Title: PSI 17/18 - Quantum Field Theory II - Lecture 9

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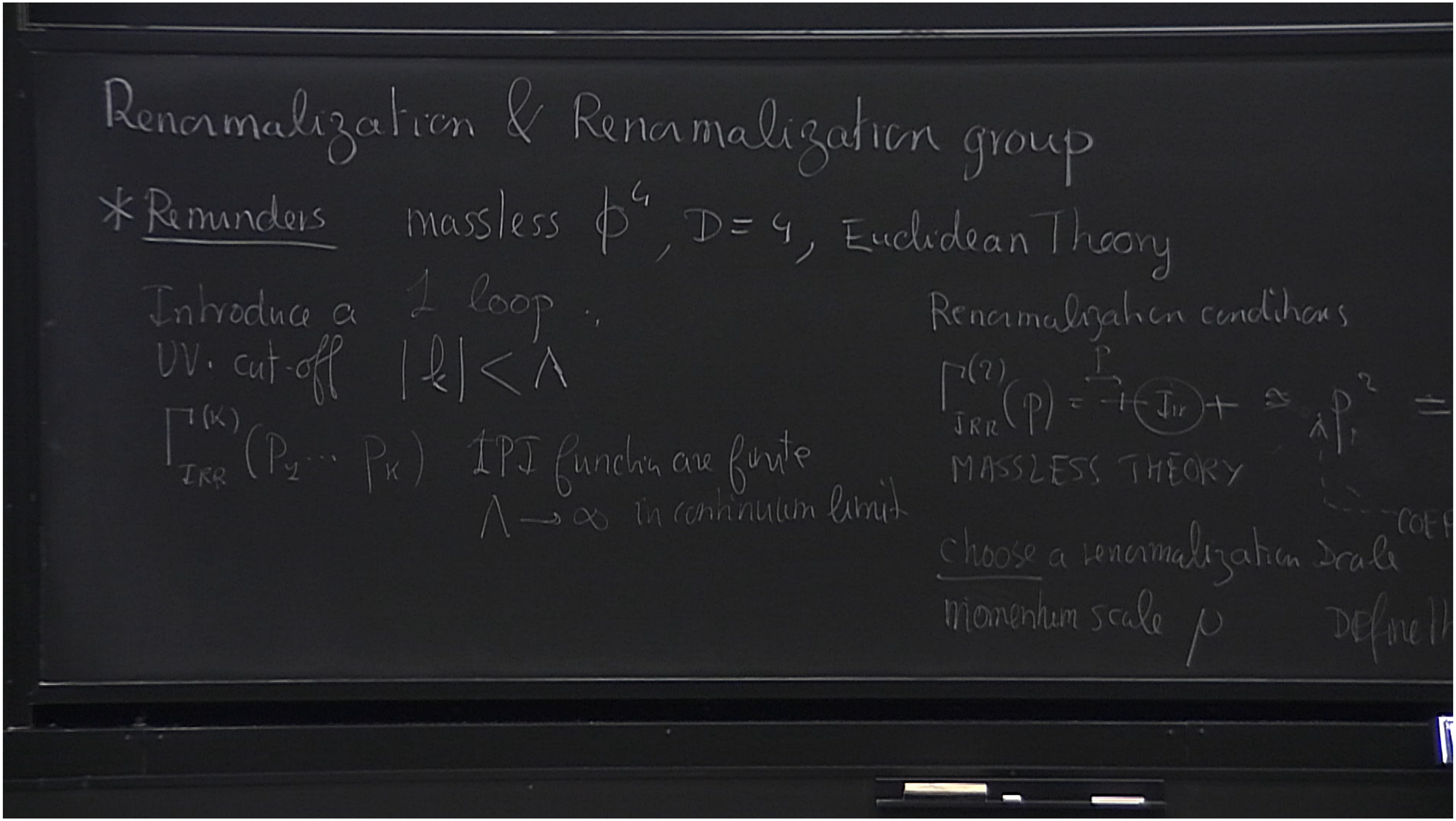
Abstract:



# Renormalization & Renormalization group

* Reminders   massless $\phi^4$, $D = 4$, Euclidean Theory

Introduce a UV. cut-off $|k| < \Lambda$

1 loop :

