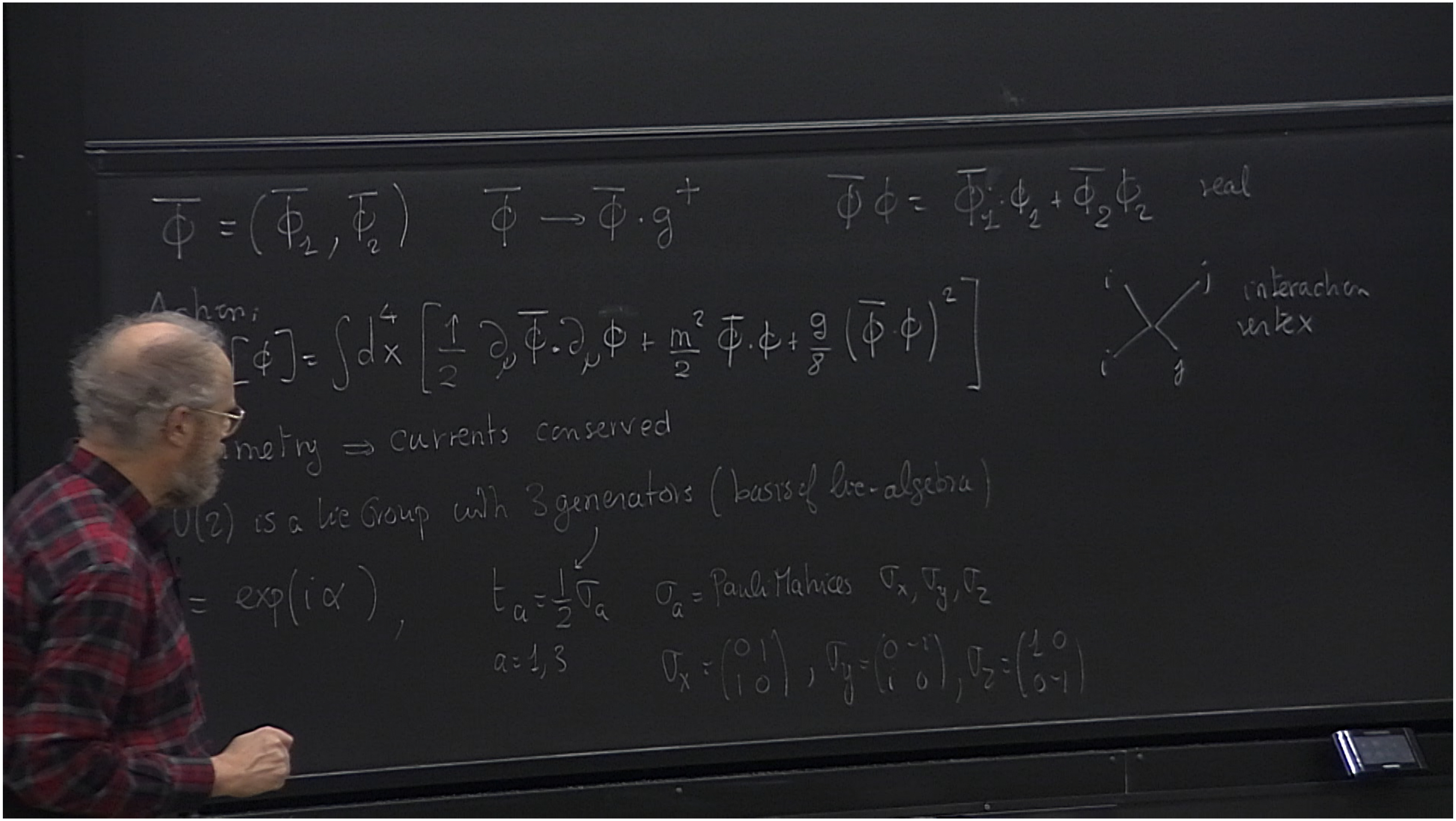
Symmetry ⇒ currents conserved

SU(2) is a Lie Group with 3 generators (basis of Lie-algebra)

$g = \exp(i\alpha)$, $t_a = \frac{1}{2}\sigma_a$ σ_a = Pauli Matrices $\sigma_x, \sigma_y, \sigma_z$

$a = 1, 3$



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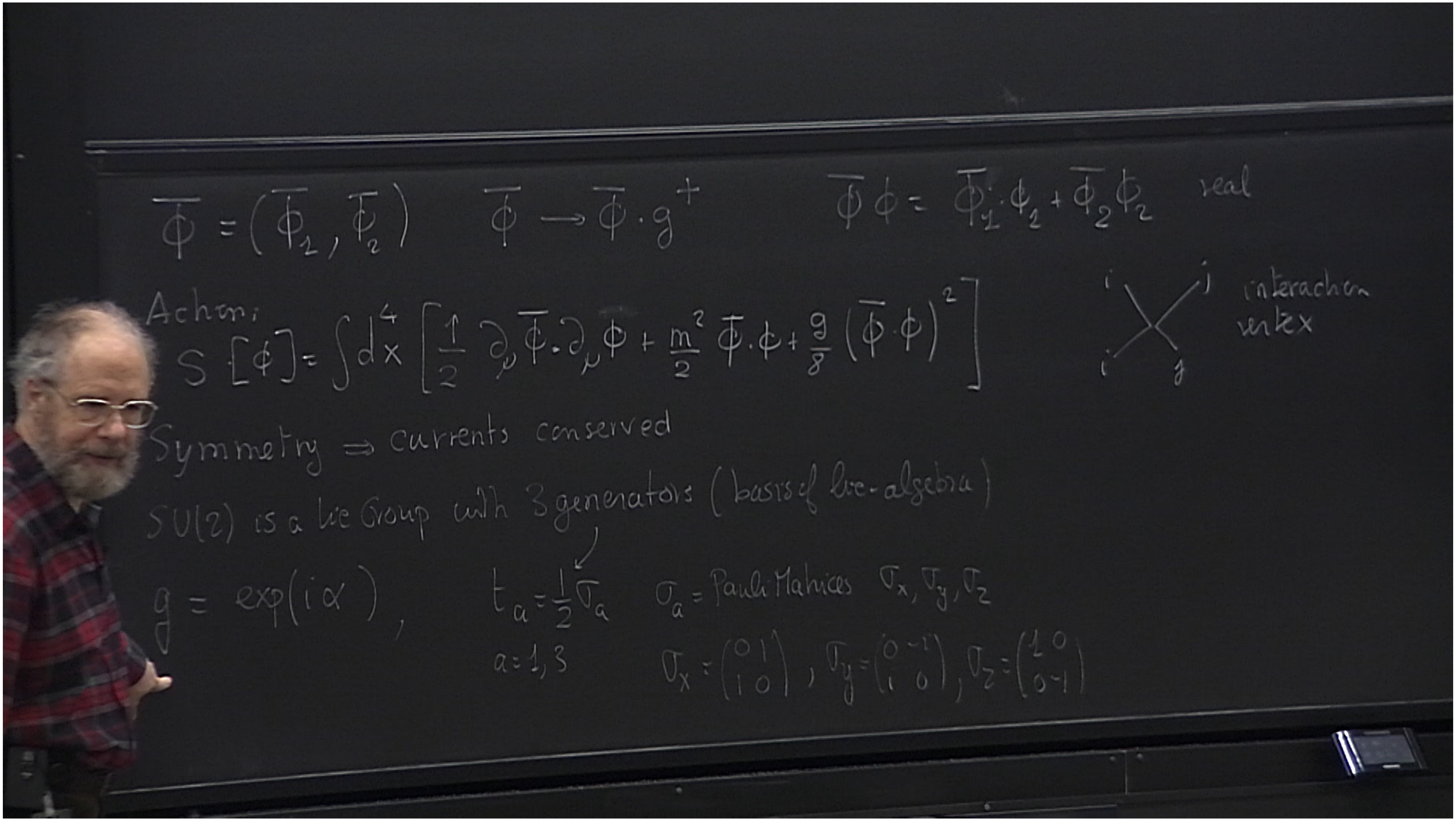
metry ⇒ currents conserved

U(2) is a Lie Group with 3 generators (basis of Lie-algebra)

$= \exp(i\alpha)$, $\quad t_a = \frac{1}{2}\sigma_a \quad \sigma_a$ = Pauli Matrices $\sigma_x, \sigma_y, \sigma_z$

$a = 1,3 \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$



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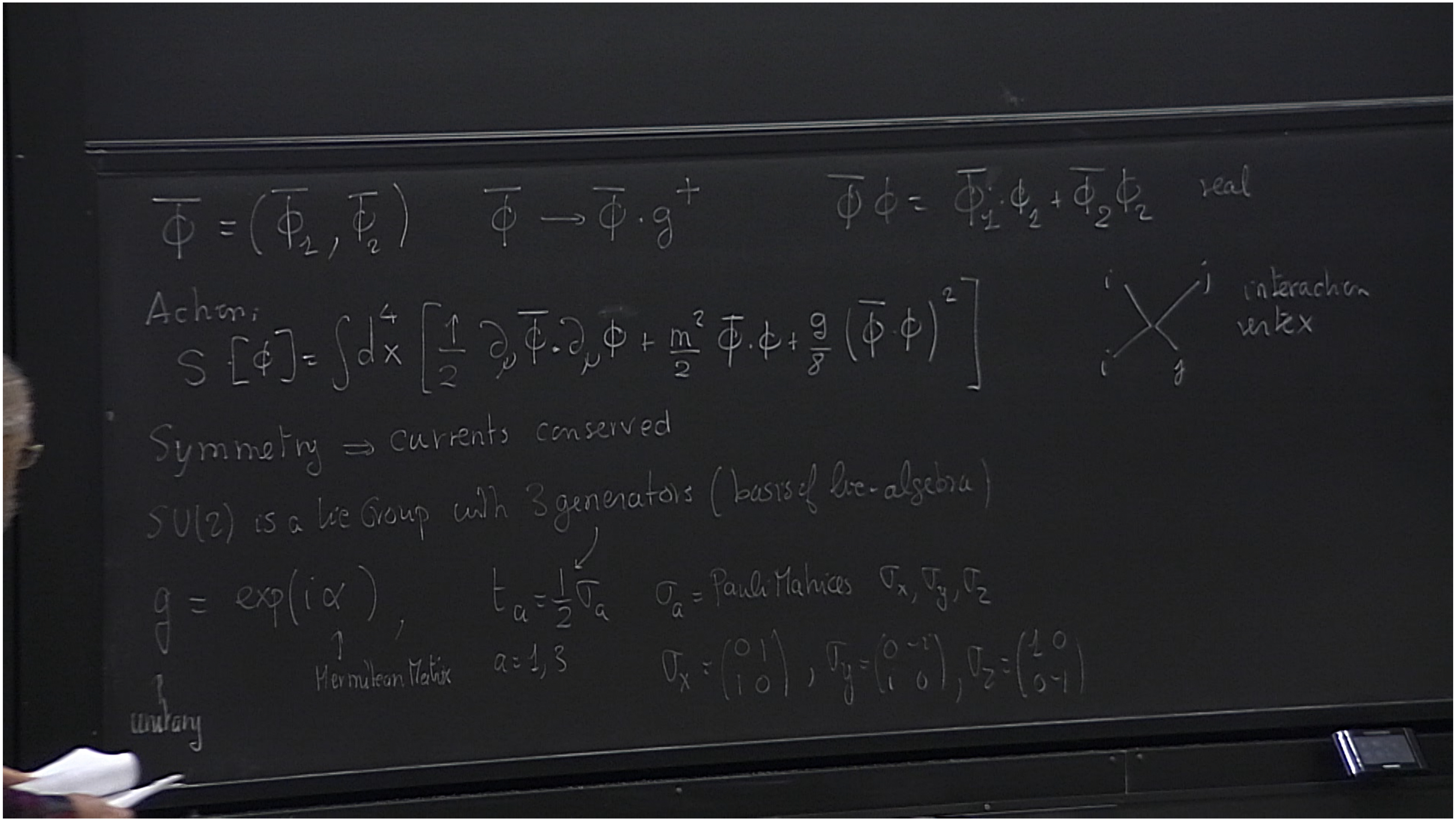

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↑ Hermitean Matrix $a = 1,3$

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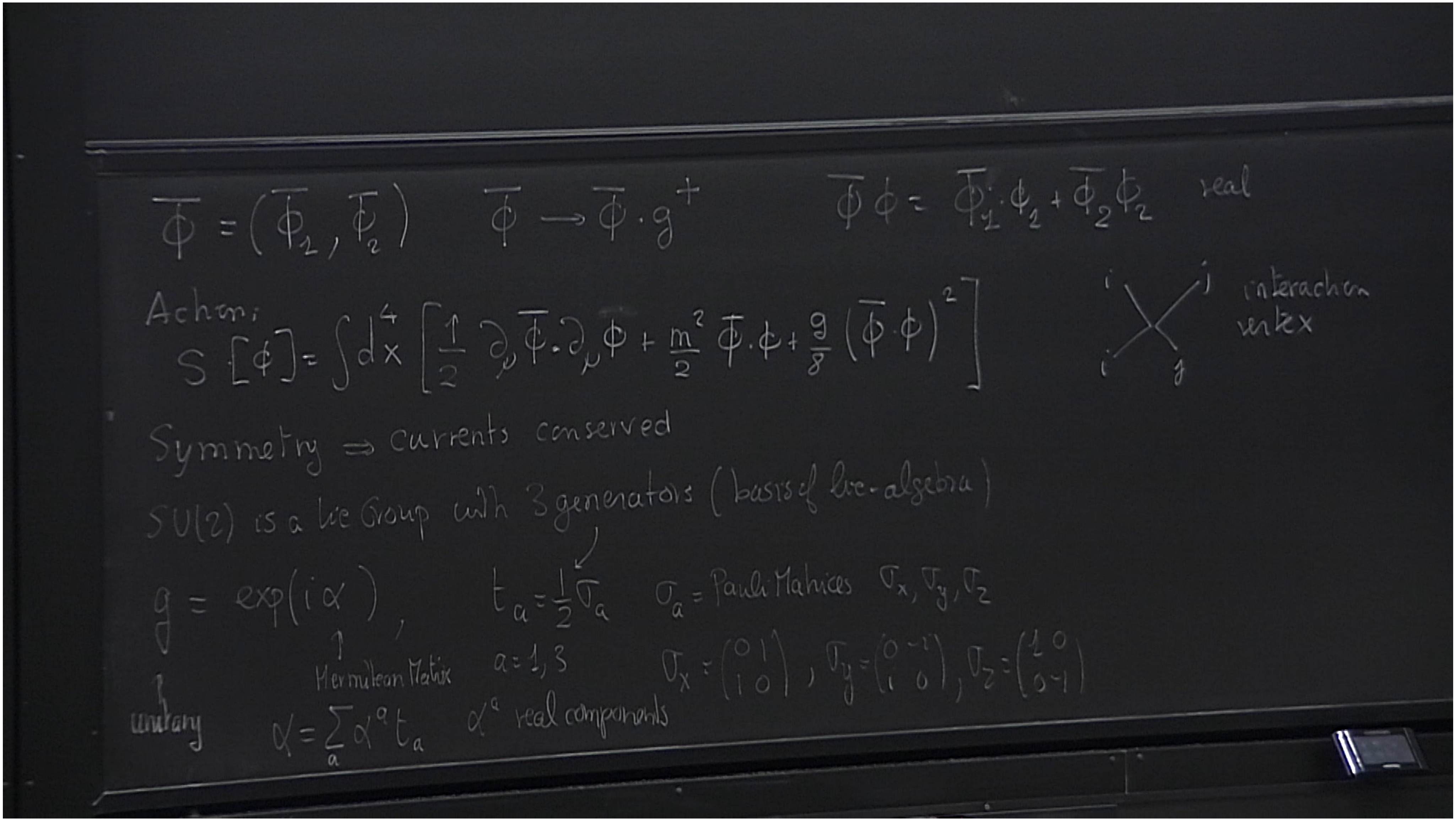
unitary



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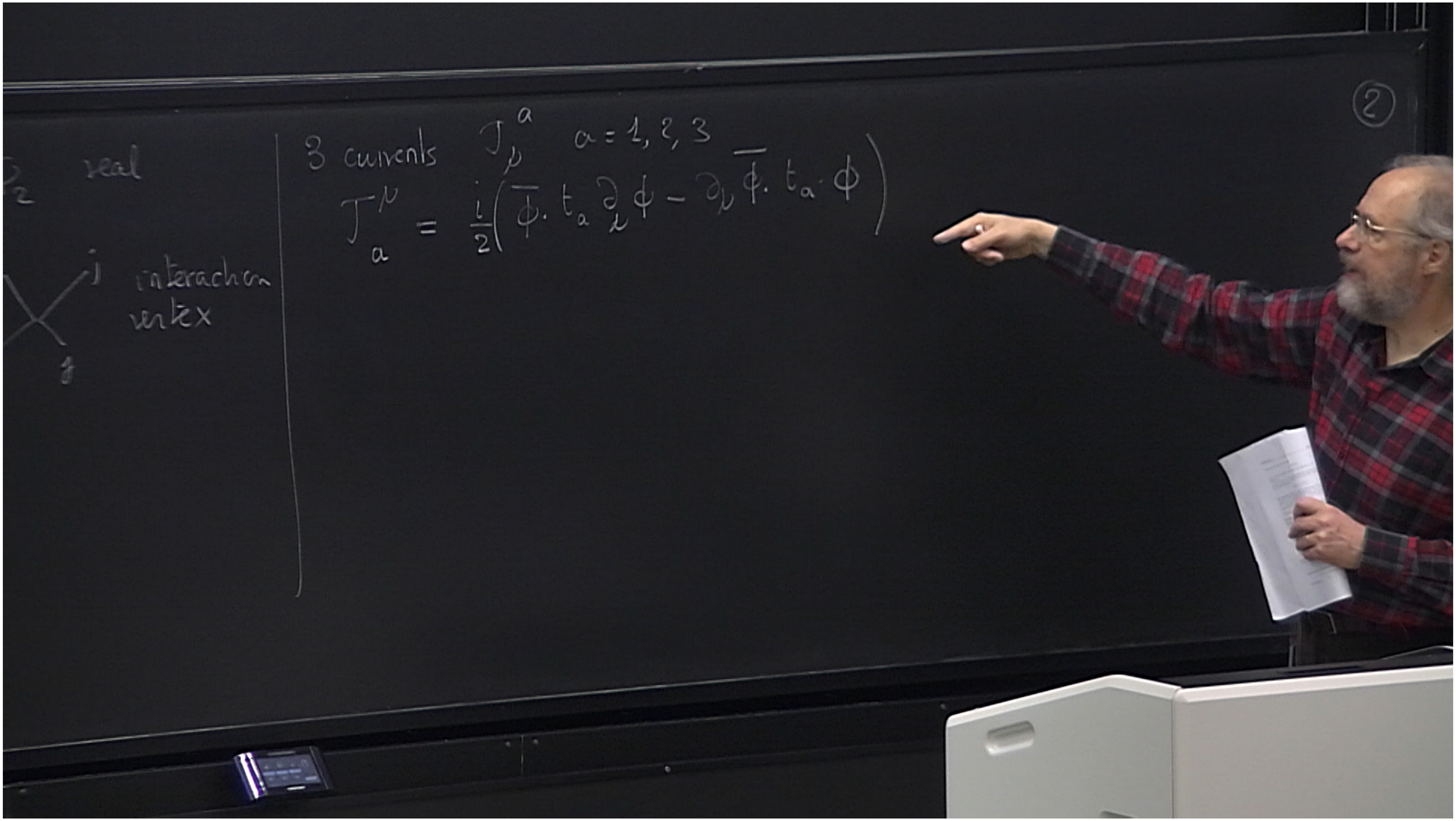
Hermitean Matrix $a = 1, 3$ $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

unitary $\alpha = \sum_a \alpha^a t_a$ α^a real components



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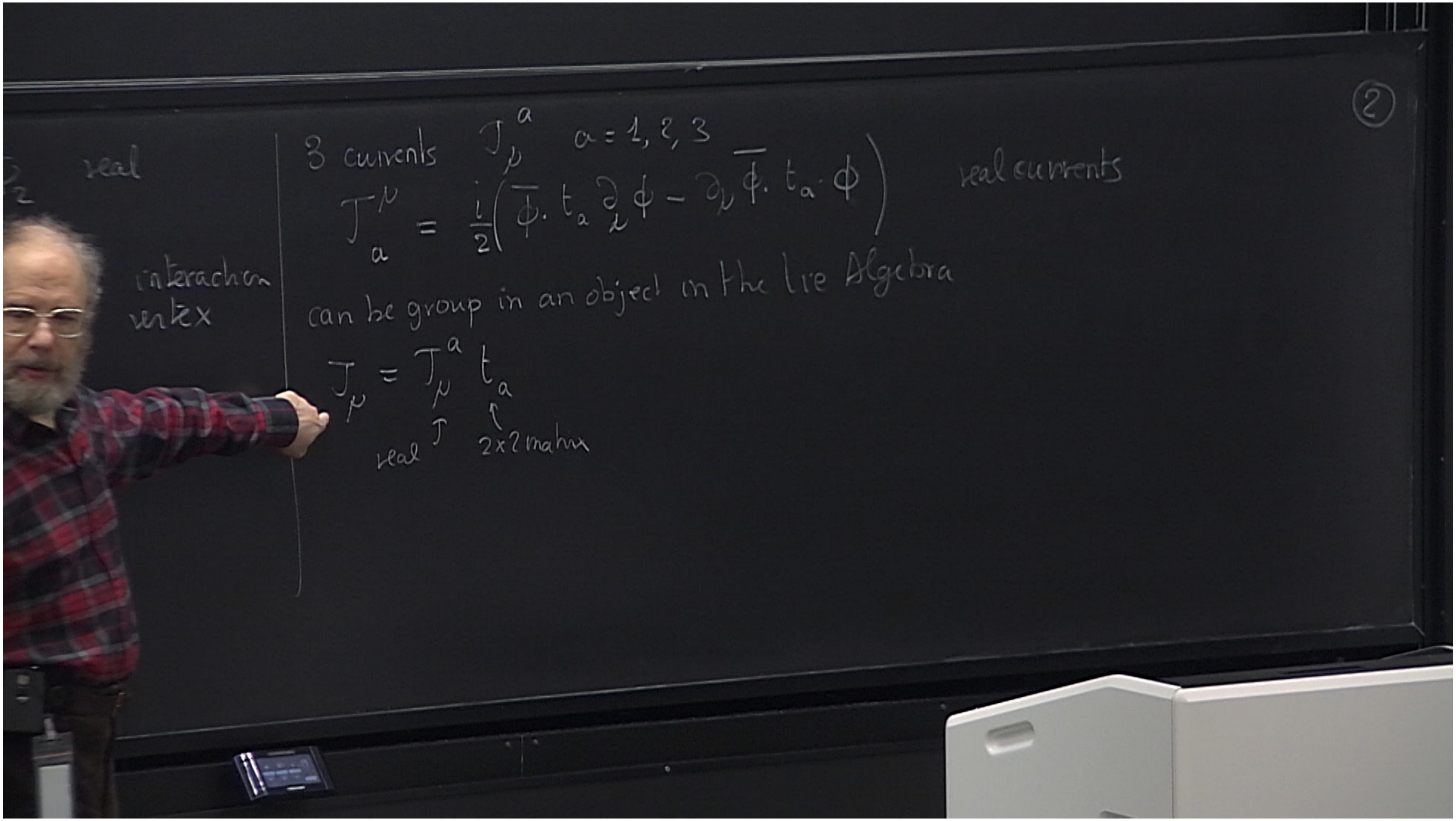
$_2$ real

interaction vertex

3 currents J_ν^a $a = 1, 2, 3$

$$J_a^\nu = \frac{i}{2}\left(\bar{\phi} \cdot t_a \partial_\nu \phi - \partial_\nu \bar{\phi} \cdot t_a \cdot \phi\right)$$



②

real

interaction vertex

3 currents J_μ^a $a = 1, 2, 3$

$$J^\mu_a = \frac{i}{2}\left(\bar{\phi}\cdot t_a \partial_\mu \phi - \partial_\mu \bar{\phi}\cdot t_a \cdot \phi\right)$$

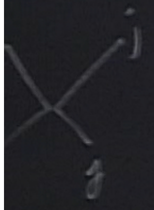
real currents

can be group in an object in the lie Algebra

$$J_\mu = \underset{\substack{\uparrow \\ \text{real}}}{J_\mu^a}\ \underset{\substack{\uparrow \\ 2\times 2 \text{ matrix}}}{t_a}$$

②

$_2$ real



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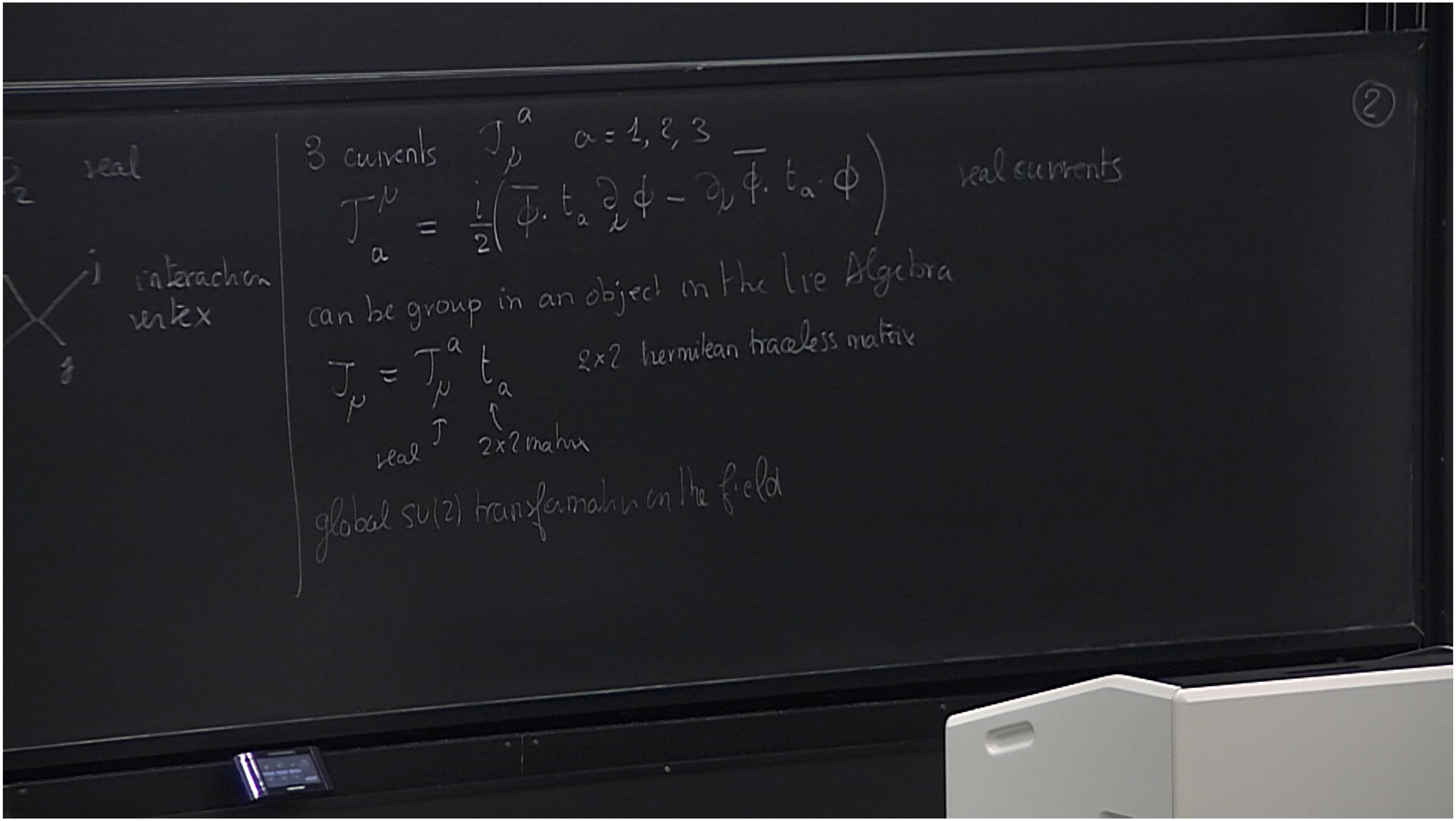
2×2 hermitean traceless matrix



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real currents

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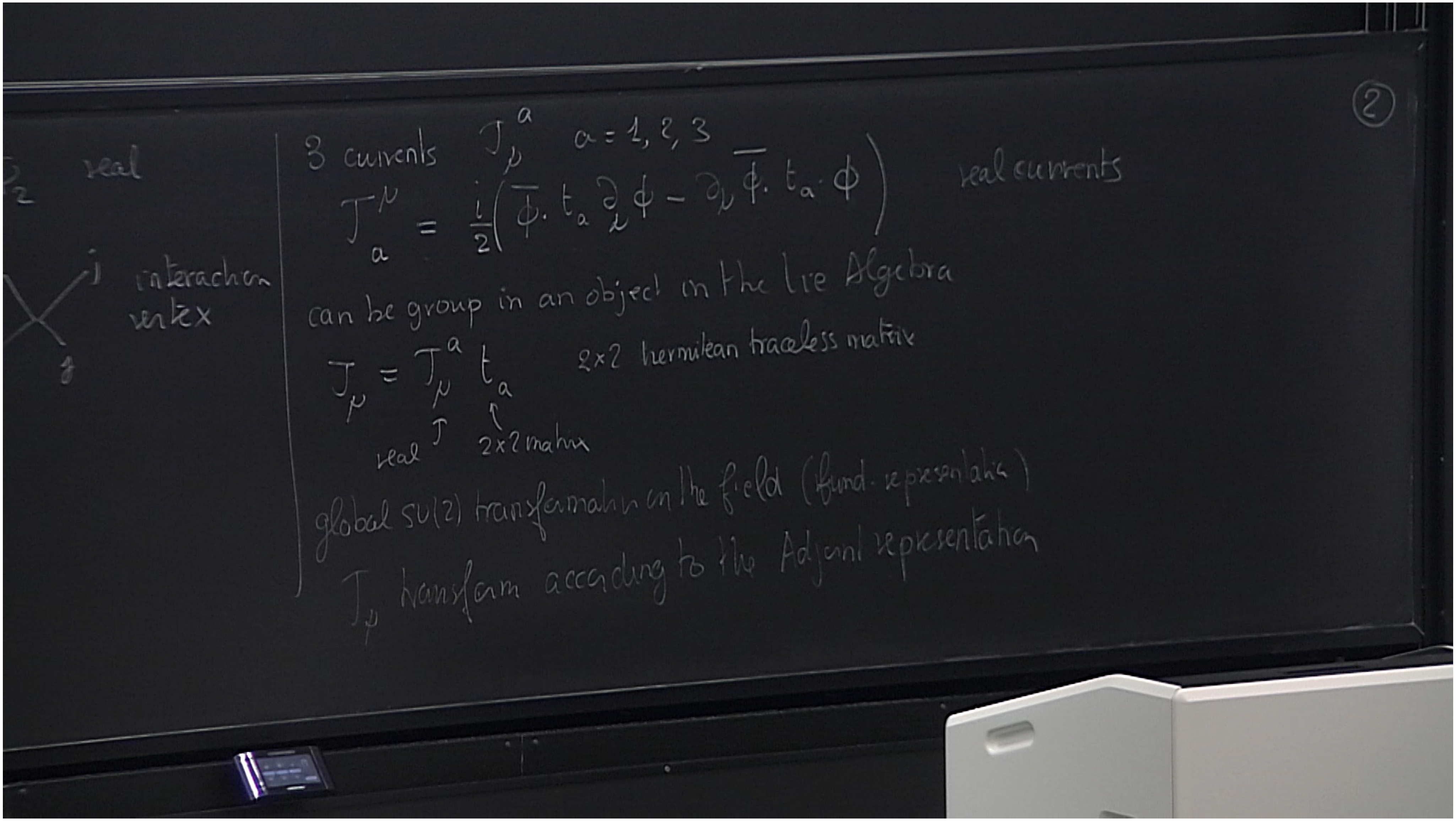
$$J_\mu = \underset{\text{real}}{J_\mu^a}\ \underset{\text{2x2 matrix}}{t_a}$$

2x2 hermitean traceless matrix

global SU(2) transformation on the field



②

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3 currents J_μ^a $a = 1, 2, 3$

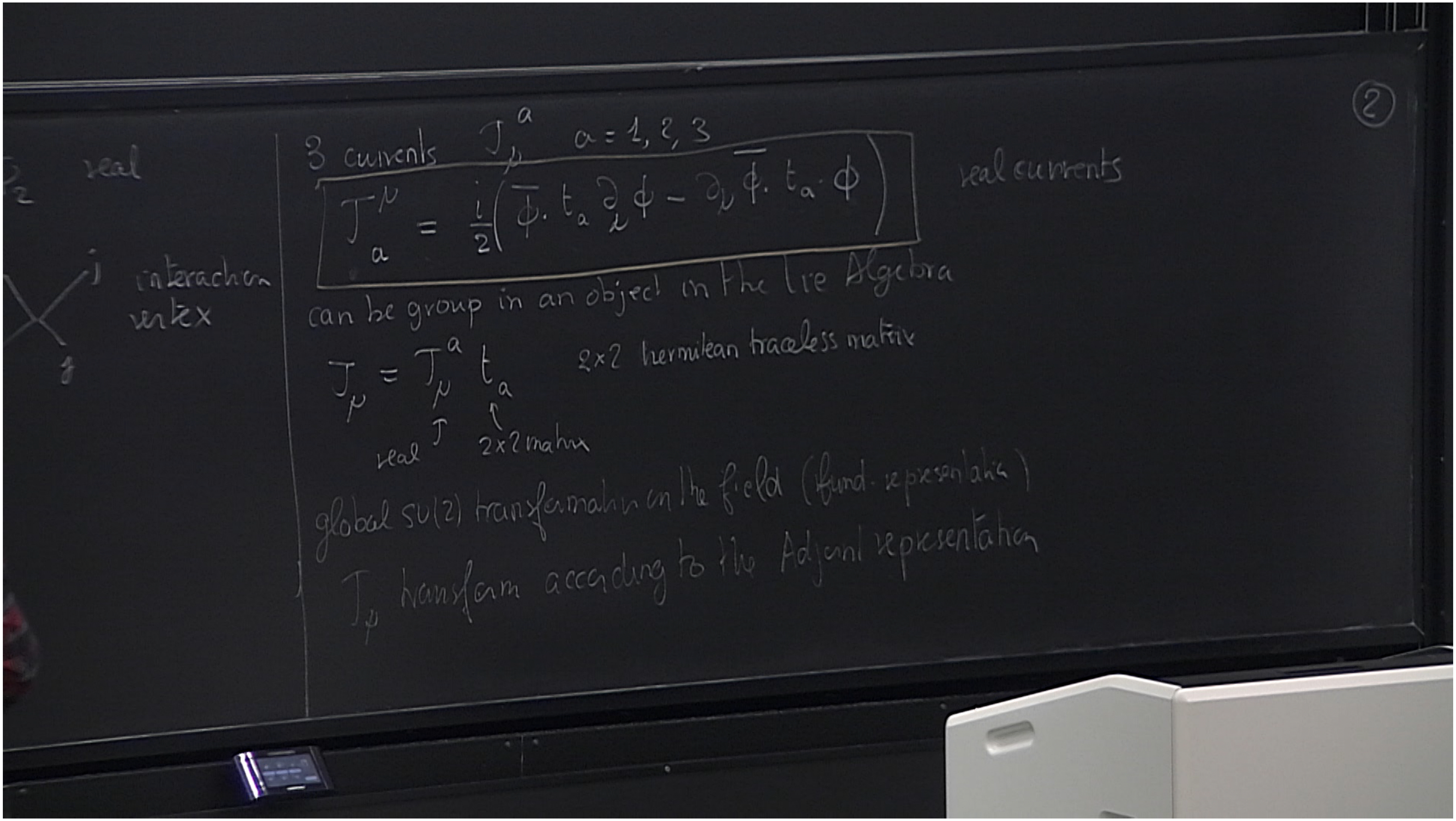
$$J_a^\nu = \frac{i}{2}\left(\bar{\phi}\cdot t_a \partial_\nu \phi - \partial_\nu \bar{\phi}\cdot t_a \cdot \phi\right)$$ real currents

can be group in an object in the lie Algebra

$$J_\mu = \underset{\substack{\uparrow \\ \text{real}}}{J_\mu^a}\ \underset{\substack{\uparrow \\ 2\times 2 \text{ matrix}}}{t_a}$$ 2×2 hermitean traceless matrix

global SU(2) transformation on the field (fund. representation)

J_μ transform according to the Adjoint representation



ϕ_2 real

interaction vertex

3 currents J_μ^a $a = 1, 2, 3$

$$\boxed{J^\nu_a = \frac{i}{2}\left(\bar{\phi}.\, t_a \partial_\nu \phi - \partial_\nu \bar{\phi}.\, t_a \cdot \phi\right)}$$

real currents

can be group in an object in the Lie Algebra

$$J_\mu = \underset{\text{real}}{J_\mu^a}\ \underset{2\times 2 \text{ matrix}}{t_a}$$

2×2 hermitean traceless matrix

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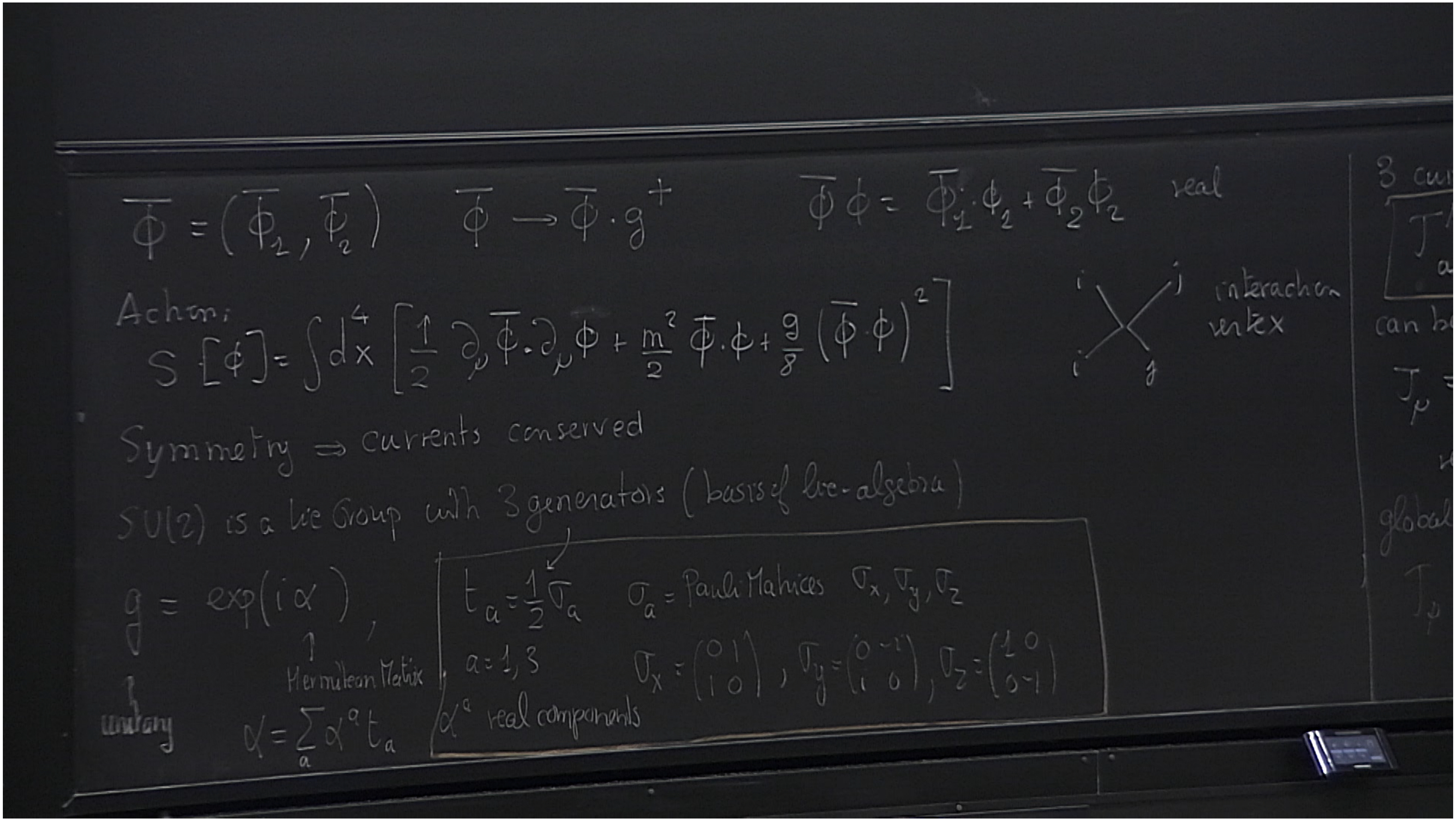

