Title: PSI 17/18 - Quantum Field Theory II - Lecture 11

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Abstract:

Fermionic Path & Functional integrals : HEP & Condensed M

* Grassmann Algebras & Berezin calculus
* Dirac Field

Scalar Field S_{pi}

Complex Grassmann Algebra G_N over a field $K = \mathbb{C}$ (or $\mathbb{R}$) finite dim.

Real ;

Sp

① generated by pairs $(\theta_i, \bar{\theta}_i)$ $i = 1, N$ $2N$ generators

θ's anticommute $\{\theta_i, \theta_j\} = \theta_i\theta_j + \theta_j\theta_i = 0$

$\{\theta, \bar{\theta}\}, \{\bar{\theta}, \theta\}, \{\bar{\theta}, \bar{\theta}\}$ $\Rightarrow$ the θ's are nilpotent

$\theta_i^2 = 0$ $\bar{\theta}_i^2 = 0$

② gene

$g \in$

HEP & Condensed Matter

Scalar Field $S_{spin} = 0$ $\quad \Phi \longleftrightarrow \phi$ commuting

operator random variable

$S_{spin} = \frac{1}{2}$ $\quad \Psi \longleftrightarrow ? \quad ?$

(or $\mathbb{R}$) finite dim.

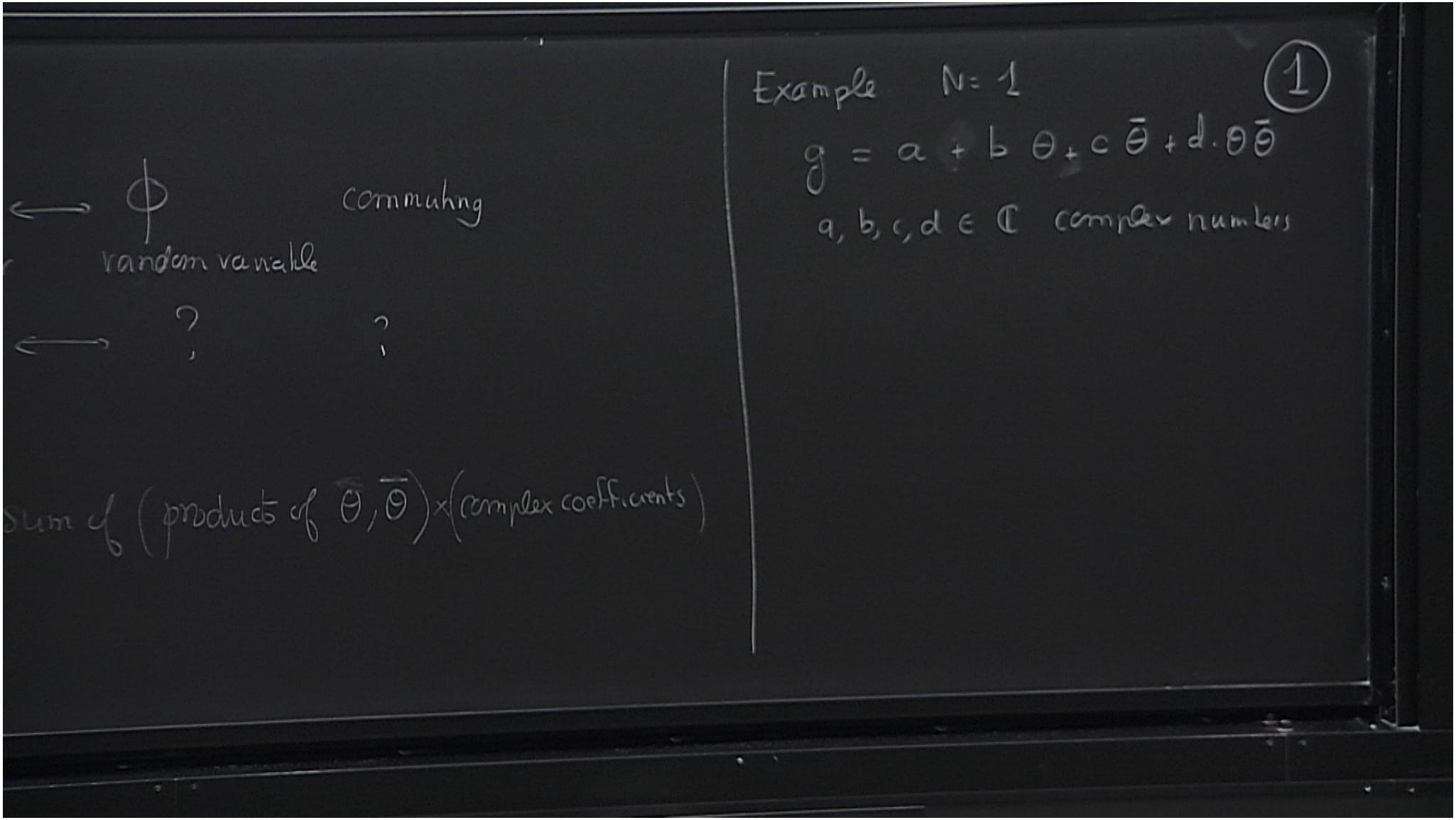
② general element

$g \in G_N$ $\quad g =$ sum of (products of $\theta, \bar{\theta}$) × (complex coefficients)

ors

θ's are nilpotent

$\theta_i^2 = 0 \quad \bar{\theta}_i^2 = 0$


⟷ ϕ commuting
random variable
⟷ ? ?
Sum of (products of θ, θ̄) × (complex coefficients)
Example N = 1
1
g = a + b θ + c θ̄ + d · θθ̄
a, b, c, d ∈ ℂ complex numbers

Fermionic Path & Functional integrals : HEP & Conden

* Grassmann Algebras & Berezin calculus — Scalar Field
* Dirac Field

Complex Grassmann Algebra G_N over a fied $K = \mathbb{C}$ (or $\mathbb{R}$) finite dim
Real ;