$\Gamma^{(K)}_{IRR}(P_1 \cdots P_K)$ 1PI function are finite

$\Lambda \to \infty$ in continuum limit

Renormalization conditions

$\Gamma^{(2)}_{IRR}(p) = \overset{p}{\longrightarrow}\!\!-\!|\!-\!\bigcirc\!\!\!\!\!\!\!\text{1PI}\;\,-\!|\!- \simeq {}_{\wedge}p_1^{\,2} \doteq$

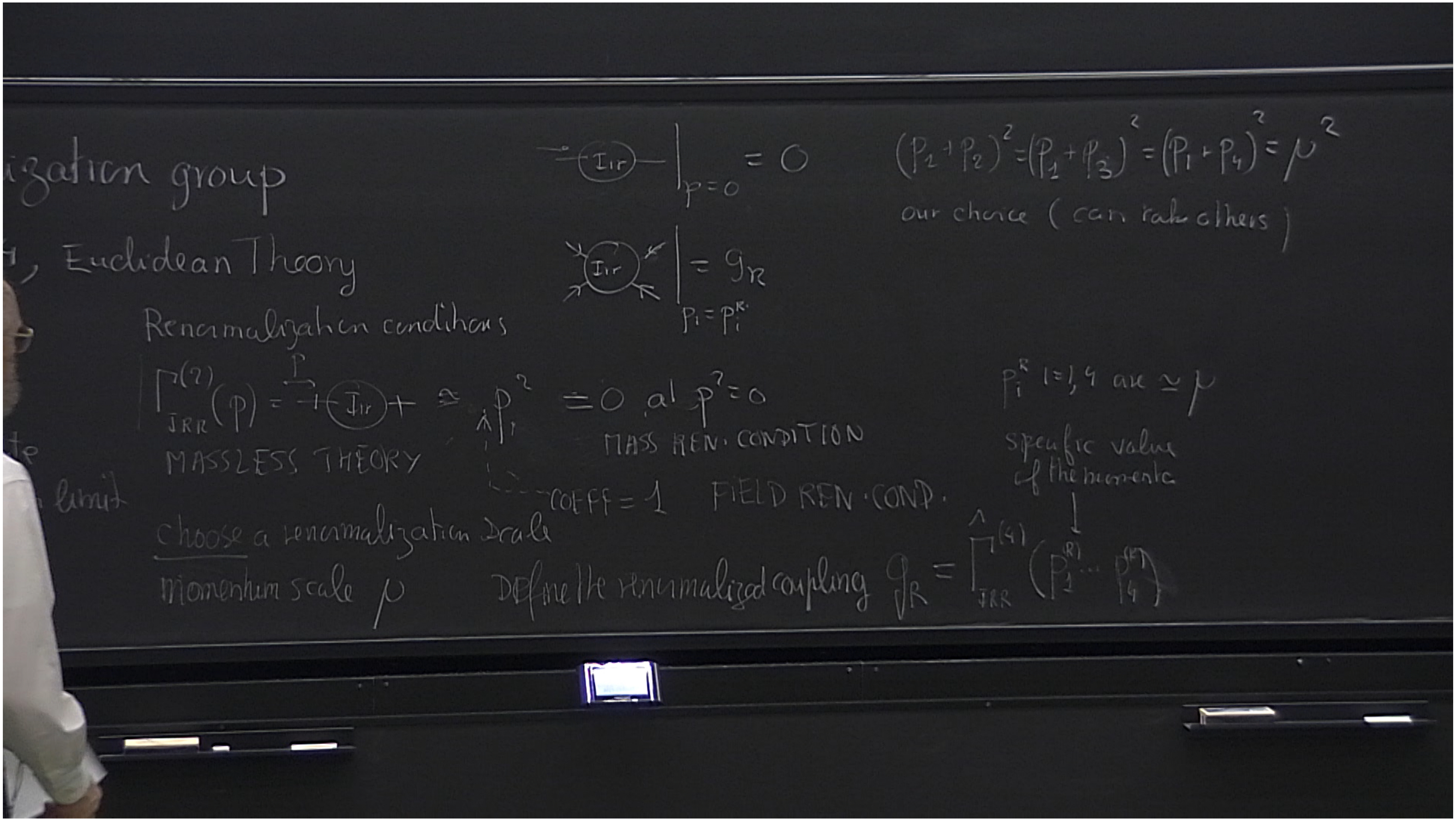
MASSLESS THEORY

COEF

Choose a renormalization scale

momentum scale $\mu$

Define th



ization group

, Euclidean Theory

Renormalization conditions

$\Gamma^{(2)}_{IRR}(p) = -\!\!\!-\!\!\!\bigcirc\!\!\!-\!\!\!- + \simeq \hat{p}^2 \doteq 0$ at $p^2 = 0$