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interaction vertex

Symmetry ⇒ currents conserved

SU(2) is a Lie Group with 3 generators (basis of Lie-algebra)

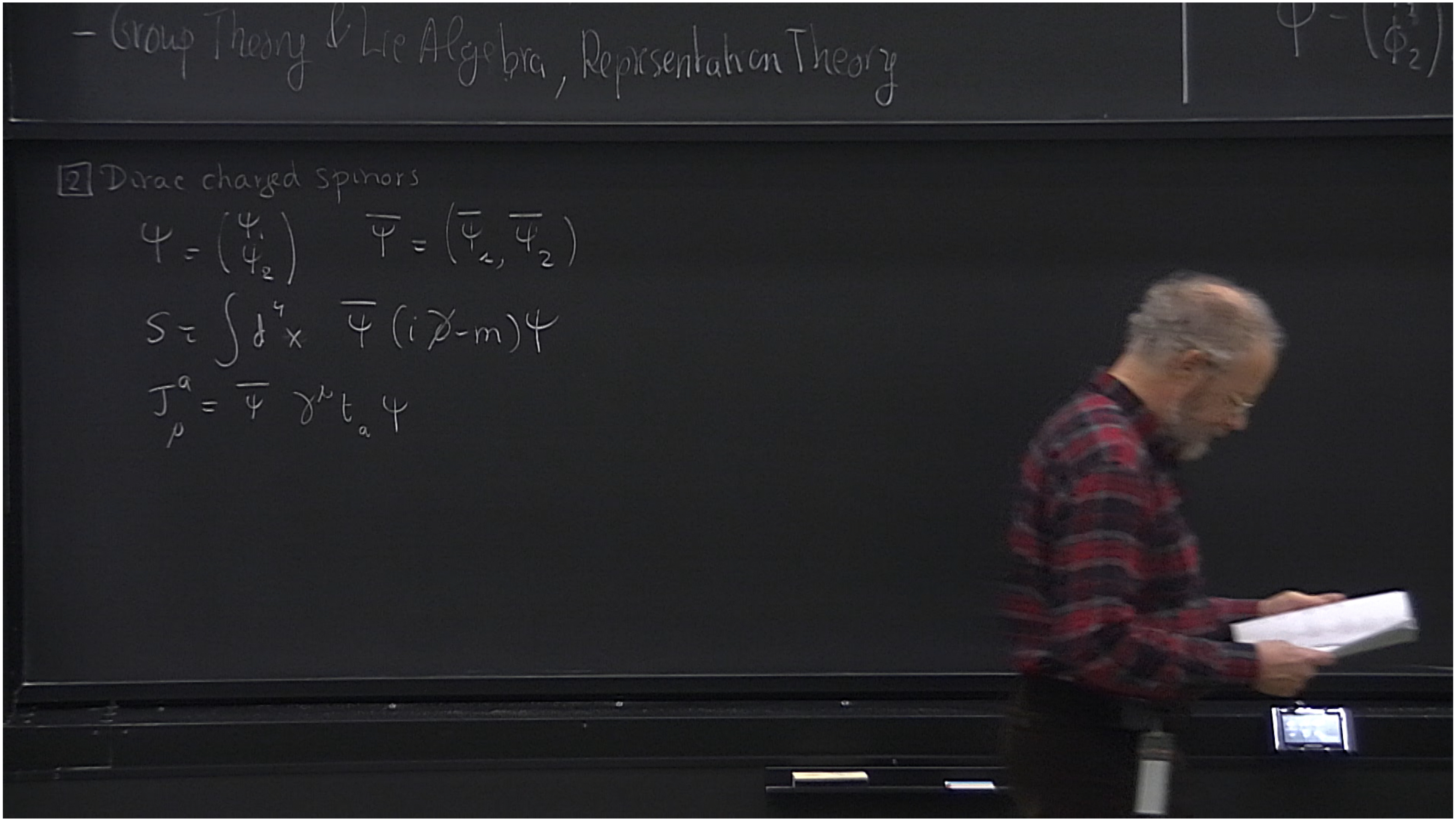
$g = \exp(i\alpha)$, unitary, Hermitean Matrix

$\alpha = \sum_a \alpha^a t_a$

$t_a = \frac{1}{2} \sigma_a$ σ_a = Pauli Matrices $\sigma_x, \sigma_y, \sigma_z$

$a = 1, 3$ $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

α^a real components



– Group Theory & Lie Algebra, Representation Theory

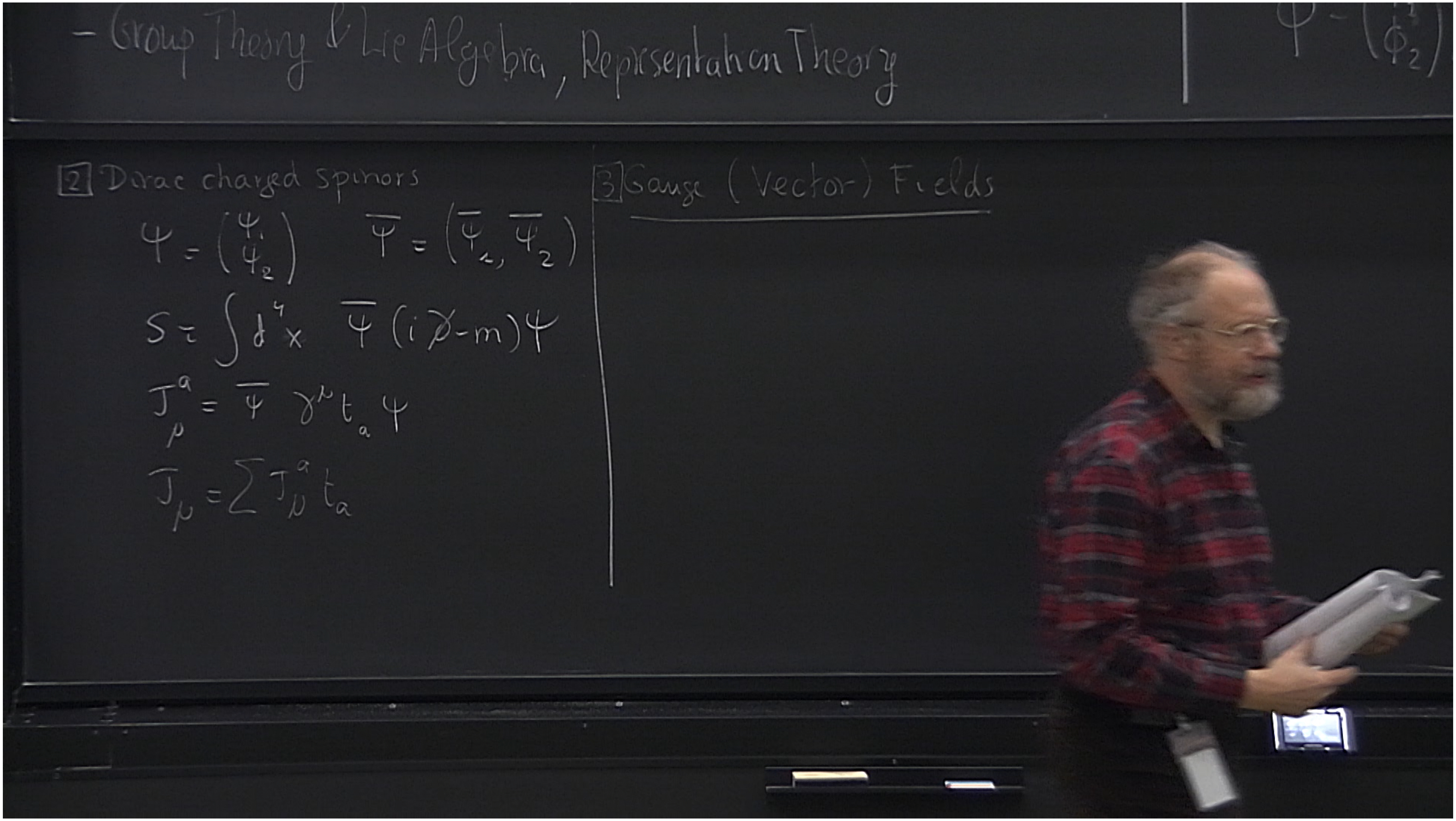
$\phi = \begin{pmatrix} \\ \phi_2 \end{pmatrix}$

$\boxed{2}$ Dirac charged spinors

$$\Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} \qquad \overline{\Psi} = \left(\overline{\Psi}_1, \overline{\Psi}_2 \right)$$

$$S = \int d^4x \ \overline{\Psi} \left(i \not{\partial} - m \right) \Psi$$

$$J^a_\mu = \overline{\Psi} \ \gamma^\mu t_a \Psi$$



- Group Theory & Lie Algebra, Representation Theory

$\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$

$\boxed{2}$ Dirac charged spinors

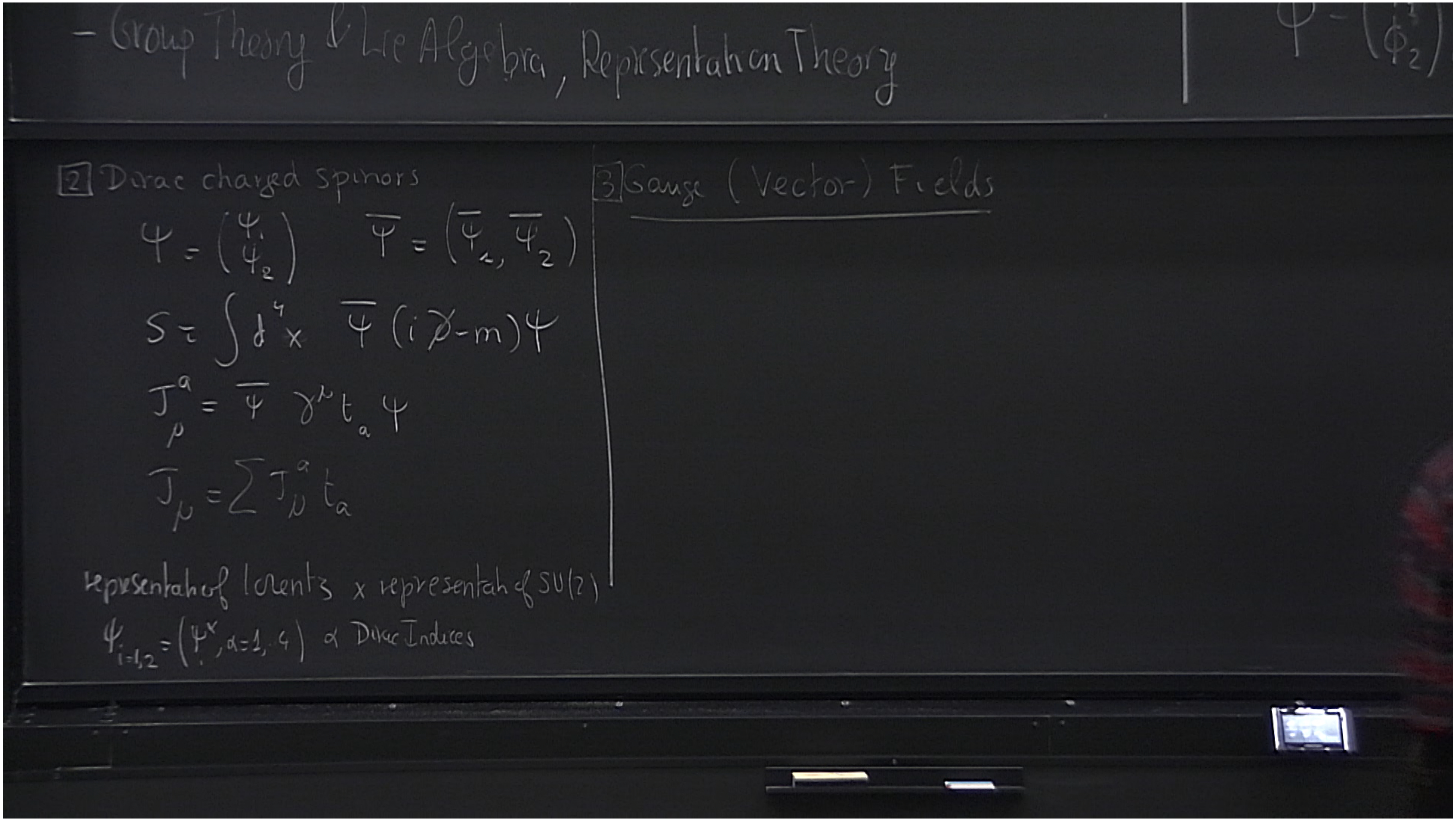
$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \qquad \overline{\Psi} = \left(\overline{\psi}_1, \overline{\psi}_2 \right)$$

$$S = \int d^4x \; \overline{\Psi} (i \not{\partial} - m) \Psi$$

$$J^a_\mu = \overline{\Psi} \; \gamma^\mu t_a \Psi$$

$$J_\mu = \sum J^a_\mu t_a$$

$\boxed{3}$ Gauge (Vector) Fields



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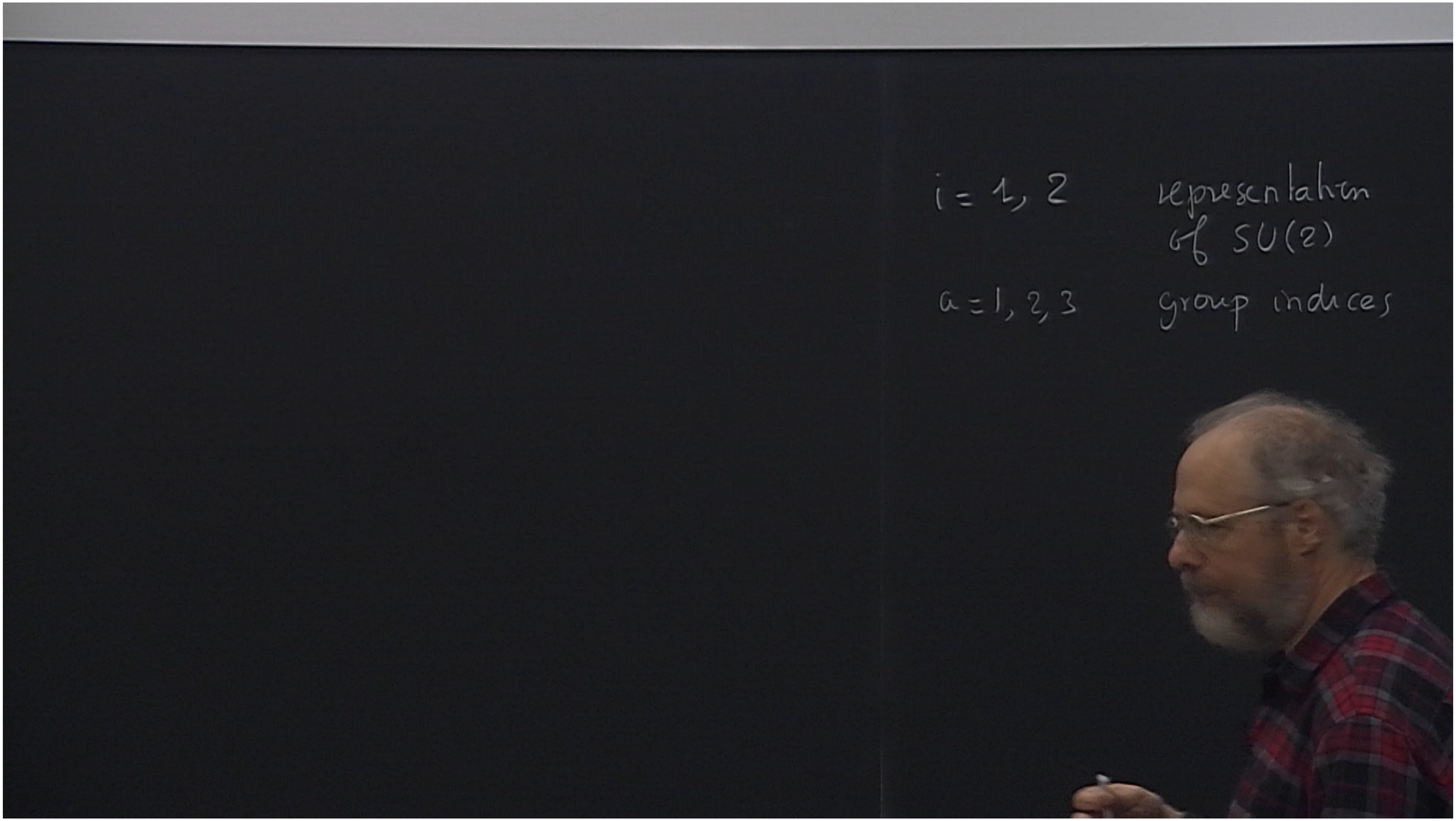
representat of Lorentz x representat of SU(2)

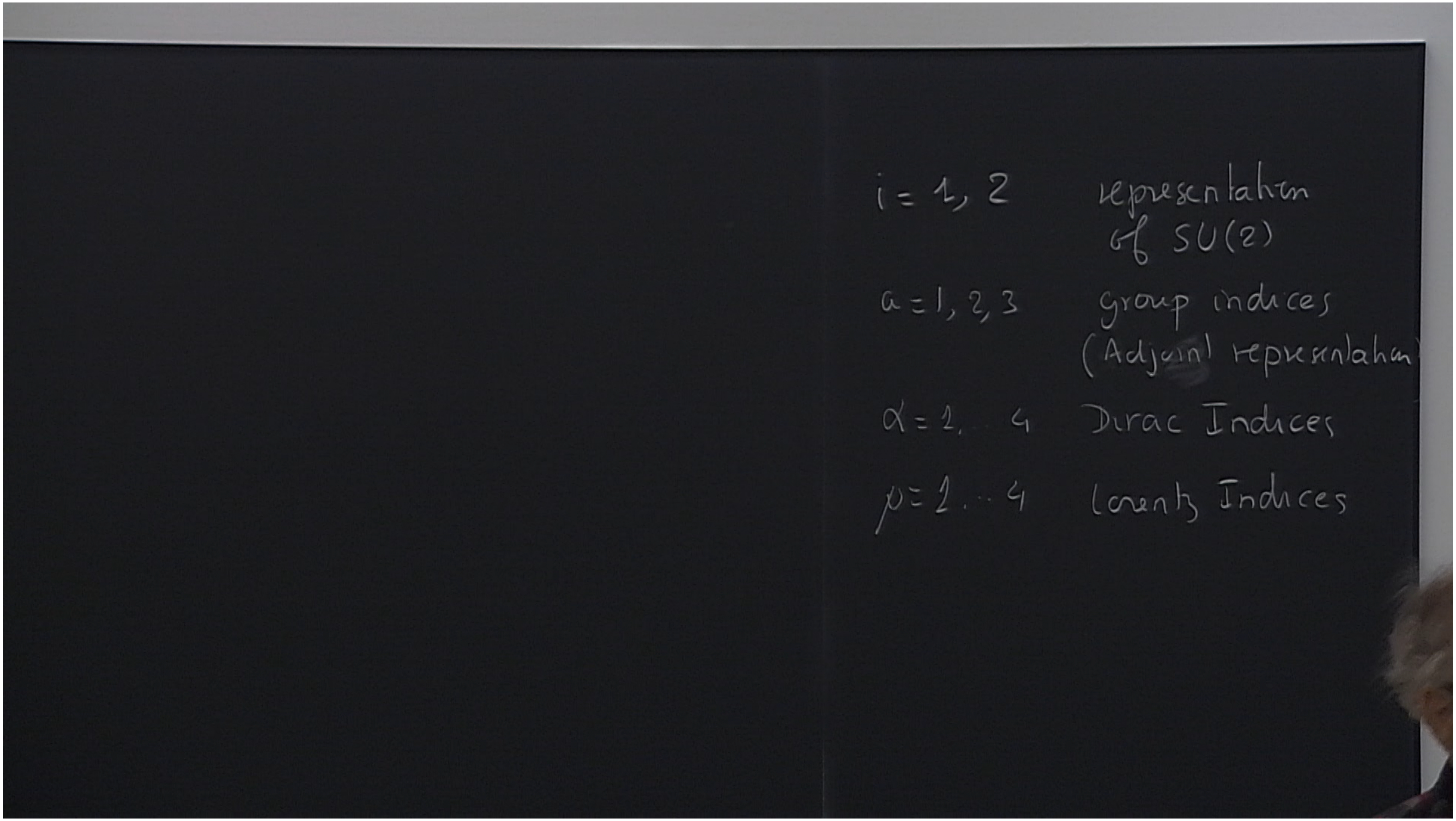
$\Psi_{i=1,2} = \left(\Psi^\alpha_i , \alpha = 1, \cdot 4 \right)$ α Dirac Indices

$\boxed{3}$ Gauge (Vector) Fields



$i = 1, 2$ representation of $SU(2)$

$a = 1, 2, 3$ group indices

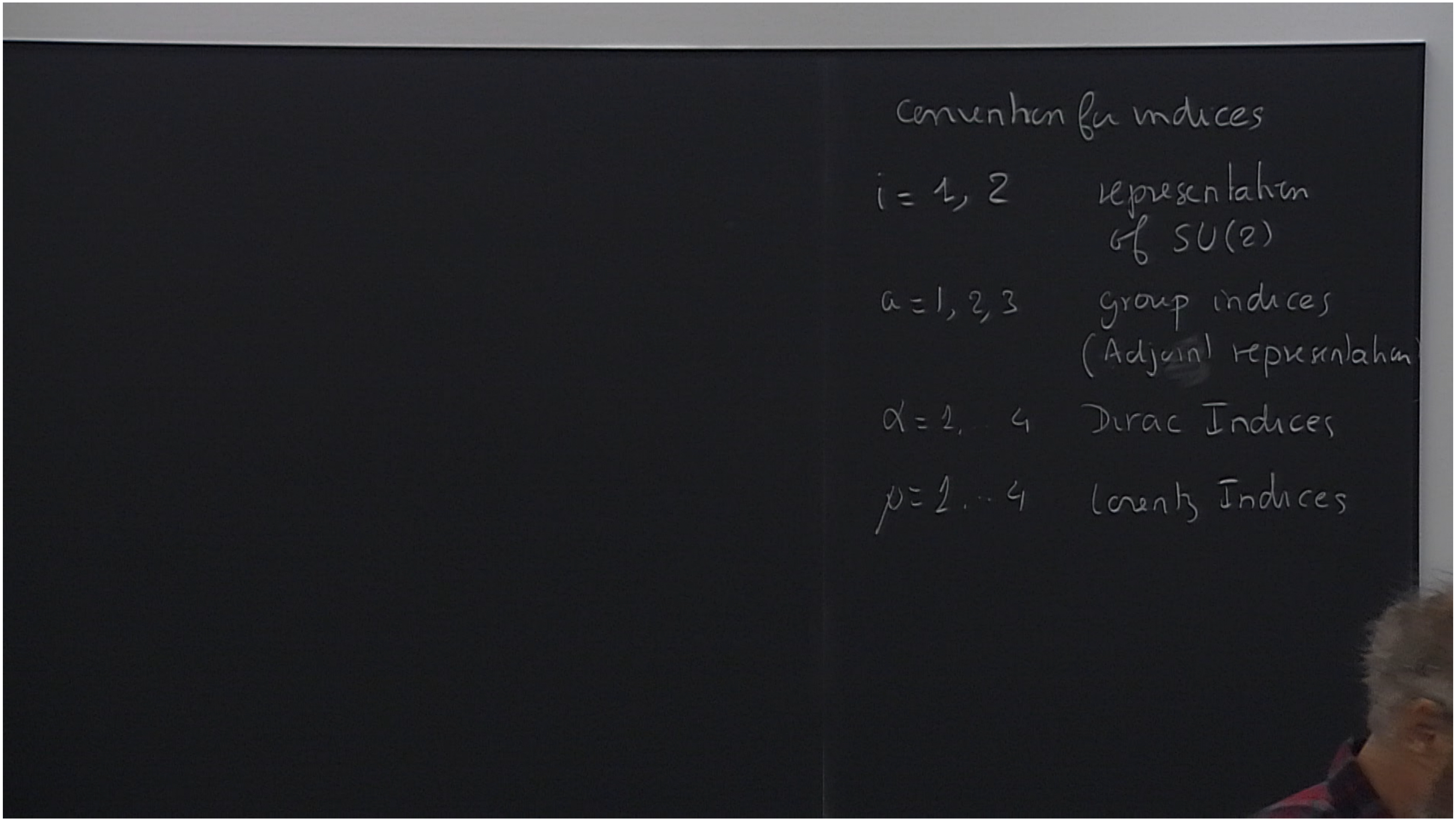


$i = 1, 2$ representation of SU(2)

$a = 1, 2, 3$ group indices (Adjoint representation)

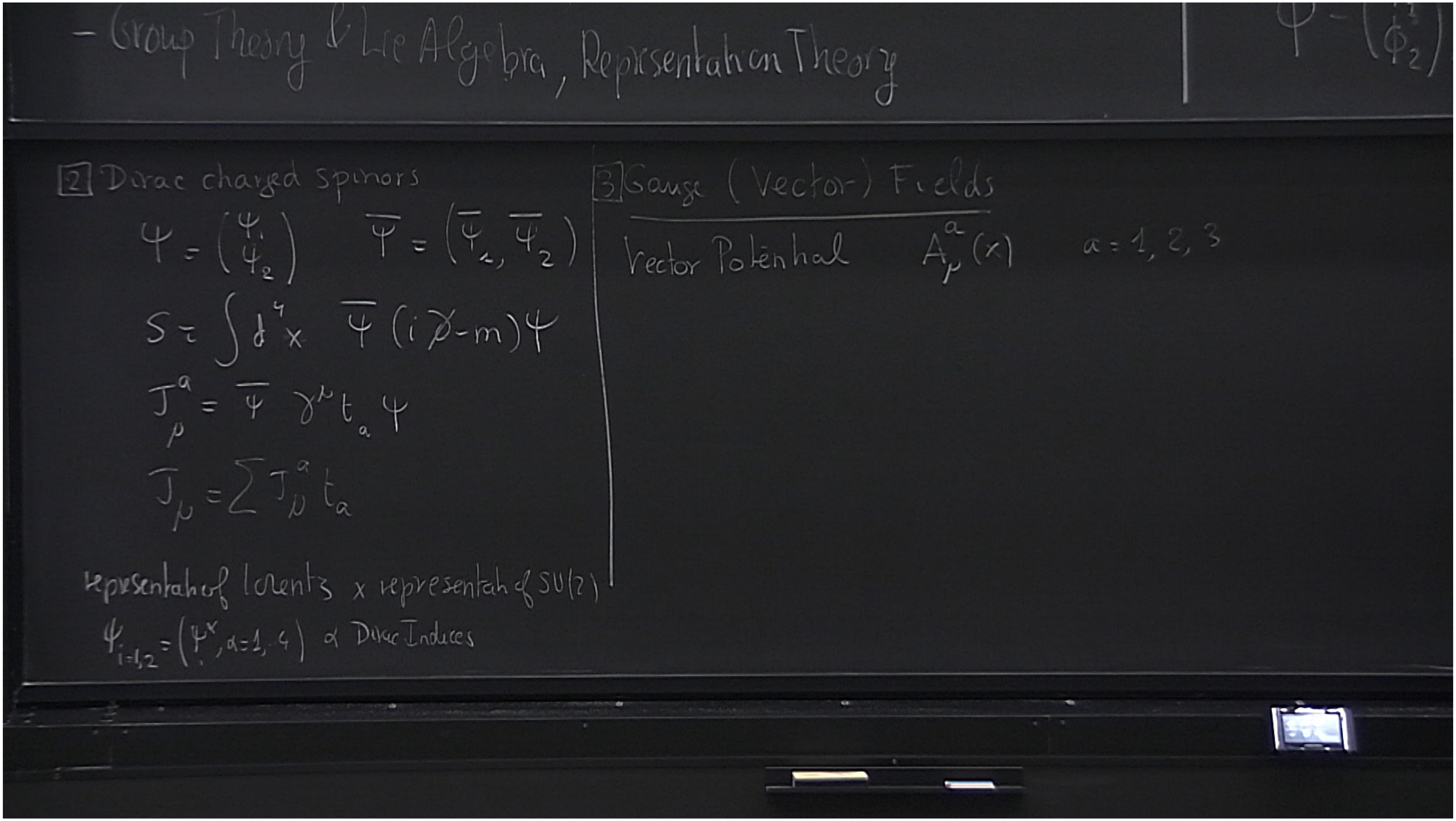
$\alpha = 1, \dots 4$ Dirac Indices

$\mu = 1 \dots 4$ Lorentz Indices



Convention for indices

$i = 1, 2$	representation of SU(2)
$a = 1, 2, 3$	group indices (Adjoint representation)
$\alpha = 1, .. 4$	Dirac Indices
$\mu = 1 \cdots 4$	Lorentz Indices



- Group Theory & Lie Algebra, Representation Theory

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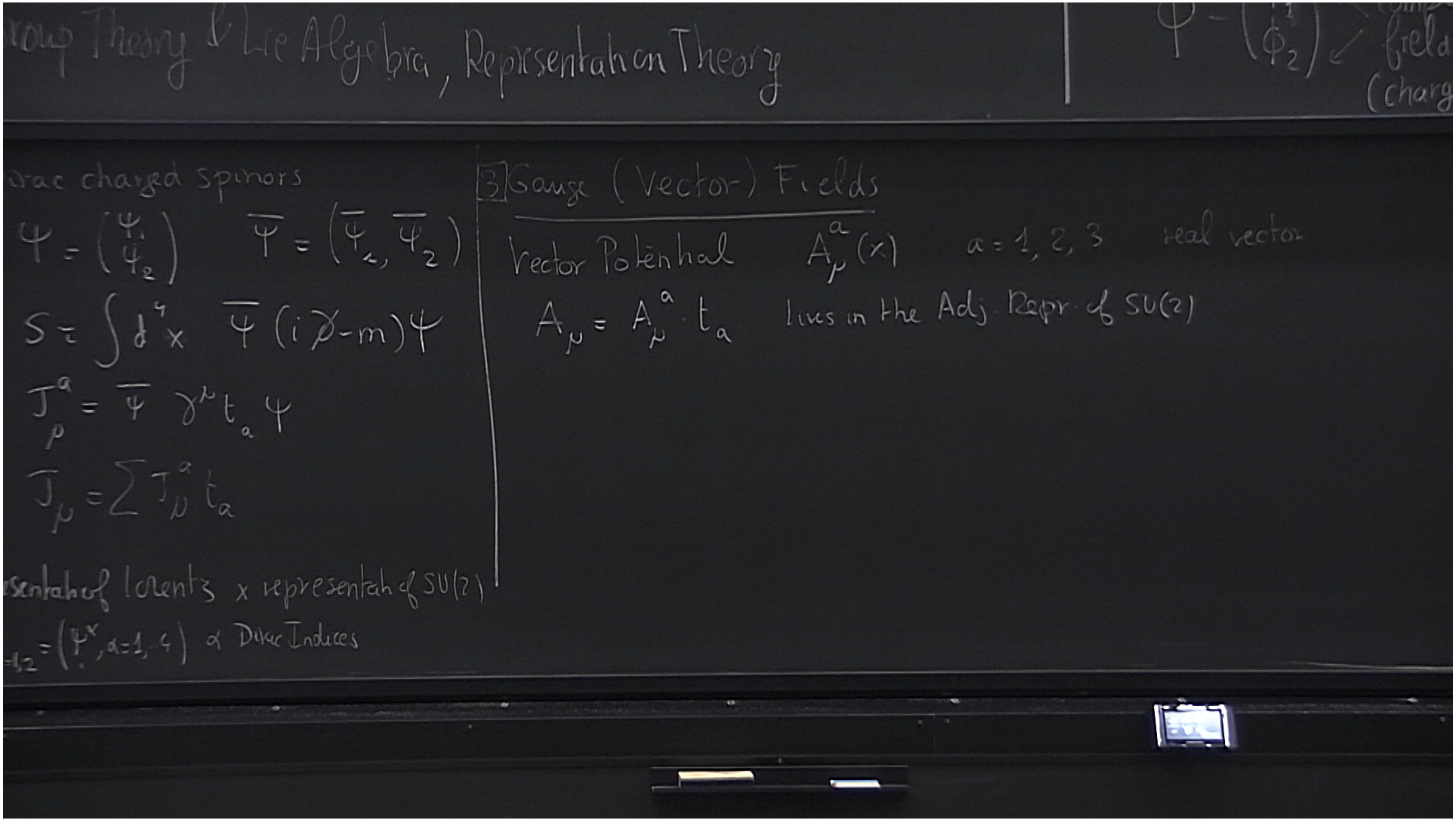
$\boxed{3}$ Gauge (Vector) Fields

Vector Potential $A^a_\mu(x)$ $a = 1, 2, 3$



roup Theory & Lie Algebra, Representation Theory

$\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$ ↙ field (charg

irac charged spinors

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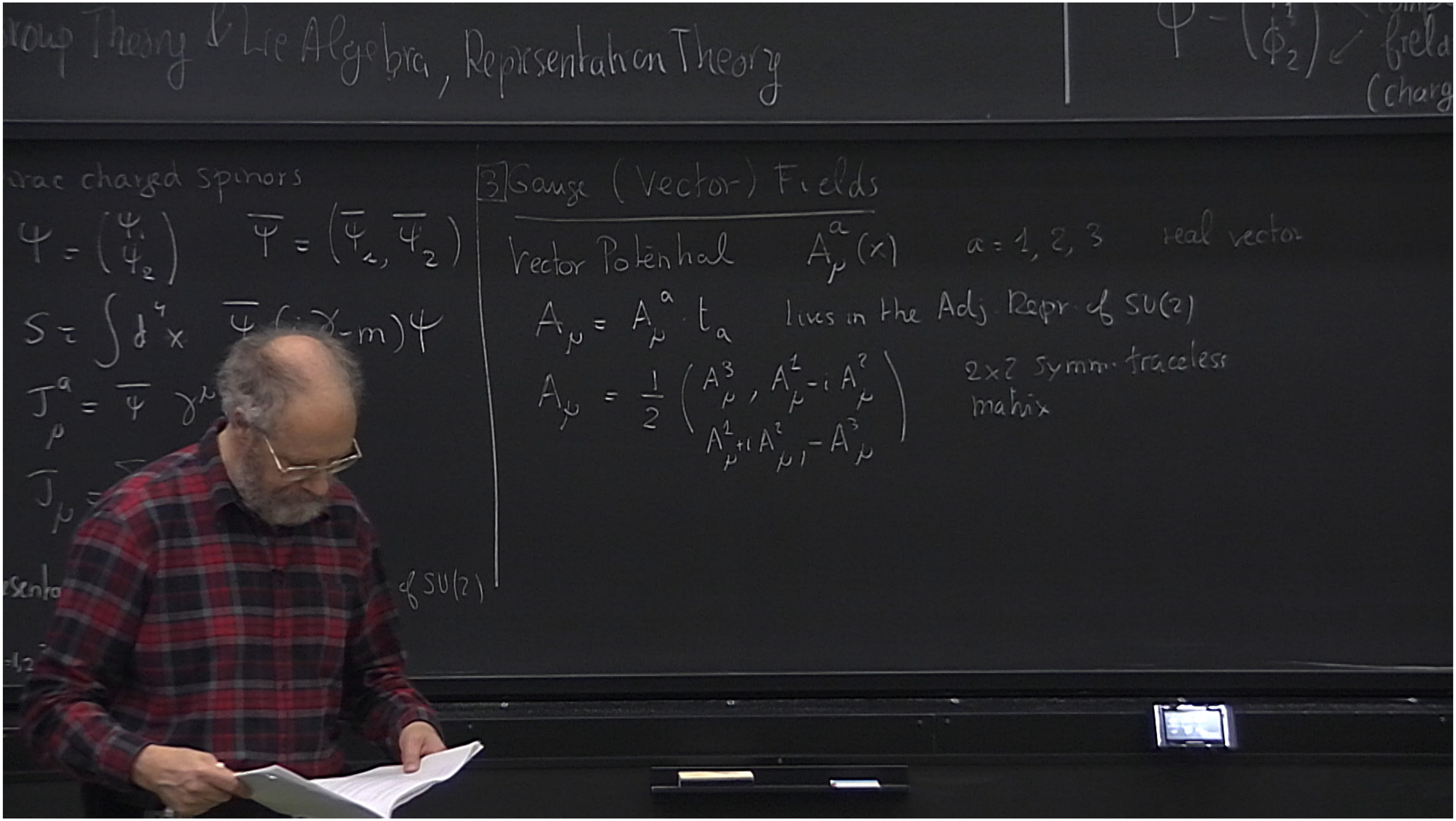
esentah of lorentz x representah of SU(2)

$_{=1,2} = (\Psi^\alpha, \alpha = 1, .. 4)$ α Dirac Indices

$\boxed{3}$ Gauge (Vector) Fields

Vector Potential $\quad A^a_\mu(x) \quad a = 1, 2, 3 \quad$ real vector

$A_\mu = A^a_\mu \cdot t_a$ lives in the Adj. Repr. of SU(2)



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esenta of SU(2)