① generated by pairs $(\theta_i, \bar{\theta}_i)$ $i = 1, N$ $2N$ generators

θ's anticommute $\{\theta_i, \theta_j\} = \theta_i\theta_j + \theta_j\theta_i = 0$ ⇒ the θ's are nilpotent

Associative $\{\theta, \bar{\theta}\}, \{\bar{\theta}, \theta\}, \{\bar{\theta}, \bar{\theta}\}$ $\theta_i^2 = 0$ $\bar{\theta}_i^2 = 0$

②

ordered Mathe

associativity $(g_1 \cdot g_2) \cdot g_3 = g_1 (g_2 \cdot g_3) = g_1 g_2 g_3$

Example N=

$g = a +$

$a, b, c, d \in$

Field $S_{pin} = 0$ $\quad \Phi \longleftrightarrow \phi$ commuting

operator random variable

$S_{pin} = 1/2$ $\quad \psi \longleftrightarrow ? \quad ?$

dim.

② general element

$g \in G_N$ $\quad g =$ sum of (products of $\theta, \bar{\theta}$) × (complex coefficients)

associativity $(g_1 \cdot g_2) \cdot g_3 = g_1 (g_2 \cdot g_3) = g_1 g_2 g_3$

$\longleftrightarrow$ Φ commuting

ator random variable

$\longleftrightarrow$? ?

= Sum of (products of $\theta, \bar{\theta}$) × (complex coefficients)

Example $N = 1$ ①

$$g = a + b\,\theta + c\,\bar{\theta} + d\cdot\theta\bar{\theta}$$

$a, b, c, d \in \mathbb{C}$ complex numbers

General case

$I = \{ i_1 < i_2 < \cdots < i_K \}$ $\quad 1 \le i_a \le N$

$J = \{ j_1 < j_2 < \cdots < j_H \}$ $\quad 1 \le j_b \le N$

$C_{IJ} \in \mathbb{C}$ $\quad K = |I|,\ H = |J|,\ 0 \le K, H \le N$

$$g = \sum_{I,J} C_{IJ}\, \theta_{i_1} \theta_{i_2} \cdots \theta_{i_K} \bar{\theta}_{j_1} \cdots \bar{\theta}_{j_K}$$

Functional calculus with Grassmann algebra. Very abstract definition

Derivation w.r.t. $\theta_i, \overline{\theta}_i$

$$\frac{\partial}{\partial\theta_i} 1 = 0 \,, \quad \frac{\partial}{\partial\theta_i}\theta_i = 1 \,, \quad \frac{\partial}{\partial\theta_i}\theta_j = 0 \;\; j \neq i \,, \quad \frac{\partial}{\partial\theta_i}\overline{\theta}_j = 0$$

$\frac{\partial}{\partial\theta_i}(\cdots\theta_i\cdots)$: First move the θ_i to the left ← use anticommutation relations
Then apply the above rule

Same for

$$\frac{\partial}{\partial\overline{\theta}_i} 1 = 0 \,, \quad \frac{\partial}{\partial\overline{\theta}_i}\overline{\theta}_j = \delta_{ij} \,, \quad \frac{\partial}{\partial\overline{\theta}_i}\theta_j = 0$$

Example

Functional calculus with Grassmann algebra. Very abstract definition

Derivation w.r.t. $\theta_i, \overline{\theta}_i$

$$\frac{\partial}{\partial \theta_i} 1 = 0 \,, \quad \frac{\partial}{\partial \theta_i} \theta_i = 1 \,, \quad \frac{\partial}{\partial \theta_i} \theta_j = 0 \;\; j \neq i \,, \quad \frac{\partial}{\partial \theta_i} \overline{\theta}_j = 0$$