MASSLESS THEORY

MASS REN. CONDITION

COEFF = 1 FIELD REN. COND.

te

limit

$-\!\!\!\bigcirc\!\!\!- \Big|_{p=0} = 0$

$\Big|_{p_i = p_i^R} = g_R$

$(p_2 + p_2)^2 = (p_1 + p_3)^2 = (p_1 + p_4)^2 = \mu^2$

our choice (can take others)

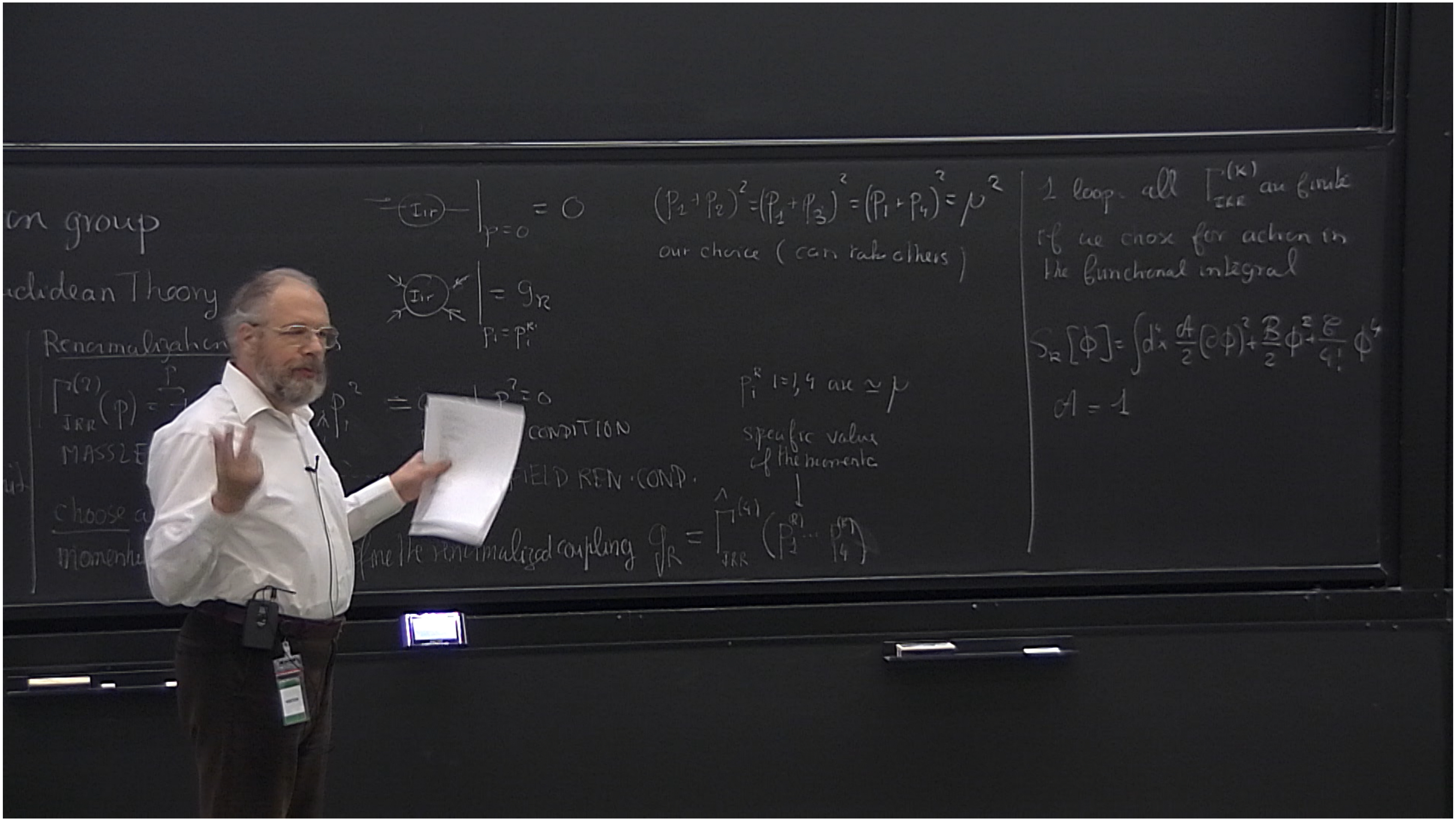
$p_i^R$ $i = 1, 4$ are $\simeq \mu$

specific value of the momenta

Choose a renormalization scale

momentum scale $\mu$

Define the renormalized coupling $g_R = \hat{\Gamma}^{(4)}_{IRR}(p_1^{(R)} \dots p_4^{(R)})$



...on group

...uclidean Theory

Renormalization ...