=1,2

$\boxed{3}$ Gauge (Vector) Fields

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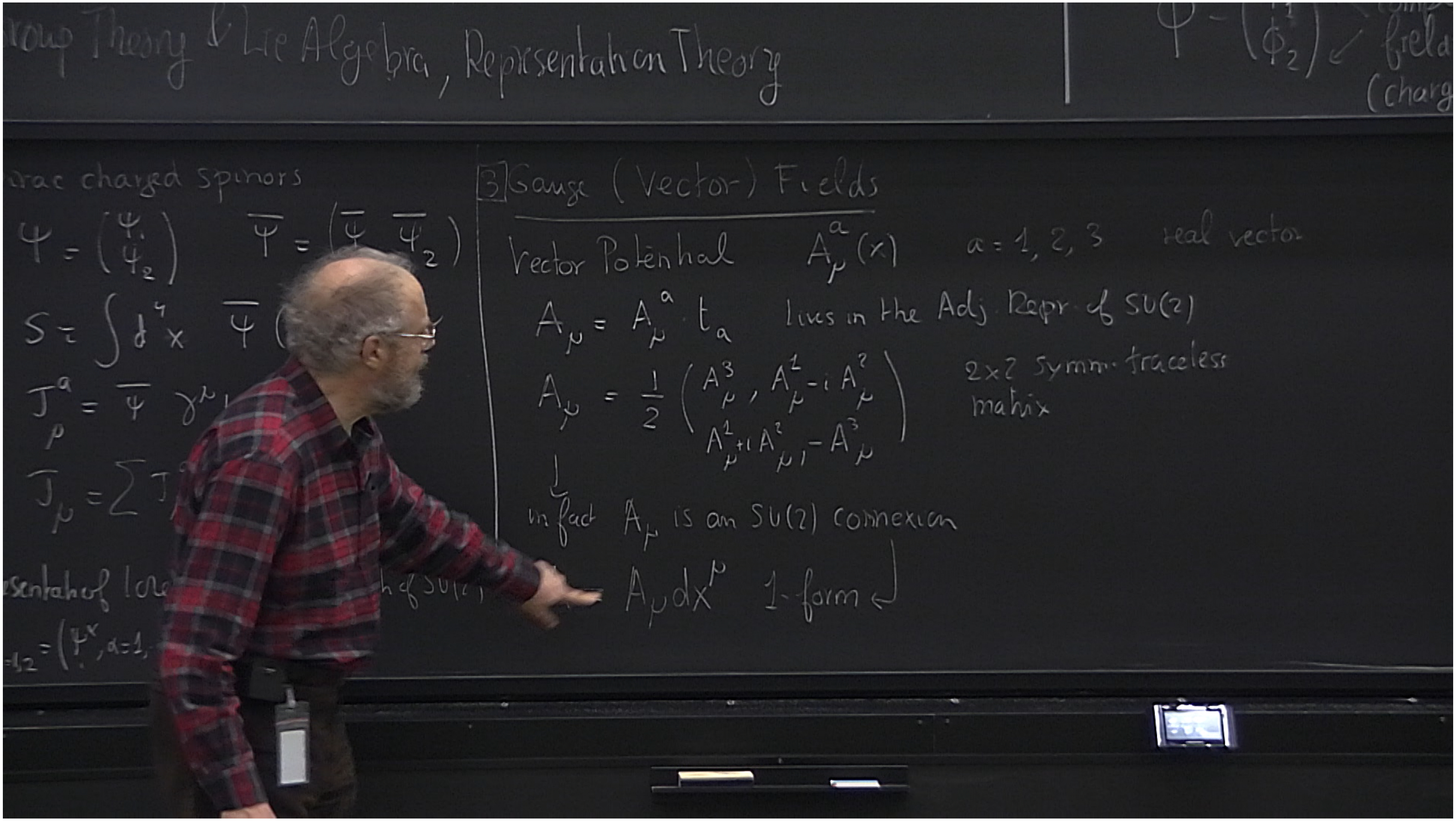
$A_\mu = A^a_\mu \cdot t_a$ lives in the Adj. Repr. of SU(2)

$$A_\mu = \frac{1}{2} \begin{pmatrix} A^3_\mu , & A^1_\mu - i A^2_\mu \\ A^1_\mu + i A^2_\mu , & - A^3_\mu \end{pmatrix}$$

2×2 symm. traceless matrix



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$J_\mu = \sum J$

esentation of lo h of SU(2)

$=1,2 = (\psi^\alpha, \alpha = 1,$

3 Gauge (Vector) Fields

Vector Potential $A^a_\mu(x)$ $a = 1, 2, 3$ real vector

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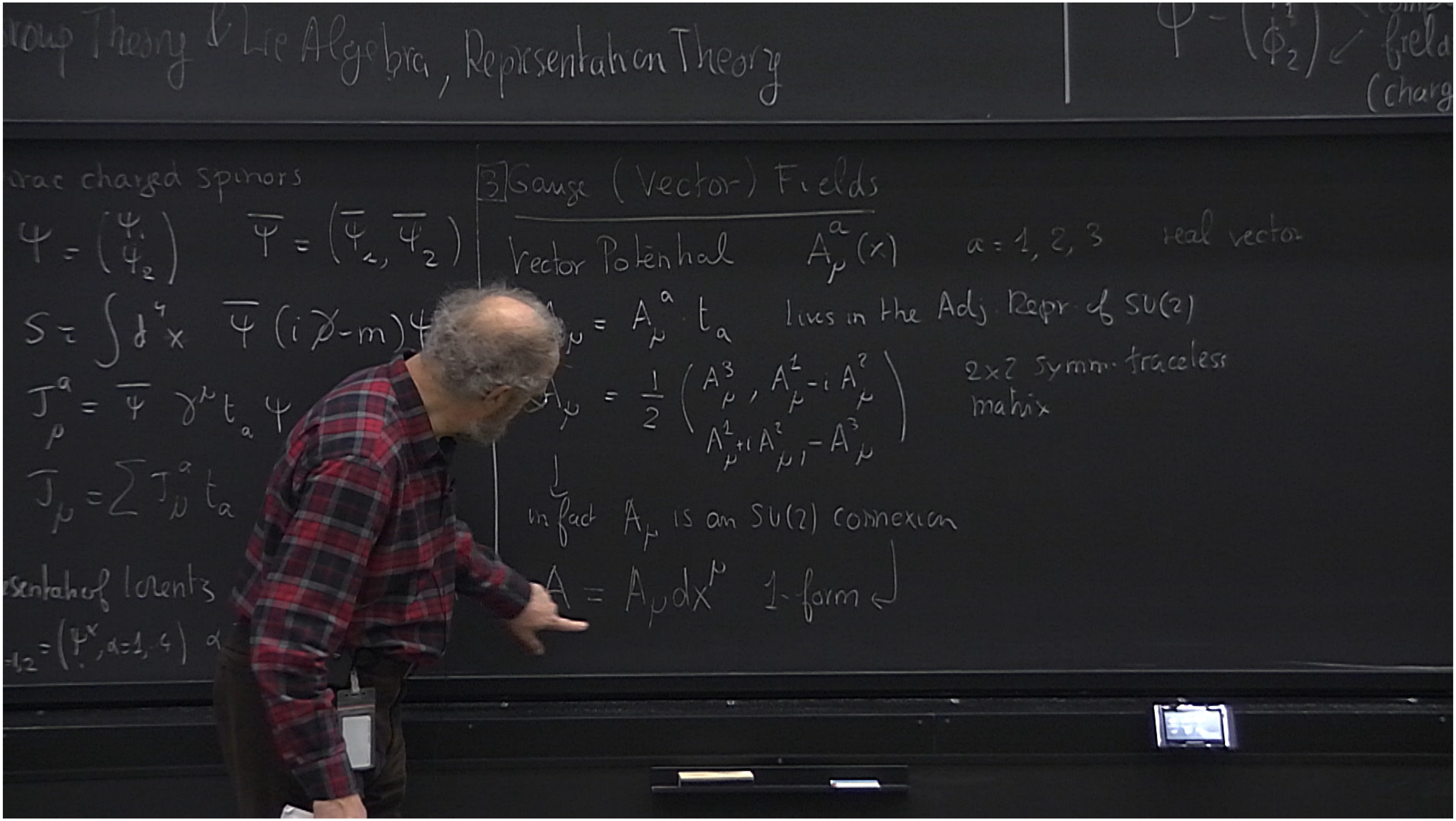
2×2 symm. traceless matrix

$\downarrow$

in fact A_μ is an SU(2) connexion

$A_\mu dx^\mu$ 1-form



roup Theory & Lie Algebra, Representation Theory

$\Phi = \begin{pmatrix}\phi_1 \\ \phi_2\end{pmatrix}$ field (charg

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sentation of Lorentz

$=_{1,2} = (\psi^\alpha, \alpha = 1, \ldots 4)$ α

[3] Gauge (Vector) Fields

Vector Potential $A^a_\mu(x)$ $a = 1, 2, 3$ real vector

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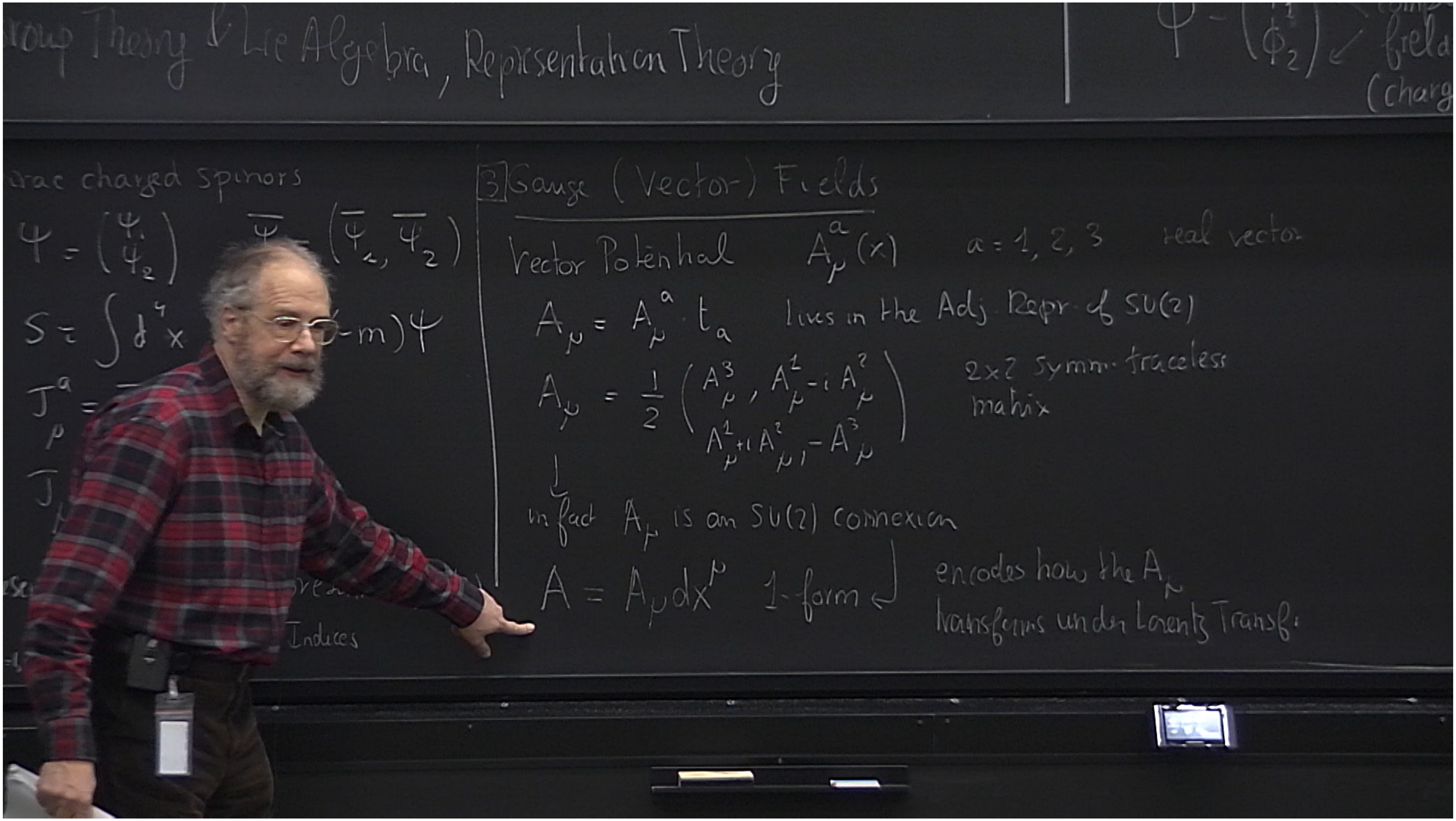
$$A_\mu = \frac{1}{2}\begin{pmatrix} A^3_\mu, & A^1_\mu - iA^2_\mu \\ A^1_\mu + iA^2_\mu, & -A^3_\mu \end{pmatrix}$$ 2x2 symm. traceless matrix

$\downarrow$ in fact A_μ is an SU(2) connexion

$A = A_\mu dx^\mu$ 1-form ↲



roup Theory & Lie Algebra, Representation Theory

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$S = \int d^4x \; \ldots - m)\psi$

$J^a_\mu = \ldots$

$J_\mu \ldots$

Indices

$\boxed{3}$ Gauge (Vector-) Fields

Vector Potential $A^a_\mu(x)$ $a = 1, 2, 3$ real vector

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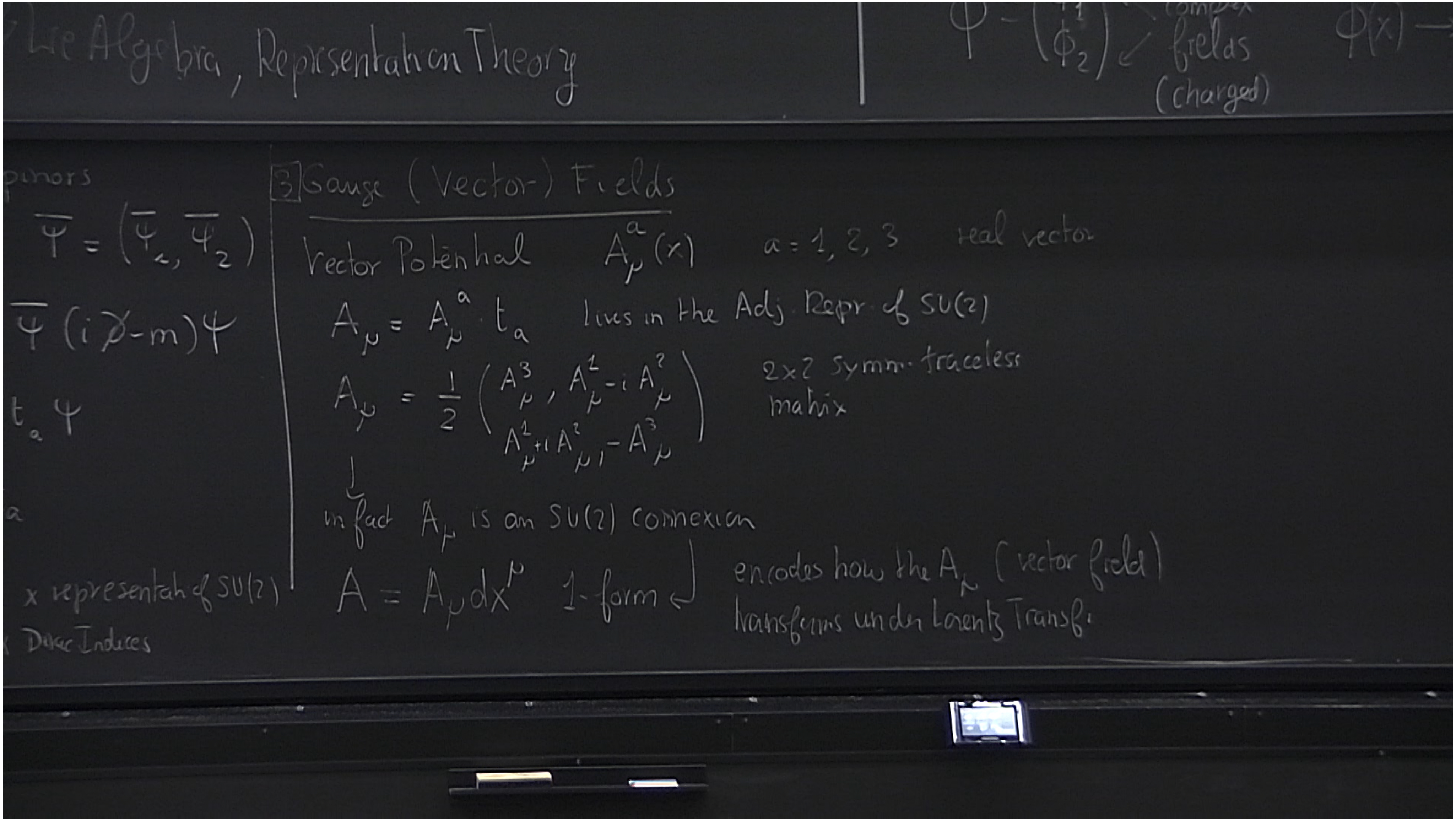
↓

in fact A_μ is an SU(2) connexion

$A = A_\mu dx^\mu$ 1-form ↲ encodes how the A_μ transforms under Lorentz Transf.



Lie Algebra, Representation Theory

$\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$ complex fields (charged) $\phi(x) \to$

pinors

$\overline{\Psi} = (\overline{\Psi}_1, \overline{\Psi}_2)$

$\overline{\Psi}(i\not{\partial} - m)\Psi$

$t_a \Psi$

a

x representation of SU(2)

x Dirac Indices

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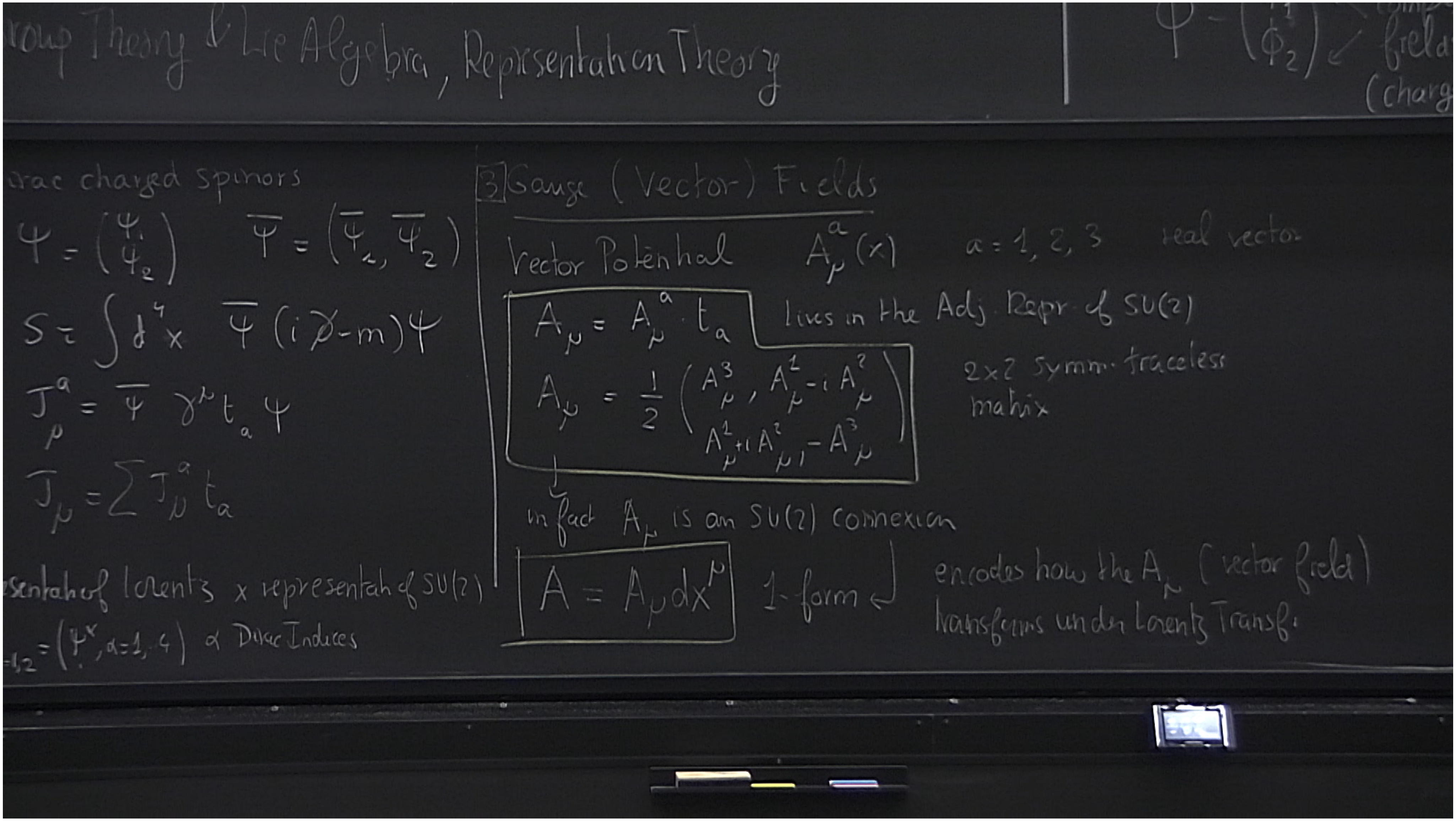

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3 Gauge (Vector) Fields

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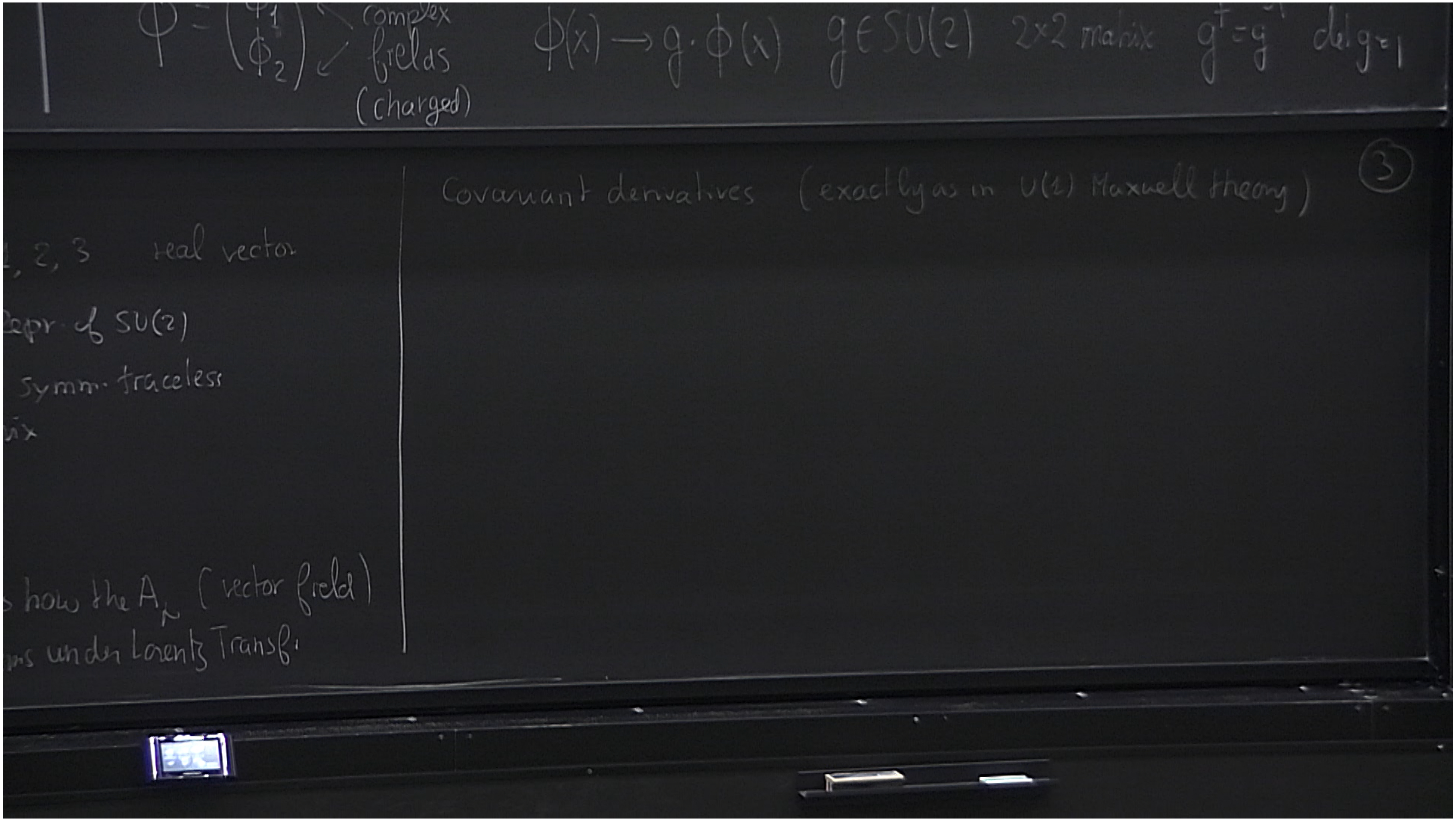

$\phi = \begin{pmatrix}\phi_1 \\ \phi_2\end{pmatrix}$ complex fields (charged) $\phi(x) \to g \cdot \phi(x)$ $g \in SU(2)$ 2×2 matrix $g^\dagger = g^{-1}$ $\det g = 1$

1, 2, 3 real vector

Repr. of SU(2)

symm. traceless

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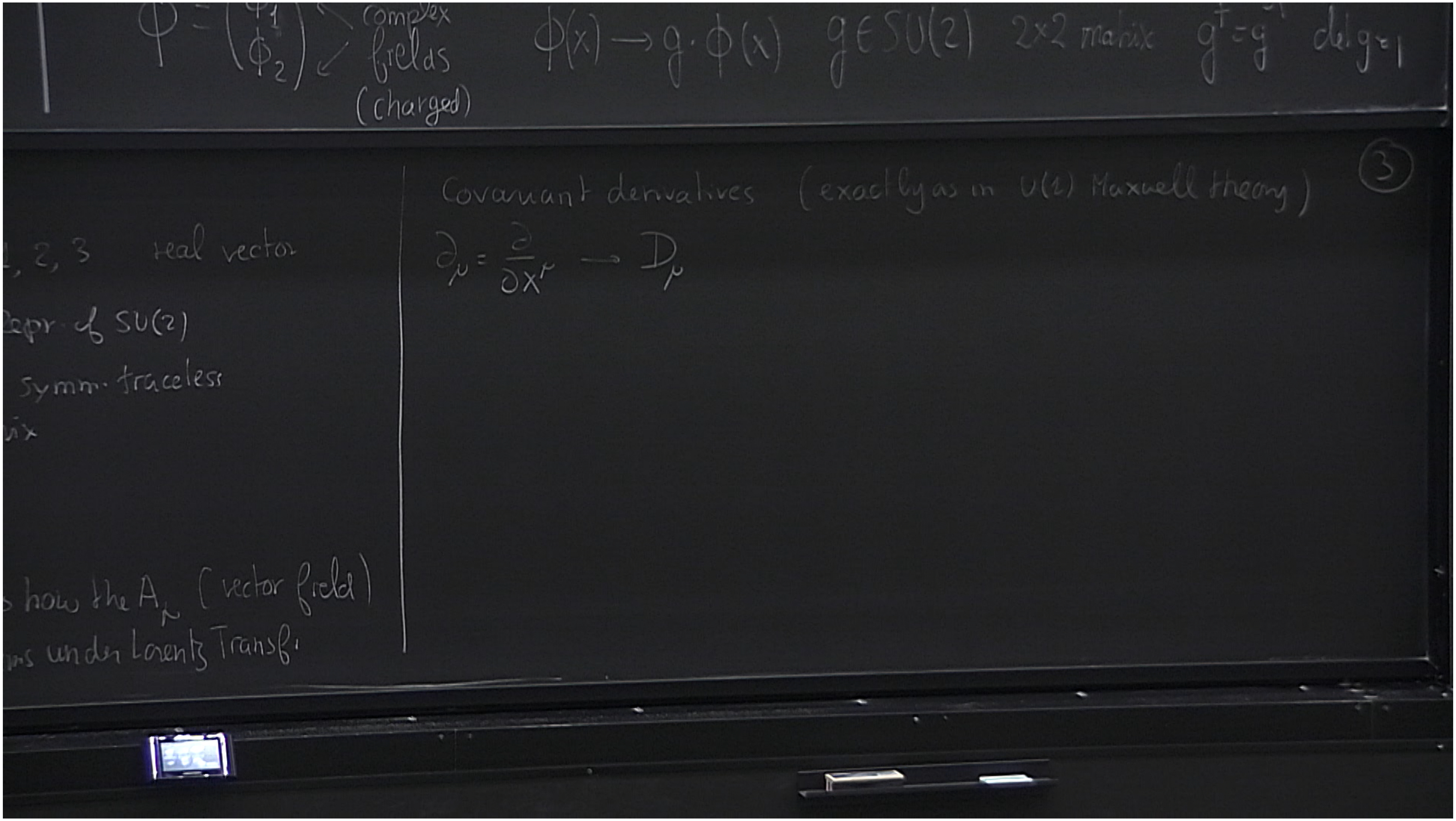
how the A_μ (vector field)

ms under Lorentz Transf.

③

Covariant derivatives (exactly as in U(1) Maxwell theory)



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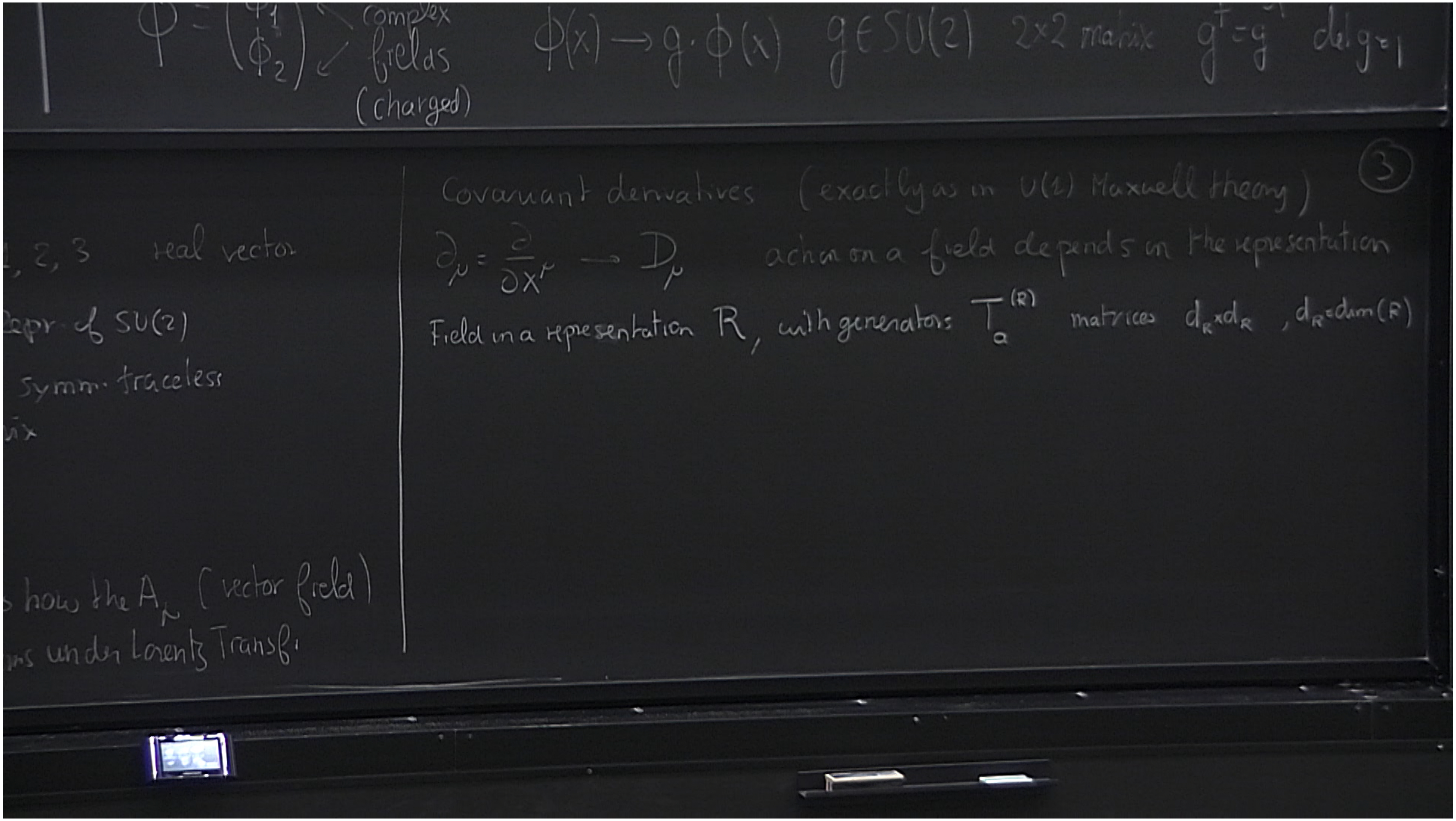
$$\partial_\mu = \frac{\partial}{\partial x^\mu} \longrightarrow D_\mu$$



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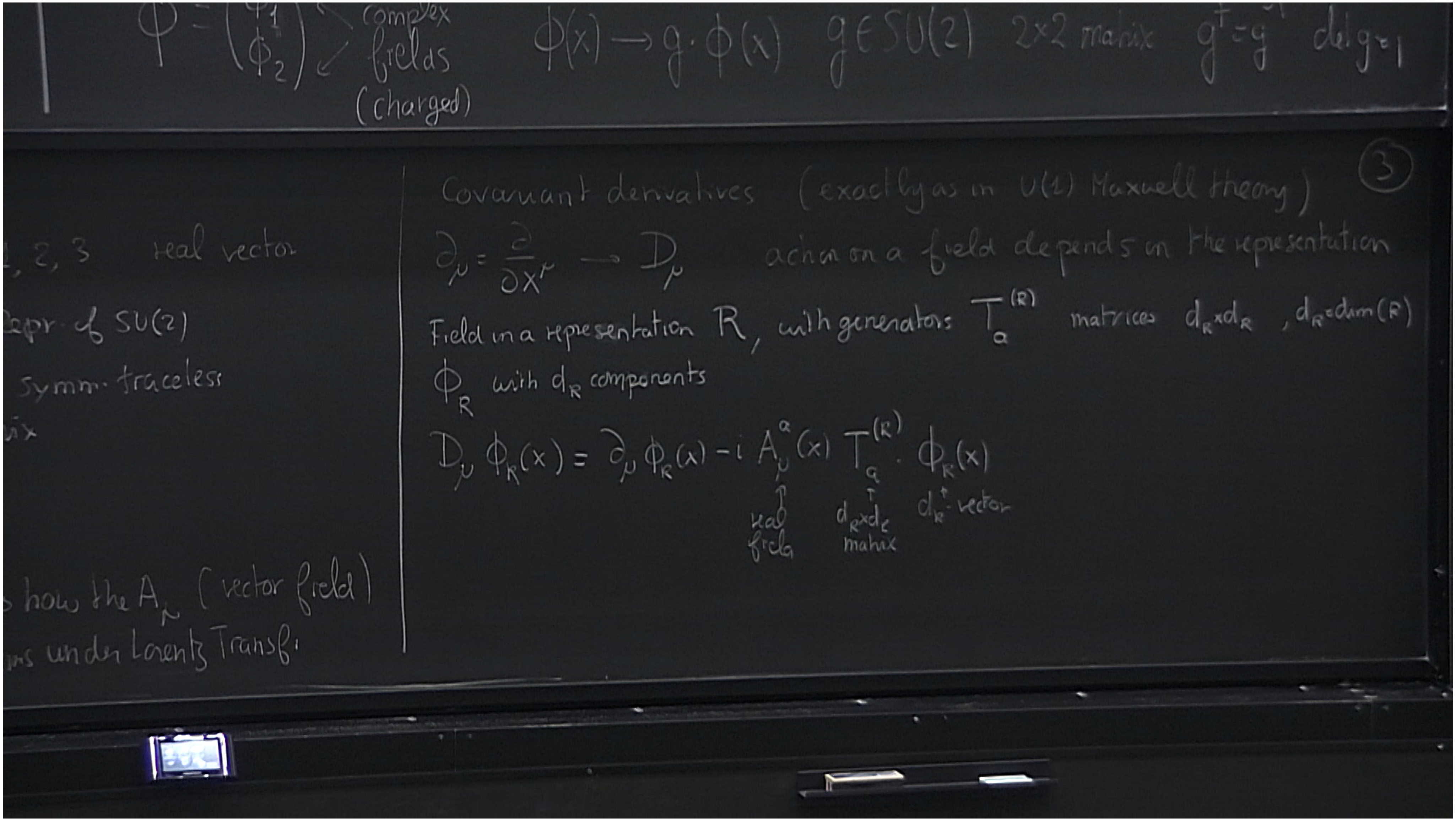
$\partial_\mu = \frac{\partial}{\partial x^\mu} \to D_\mu$ action on a field depends on the representation

Field in a representation R, with generators $T_a^{(R)}$ matrices $d_R \times d_R$, $d_R = \dim(R)$