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Same for

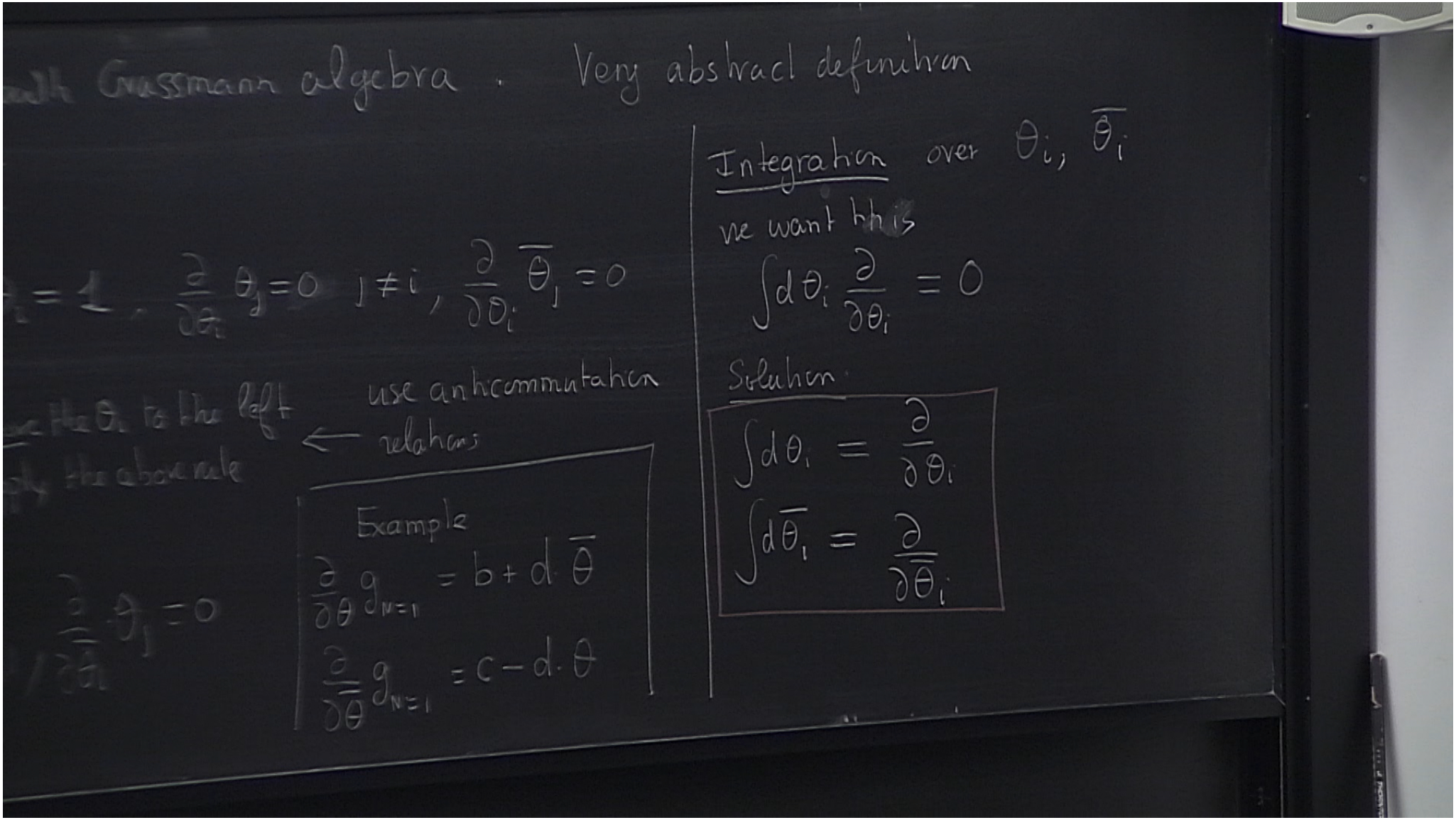
$$\frac{\partial}{\partial \overline{\theta}_i} 1 = 0 \,, \quad \frac{\partial}{\partial \overline{\theta}_i} \overline{\theta}_j = \delta_{ij} \,, \quad \frac{\partial}{\partial \overline{\theta}_i} \theta_j = 0$$

Example

$$\frac{\partial}{\partial \theta} g_{N=1} = b + d \cdot \overline{\theta}$$

$$\frac{\partial}{\partial \overline{\theta}} g_{N=1} = c - d \cdot \theta$$

Integration over $\theta_i, \overline{\theta}_i$


Grassmann algebra . Very abstract definition
$\frac{\partial}{\partial\theta_i}\theta_j = 0 \quad j \neq i, \quad \frac{\partial}{\partial\theta_i}\bar{\theta}_j = 0$
use anticommutation relations
Example
$\frac{\partial}{\partial\theta} g_{N=1} = b + d\cdot\bar{\theta}$
$\frac{\partial}{\partial\bar{\theta}} g_{N=1} = c - d\cdot\theta$
Integration over θ_i, $\bar{\theta}_i$
we want this
$\int d\theta_i \frac{\partial}{\partial\theta_i} = 0$
Solution
$\int d\theta_i = \frac{\partial}{\partial\theta_i}$
$\int d\bar{\theta}_i = \frac{\partial}{\partial\bar{\theta}_i}$

Functional calculus with Grassmann algebra. Very abstract defini

Derivation w.r.t. $\theta_i, \bar{\theta}_i$

$$\frac{\partial}{\partial \theta_i} 1 = 0 \quad , \quad \frac{\partial}{\partial \theta_i} \theta_i = 1 \quad , \quad \frac{\partial}{\partial \theta_i} \theta_j = 0 \quad j \neq i \quad , \quad \frac{\partial}{\partial \theta_i} \bar{\theta}_j = 0$$

$\frac{\partial}{\partial \theta_i}(\cdots \theta_i \cdots)$: First move the θ_i to the left ← use anticommutation relations
Then apply the above rule

Same for $\bar{\theta}$

$$\frac{\partial}{\partial \bar{\theta}_i} 1 = 0 \quad , \quad \frac{\partial}{\partial \bar{\theta}_i} \bar{\theta}_j = \delta_{ij} \quad , \quad \frac{\partial}{\partial \bar{\theta}_i} \theta_j = 0$$