$\Gamma^{(2)}_{IRR}(p) = \dots$

MASSLE...

choose a...

momentu...

$\text{-}(Ir)\text{-}\Big|_{p=0} = 0$

$(Ir)\Big|_{p_i = p_i^R} = g_R$

$\hat{p}_1^2 = 0 \quad p^2 = 0$

CONDITION

FIELD REN. COND.

...fine the renormalized coupling $g_R = \hat{\Gamma}^{(4)}_{IRR}(p_1^{(R)} \dots p_4^{(R)})$

$(p_1+p_2)^2 = (p_1+p_3)^2 = (p_1+p_4)^2 = \mu^2$

our choice ( can take others )

$p_i^R \; i=1,4$ are $\simeq \mu$

specific value of the momenta

1 loop: all $\Gamma^{(k)}_{IRR}$ are finite if we chose for action in the functional integral

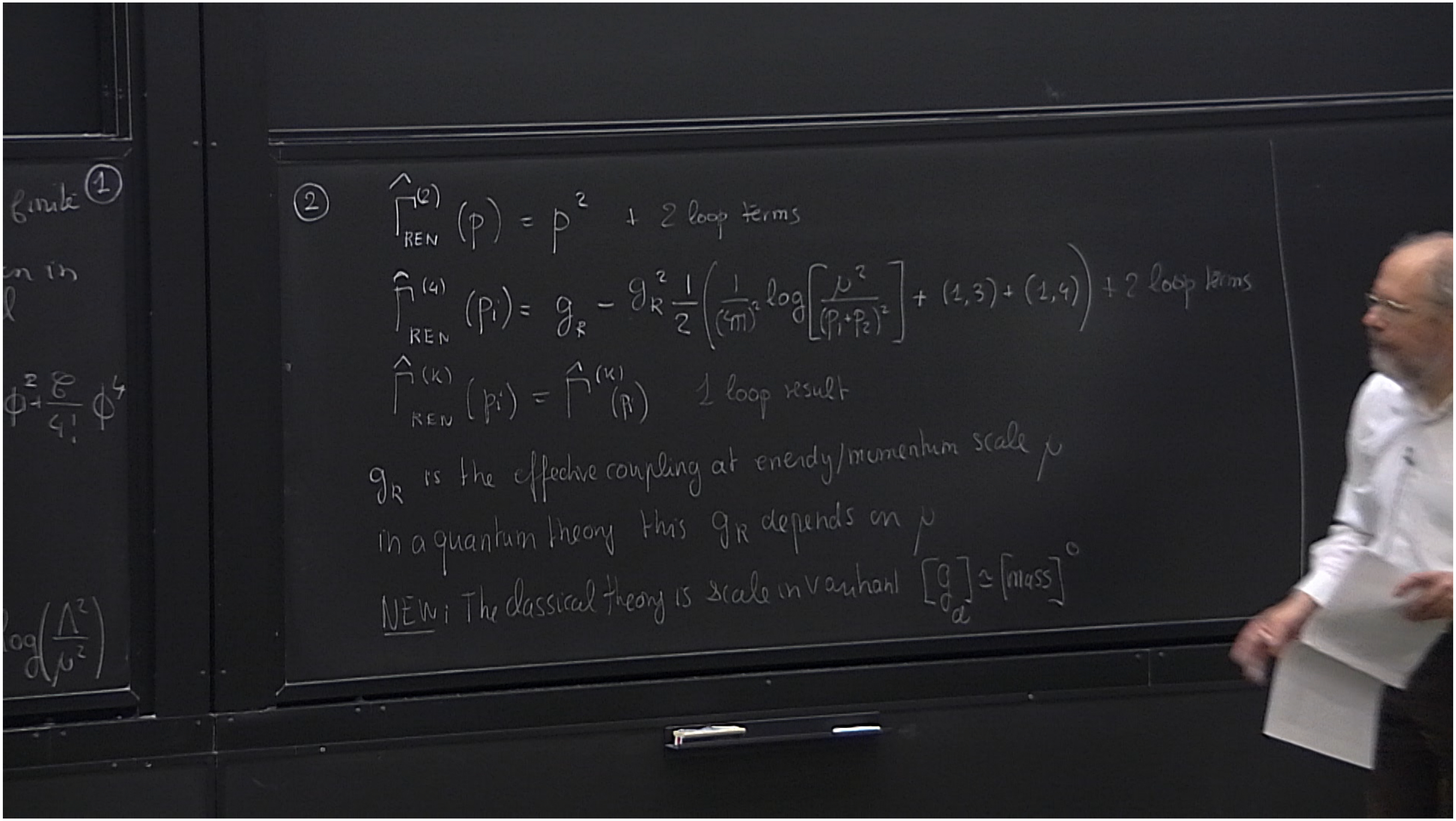
$$S_R[\phi] = \int d^4x \, \frac{A}{2}(\partial\phi)^2 + \frac{B}{2}\phi^2 + \frac{C}{4!}\phi^4$$