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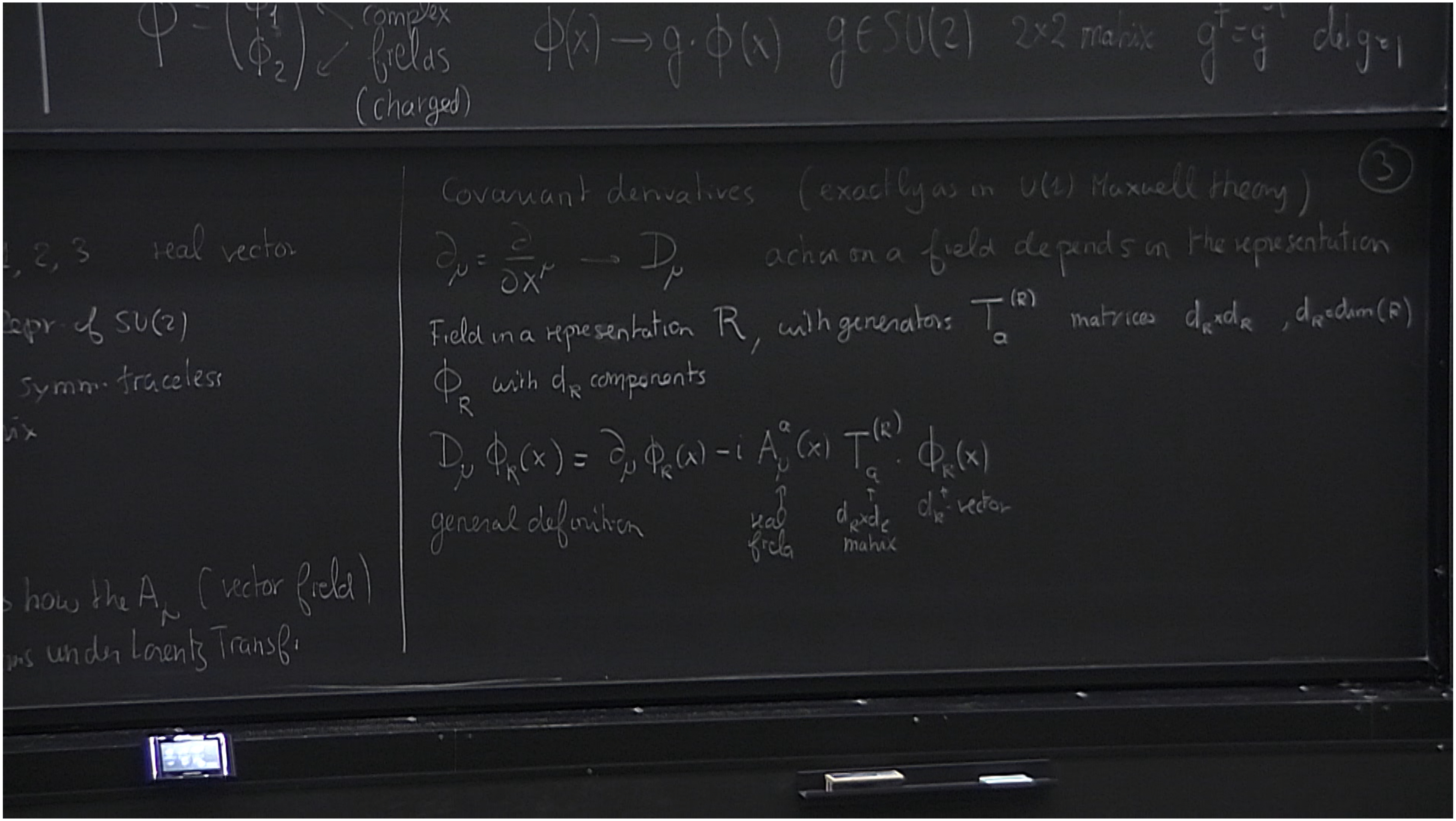
ϕ_R with d_R components

$$D_\mu \phi_R(x) = \partial_\mu \phi_R(x) - i \underset{\substack{\uparrow \\ \text{real} \\ \text{field}}}{A^a_\mu(x)} \underset{\substack{\uparrow \\ d_R \times d_R \\ \text{matrix}}}{T^{(R)}_a} \cdot \underset{\substack{\uparrow \\ d_R\text{-vector}}}{\phi_R(x)}$$



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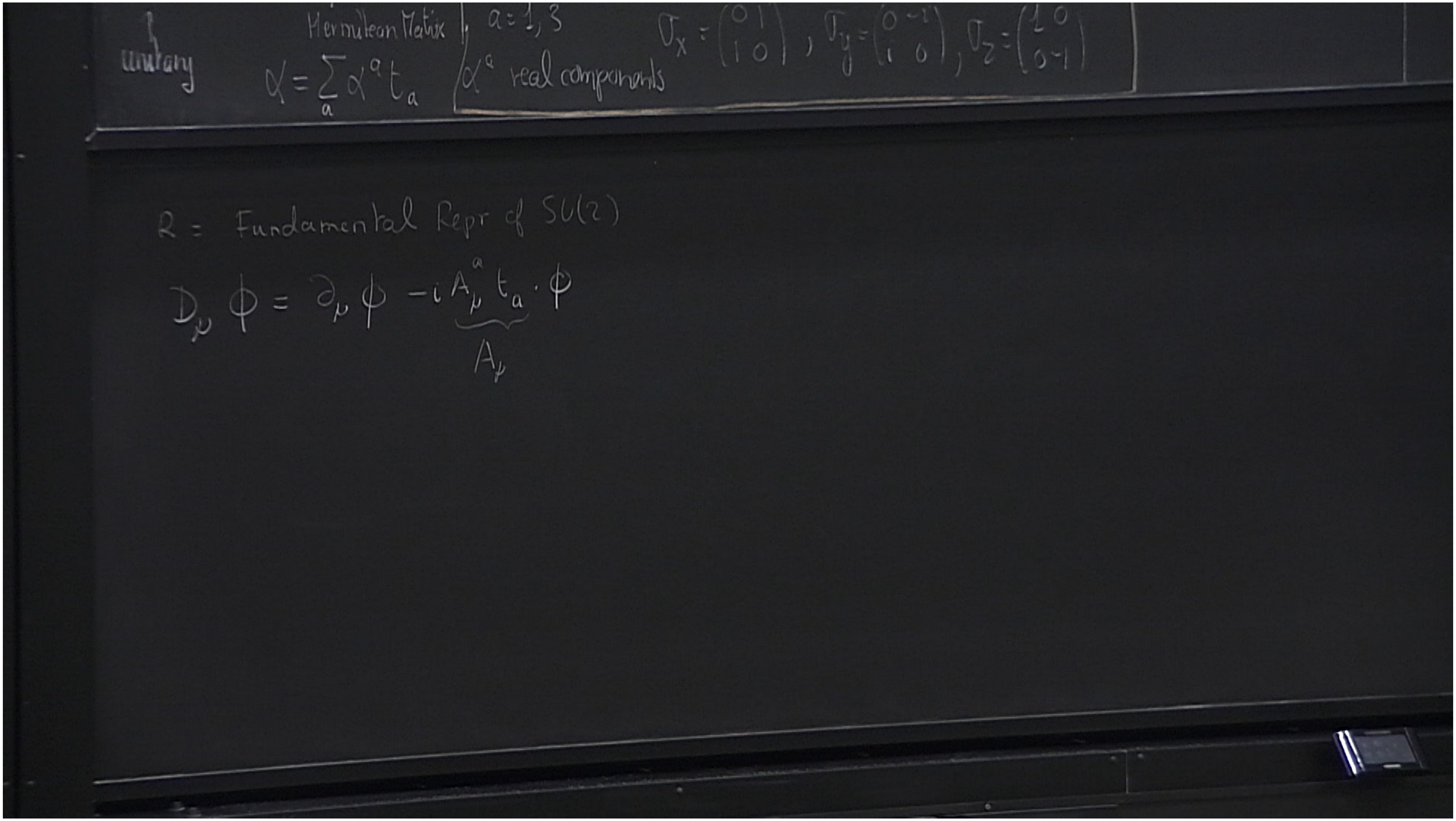
general definition



unitary

Hermitean Matrix

$a = 1, 3$

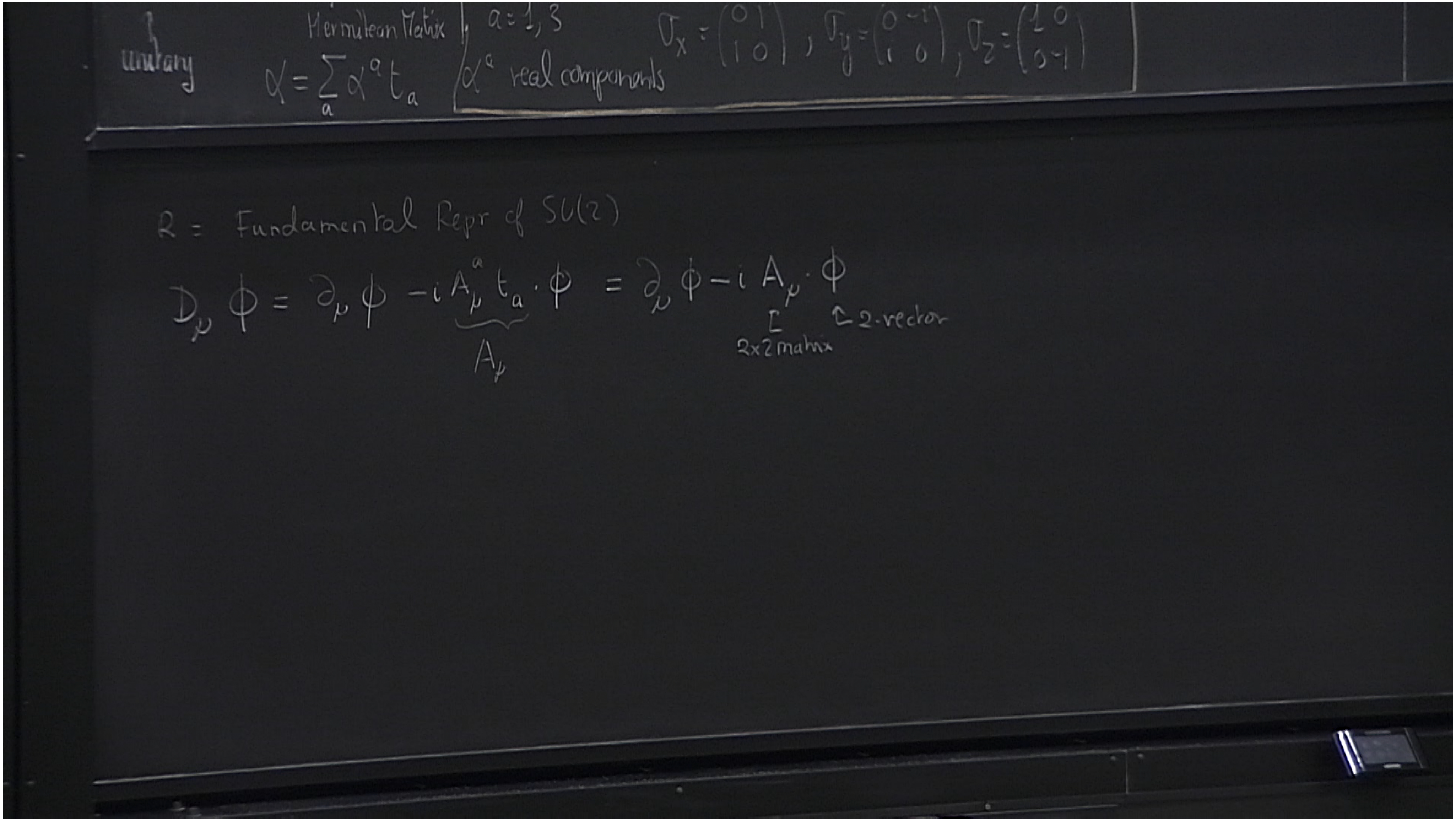
$\alpha = \sum_a \alpha^a t_a$

α^a real components

$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

R = Fundamental Repr of SU(2)

$$D_\mu \phi = \partial_\mu \phi - i \underbrace{A_\mu^a t_a}_{A_\mu} \cdot \phi$$



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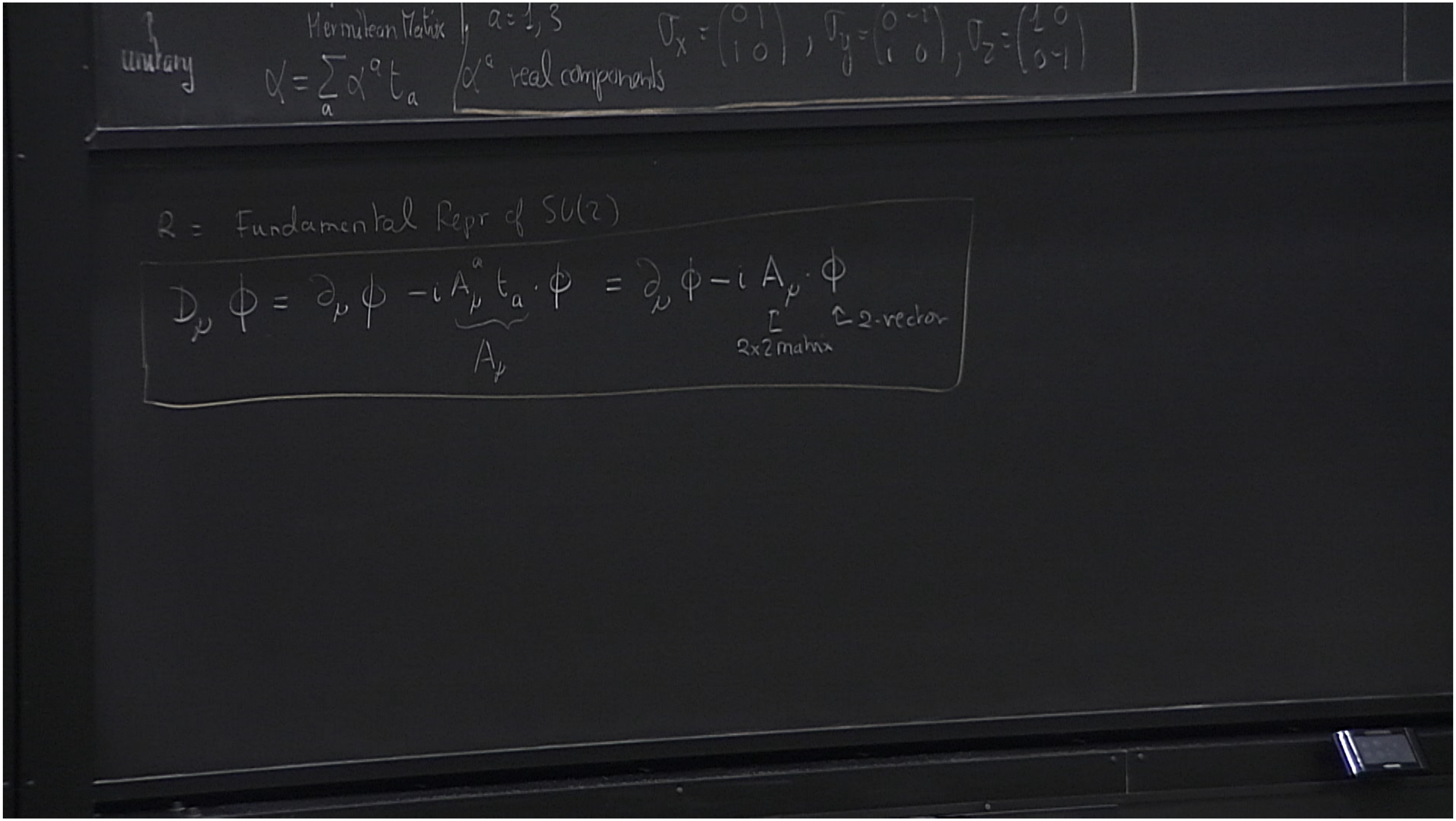
$$D_\mu \phi = \partial_\mu \phi - i \underbrace{A_\mu^a t_a}_{A_\mu} \cdot \phi = \partial_\mu \phi - i \underset{\substack{\uparrow \\ 2\times 2 \text{ matrix}}}{A_\mu} \cdot \phi \quad \leftarrow 2\text{-vector}$$



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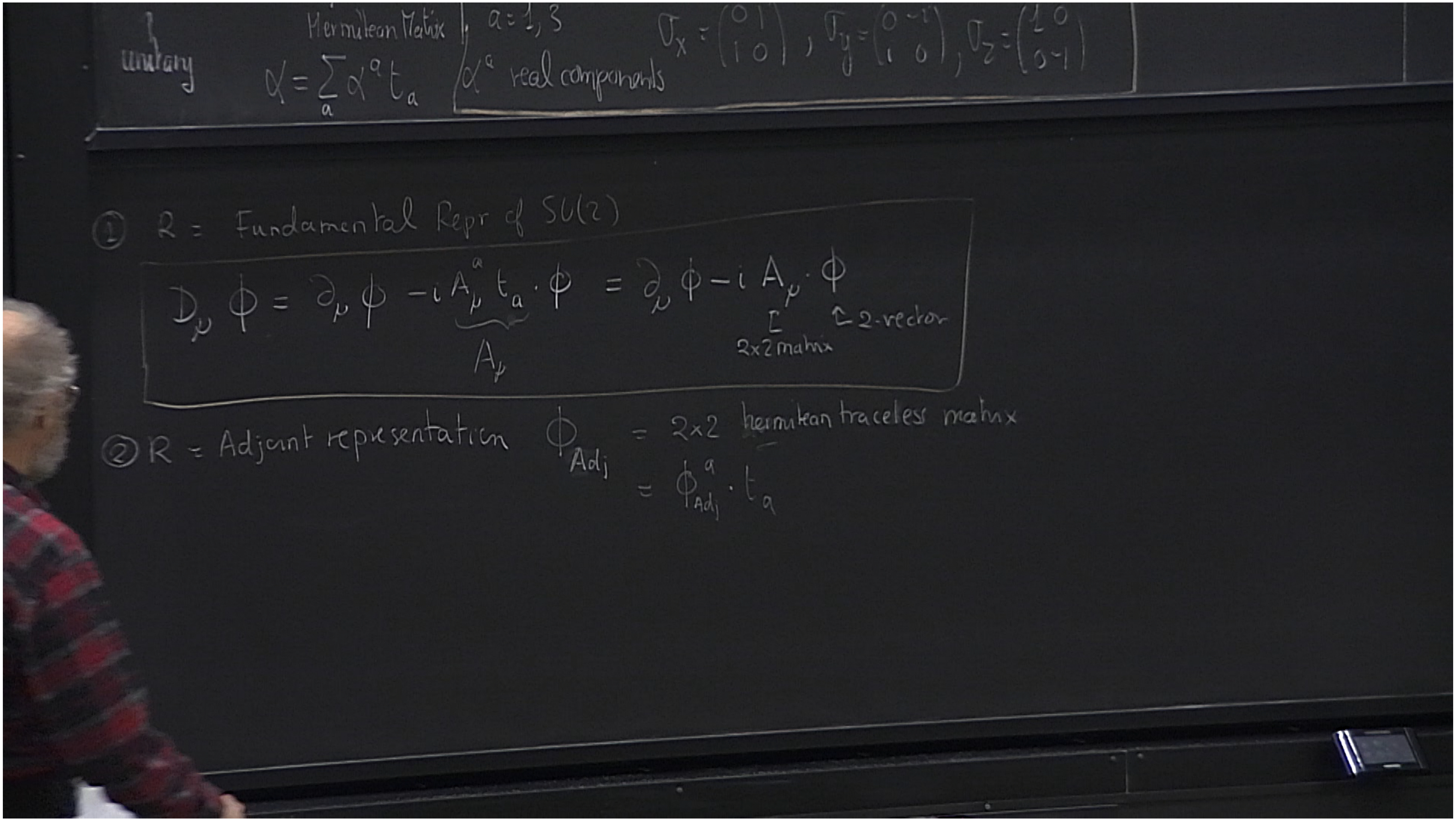

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① R = Fundamental Repr of SU(2)

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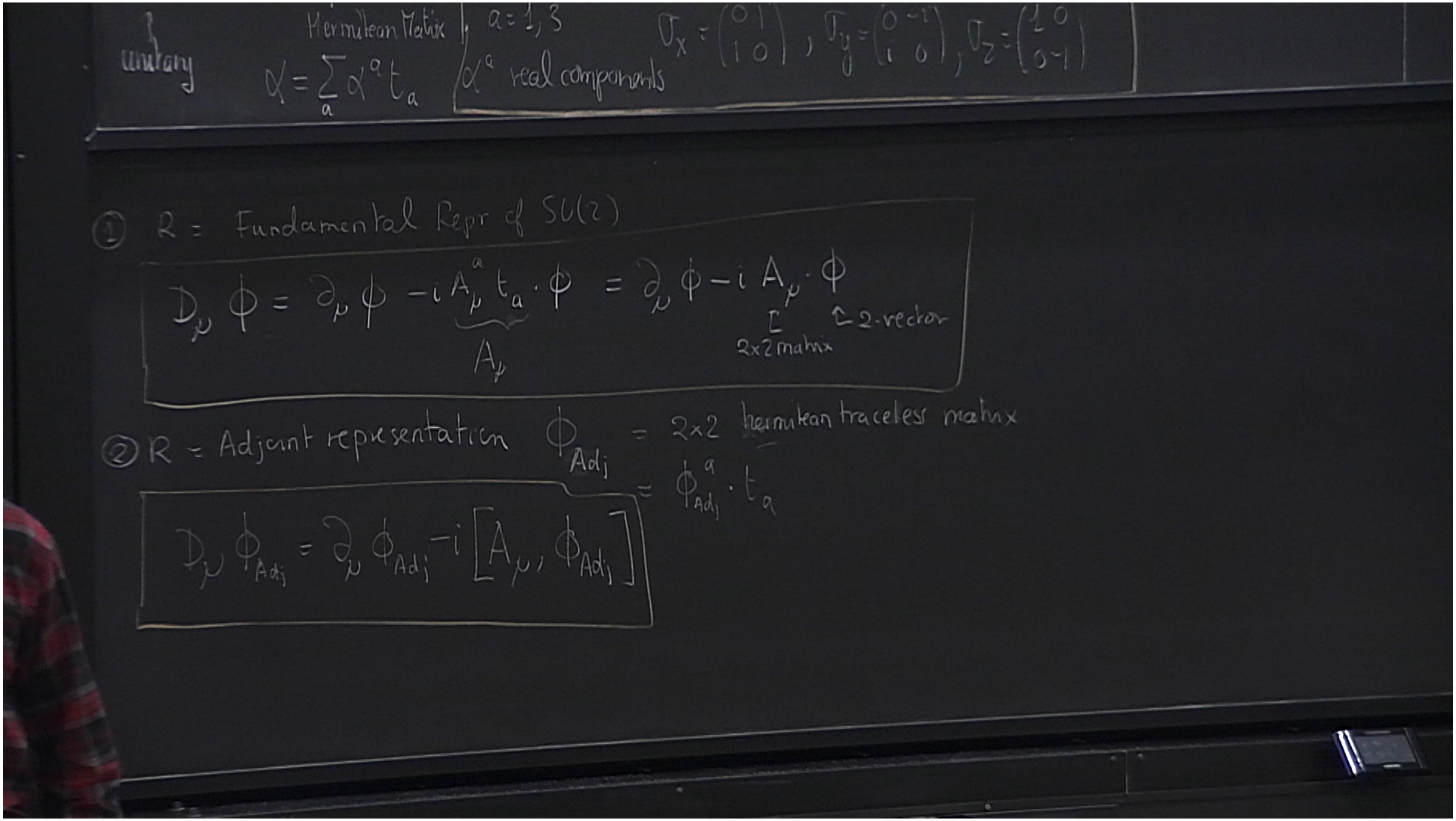
② R = Adjoint representation ϕ_{Adj} = 2×2 hermitean traceless matrix

$= \phi^a_{Adj} \cdot t_a$



unitary Hermitean Matrix $a=1,3$ $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

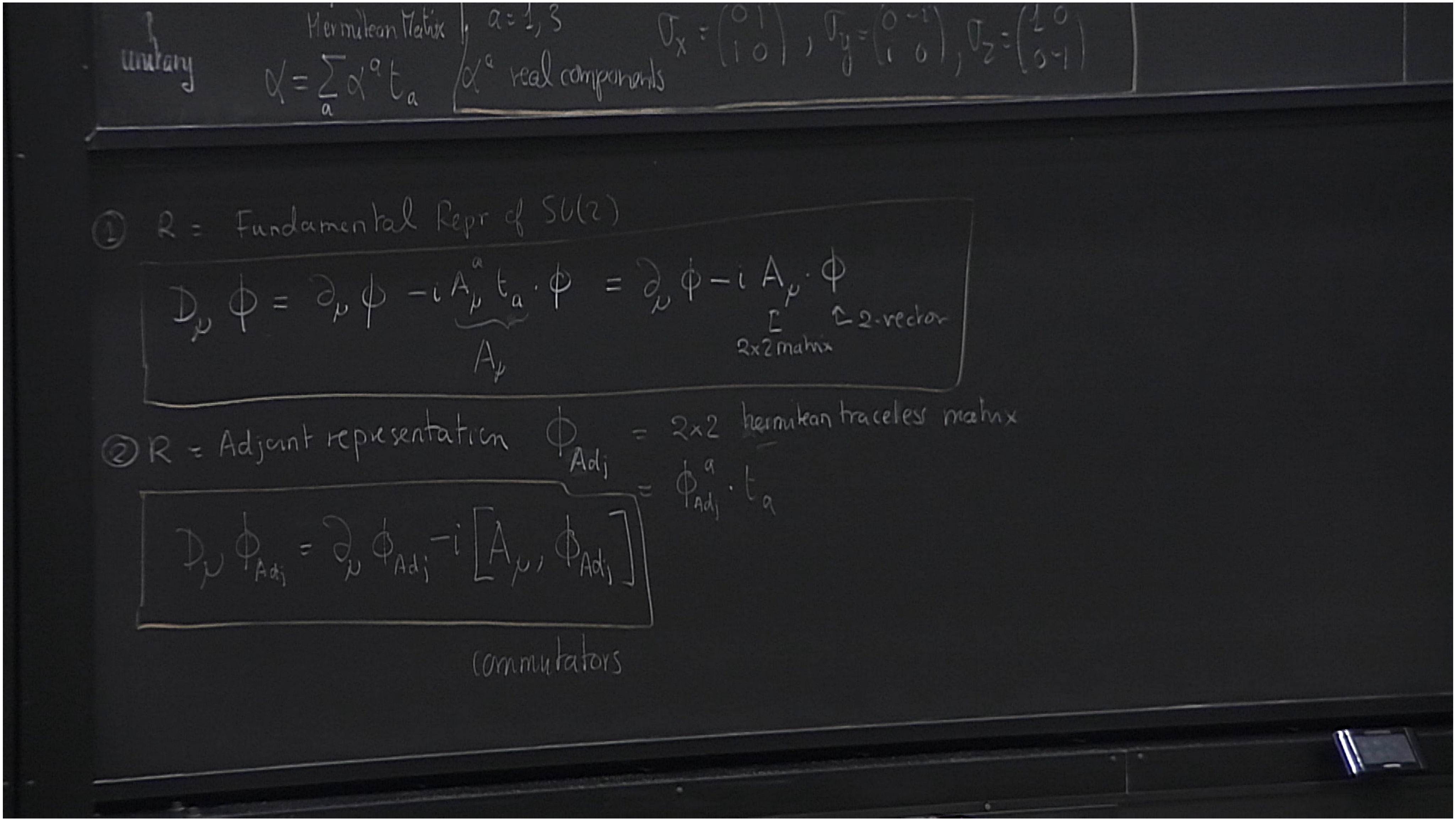
$\alpha = \sum_a \alpha^a t_a$ α^a real components

① R = Fundamental Repr of $SU(2)$

$$\boxed{D_\mu \phi = \partial_\mu \phi - i \underbrace{A_\mu^a t_a}_{A_\mu} \cdot \phi = \partial_\mu \phi - i \underset{\text{2x2 matrix}}{A_\mu} \cdot \underset{\text{2-vector}}{\phi}}$$

② R = Adjoint representation ϕ_{Adj} = 2x2 hermitean traceless matrix $= \phi^a_{Adj} \cdot t_a$

$$\boxed{D_\mu \phi_{Adj} = \partial_\mu \phi_{Adj} - i \left[A_\mu , \phi_{Adj} \right]}$$



unitary

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$a = 1, 3$

α^a real components

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

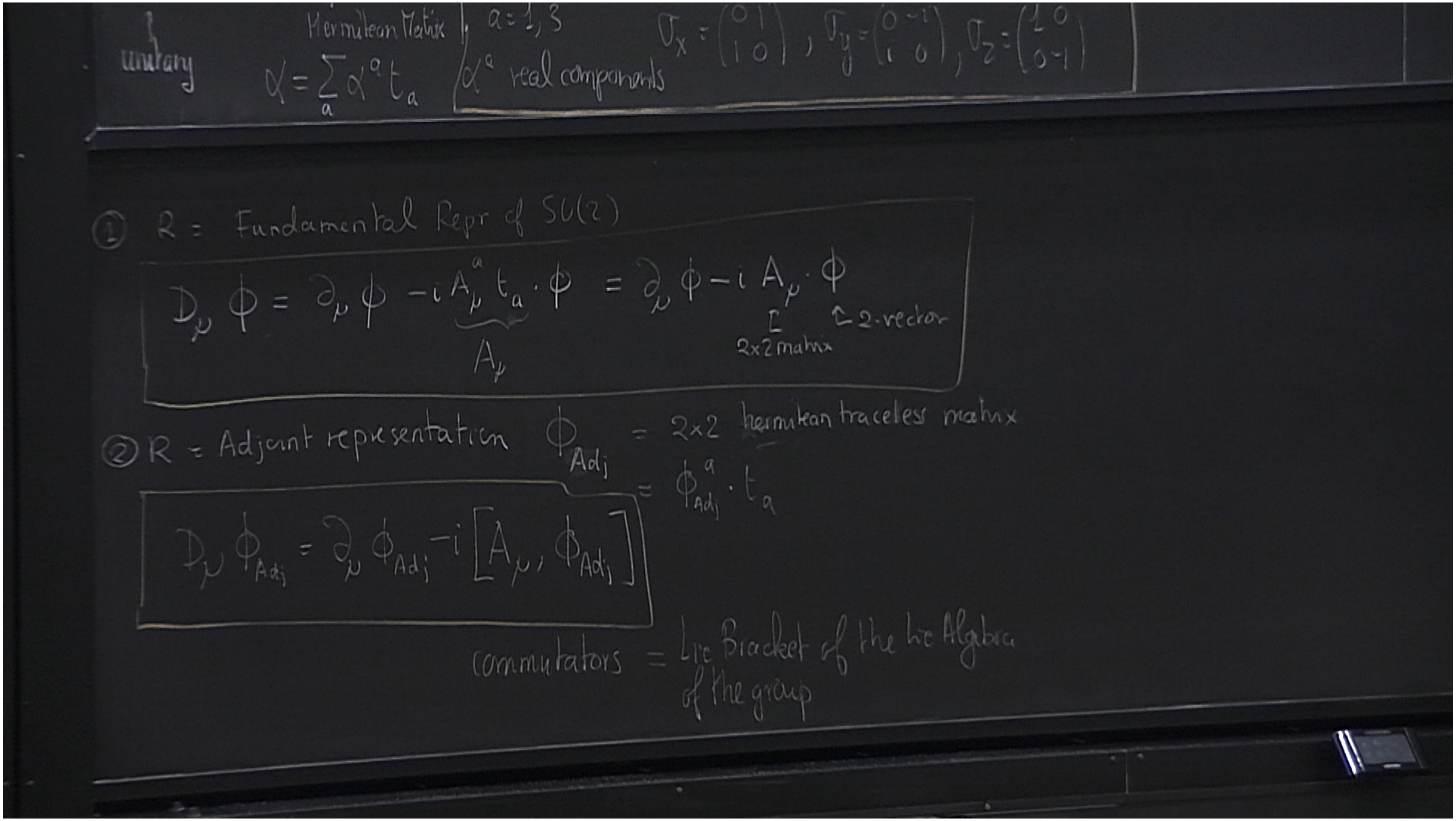
① R = Fundamental Repr of SU(2)

$$\boxed{D_\mu \phi = \partial_\mu \phi - i \underbrace{A_\mu^a t_a}_{A_\mu} \cdot \phi = \partial_\mu \phi - i \underset{2\times 2 \text{ matrix}}{A_\mu} \cdot \underset{\text{2-vector}}{\phi}}$$

② R = Adjoint representation ϕ_{Adj} = 2×2 hermitean traceless matrix $= \phi_{Adj}^a \cdot t_a$

$$\boxed{D_\mu \phi_{Adj} = \partial_\mu \phi_{Adj} - i \left[A_\mu, \phi_{Adj} \right]}$$

commutators



unitary

Hermitean Matrix | $a = 1, 3$ $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$\alpha = \sum_a \alpha^a t_a$ / α^a real components

① R = Fundamental Repr of $SU(2)$

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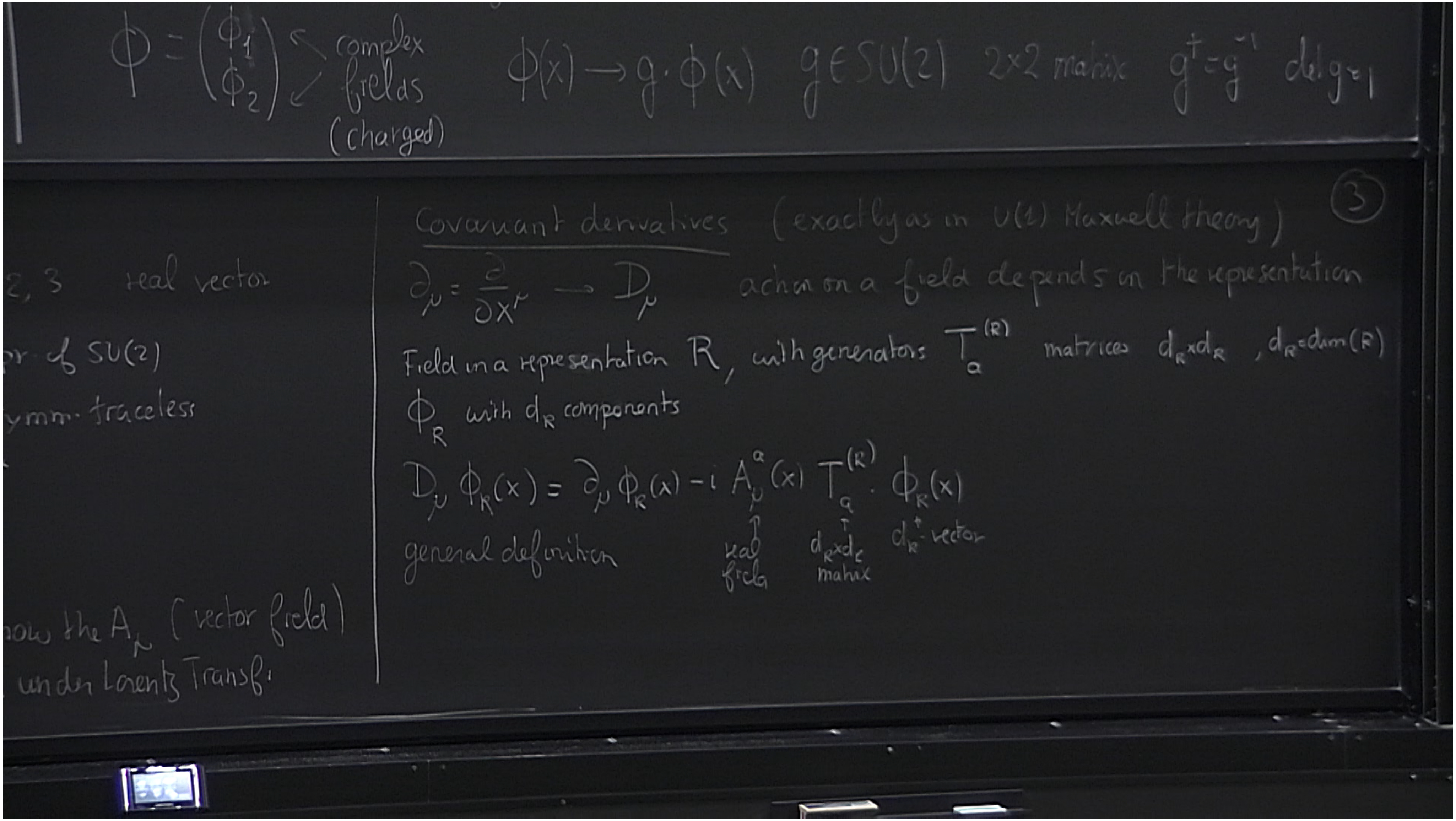
commutators = Lie Bracket of the Lie Algebra of the group



$\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$ complex fields (charged) $\phi(x) \to g \cdot \phi(x)$ $g \in SU(2)$ 2×2 matrix $g^\dagger = g^{-1}$ $\det g = 1$

2, 3 real vector

r. of SU(2)

ymm. traceless

ow the A_μ (vector field)

under Lorentz Transf.