Example

$$\frac{\partial}{\partial \theta} g_{N=1} = b + d \cdot \bar{\theta}$$

$$\frac{\partial}{\partial \bar{\theta}} g_{N=1} = c - d \cdot \theta$$

Integration

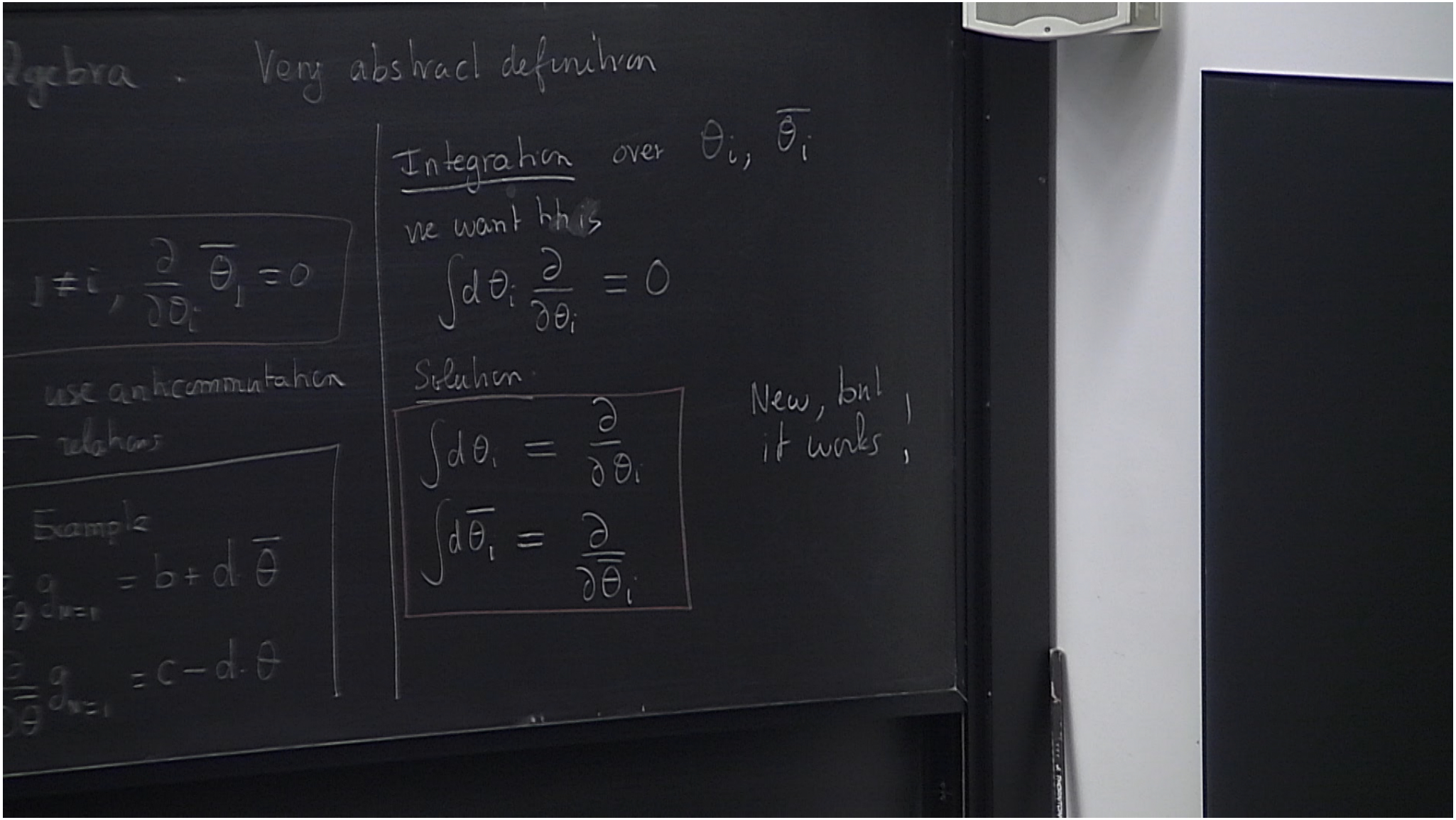
we want this

$$\int d\theta_i \frac{\partial}{\partial \theta_i}$$

Solution

$$\int d\theta_i =$$

$$\int d\bar{\theta}_i =$$


Algebra. Very abstract definition
$j \neq i, \quad \frac{\partial}{\partial \theta_i} \bar{\theta}_j = 0$
use anticommutation relations
Example
$g_{d\bar{\theta}=1} = b + d\,\bar{\theta}$
$\frac{\partial}{\partial \bar{\theta}} g_{d\bar{\theta}=1} = c - d\,\theta$
Integration over θ_i, $\bar{\theta}_i$
we want this
$\int d\theta_i \frac{\partial}{\partial \theta_i} = 0$
Solution
$\int d\theta_i = \frac{\partial}{\partial \theta_i}$
$\int d\bar{\theta}_i = \frac{\partial}{\partial \bar{\theta}_i}$
New, but it works!

Property: the $\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\bar{\theta}}, \int d\theta_i, \int d\bar{\theta}_i$ anticommute too

$\left\{\frac{\partial}{\partial\theta_i}, \frac{\partial}{\partial\theta_j}\right\} = 0$, etc. --

⚠ Beware of the order of operations in your algebra

only additions commute

⚠ $\frac{\partial}{\partial\theta_i} C = 0$, but $\int d\theta_i C = 0$ also

$C \in \mathbb{C}$

change of basis

$\{\theta_i, \bar{\theta}_i\} \longrightarrow \{v_i, \bar{v}_i\}$ linear change of basis

$v_i = J_{ij}\theta_j$, $\bar{v}_i = \bar{J}_{ij}\bar{\theta}_j$

$J = \{J_{ij}\}$ some complex matrix

$\frac{\partial}{\partial v_i} = (J^{-1})_{ij}\frac{\partial}{\partial \theta_j}$ usual Jacobian rule for derivatives

but for integrations, you get $\det(\ )^{-1}$ instead of $\det(\ )$

Gaussian integrations with Grassmann variables

$$\int d\bar{\theta}_1 \, d\theta_1 \cdots \int d\bar{\theta}_N \, d\theta_N \; \exp\left(- \bar{\theta}_i \, A_{ij} \, \theta_j \right)$$

$(A_{ij}) = A$ complex Hermitean matrix

$$= \int d\bar{\theta} \, d\theta \; \exp\left(- \bar{\theta} \cdot A \cdot \theta \right)$$

shul hand notation

$\bar{\theta} = \{ \bar{\theta}_1 \cdots \bar{\theta}_N \}$

$\theta = \{ \theta_1 \cdots \theta_N \}$

Gaussian integrations with Grassmann variables

$$\int d\bar{\theta}_1 \, d\theta_1 \cdots \int d\bar{\theta}_N \, d\theta_N \; \exp\left(-\bar{\theta}_i A_{ij} \theta_j\right) = \mathcal{J}$$