$A = 1$

② $\hat{\Gamma}^{(2)}_{REN}(p) = p^2$ + 2 loop terms

$\hat{\Gamma}^{(4)}_{REN}(p_i) = g_R - g_R^2 \frac{1}{2}\left(\frac{1}{(4\pi)^2}\log\left[\frac{\mu^2}{(p_1+p_2)^2}\right] + (1,3) + (1,4)\right)$ + 2 loop terms

$\hat{\Gamma}^{(k)}_{REN}(p_i) = \hat{\Gamma}^{(k)}(p_i)$ 1 loop result



finite ①

n in

l

$\phi^2 + \frac{g}{4!}\phi^4$

$\log\left(\frac{\Lambda^2}{\mu^2}\right)$

② $\hat{\Gamma}^{(2)}_{REN}(p) = p^2$ + 2 loop terms

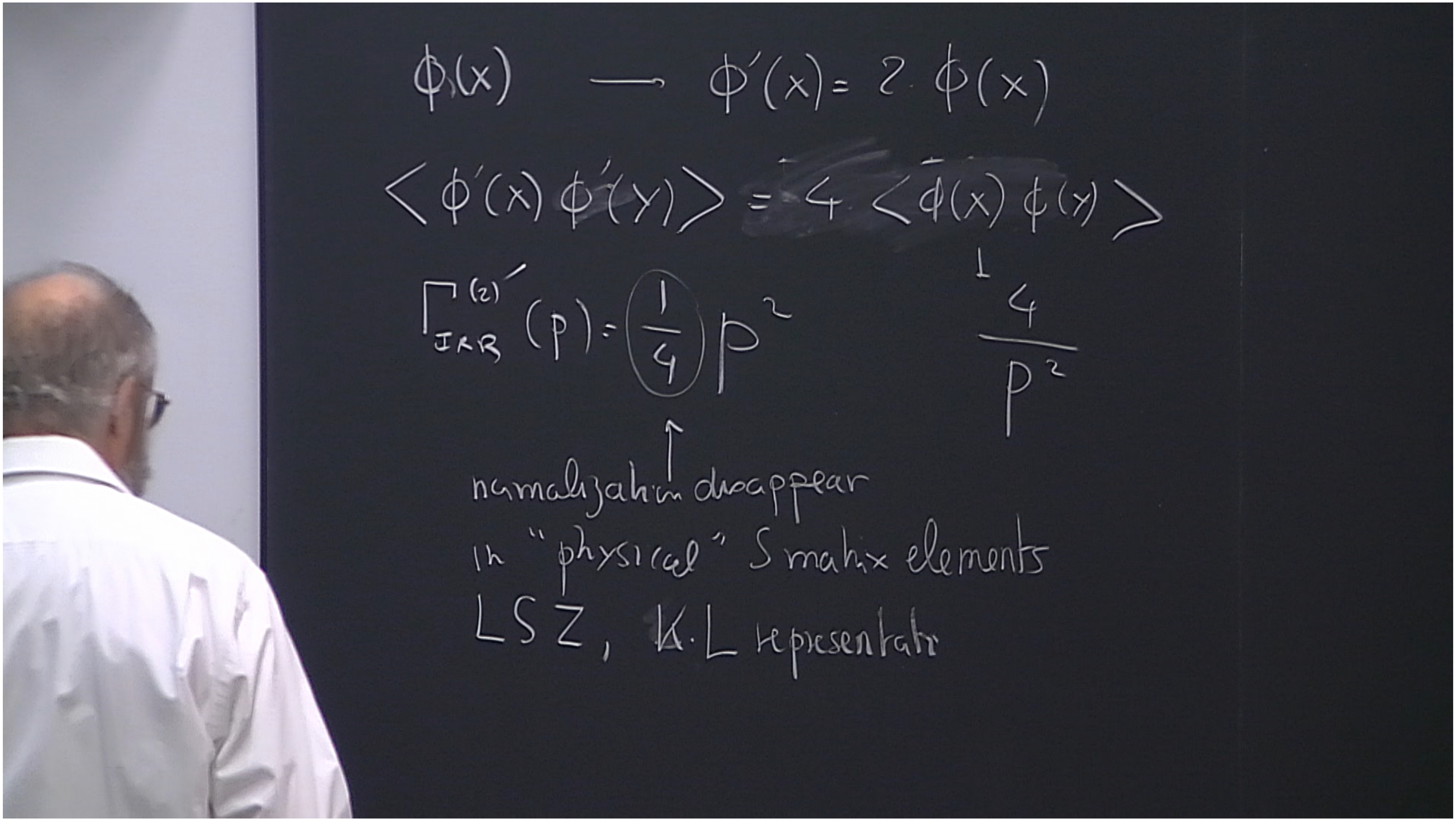
$\hat{\Gamma}^{(4)}_{REN}(p_i) = g_R - g_R^2\frac{1}{2}\left(\frac{1}{(4\pi)^2}\log\left[\frac{\mu^2}{(p_1+p_2)^2}\right] + (1,3) + (1,4)\right)$ + 2 loop terms