③

Covariant derivatives (exactly as in U(1) Maxwell theory)

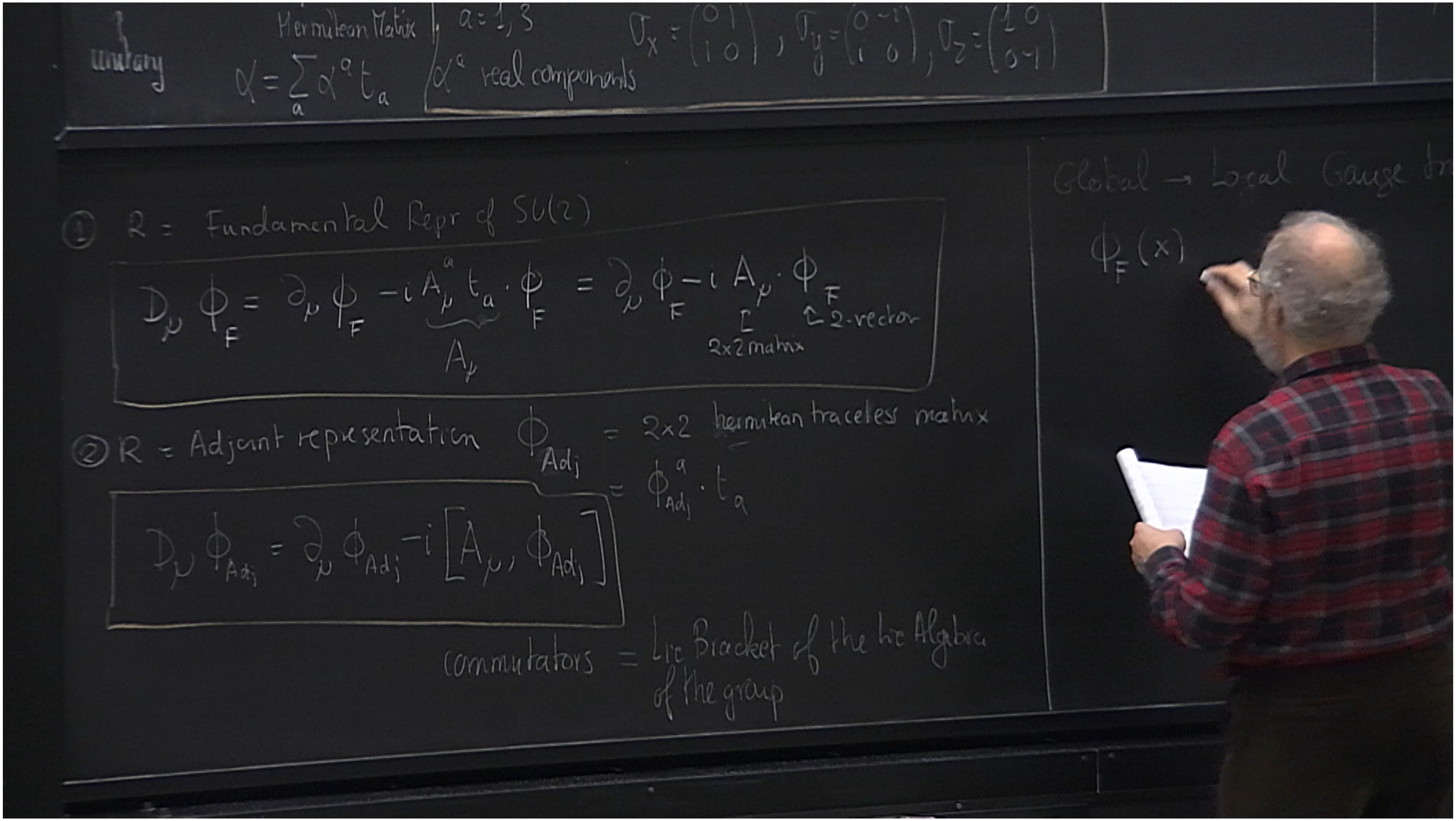
$\partial_\mu = \frac{\partial}{\partial x^\mu} \longrightarrow D_\mu$ action on a field depends on the representation

Field in a representation R, with generators $T_a^{(R)}$ matrices $d_R \times d_R$, $d_R = \dim(R)$

ϕ_R with d_R components

$$D_\mu \phi_R(x) = \partial_\mu \phi_R(x) - i \underset{\substack{\uparrow \\ \text{real} \\ \text{field}}}{A_\mu^a(x)} \underset{\substack{\uparrow \\ d_R \times d_R \\ \text{matrix}}}{T_a^{(R)}} \cdot \underset{\substack{\uparrow \\ d_R\text{-vector}}}{\phi_R(x)}$$

general definition



unitary Hermitean Matrix, $a = 1,3$ $\sigma_x = \begin{pmatrix}0&1\\1&0\end{pmatrix}, \sigma_y = \begin{pmatrix}0&-i\\i&0\end{pmatrix}, \sigma_z = \begin{pmatrix}1&0\\0&-1\end{pmatrix}$

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① $R =$ Fundamental Repr of SU(2)

$$\boxed{D_\mu \phi_F = \partial_\mu \phi_F - i \underbrace{A^a_\mu t_a}_{A_\mu} \cdot \phi_F = \partial_\mu \phi_F - i \underset{\text{2x2 matrix}}{A_\mu} \cdot \underset{\text{2-vector}}{\phi_F}}$$

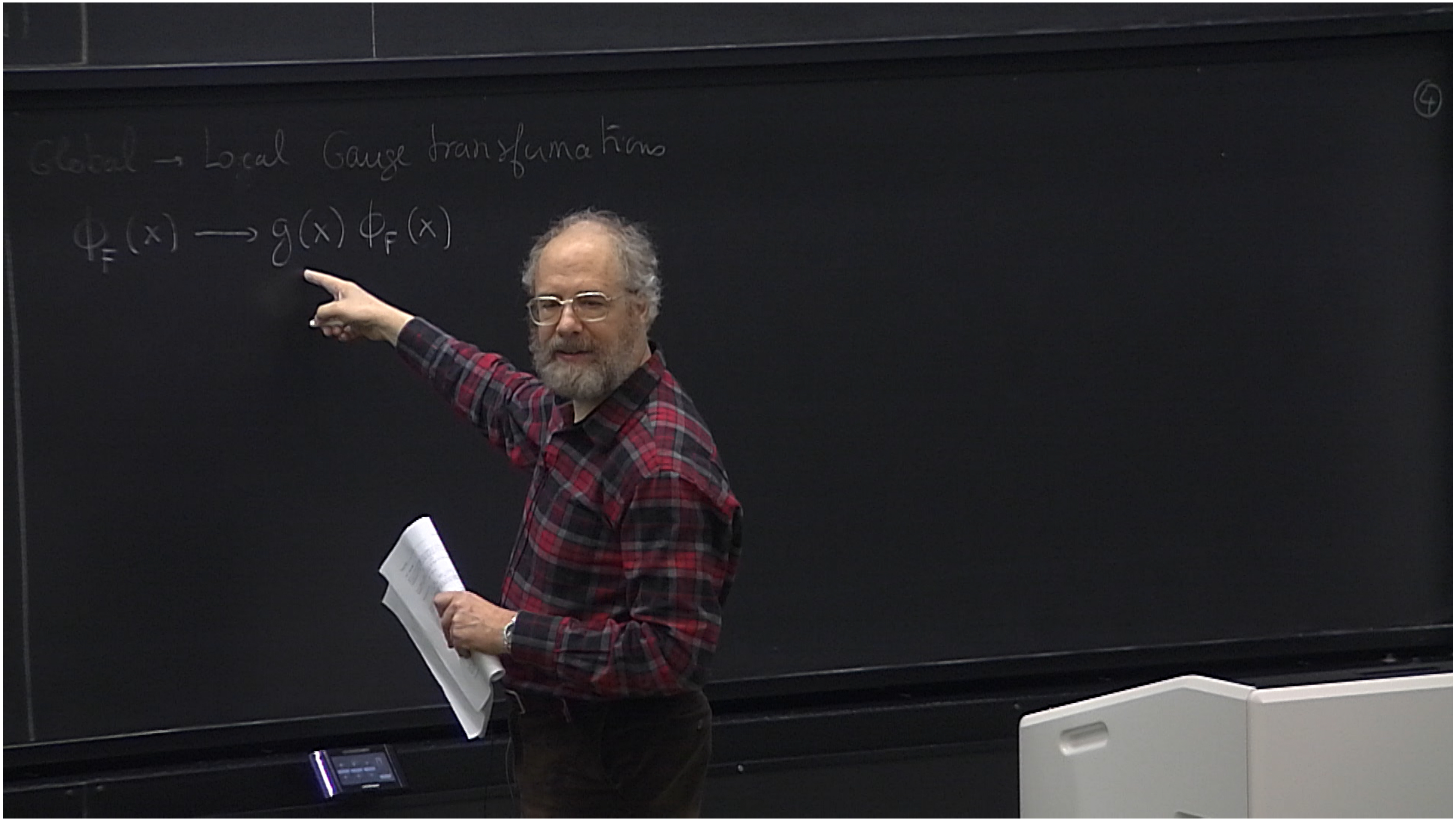
② $R =$ Adjoint representation $\phi_{Adj} = 2\times 2$ hermitean traceless matrix $= \phi^a_{Adj} \cdot t_a$

$$\boxed{D_\mu \phi_{Adj} = \partial_\mu \phi_{Adj} - i\left[A_\mu, \phi_{Adj}\right]}$$

commutators = Lie Bracket of the Lie Algebra of the group

Global → Local Gauge tr

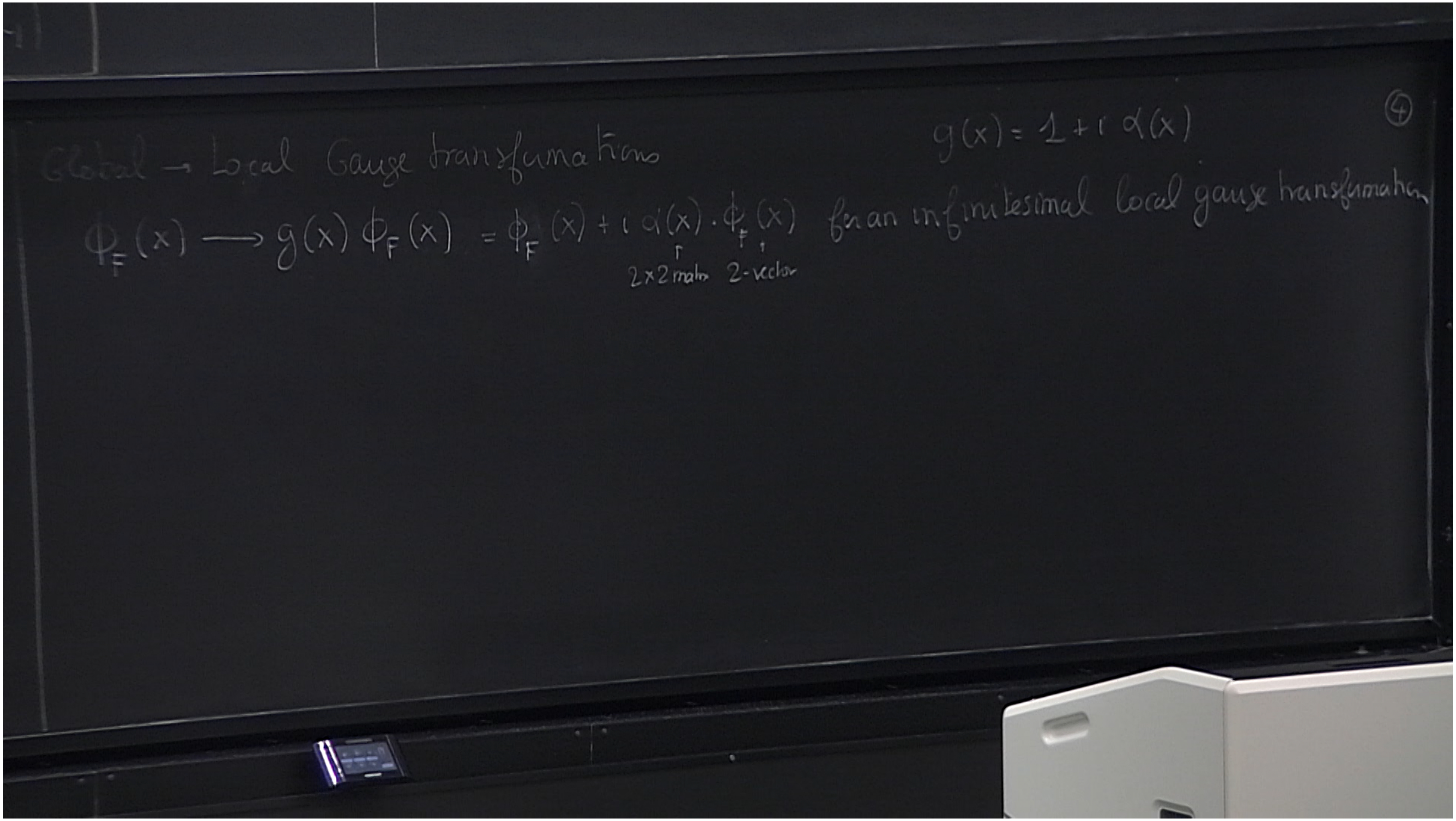
$\phi_F(x)$



④

Global → Local Gauge transformations

$$\phi_F(x) \longrightarrow g(x)\,\phi_F(x)$$



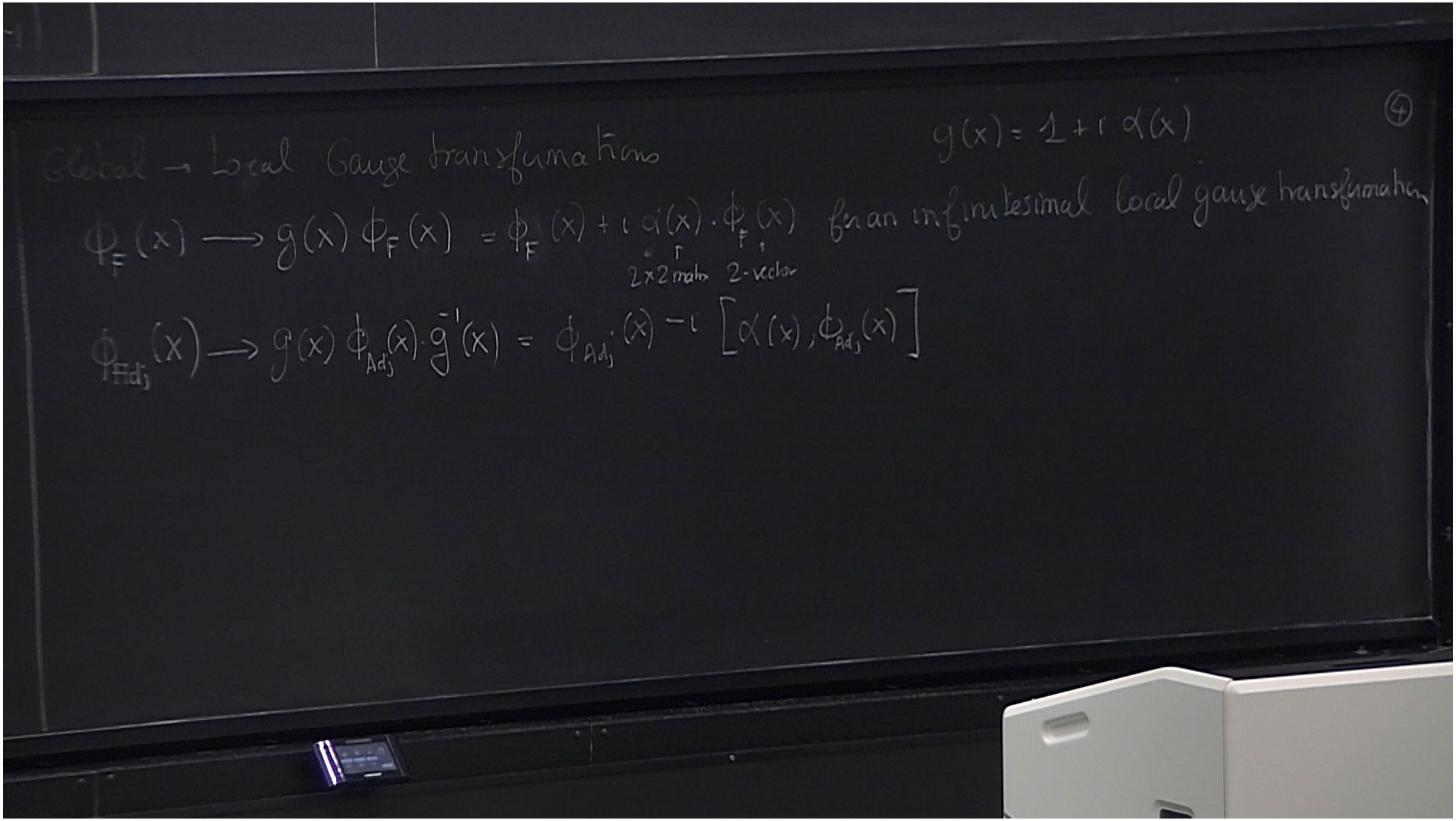
④

$$g(x) = 1 + i\,\alpha(x)$$

Global → Local Gauge transformations

$$\phi_F(x) \longrightarrow g(x)\,\phi_F(x) = \phi_F(x) + i\,\underset{\substack{\uparrow \\ 2\times 2\text{ matrix}}}{\alpha(x)} \cdot \underset{\substack{\uparrow \\ 2\text{-vector}}}{\phi_F(x)}$$

for an infinitesimal local gauge transformation



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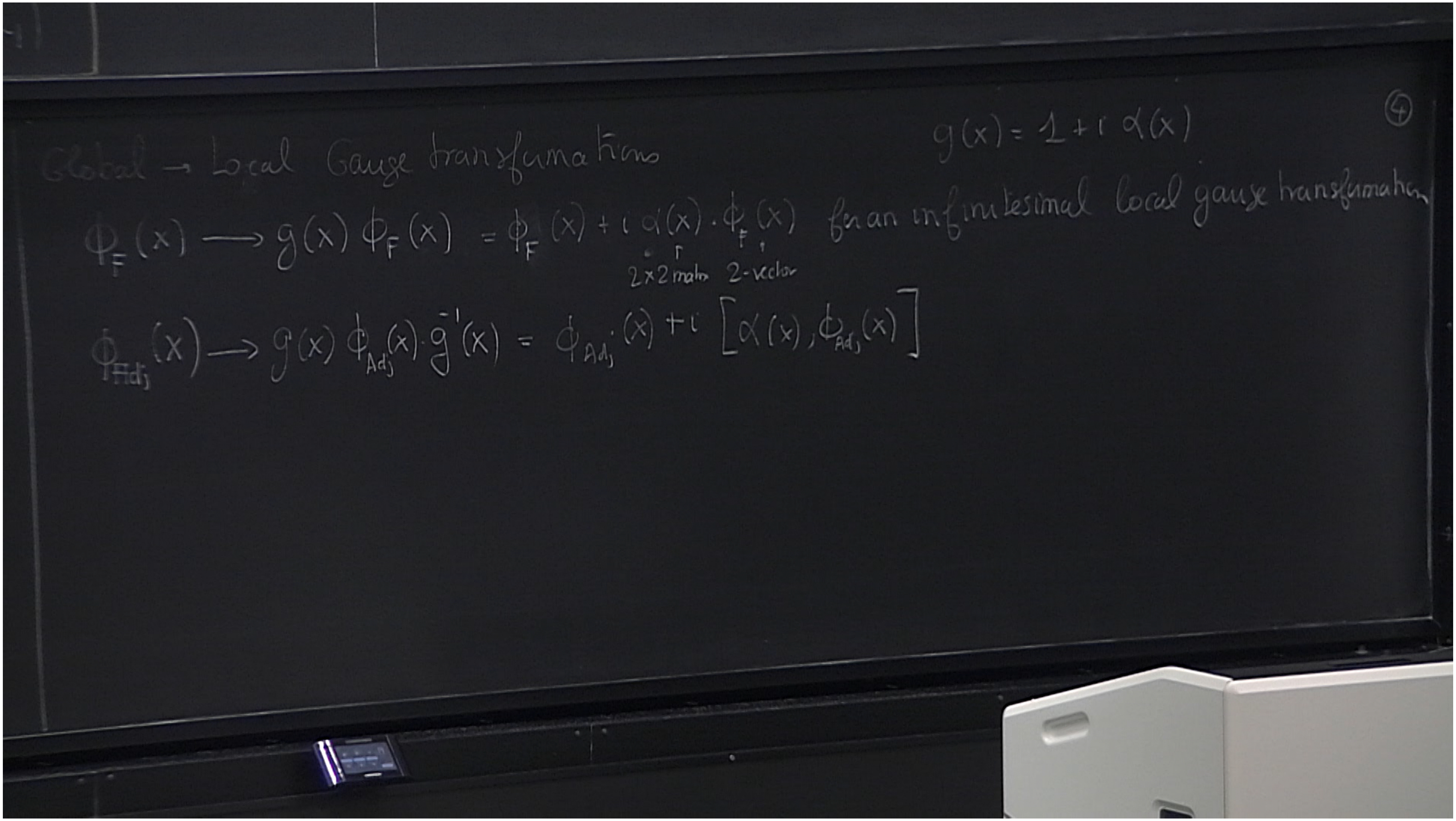
for an infinitesimal local gauge transformation

$$\phi_{Adj}(x) \longrightarrow g(x)\,\phi_{Adj}(x) \cdot g^{-1}(x) = \phi_{Adj}(x) - i\left[\alpha(x), \phi_{Adj}(x)\right]$$



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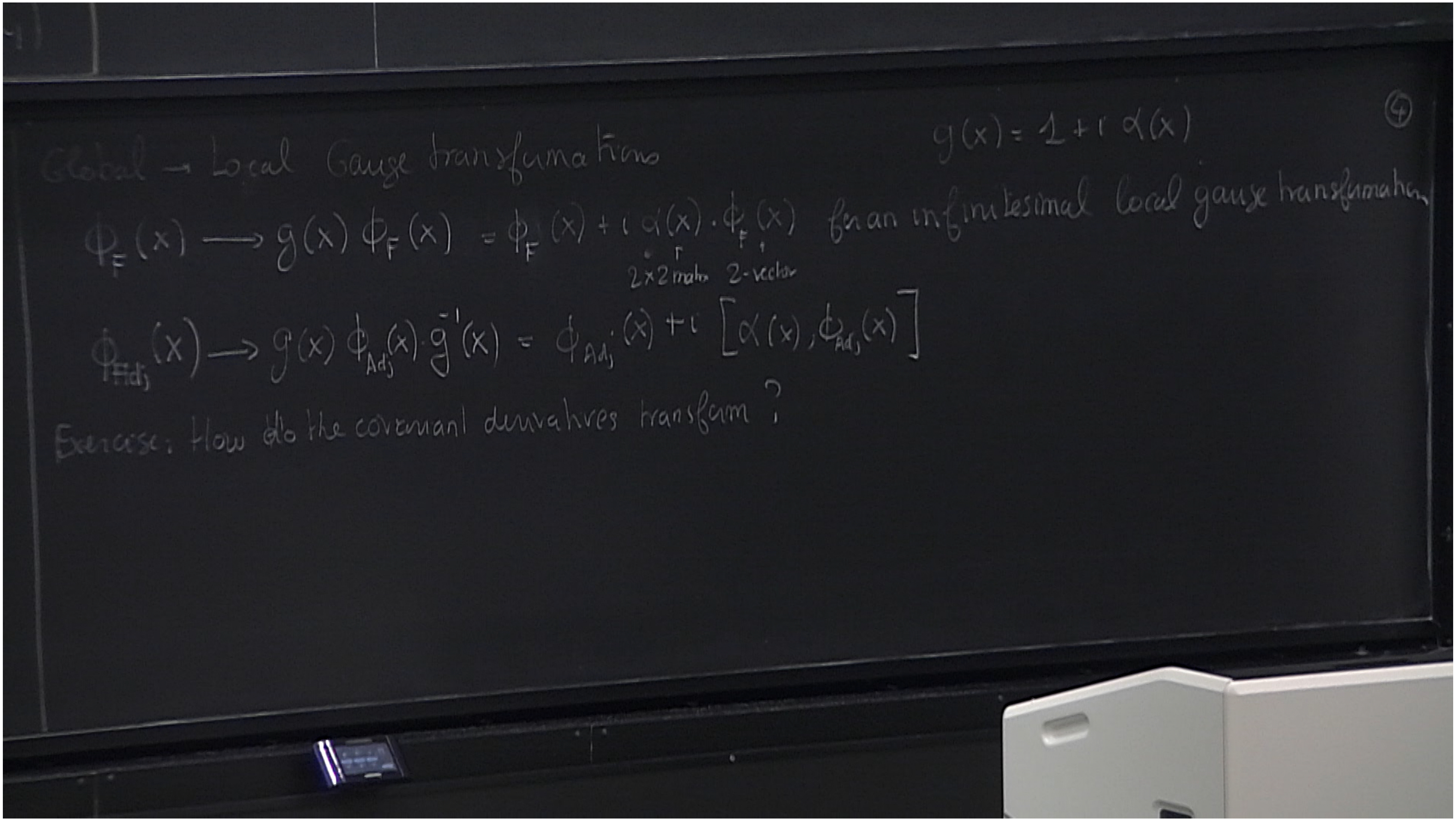

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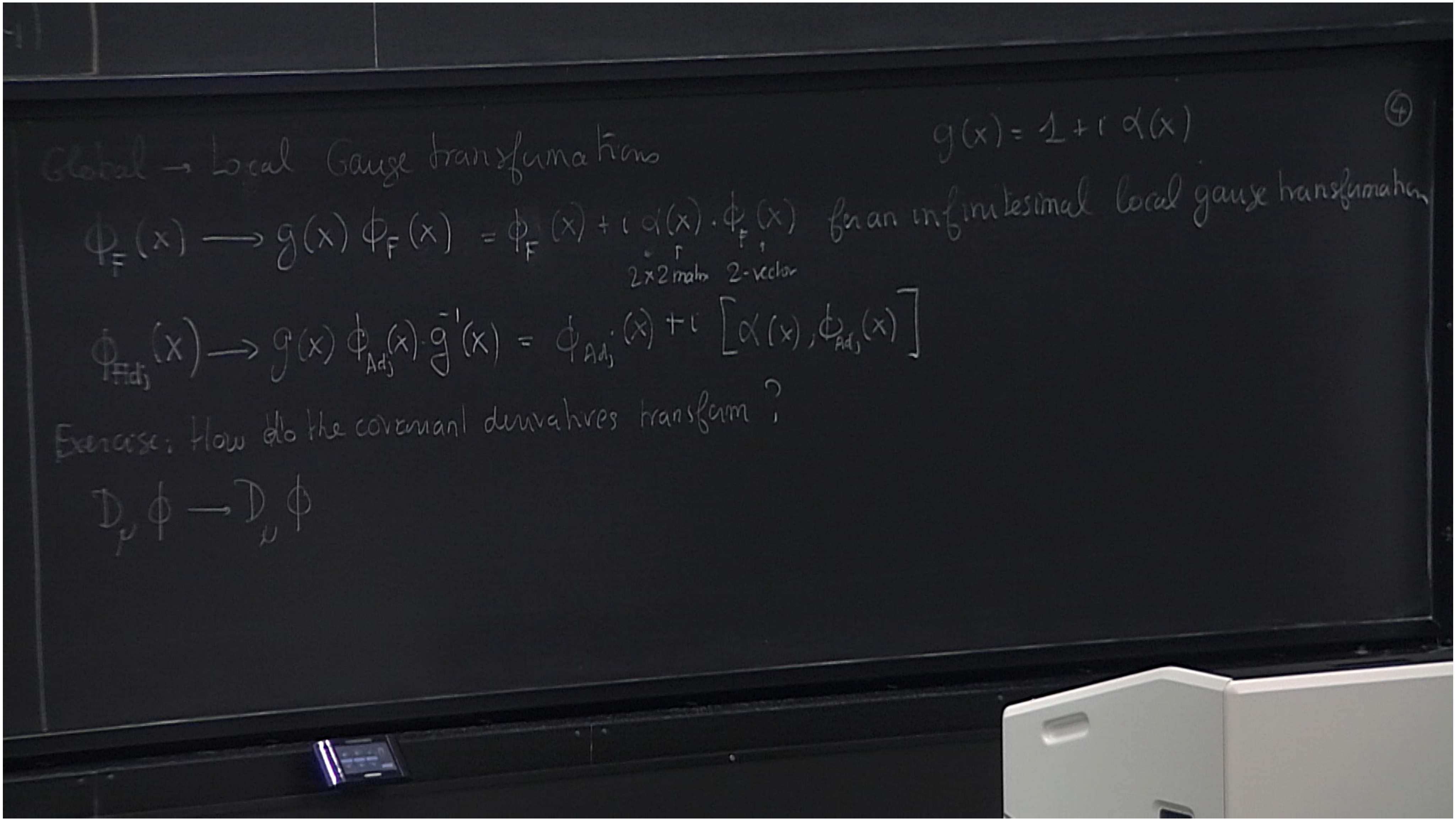
Exercise: How do the covariant derivatives transform?



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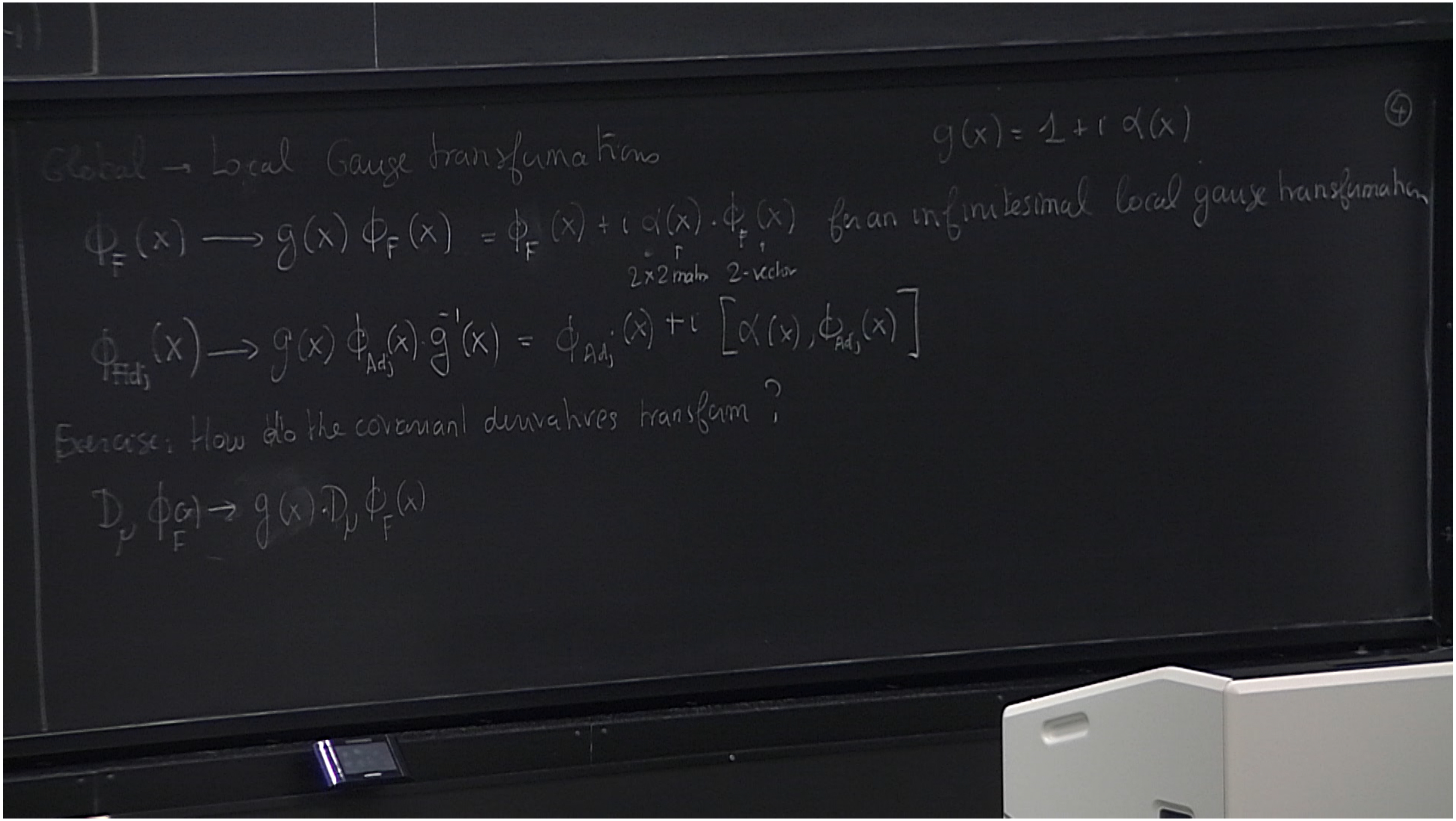
Exercise: How do the covariant derivatives transform?

$$D_\mu \phi \longrightarrow D_\mu \phi$$



④

$g(x) = 1 + i\,\alpha(x)$

Global → Local Gauge transformations

$$\phi_F(x) \longrightarrow g(x)\,\phi_F(x) = \phi_F(x) + i\,\underset{\substack{\uparrow\\ 2\times2\text{ matrix}}}{\alpha(x)}\cdot\underset{\substack{\uparrow\\ 2\text{-vector}}}{\phi_F(x)}$$ for an infinitesimal local gauge transformation

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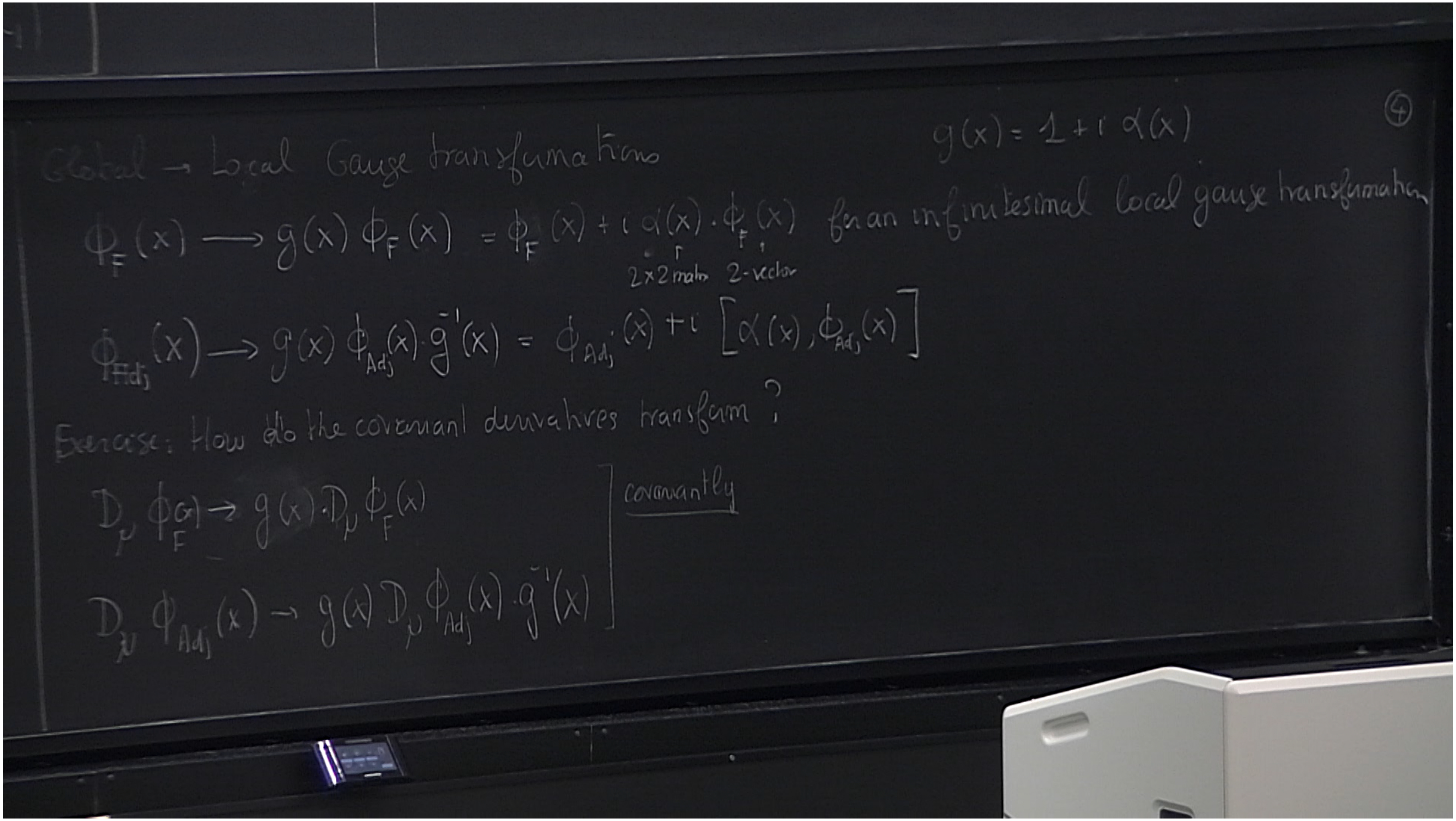
Exercise: How do the covariant derivatives transform?

$$D_\mu \phi_F(x) \to g(x)\cdot D_\mu \phi_F(x)$$



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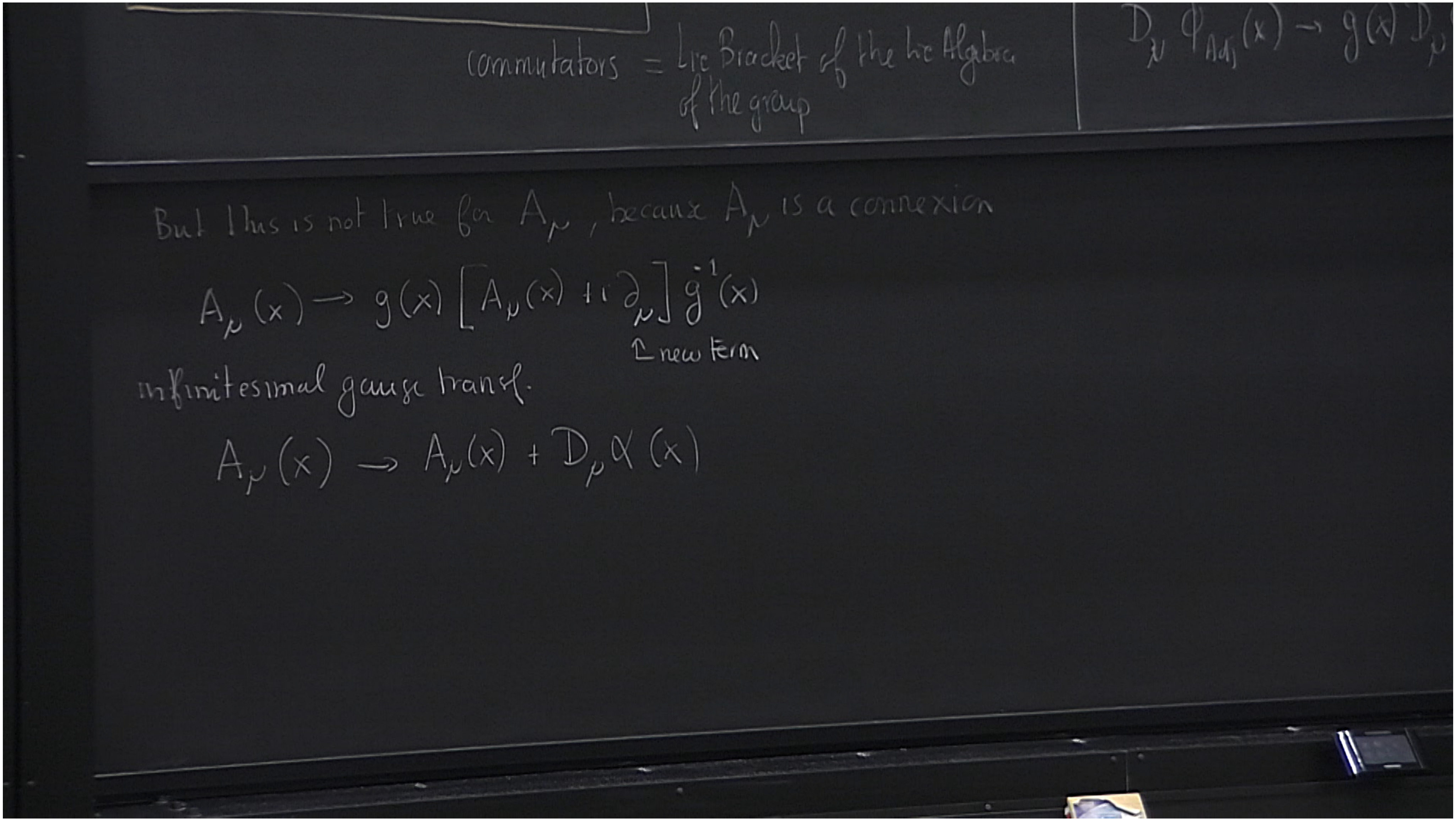
Exercise: How do the covariant derivatives transform?