$(A_{ij}) = A$ complex Hermitian matrix

$$= \int d\bar{\theta} \, d\theta \; \exp\left(-\bar{\theta} \cdot A \cdot \theta\right)$$

shrinl hand notation

$\bar{\theta} = \{\bar{\theta}_1 \cdots \bar{\theta}_N\}$

$\theta = \{\theta_1 \cdots \theta_N\}$

How to compute

$$\exp\left(-\bar{\theta} \cdot A \cdot \theta\right) = \sum_{K=0}^{N} \frac{1}{K!} \left(-\bar{\theta} A \theta\right)^K$$

polynomial

Apply the rule of integrations, only $K = N$ survive,

At the end: $\boxed{\mathcal{J} = \det(A)}$



complex numbers $z = (z_1 \cdots z_N)$, $\bar{z} = (\bar{z}_1 \cdots \bar{z}_N)$

$$\int \frac{d\bar{z}\, dz}{(2\pi)^N} \cdot \exp(-\bar{z} \cdot A z) = \det(A)^{-1}$$

"anticommuting numbers"

$$\int d\bar{\theta} \cdot d\theta \cdot \exp(-\bar{\theta} \cdot A \cdot \theta) = \det(A)$$

Example:

$N=1 \qquad A=(a)$

$$\exp(-\bar{\theta}\theta a) = 1 - a\bar{\theta}\theta$$

$$\int d\bar{\theta} \int d\theta \left(1 - a\bar{\theta}\theta\right) = a$$

$N=2: \quad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \det A = a_{11}a_{22} - a_{12}a_{21}$

$$\exp(-\bar{\theta} A \theta) = \exp\left(-\bar{\theta}_1\theta_1 a_{11} - \bar{\theta}_1\theta_2 a_{12} - \bar{\theta}_2\theta_1 a_{21} - \bar{\theta}_2\theta_2 a_{22}\right)$$

$$= \ldots + \left(a_{11}a_{22}\,\bar{\theta}_1\theta_1\bar{\theta}_2\theta_2 + a_{12}a_{21}\,\bar{\theta}_1\theta_2\bar{\theta}_2\theta_1\right)$$

Wick theorem

$$\langle \theta_i \bar{\theta}_j \rangle = \frac{\int d\bar{\theta} d\theta \cdot \exp(-\bar{\theta} \cdot A \theta) \cdot \theta_i \bar{\theta}_j}{\int d\bar{\theta} d\theta \cdot \exp(-\bar{\theta} \cdot A \cdot \theta)} = \frac{\text{complex nb}}{\text{complex nb}}$$

2 point expectation value $= \frac{\text{Minor}(A)_{ij}}{\det(A)} = \boxed{(A^{-1})_{ij} = \langle \theta_i \bar{\theta}_j \rangle}$

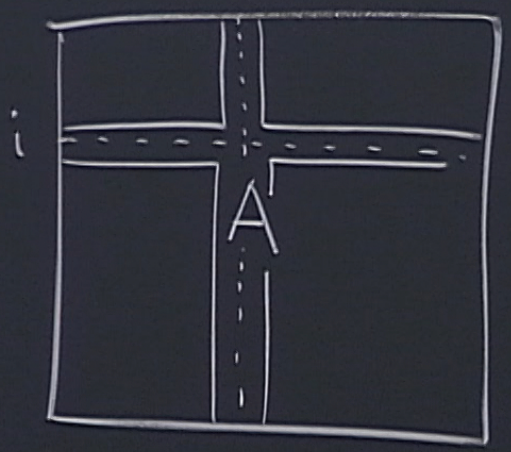
Wick theorem

$$\langle \theta_i \bar{\theta}_j \rangle := \frac{\int d\bar{\theta} d\theta \cdot \exp(-\bar{\theta} \cdot A \theta) \cdot \theta_i \bar{\theta}_j}{\int d\bar{\theta} d\theta \cdot \exp(-\bar{\theta} \cdot A \theta)}$$

complex nb / complex nb

2 point expectation value definition $= \frac{\text{Minor}(A)_{ij}}{\det(A)} = (\bar{A}^{-1})_{ij} = \langle \theta_i \bar{\theta}_j \rangle$

result of the calculation



$$\text{Minor}(A)_{i,j} = (-1)^{(i+j)} \det\left(A_{\substack{\text{row } i \\ \text{and column } j \text{ removed}}} \right)$$

$$(A^{-1})_{i,j} = \frac{\text{Minor}_{i,j}(A)}{\det(A)}$$

$$\int \frac{d\bar{z}\,dz}{(2\pi)^N} \cdot \exp\left(-\bar{z} \cdot A z\right) = \det(A)^{-1}$$

"anticommuting numbers"

$$\int d\bar{\theta} \cdot d\theta \cdot \exp\left(-\bar{\theta} \cdot A \cdot \theta\right) = \det(A)$$

$$\langle z_i \bar{z}_j \rangle = (A^{-1})_{ij}$$

$$\langle \theta_i \bar{\theta}_j \rangle = (A^{-1})_{ij}$$

N = 1

exp(

$\int d\bar{\theta} \int d$

N = 2

exp(-$\bar{\theta}$ A

$(2\pi)^N$

anticommuting numbers"

$$\int d\bar{\theta}\, d\theta \cdot \exp\left(-\bar{\theta} \cdot A \cdot \theta\right) = \det(A)$$