$\hat{\Gamma}^{(K)}_{REN}(p_i) = \hat{\Gamma}^{(K)}(p_i)$ 1 loop result

$g_R$ is the effective coupling at energy/momentum scale $\mu$

in a quantum theory this $g_R$ depends on $\mu$

NEW: The classical theory is scale invariant $[g_d] \simeq [\text{mass}]^0$



$$\phi(x) \longrightarrow \phi'(x) = 2\cdot\phi(x)$$

$$\langle \phi'(x)\,\phi'(y)\rangle = 4\cdot\langle \phi(x)\,\phi(y)\rangle$$

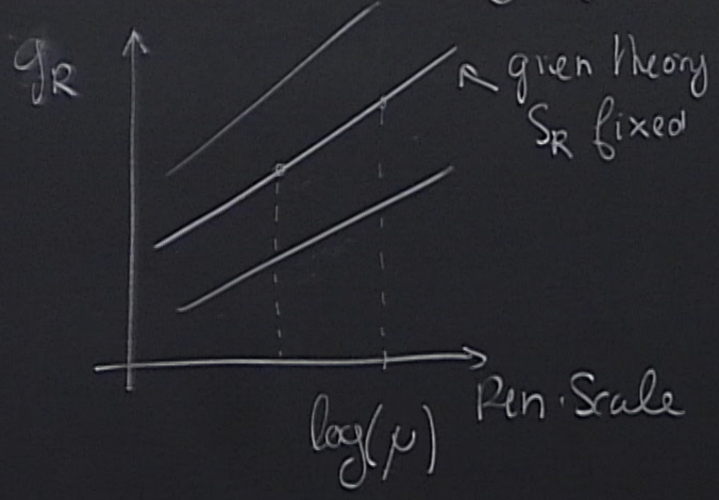
$$\Gamma^{(2)'}_{IRR}(p) = \boxed{\frac{1}{4}}\,p^2 \qquad \overset{1}{\phantom{x}}\frac{4}{p^2}$$