$$D_\mu \phi_F(x) \to g(x) \cdot D_\mu \phi_F(x)$$

$$D_\mu \phi_{Adj}(x) \to g(x)\, D_\mu \phi_{Adj}(x) \cdot g^{-1}(x)$$

covariantly



commutators = Lie Bracket of the Lie Algebra of the group

$D_\nu \phi_{Adj}(x) \to g(x) D_\nu$

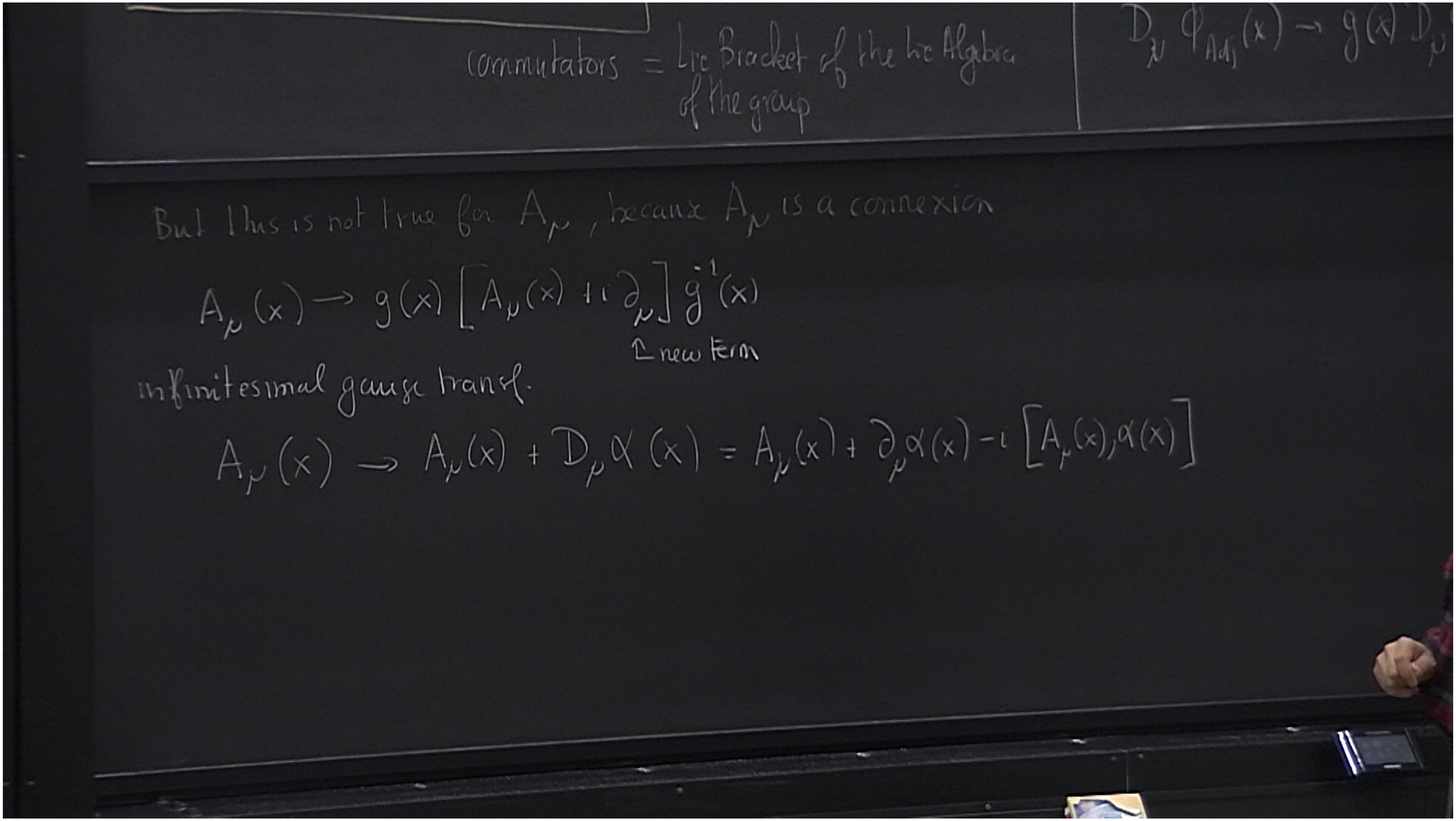
But this is not true for A_μ, because A_μ is a connexion

$$A_\mu(x) \to g(x)\left[A_\mu(x) + i\partial_\mu\right] g^{-1}(x)$$

↑ new term

infinitesimal gauge transf.

$$A_\mu(x) \to A_\mu(x) + D_\mu \alpha(x)$$



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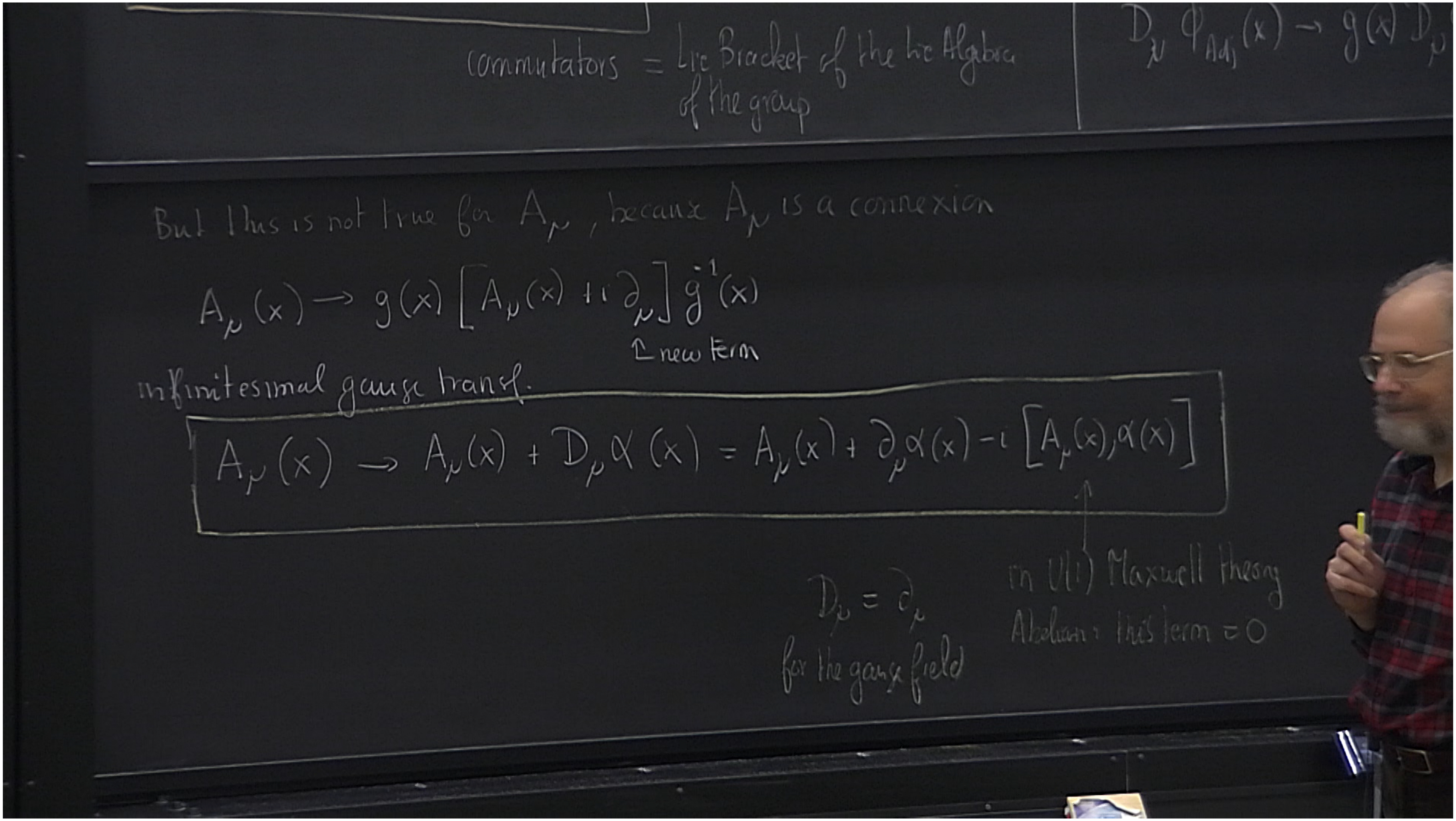

commutators = Lie Bracket of the Lie Algebra of the group

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But this is not true for A_μ, because A_μ is a connexion

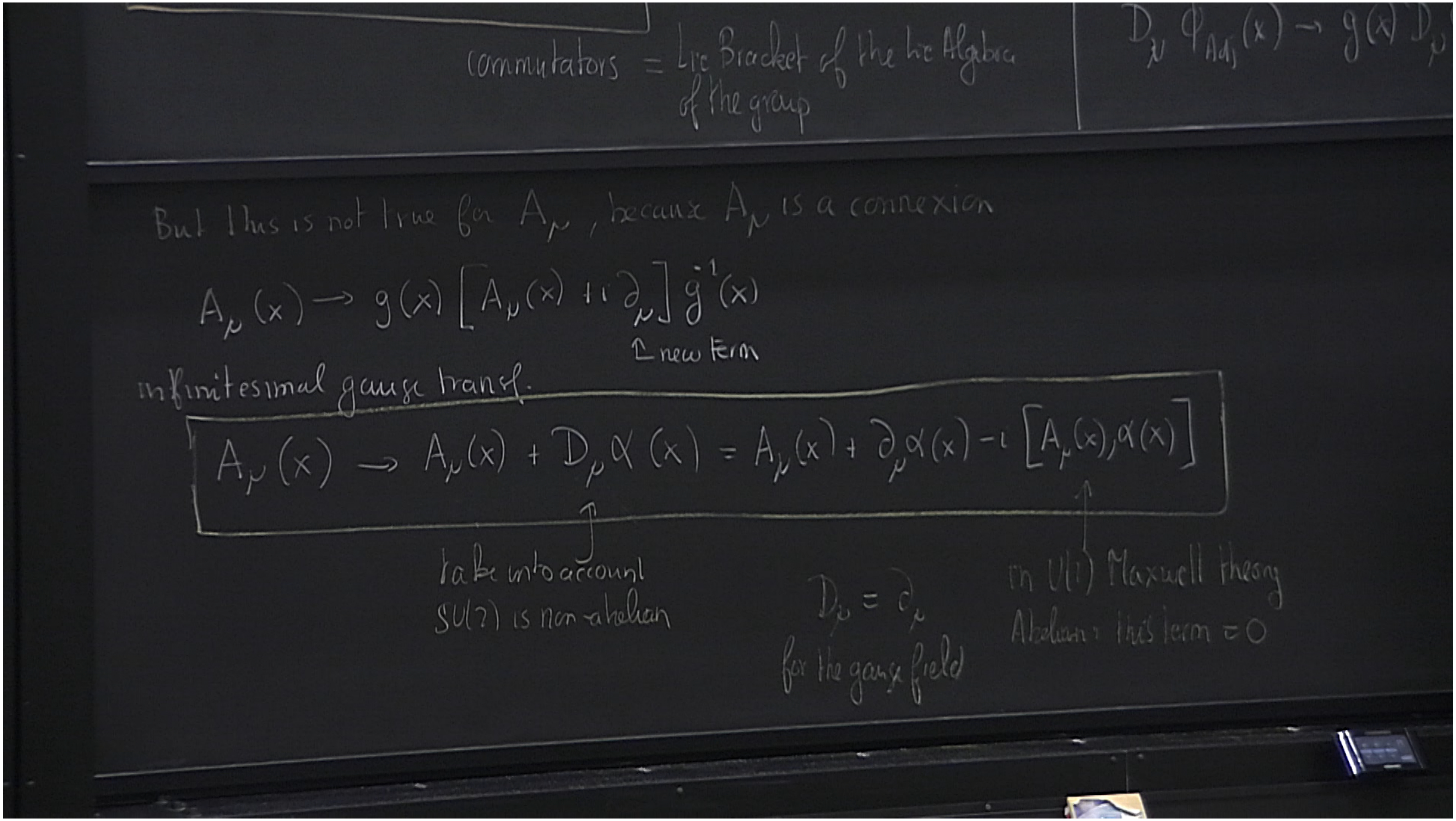
$$A_\mu(x) \to g(x) \left[A_\mu(x) + i \underset{\uparrow \text{ new term}}{\partial_\mu} \right] g^{-1}(x)$$

infinitesimal gauge transf.

$$\boxed{A_\mu(x) \to A_\mu(x) + D_\mu \alpha(x) = A_\mu(x) + \partial_\mu \alpha(x) - i \underset{\uparrow}{\left[A_\mu(x), \alpha(x) \right]}}$$

in U(1) Maxwell theory
Abelian: this term = 0

$D_\mu = \partial_\mu$
for the gauge field



commutators = Lie Bracket of the Lie Algebra of the group

$D_\nu \Phi_{Adj}(x) \to g(x) D_\mu$

But this is not true for A_μ, because A_μ is a connexion

$$A_\mu(x) \to g(x)\left[A_\mu(x) + i\partial_\mu\right] g^{-1}(x)$$

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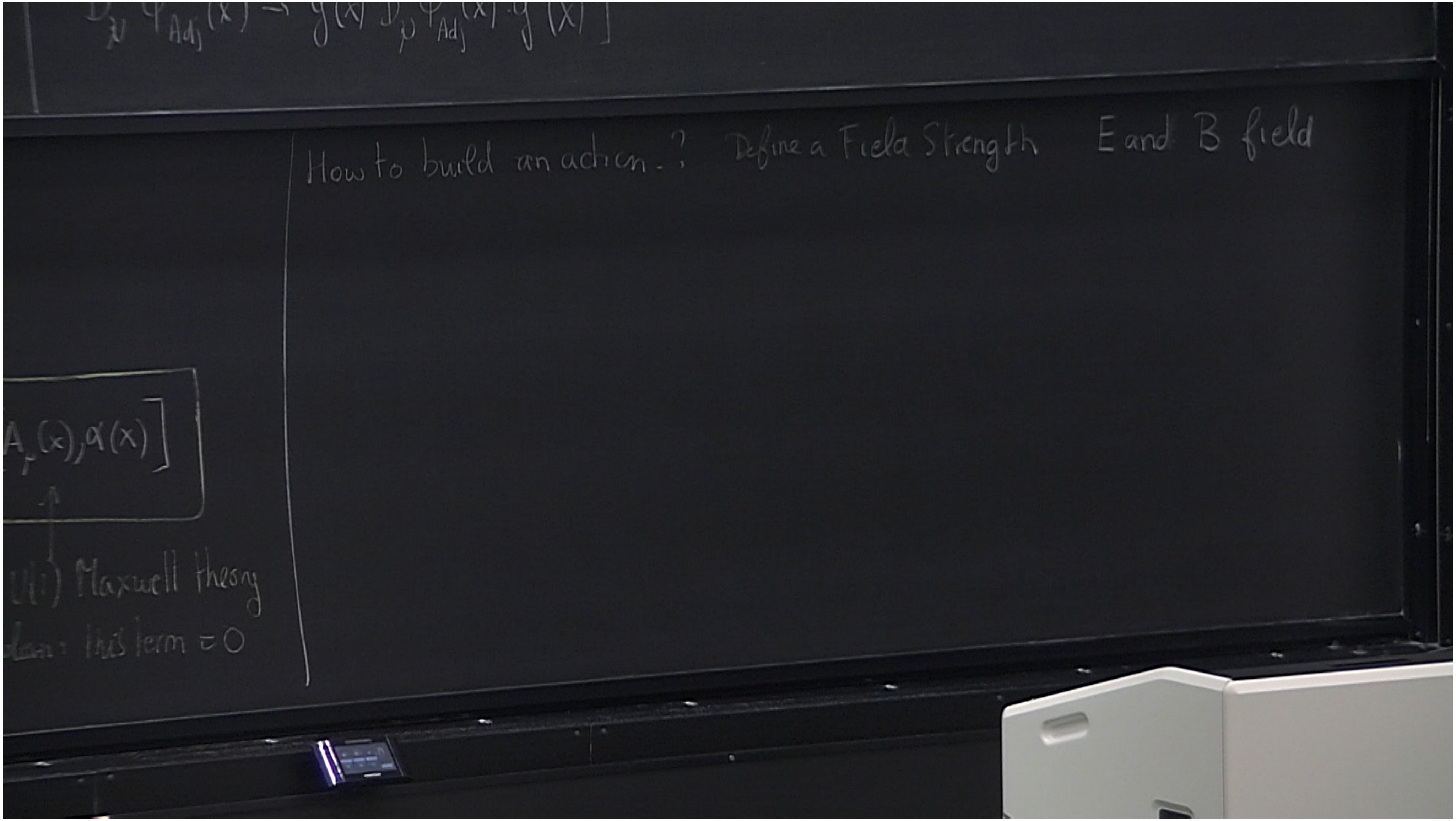
infinitesimal gauge transf.

$$\boxed{A_\mu(x) \to A_\mu(x) + D_\mu \alpha(x) = A_\mu(x) + \partial_\mu \alpha(x) - i\left[A_\mu(x), \alpha(x)\right]}$$

take into account SU(2) is non-abelian

$D_\mu = \partial_\mu$ for the gauge field

in U(1) Maxwell theory Abelian: this term = 0



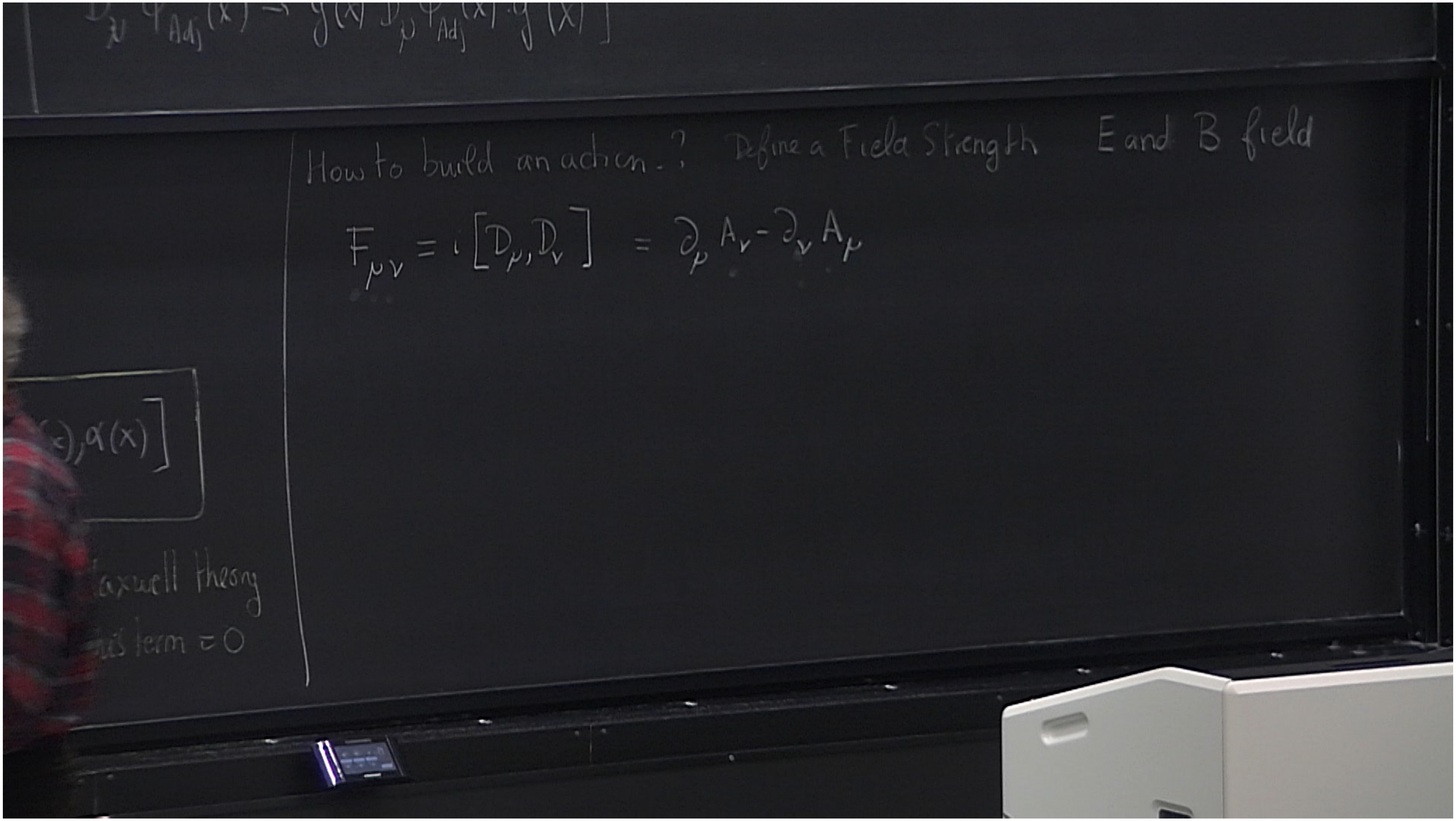
$D_\mu \varphi_{Adj}(x) \to g(x) D_\mu \varphi_{Adj}(x) \cdot g^{-1}(x)$]

How to build an action.? Define a Field Strength E and B field

$A_\mu(x), \alpha(x)$]

U(1) Maxwell theory

: this term = 0



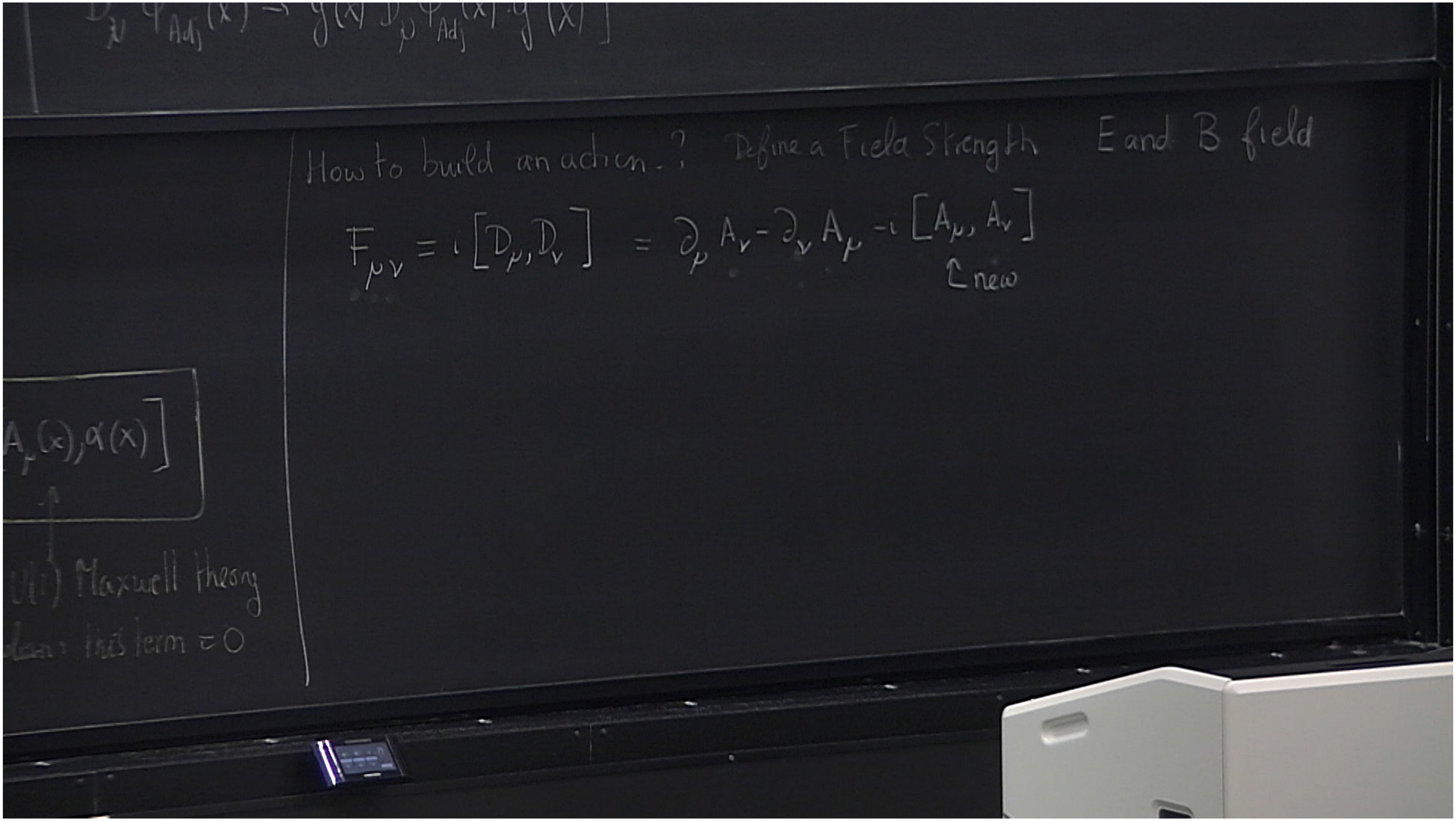
How to build an action.? Define a Field Strength   E and B field

$$F_{\mu\nu} = i\left[D_\mu, D_\nu\right] = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$\boxed{\ldots),\alpha(x)\big]}$$

axwell theory

is term = 0



$D_\nu \varphi_{Adj}(x) \to g(x) D_\mu \varphi_{Adj}(x) \cdot g(x)]$

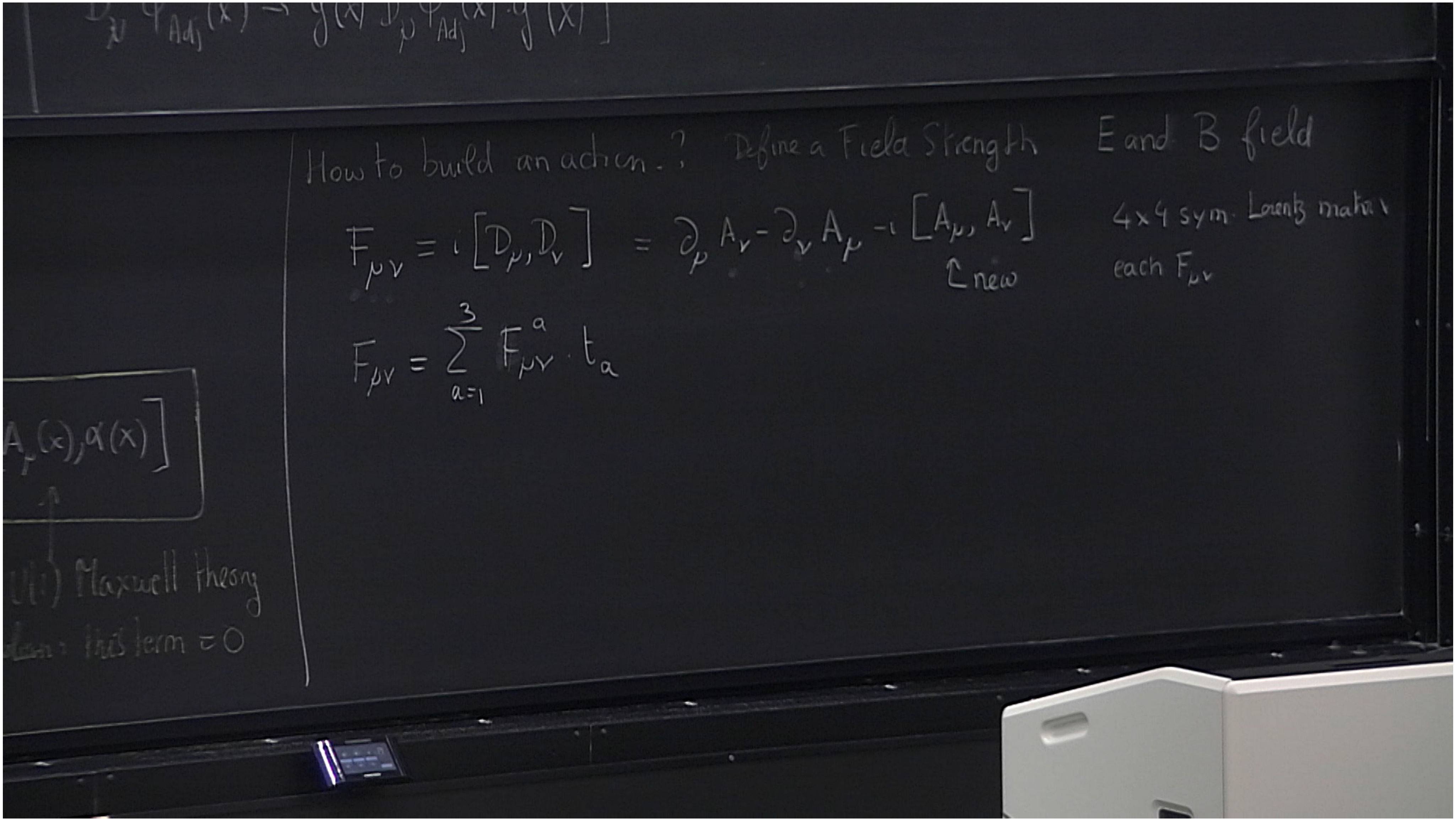
$A_\mu(x), \alpha(x)]$

U(1) Maxwell theory

this term = 0

How to build an action.? Define a Field Strength E and B field

$$F_{\mu\nu} = i [D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu - i \underset{\uparrow \text{new}}{[A_\mu, A_\nu]}$$



$D_\nu \varphi_{Adj}(x) \to g(x) D_\nu \varphi_{Adj}(x) \cdot g(x)$]

$A_\mu(x), \alpha(x)]$

U(1) Maxwell theory

: this term = 0

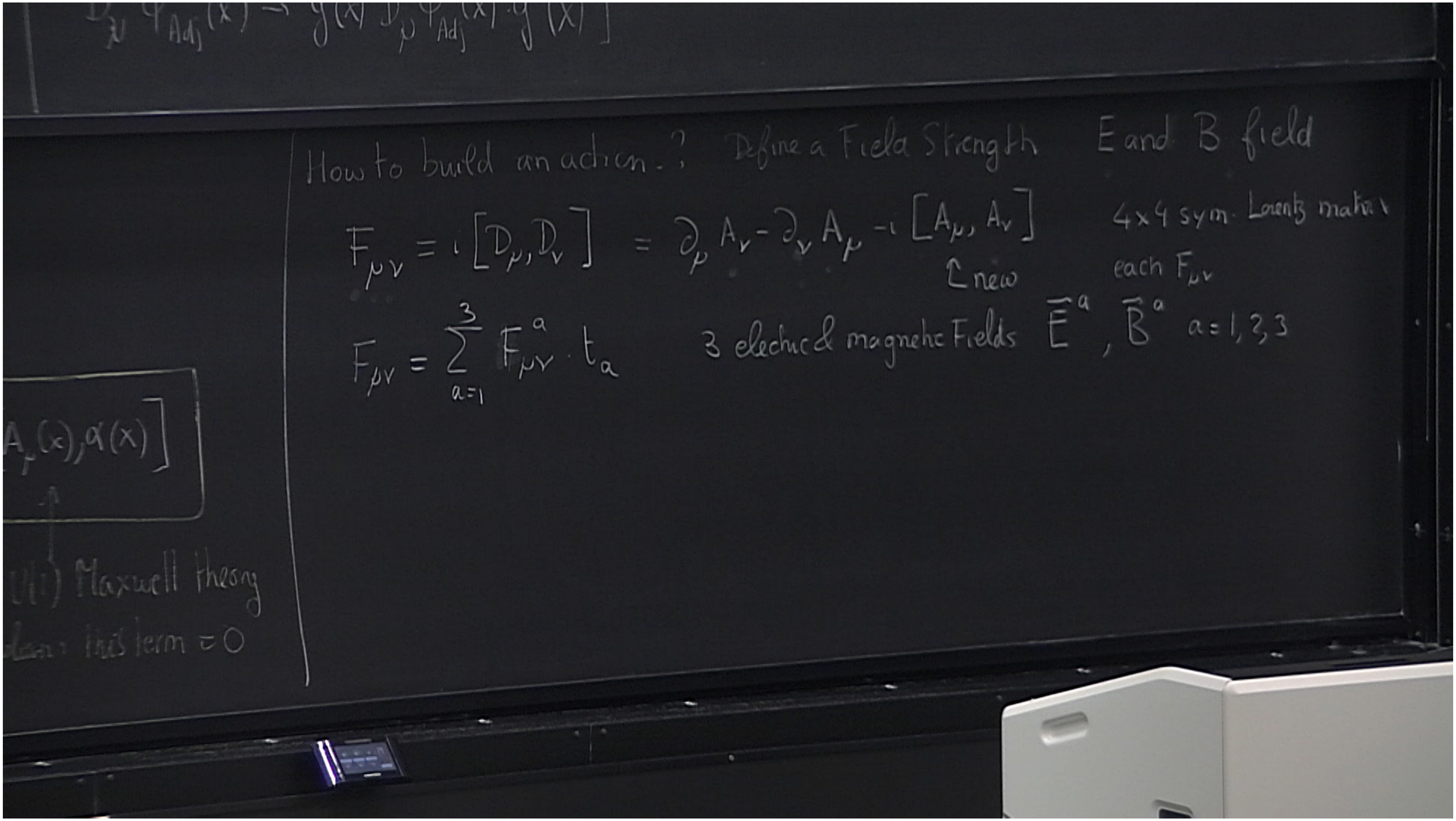
How to build an action.? Define a Field Strength E and B field

$$F_{\mu\nu} = i[D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu - i\underset{\uparrow \text{new}}{[A_\mu, A_\nu]}$$

4×4 sym. Lorentz matrix

each $F_{\mu\nu}$

$$F_{\mu\nu} = \sum_{a=1}^{3} F^a_{\mu\nu} \cdot t_a$$



$D_\nu \varphi_{Adj}(x) \to g(x) D_\mu \varphi_{Adj}(x) \cdot g(x)]$

How to build an action.? Define a Field Strength E and B field

$$F_{\mu\nu} = i[D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu \underbrace{- i[A_\mu, A_\nu]}_{\text{new}}$$

4×4 sym. Lorentz matri

each $F_{\mu\nu}$

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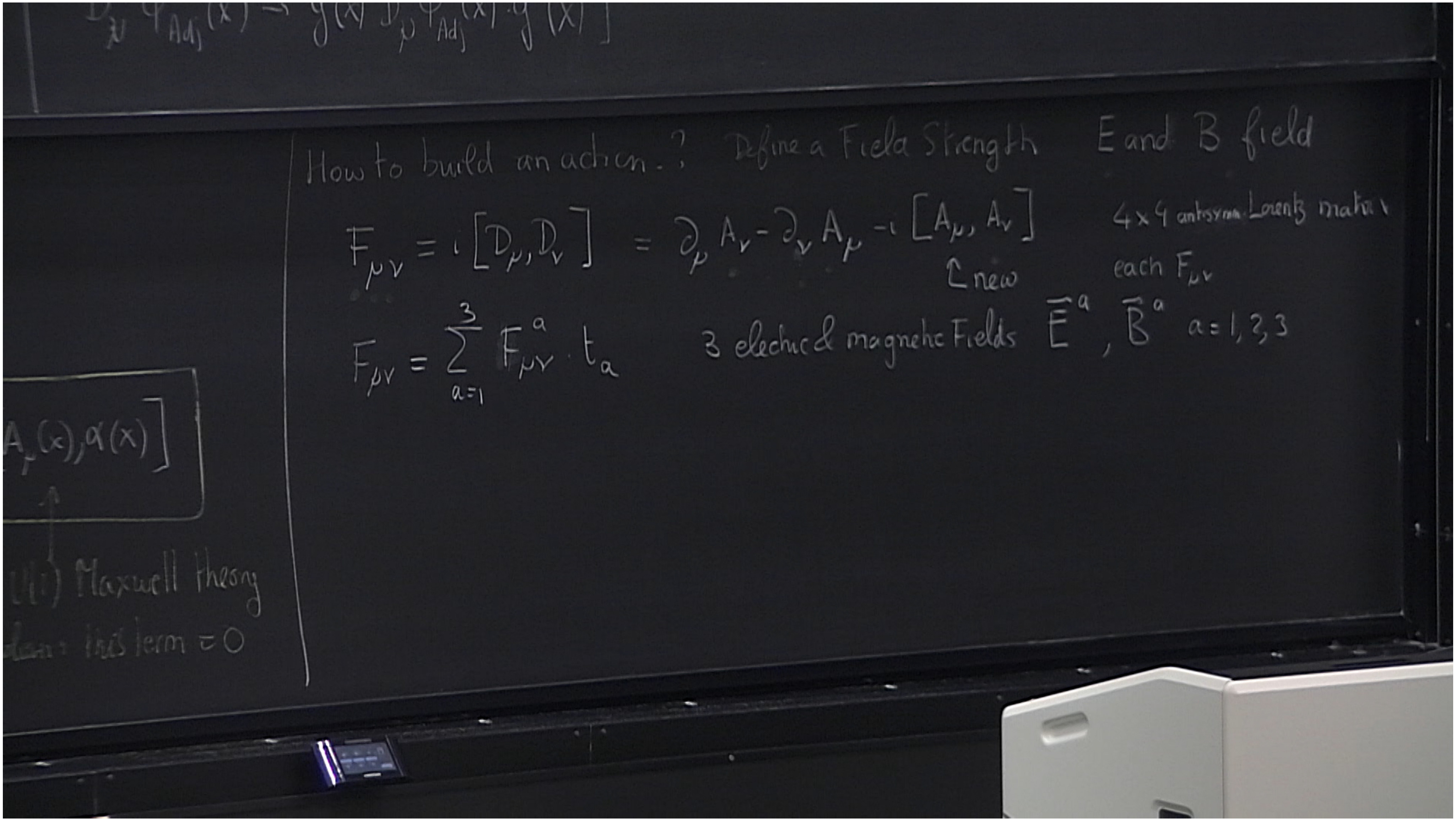
3 electric & magnetic Fields $\vec{E}^a, \vec{B}^a$ $a = 1, 2, 3$

$\boxed{A_\mu(x), \alpha(x)]}$

(iii) Maxwell theory

: this term = 0



$D_\nu \varphi_{Adj}(x) \to g(x) D_\mu \varphi_{Adj}(x) \cdot g(x)$]

$A_\mu(x), \alpha(x)]$

(ii) Maxwell theory

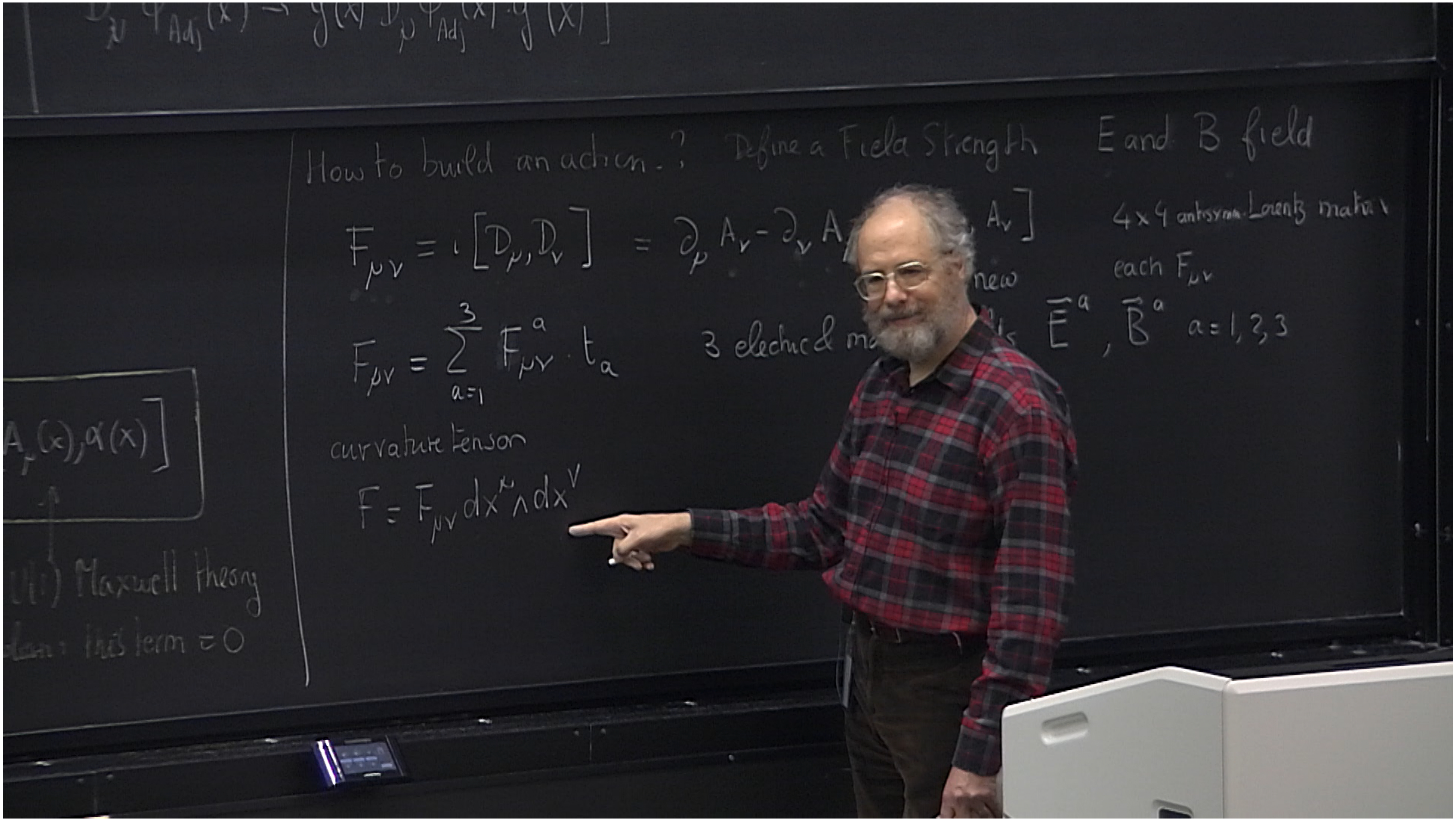
this term = 0

How to build an action..? Define a Field Strength E and B field

$F_{\mu\nu} = i\,[D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu - i\,[A_\mu, A_\nu]$ 4×4 antisymm. Lorentz matrix

new each $F_{\mu\nu}$

$F_{\mu\nu} = \sum_{a=1}^{3} F^a_{\mu\nu} \cdot t_a$ 3 electric & magnetic Fields $\vec{E}^a, \vec{B}^a$ $a = 1, 2, 3$