$$\langle z_i \bar{z}_j \rangle = (\tilde{A}^{-1})_{ij}$$

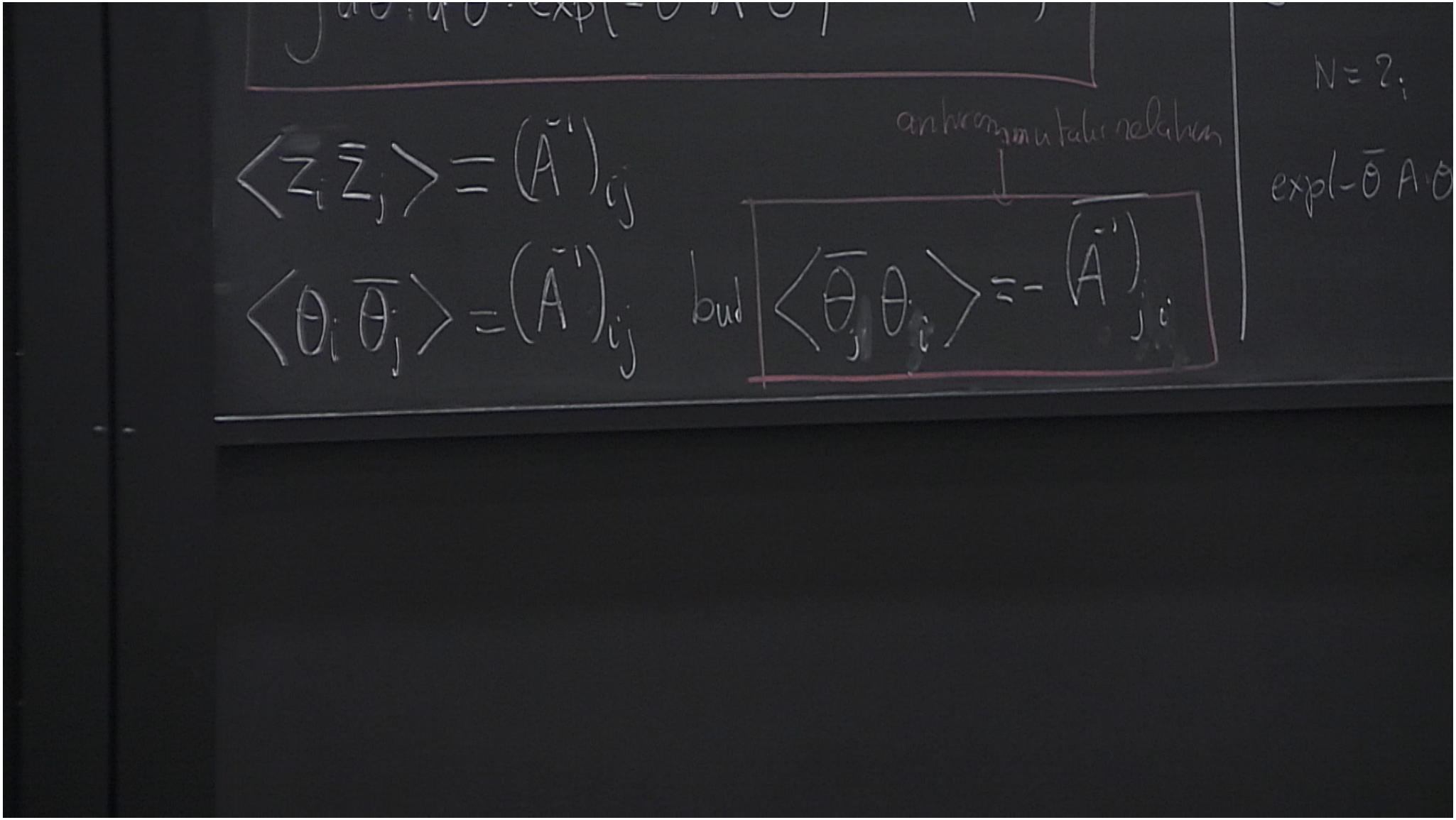
$$\langle \theta_i \bar{\theta}_j \rangle = (\tilde{A}^{-1})_{ij} \quad \text{but} \quad \langle \bar{\theta}_i \theta_j \rangle = -(\tilde{A}^{-1})_{ij}$$

$\exp(-$

$\int d\bar{\theta} \int d\theta$

$N = 2_i$

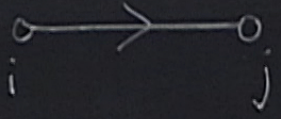
$\exp(-\bar{\theta} A$


$\langle z_i \bar{z}_j \rangle = (\tilde{A}^{-1})_{ij}$
$\langle \theta_i \bar{\theta}_j \rangle = (\tilde{A}^{-1})_{ij}$ bud
anticommutative relation
$\langle \bar{\theta}_j \theta_i \rangle = -(\tilde{A}^{-1})_{ji}$
$N = ?$
$\exp(-\bar{\theta} A \theta$

$$\int d\theta\, d\bar{\theta}\, \exp(-\bar{\theta} \cdot A \cdot \theta)$$

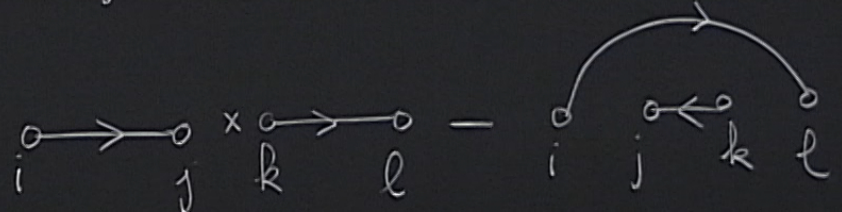
2 point expectation value definition $= \dfrac{\text{Minor}(A)_{ij}}{\det(A)} = \boxed{(A^{-1})_{ij} = \langle \theta_i \bar{\theta}_j \rangle}$

result of the calculation

$\langle \theta_i \bar{\theta}_j \rangle =$ 

4 pt function $\Rightarrow$ Wick Theorem for Dirac Fields (QFT 1 course)

$$\langle \theta_i \bar{\theta}_j \theta_k \bar{\theta}_\ell \rangle = \langle \theta_i \bar{\theta}_j \rangle \langle \theta_k \bar{\theta}_\ell \rangle - \langle \theta_i \bar{\theta}_\ell \rangle \langle \theta_k \bar{\theta}_j \rangle$$



Functional Quantization For the Dirac Field

$h_{\mu\nu} = (-1, 1, 1, 1)$ (East Coast signature)

Dirac Field $\psi = \{\psi^a(x) ; a = 1, \dots 4\}$ with $x \in \mathbb{M}^{3,3}$