$\uparrow$

normalization disappear
in "physical" S matrix elements
LSZ, K.L representation

loop terms

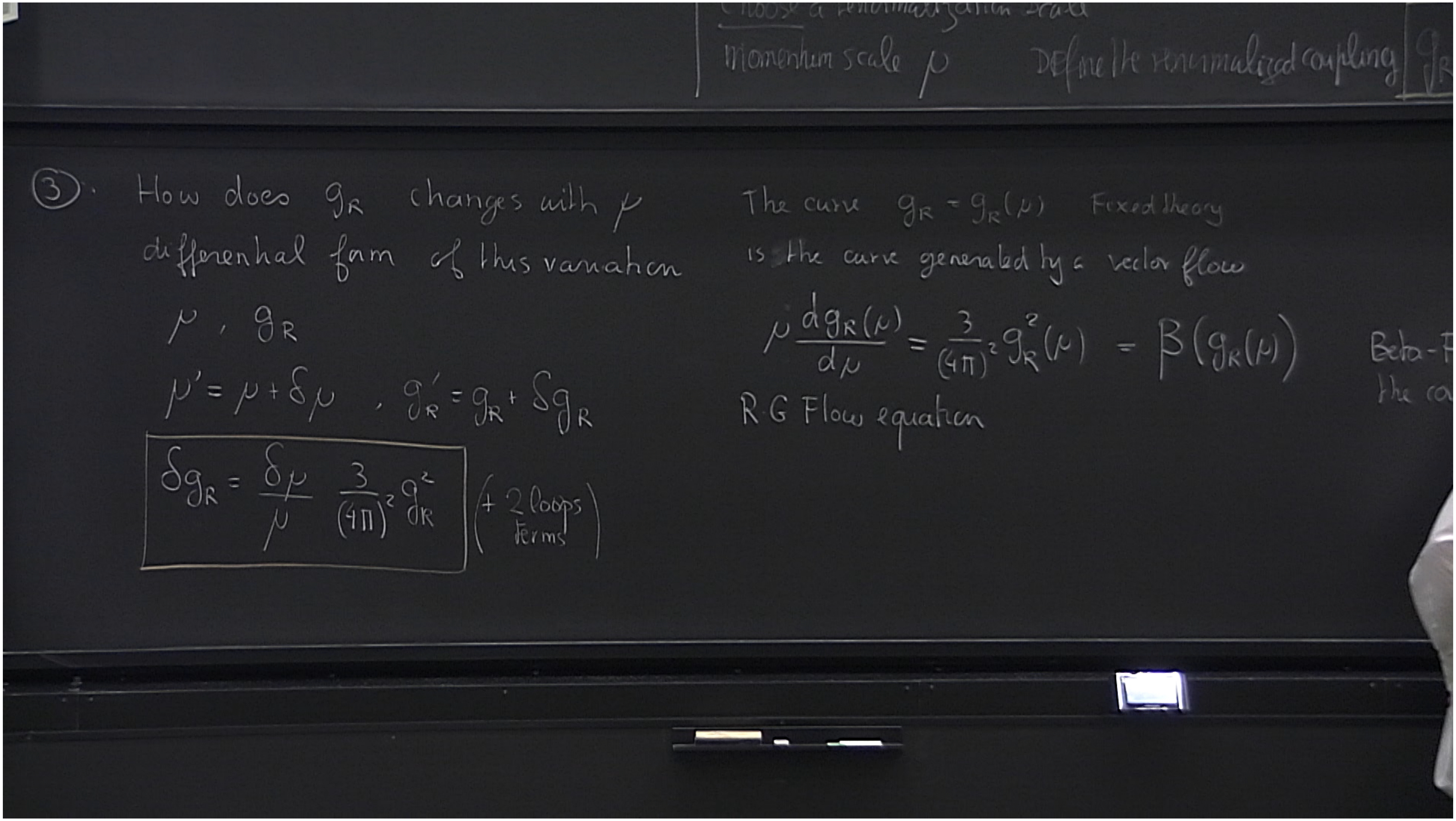
- What is the meaning of this:



- Space of massless $\phi^4$ theories as a 1 dim manifold

. choice of $\mu$ as a choice of coordinate system

$g_R$ is the coordinate in the system $\leftarrow \mu$



Choose a renormalization scale

momentum scale $\mu$ Define the renormalized coupling $g_R$

③. How does $g_R$ changes with $\mu$

differential form of this variation

$\mu$ , $g_R$

$\mu' = \mu + \delta\mu$ , $g_R' = g_R + \delta g_R$

$$\boxed{\delta g_R = \frac{\delta\mu}{\mu} \frac{3}{(4\pi)^2} g_R^2} \left( + 2 \text{ loops terms} \right)$$

The curve $g_R = g_R(\mu)$ Fixed theory

is the curve generated by a vector flow

$$\mu \frac{d g_R(\mu)}{d\mu} = \frac{3}{(4\pi)^2} g_R^2(\mu) = \beta(g_R(\mu))$$