$D_\nu \varphi_{Adj}(x) \to g(x) D_\nu \varphi_{Adj}(x) \cdot g^{-1}(x)$]

How to build an action..? Define a Field Strength E and B field

$F_{\mu\nu} = i[D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu$ … $A_\nu]$

4×4 antisym. Lorentz matrix

new

each $F_{\mu\nu}$

$F_{\mu\nu} = \sum_{a=1}^{3} F^a_{\mu\nu} \cdot t_a$

3 electric & ma…ls $\vec{E}^a, \vec{B}^a$ $a = 1, 2, 3$

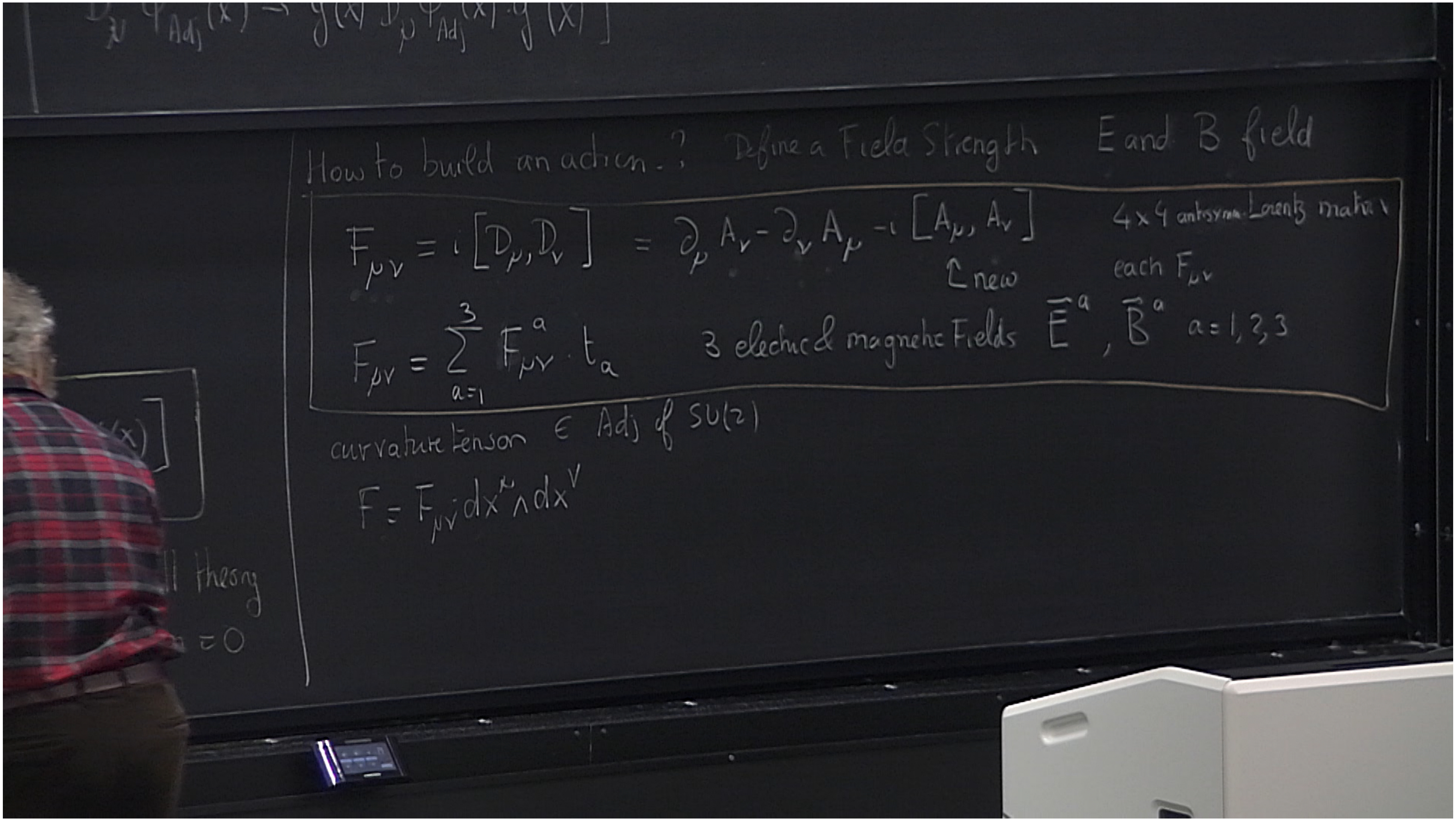
curvature tensor

$F = F_{\mu\nu} dx^\mu \wedge dx^\nu$

$A_\mu(x), \alpha(x)]$

U(1) Maxwell theory

…this term = 0



$D_\mu \varphi_{Adj}(x) \to g(x) D_\mu \varphi_{Adj}(x) \cdot g^{-1}(x)]$

How to build an action.? Define a Field Strength E and B field

$$\boxed{\begin{aligned} F_{\mu\nu} &= i\,[D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu - i\,[A_\mu, A_\nu] \quad \text{4×4 antisym. Lorentz matrix} \\ &\qquad\qquad\qquad\qquad\qquad\qquad\quad \underset{\uparrow}{}\text{new} \qquad \text{each } F_{\mu\nu} \\ F_{\mu\nu} &= \sum_{a=1}^{3} F^a_{\mu\nu} \cdot t_a \qquad \text{3 electric \& magnetic Fields } \vec{E}^a, \vec{B}^a \quad a = 1, 2, 3 \end{aligned}}$$

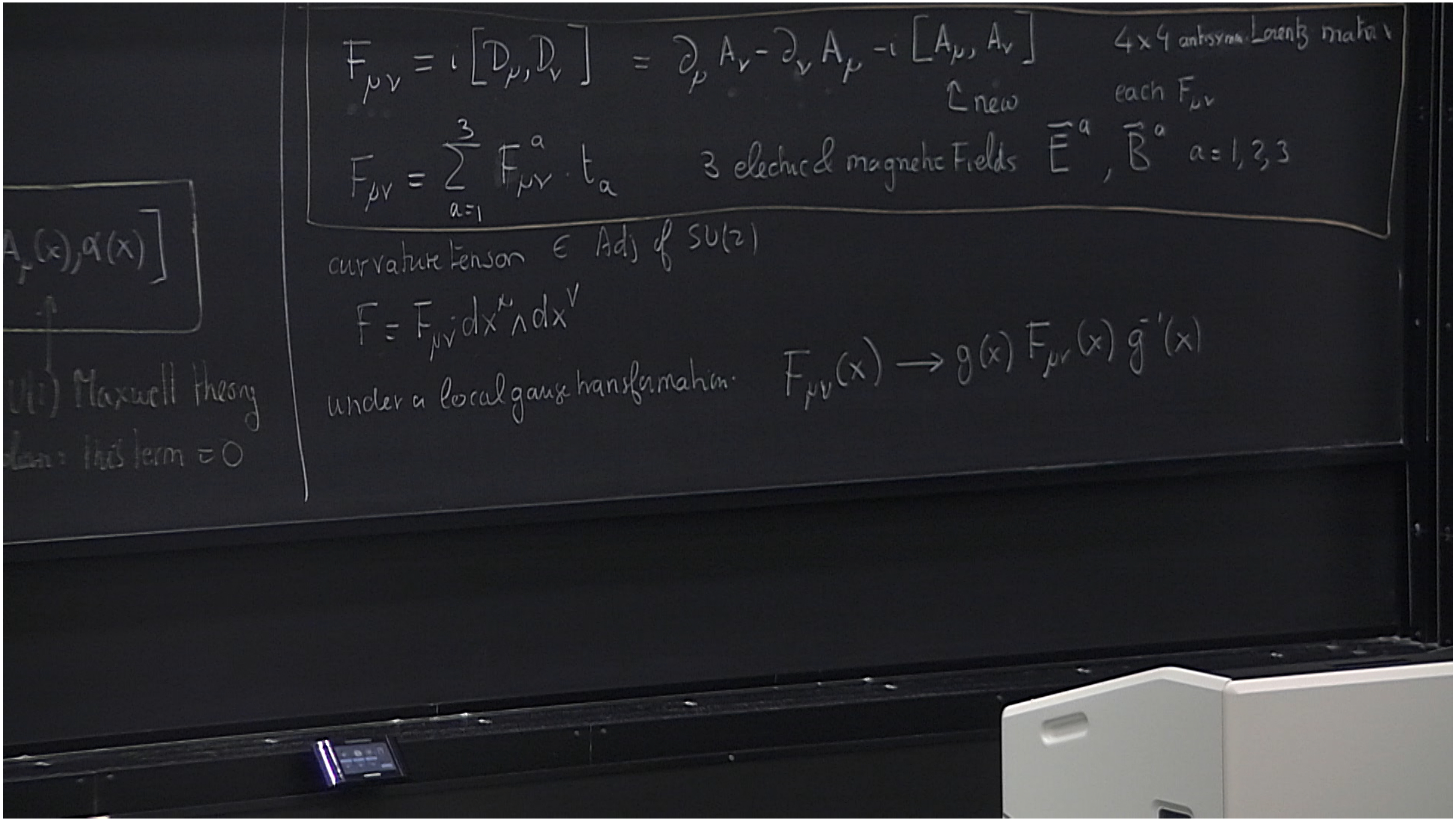
curvature tensor $\in$ Adj of $SU(2)$

$F = F_{\mu\nu}\, dx^\mu \wedge dx^\nu$

$(x)]$

ll theory

$= 0$



$A_\mu(x), \alpha(x)]$

$U(1)$ Maxwell theory

: this term $= 0$

$$F_{\mu\nu} = i[D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu \underbrace{- i[A_\mu, A_\nu]}_{\text{new}}$$

4×4 antisym. Lorentz matrix

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curvature tensor $\in$ Adj of $SU(2)$

$$F = F_{\mu\nu} dx^\mu \wedge dx^\nu$$

under a local gauge transformation: $F_{\mu\nu}(x) \longrightarrow g(x) F_{\mu\nu}(x) g^{-1}(x)$

representat of Lorentz x representat of SU(2)

$\psi_{i=1,2} = (\psi_i^\alpha, \alpha = 1,..4)$ α Dirac Indices

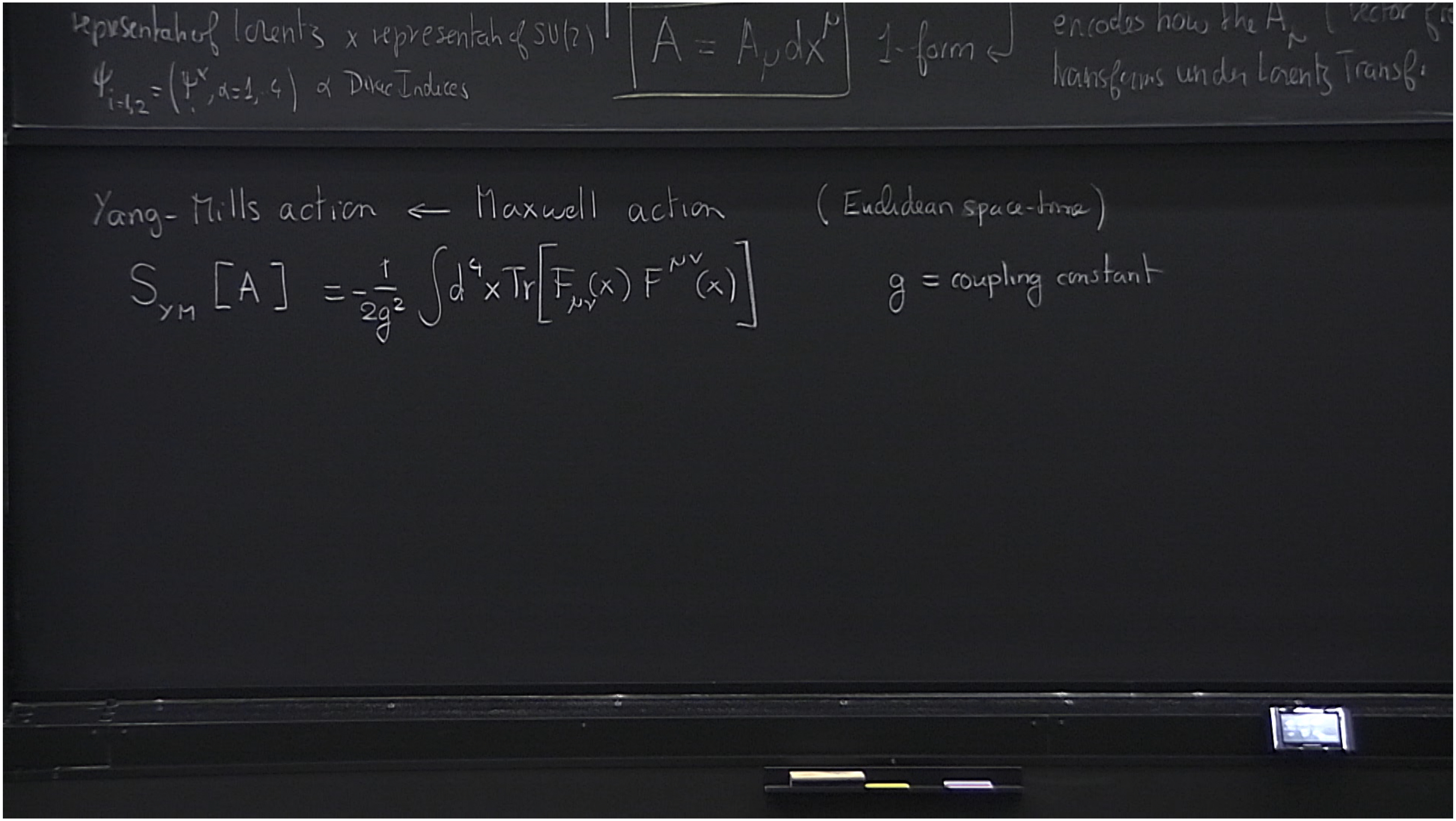
$$\boxed{A = A_\mu dx^\mu}$$ 1-form ↩

encodes how the A_μ (vector

transforms under Lorentz Transf.

Yang-Mills action ← Maxwell action (Euclidean space-time)

$$S_{YM}[A] = \int d^4x$$



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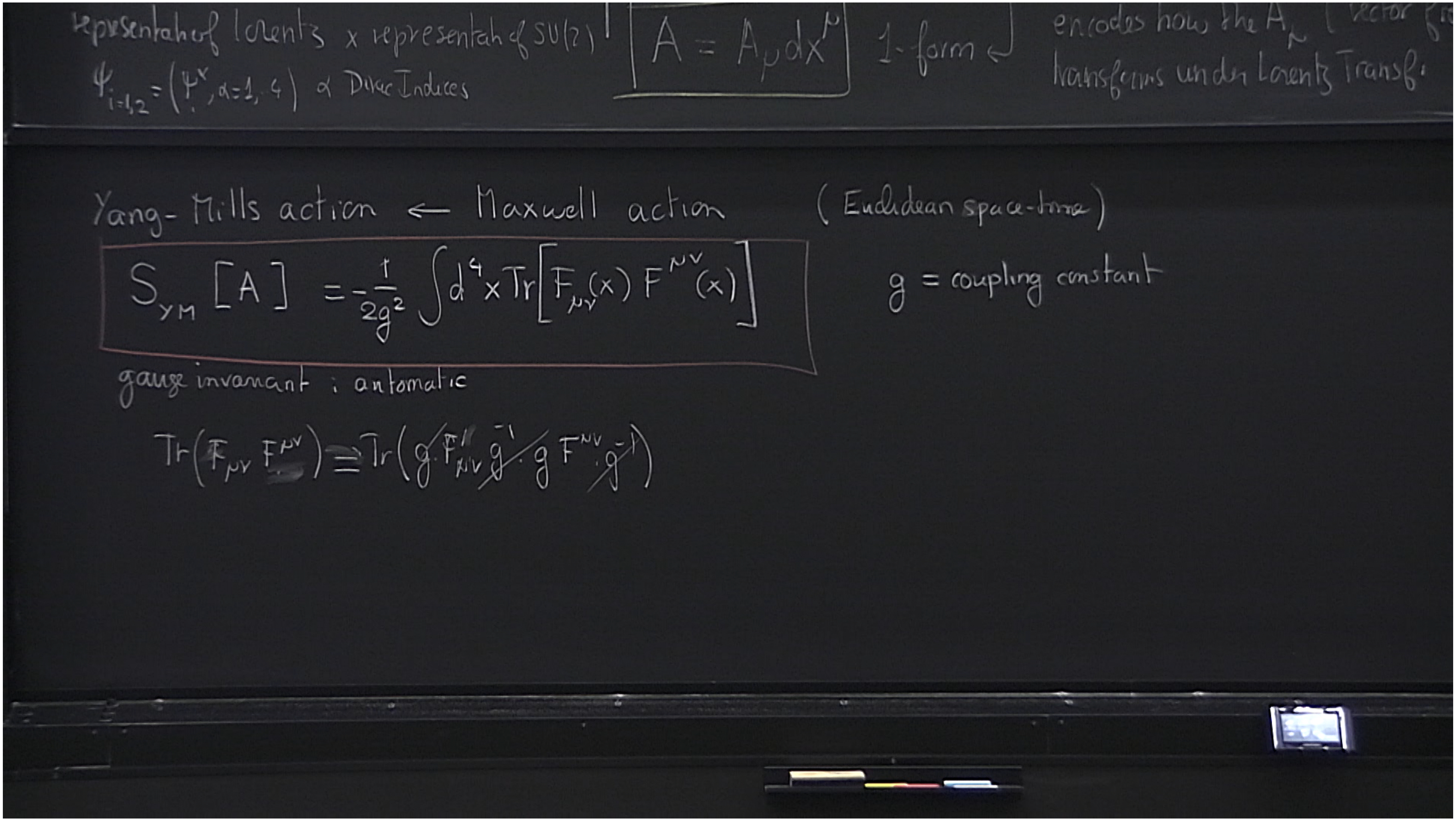
g = coupling constant



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g = coupling constant

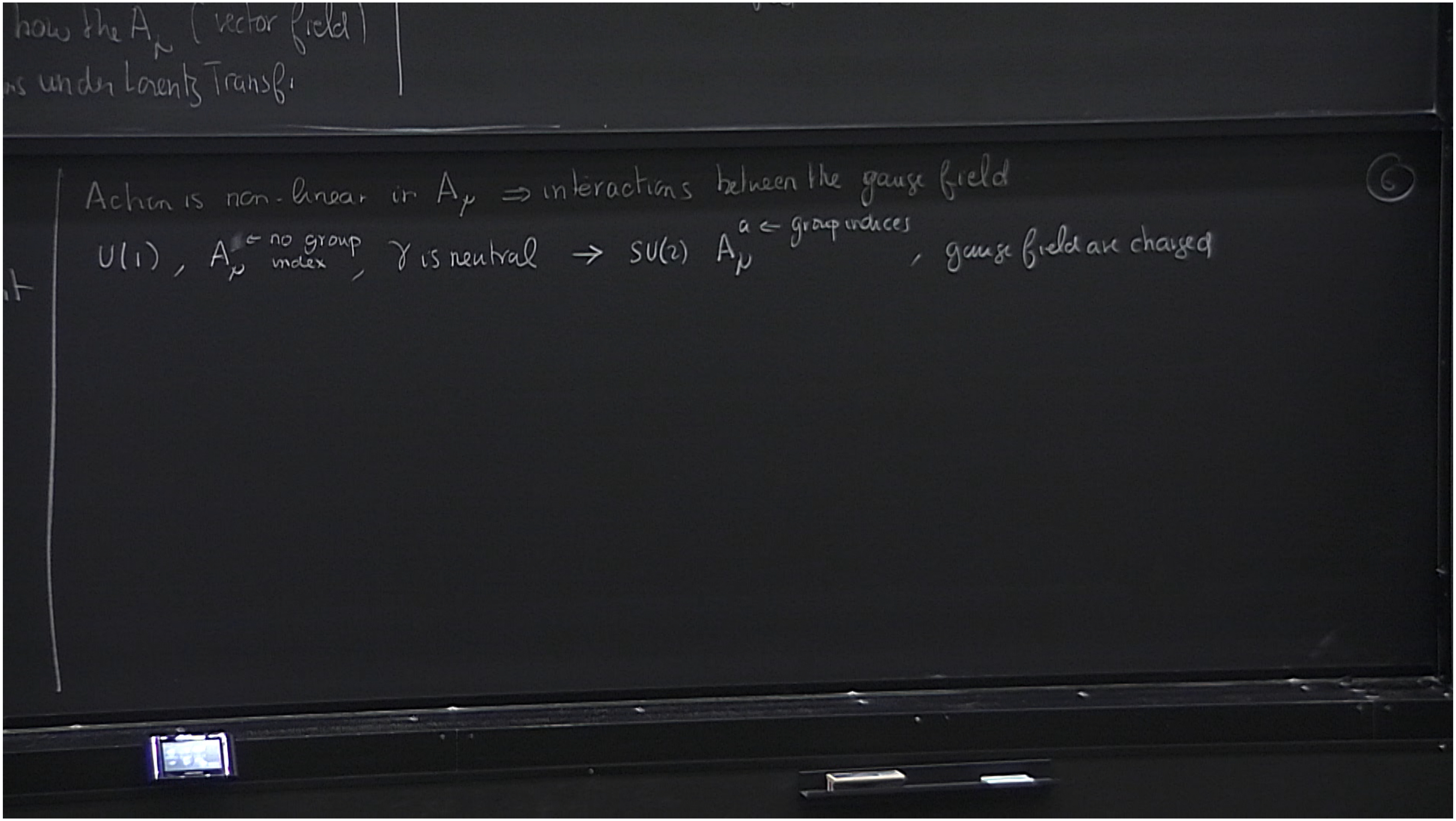
gauge invariant : automatic

$$\mathrm{Tr}(F_{\mu\nu} F^{\mu\nu}) = \mathrm{Tr}(g \cdot F'_{\mu\nu} \cancel{g^{-1} \cdot g} F^{\mu\nu} \cdot \cancel{g^{-1}})$$

how the A_μ (vector field)
ns under Lorentz Transf.

⑥

Action is non-linear in A_μ



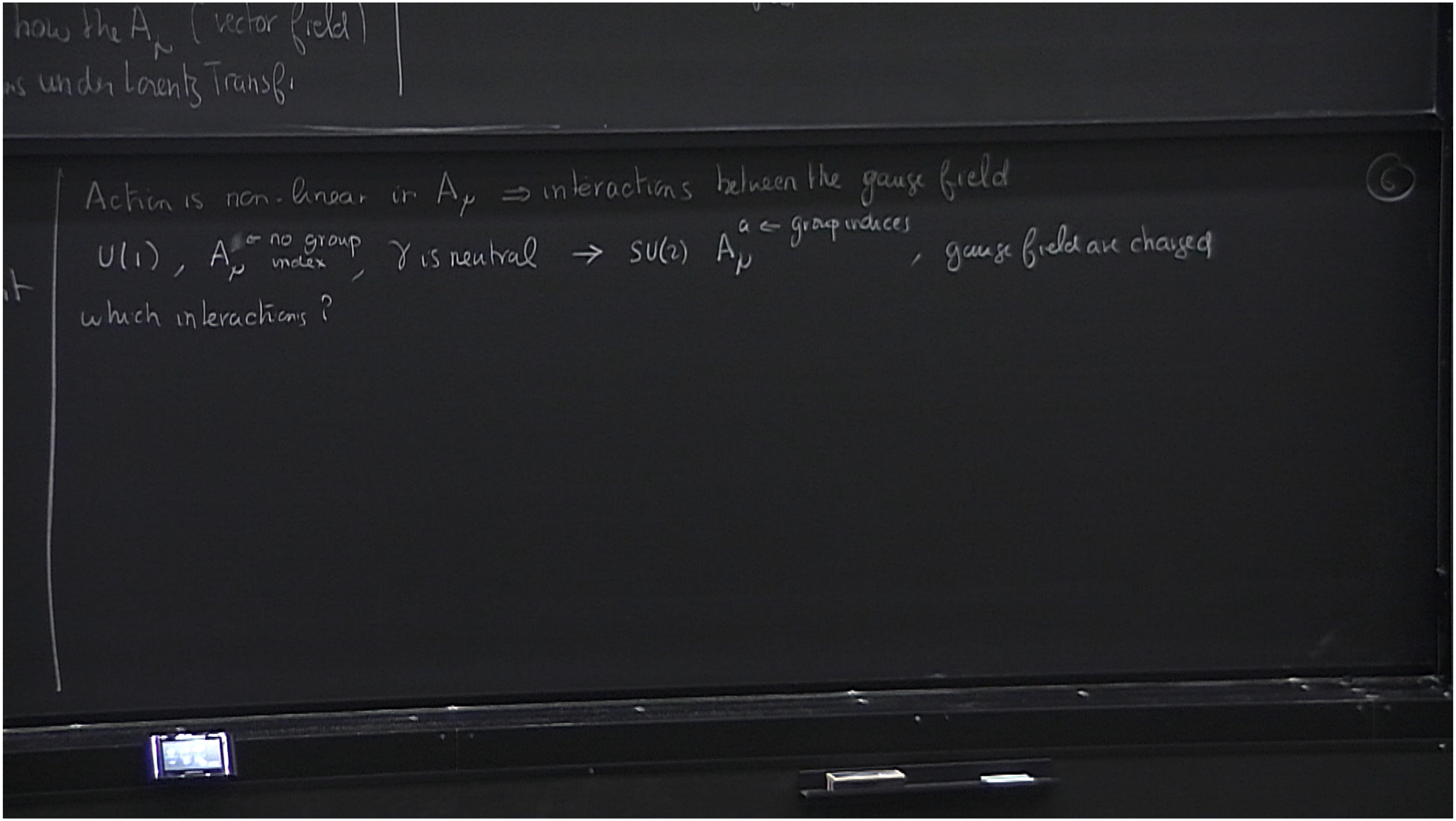
how the A_μ (vector field)

s under Lorentz Transf.

⑥

Action is non-linear in A_μ $\Rightarrow$ interactions between the gauge field

$U(1)$, A_μ ← no group index, γ is neutral → $SU(2)$ A_μ^a ← group indices, gauge field are charged



how the A_μ (vector field)

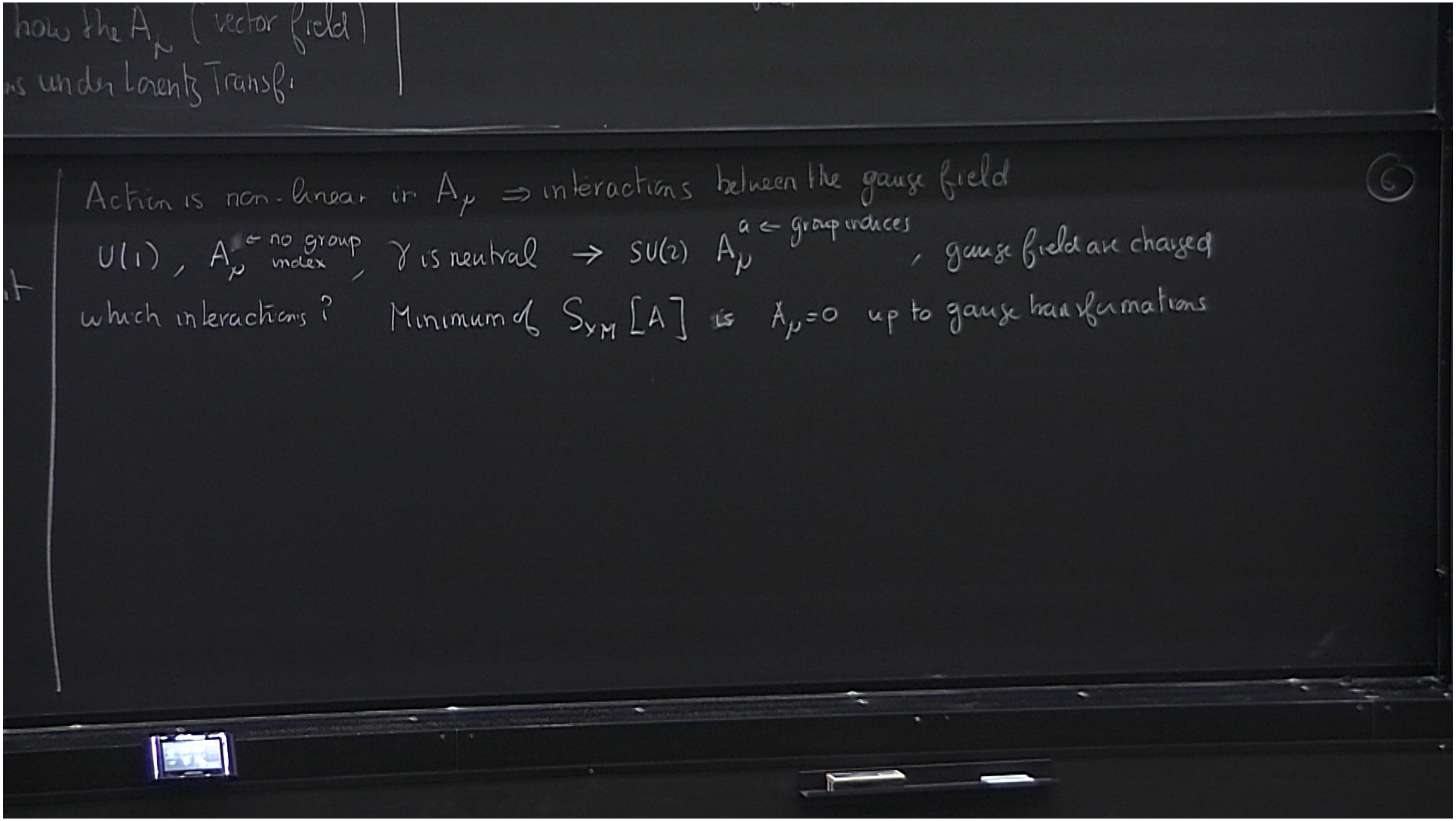
ns under Lorentz Transf.

t

Action is non-linear in A_μ ⇒ interactions between the gauge field ⑥

U(1), $A_\mu \leftarrow$ no group index, γ is neutral → SU(2) $A_\mu^{a} \leftarrow$ group indices, gauge field are charged

which interactions ?



how the A_μ (vector field)

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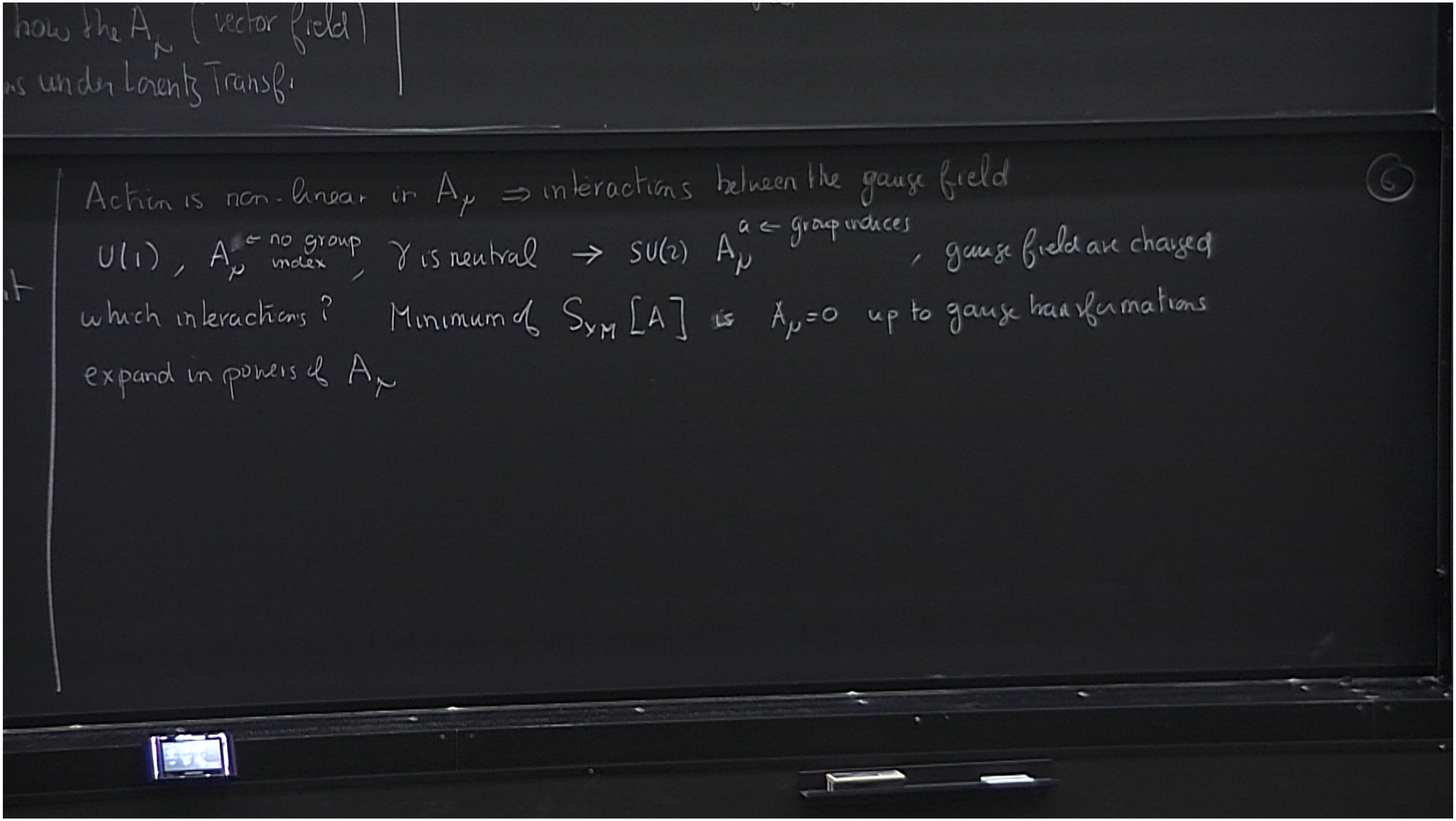
Action is non-linear in A_μ $\Rightarrow$ interactions between the gauge field ⑥

U(1), A_μ ← no group index, γ is neutral → SU(2) A_μ^a ← group indices, gauge field are charged

which interactions? Minimum of $S_{YM}[A]$ is $A_\mu = 0$ up to gauge transformations



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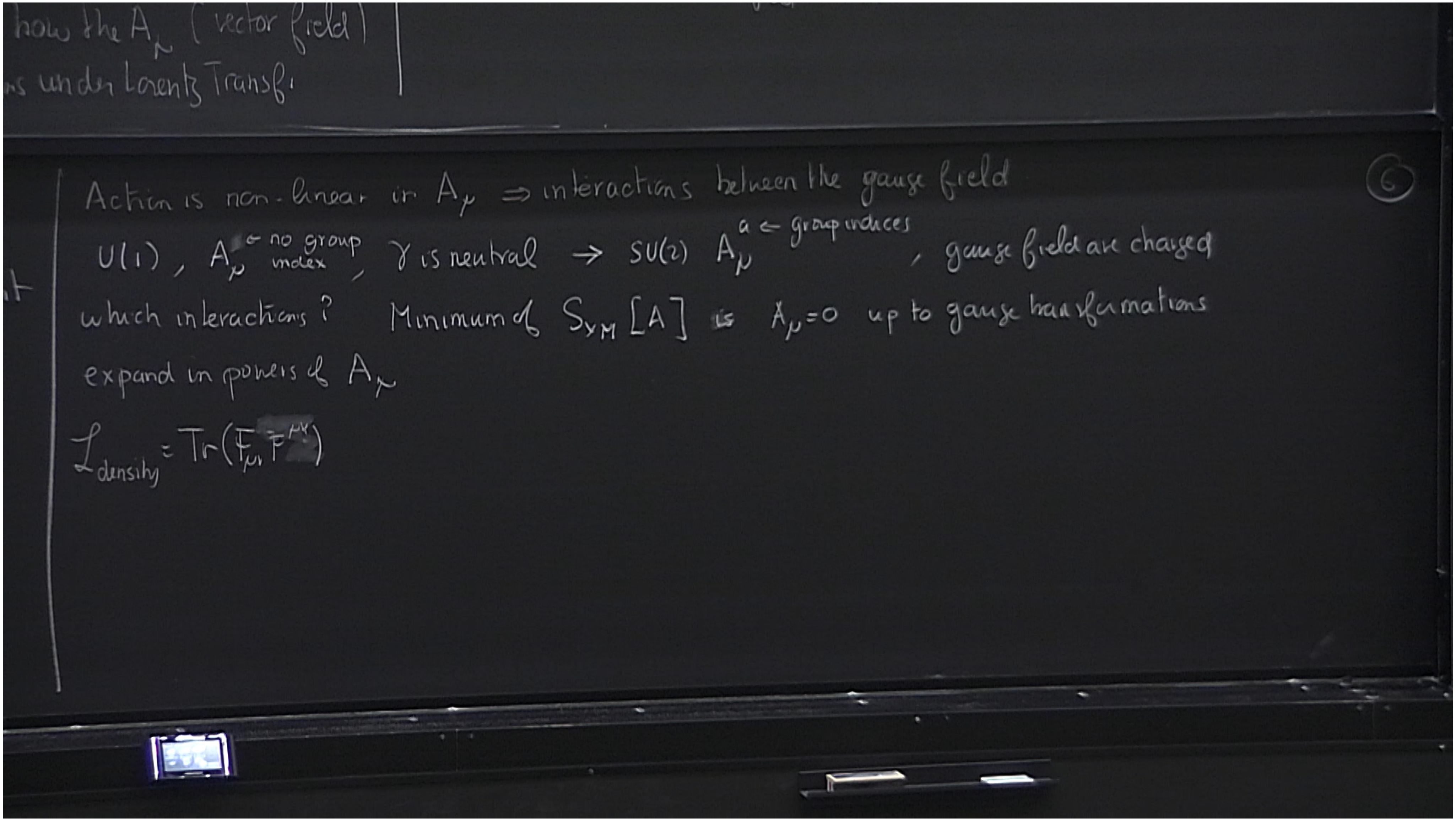
expand in powers of A_μ



how the A_μ (vector field)