$\bar{\psi} = \{\bar{\psi}^a(x) ; a = 1 \dots 4\}$ $\quad \not{\partial} = \gamma^\mu \frac{\partial}{\partial x^\mu}$

$a = 1, \dots 4$ Dirac indices

$\mu = 1, \dots 4$ Lorentz

γ matrices $\{\gamma^\mu, \gamma^\nu\} = -2h^{\mu\nu}$

Dirac Equation $(i\not{\partial} - m)\psi = 0$ $\Leftarrow$ Dirac Action $S[\bar{\psi}, \psi] = \int_{\mathbb{M}} d^4x \, \bar{\psi}(x)(i\not{\partial} - m)\psi(x)$

$\}$ with $X \in \mathbb{M}^{1,3}$ (East Coast Signature)

$4\}$ $\not{\partial} = \gamma^\mu \frac{\partial}{\partial x^\mu}$

$a = 1, 4$ Dirac indices

$\mu = 1, 4$ Lorentz

$\lfloor \frac{d}{2} \rfloor$ = # Dirac indices

Dirac Action $S[\bar{\psi}, \psi] = \int_{\mathbb{M}} d^4x \, \bar{\psi}(x)(i\not{\partial} - m)\psi(x)$

Consider a very big Grassmann Algebra $\mathcal{G}_{\text{DIRAC}}$ ∞-dimensional $\Rightarrow$

θ_i, $i = 1, N \longrightarrow \psi^a(x)$ $\quad (a, x)$: $a = 1, 4$, $x \in \mathbb{M}$

$\overline{\theta}_i$, $i = 1, N \longrightarrow \overline{\psi}^a(x)$ $\quad$ " $\quad$ "

Action $S_{\text{DIRAC}} = \int_{\mathbb{M}} d^4x \, \overline{\psi}^a(x) \left(i (\gamma_\mu)_{ab} \frac{\partial}{\partial x^\mu} - m \delta_{ab} \right) \psi^b(x)$

$= S_{\text{DIRAC}}[\overline{\psi}, \psi]$ is a quadratic form

mensional $\Rightarrow$ finite spacetime Lattice. finite dim.

Grassmann Functional Integral

$$\int D[\bar{\psi}, \psi] \exp\left(\frac{i}{\hbar} S_{\text{DIRAC}}[\bar{\psi}, \psi]\right) \simeq \det\left[i\not{\partial} - m\right]$$

with measure

$$D[\bar{\psi}, \psi] = \prod_{x \in M} \prod_{a} d\bar{\psi}^{a}(x) . d\psi^{a}(x)$$

$\det\left[i\not{\partial} - m\right]$

propagator = 2pt function

$$\langle \Psi(x)\bar{\Psi}(y)\rangle$$

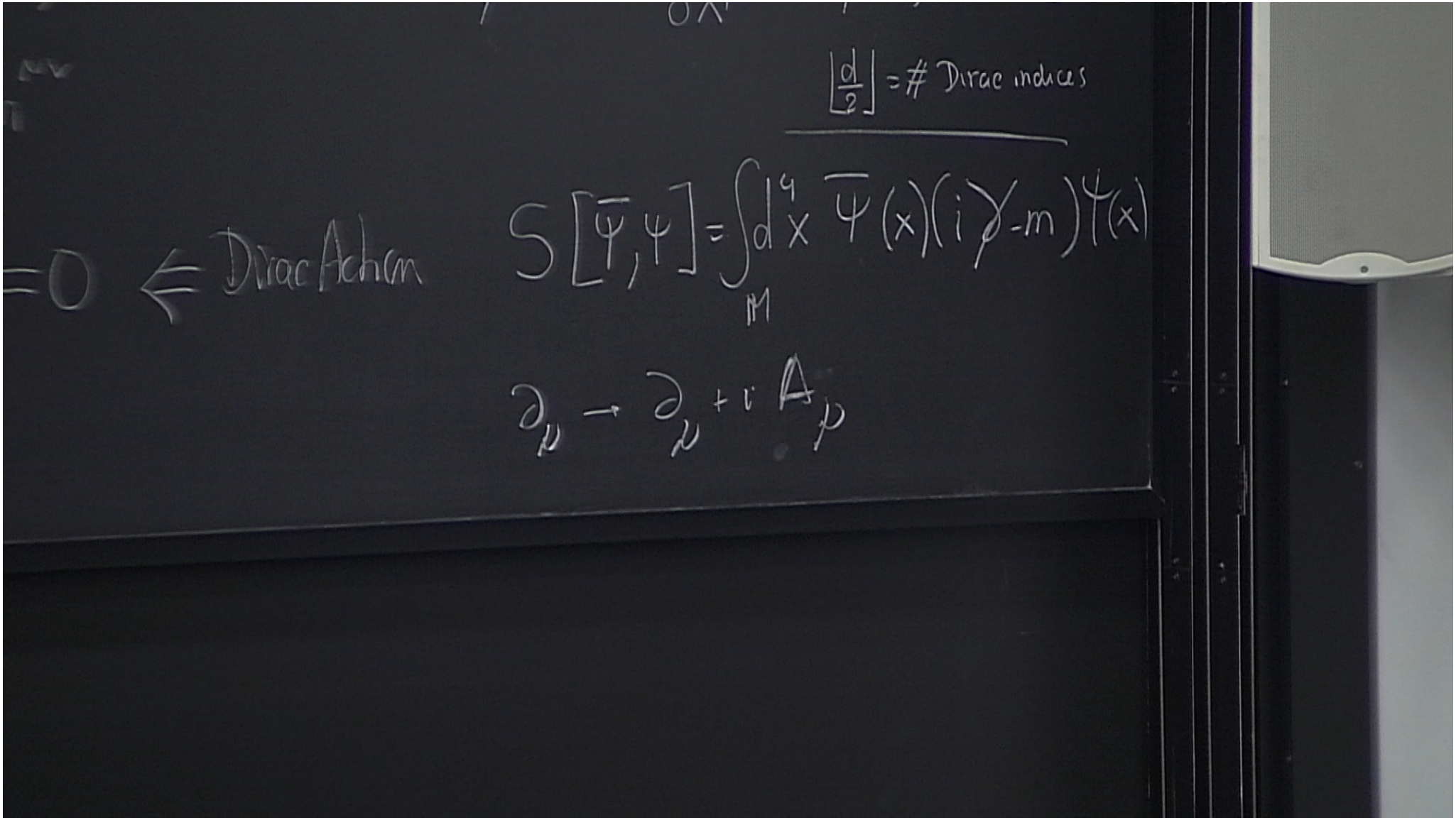
$$= \left(i\not{\partial} - m\right)^{-1}_{x,y} = G_F^{\text{DIRAC}}(x,y)$$