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t

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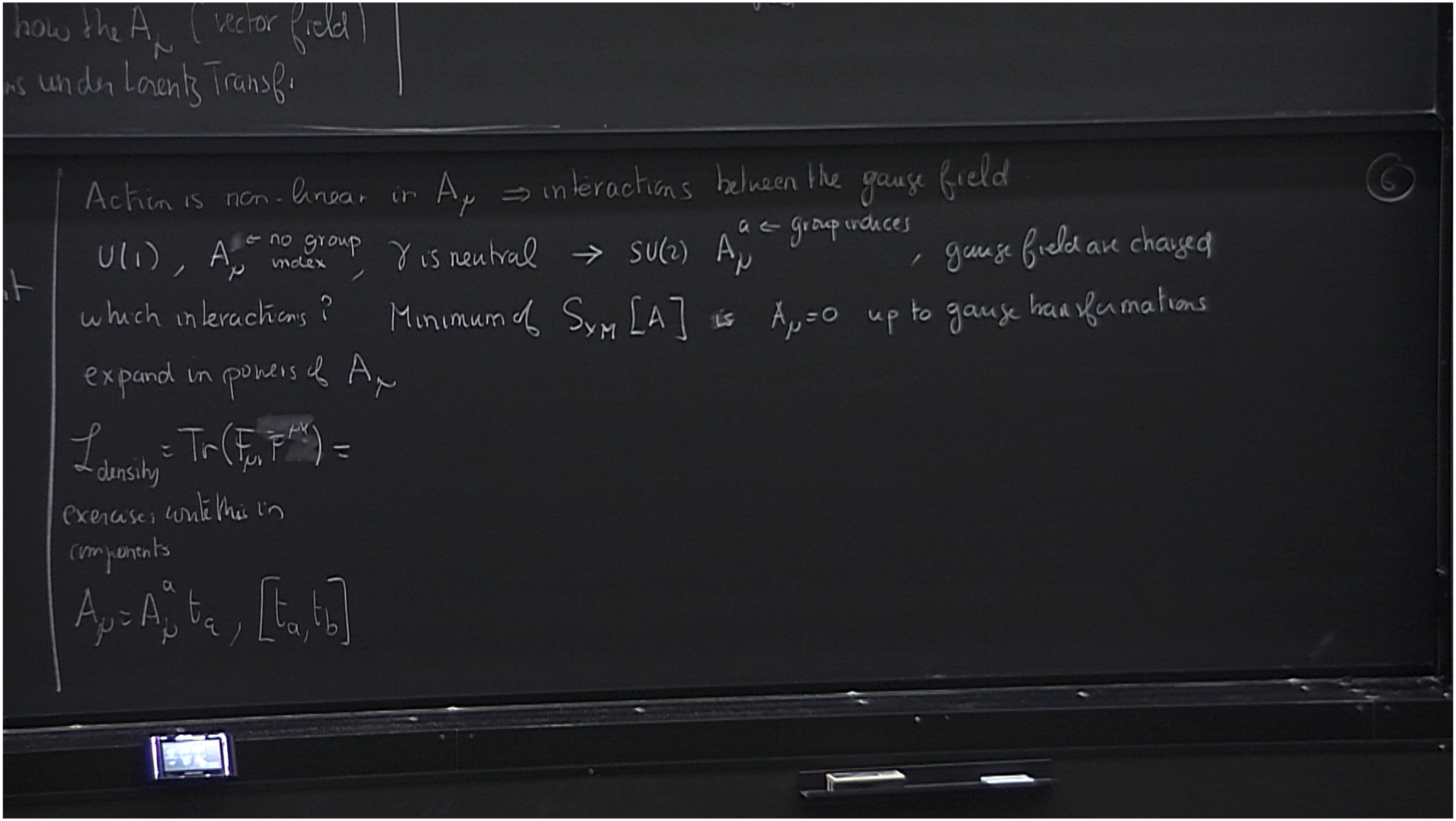
expand in powers of A_μ

$$\mathcal{L}_{\text{density}} = \mathrm{Tr}(F_{\mu\nu} F^{\mu\nu})$$



how the A_μ (vector field)

ns under Lorentz Transf.

t

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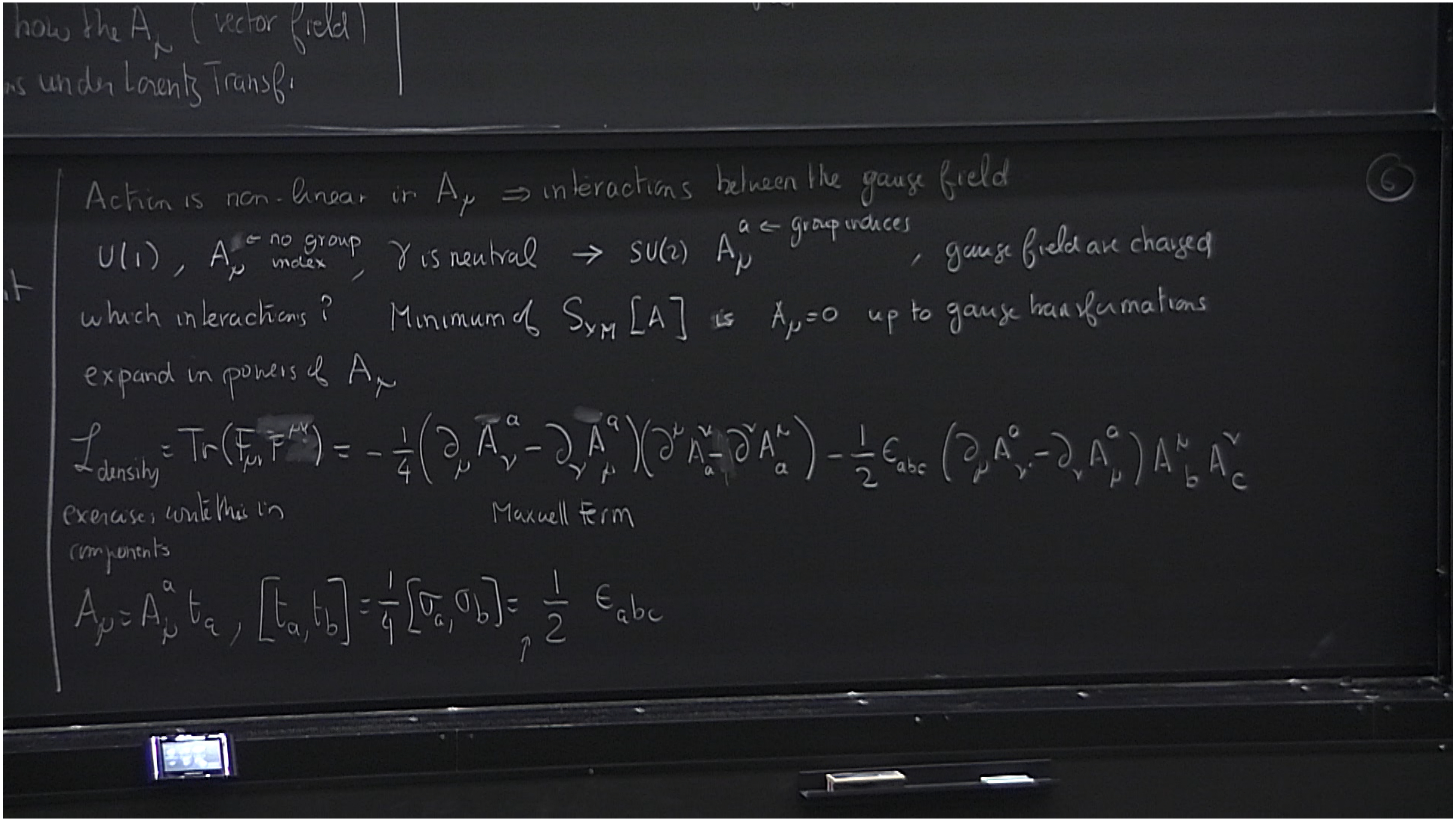
which interactions? Minimum of $S_{YM}[A]$ is $A_\mu = 0$ up to gauge transformations

expand in powers of A_μ

$$\mathcal{L}_{density} = Tr(F_{\mu\nu}F^{\mu\nu}) =$$

exercise: write this in components

$$A_\mu = A_\mu^{a} t_a , \quad [t_a, t_b]$$



how the A_μ (vector field)
s under Lorentz Transf.

Action is non-linear in A_μ $\Rightarrow$ interactions between the gauge field

⑥

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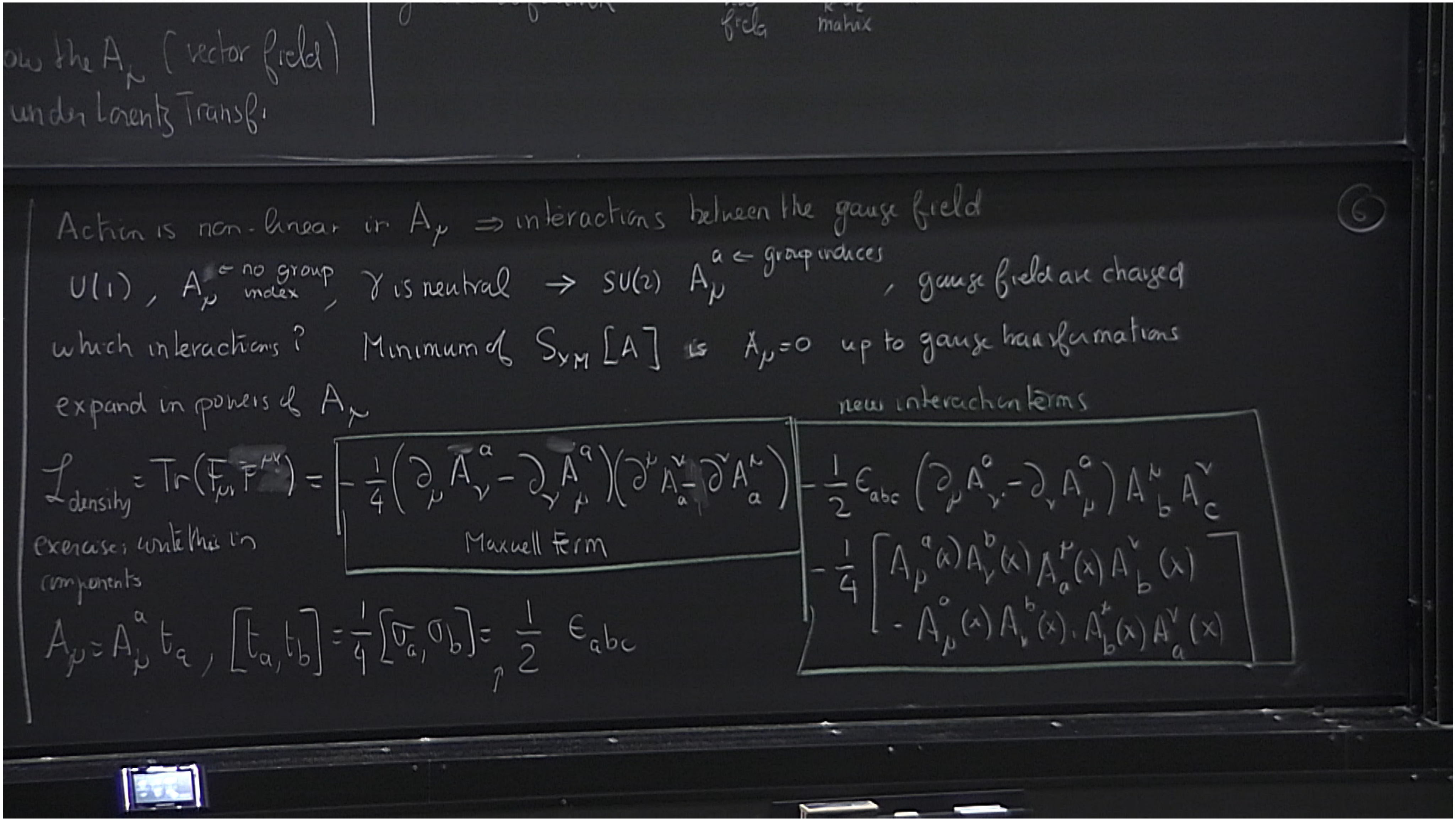
which interactions? Minimum of $S_{YM}[A]$ is $A_\mu = 0$ up to gauge transformations

expand in powers of A_μ

$$\mathcal{L}_{density} = \mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) = -\frac{1}{4}\underbrace{(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)(\partial^\mu A_a^\nu - \partial^\nu A_a^\mu)}_{\text{Maxwell term}} - \frac{1}{2}\epsilon_{abc}(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)A_b^\mu A_c^\nu$$

exercise: write this in components

$$A_\mu = A_\mu^a t_a, \quad [t_a, t_b] = \frac{1}{4}[\sigma_a, \sigma_b] \underset{\uparrow}{=} \frac{1}{2}\ \epsilon_{abc}$$



field matrix

ow the A_μ (vector field)
under Lorentz Transf.

⑥

Action is non-linear in A_μ ⇒ interactions between the gauge field

U(1), A_μ ← no group index, γ is neutral → SU(2) A_μ^a ← group indices, gauge field are charged

which interactions? Minimum of $S_{YM}[A]$ is $A_\mu = 0$ up to gauge transformations

expand in powers of A_μ

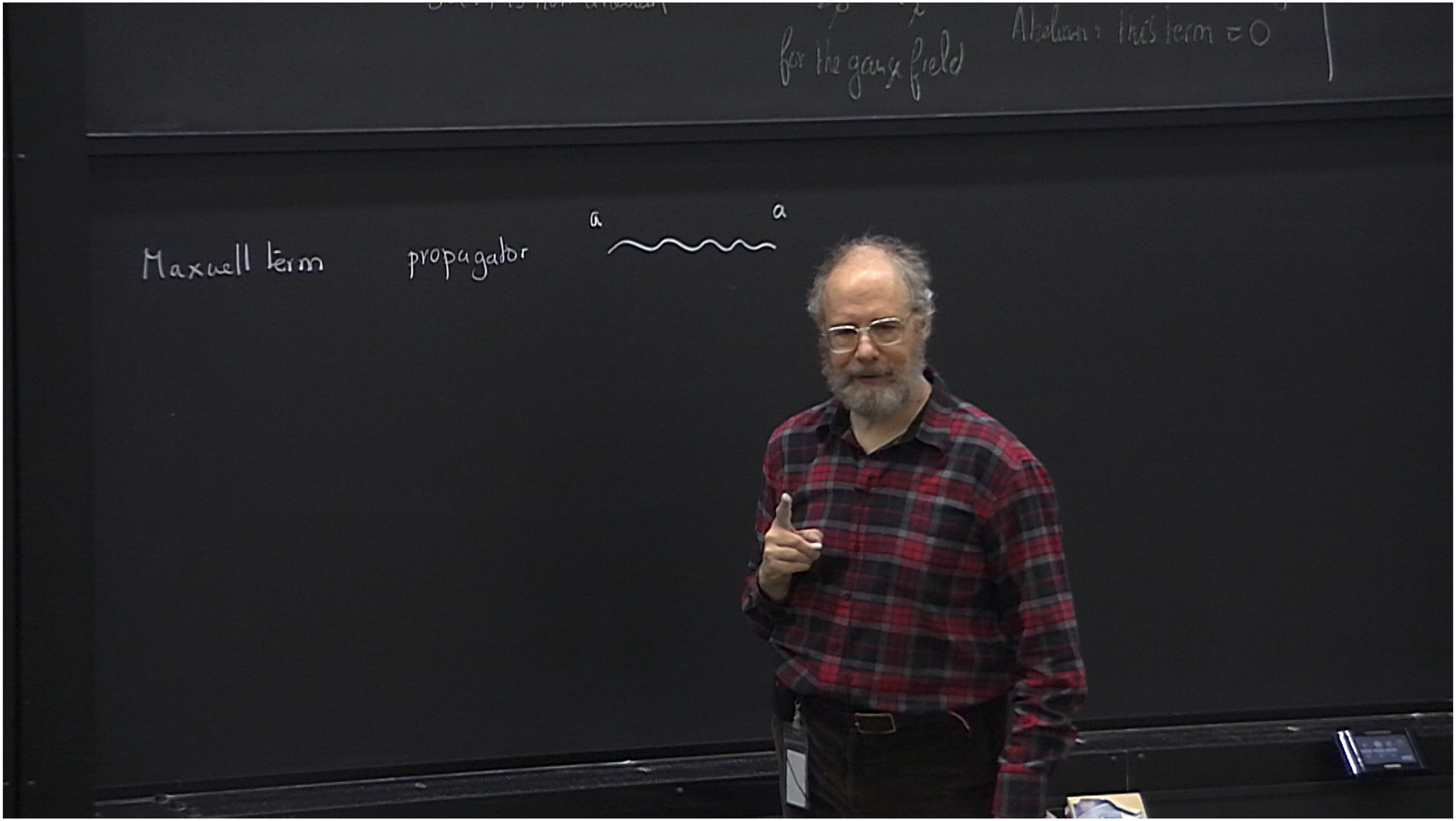
new interaction terms

$$\mathcal{L}_{density} = Tr(F_{\mu\nu}F^{\mu\nu}) = \boxed{-\frac{1}{4}(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)(\partial^\mu A_a^\nu - \partial^\nu A_a^\mu)} \boxed{-\frac{1}{2}\epsilon_{abc}(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)A_b^\mu A_c^\nu - \frac{1}{4}\left[A_\mu^a(x)A_\nu^b(x)A_a^\mu(x)A_b^\nu(x) - A_\mu^a(x)A_\nu^b(x)\cdot A_b^\mu(x)A_a^\nu(x)\right]}$$

Maxwell term

exercise: write this in components

$$A_\mu = A_\mu^a t_a, \quad [t_a, t_b] = \frac{1}{4}[\sigma_a, \sigma_b] = \frac{1}{2}\epsilon_{abc}$$


for the gauge field
Abelian: this term = 0
Maxwell term
propagator
a
a

for the gauge field

Abelian: this term = 0

Maxwell term

propagator

charge conserved

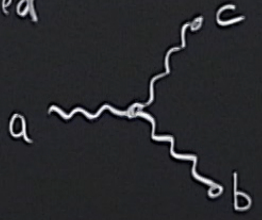
for the gauge field

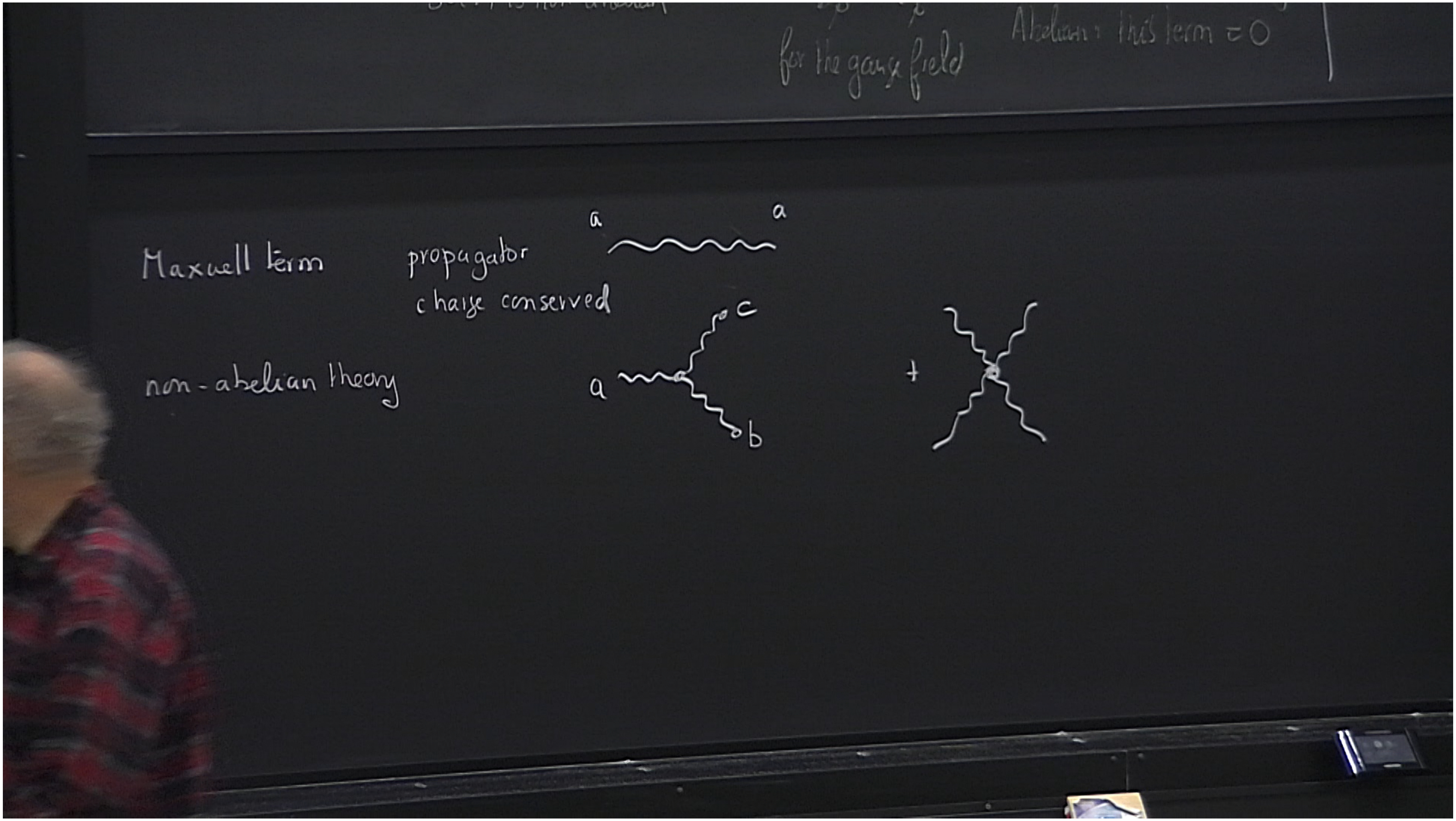
Abelian: this term = 0

Maxwell term propagator

charge conserved

non-abelian theory





for the gauge field

Abelian: this term = 0

Maxwell term propagator

charge conserved

non-abelian theory

for the gauge field

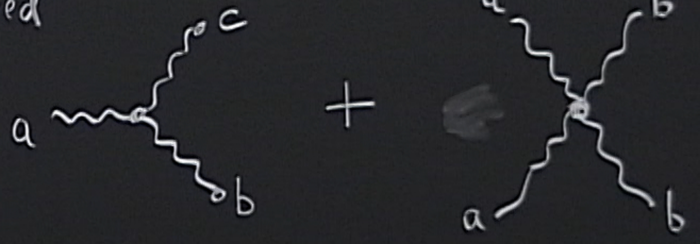
Abelian: this term = 0

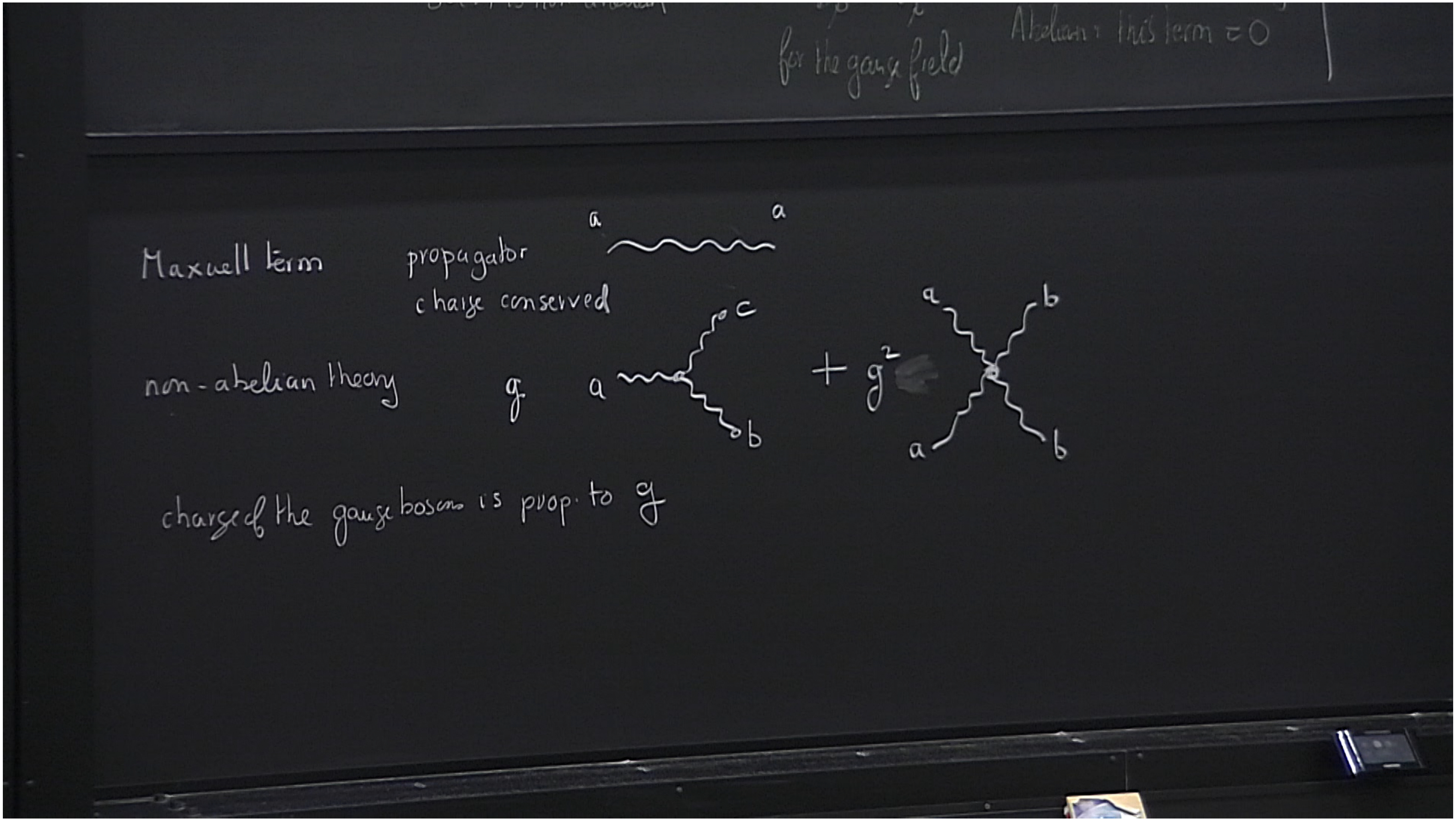
Maxwell term

propagator

charge conserved

non-abelian theory





for the gauge field

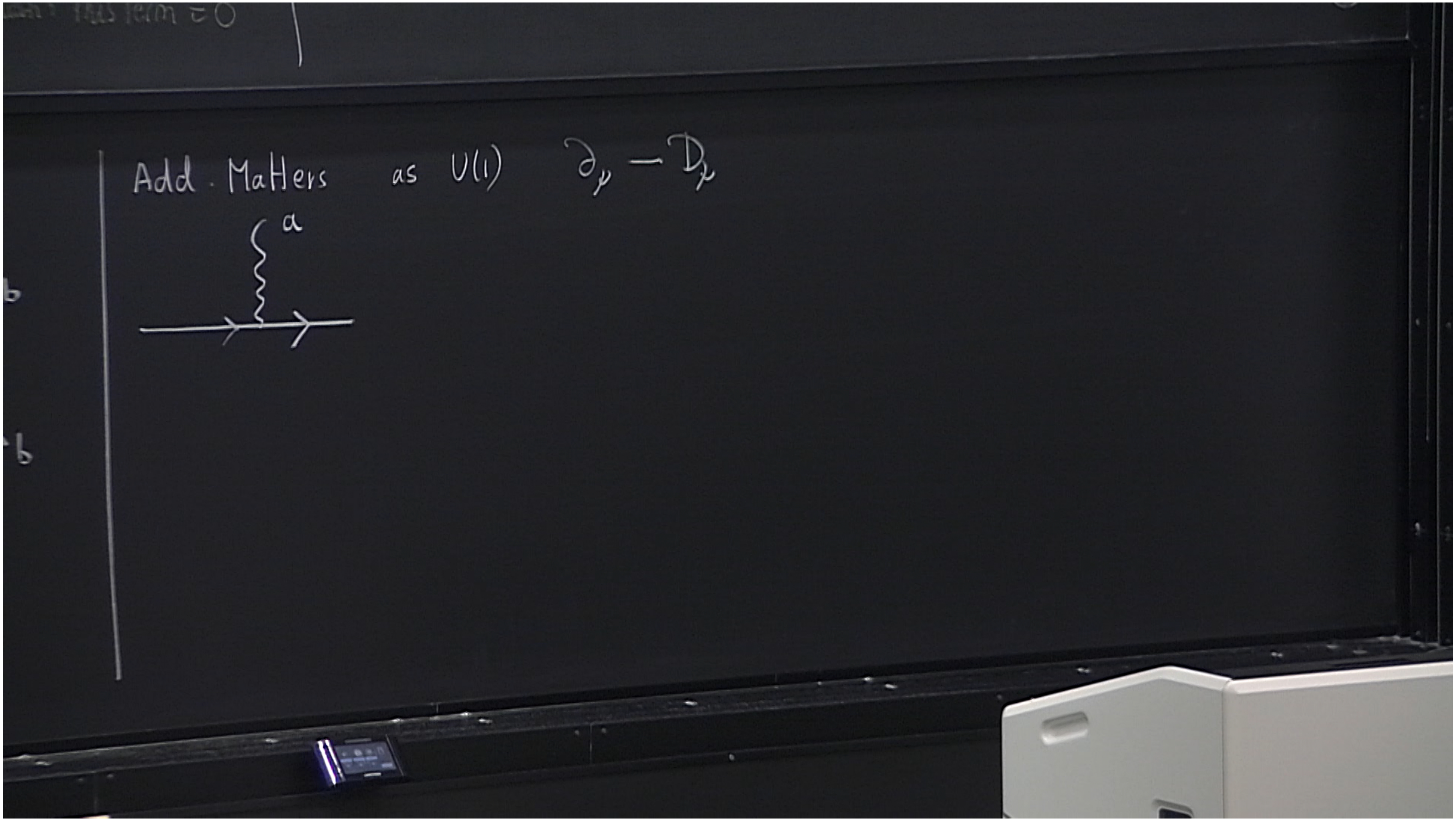
Abelian: this term = 0

Maxwell term propagator

charge conserved

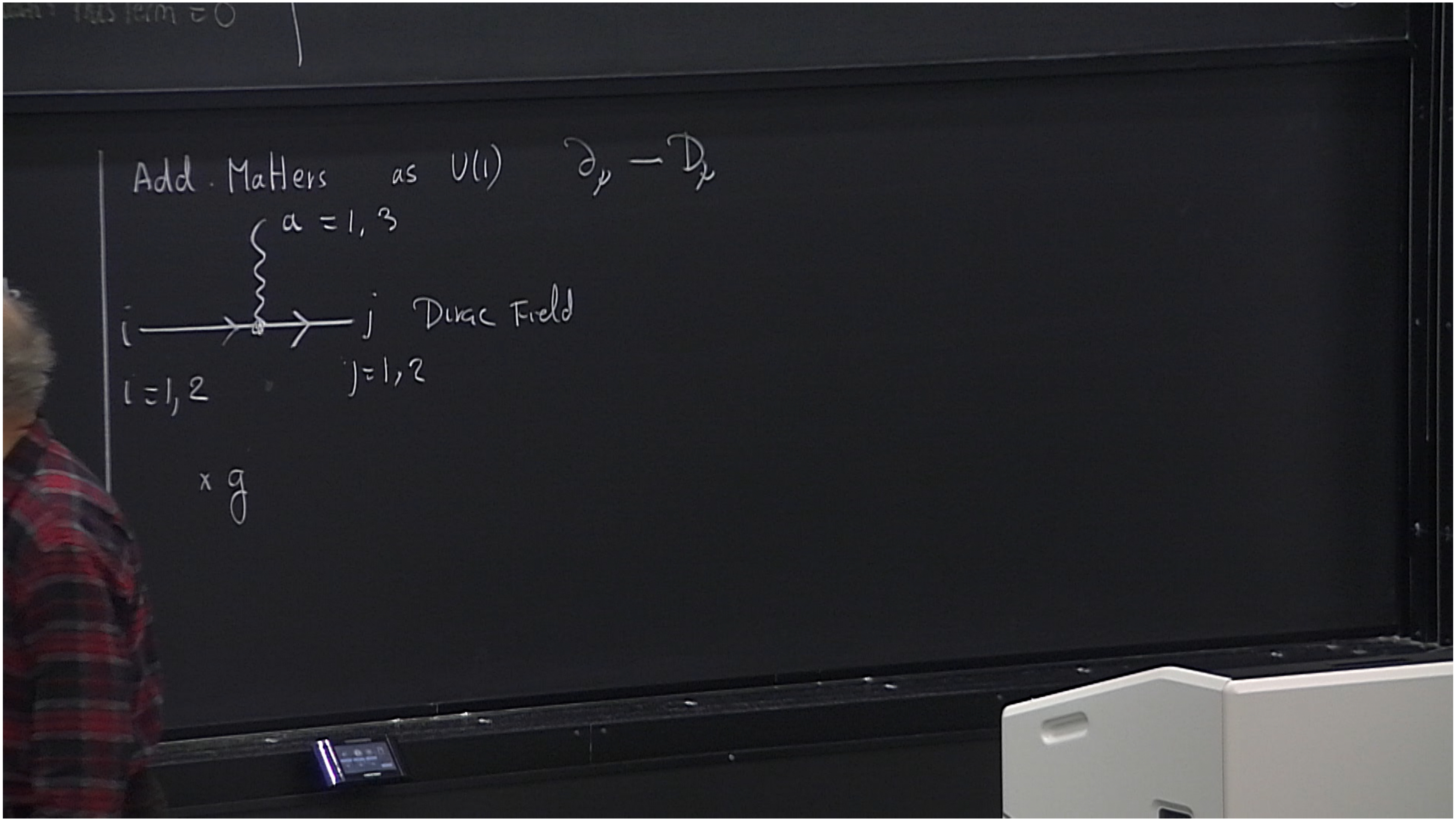
non-abelian theory g + g^2

charge of the gauge bosons is prop. to g



this term = 0

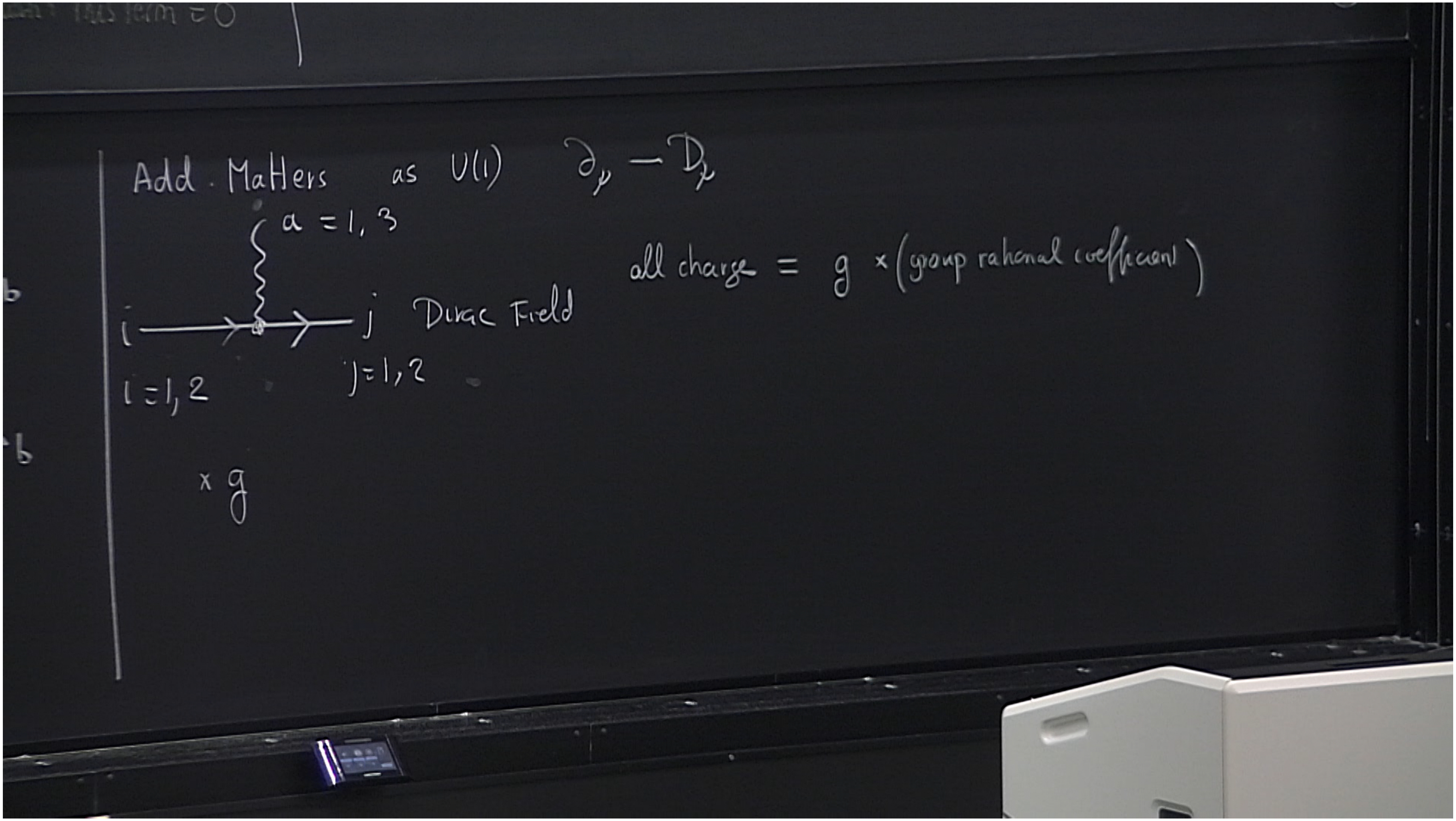
Add Matters as U(1) $\partial_\mu \to D_\mu$



mass term = 0

Add Matters as U(1) $\partial_\mu \to D_\mu$

$\times g$



this term = 0

Add Matters as U(1) $\partial_\mu \to D_\mu$

$a = 1, 3$

Dirac Field

$i = 1, 2$ $j = 1, 2$

$\times g$

all charge = $g \times$ (group rational coefficient)