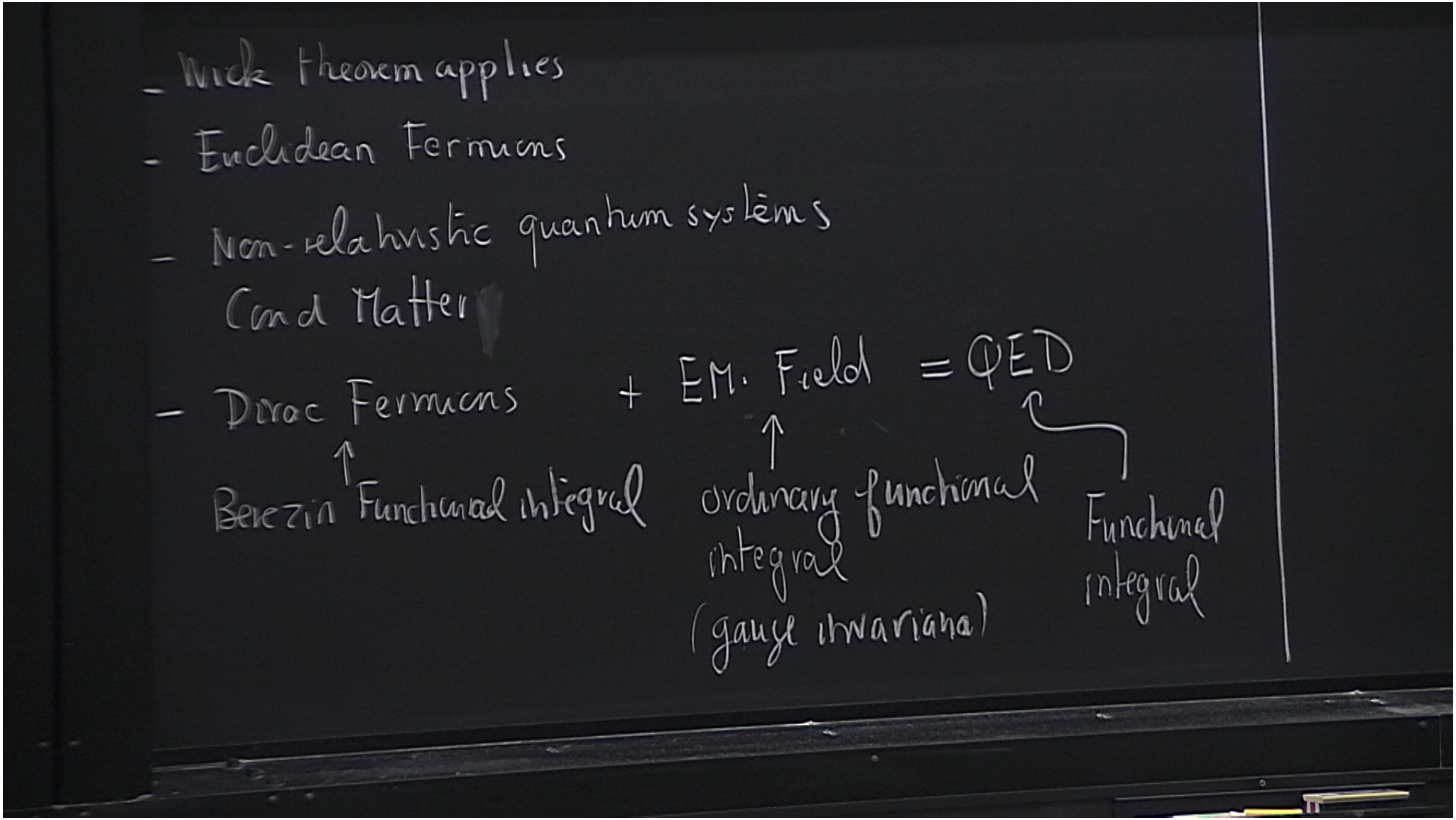
$$= \int \frac{d^4p}{(4\pi)^2}\left[\frac{i}{-\not{p} + m - i\epsilon_+}\right] e^{i\,p\cdot(x-y)}$$

$$= \langle 0|T\left[\Psi(x)\bar{\Psi}(y)\right]|0\rangle$$

Time ordered product — Field operator Dirac

2K


$\left\lfloor \frac{d}{2} \right\rfloor$ = # Dirac indices
$S[\bar{\psi},\psi] = \int_M d^4x\, \bar{\psi}(x)(i\not{\partial} - m)\psi(x)$
$\Leftarrow$ Dirac Action
$= 0$
$\partial_\mu \to \partial_\mu + i A_\mu$


- Wick theorem applies
- Euclidean Fermions
- Non-relativistic quantum systems
Cond Matter
- Dirac Fermions + EM. Field = QED
Berezin Functional integral
ordinary functional integral
(gauge invariant)
Functional integral