Beta-F

The ca

RG Flow equation

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R.G Flow equation

Beta-

the ca

## Beta-Function for the coupling $g_R$

$$\Downarrow$$

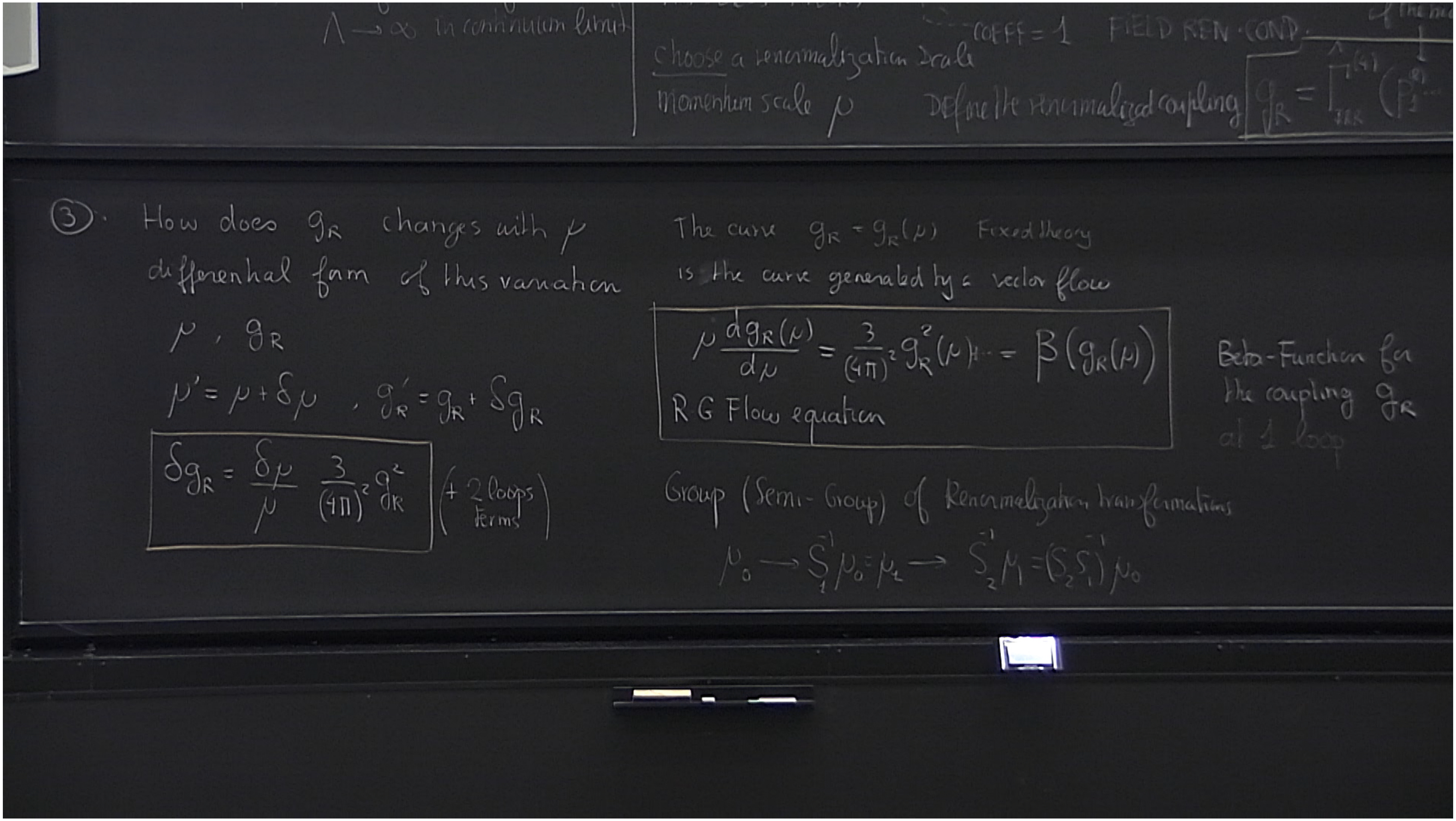
$\delta g_R , \delta \mu$ so that $\delta \mathcal{C} = 0$

$$0 = \delta g_R \left( 1 + 3 g_R \frac{1}{(4\pi)^2} \log\left(\frac{\Lambda^2}{\mu^2}\right)\right) + \frac{\delta\mu}{\mu}\left( - g_R^2 \frac{3}{(4\pi)^2} \right)$$

$$\Downarrow$$

$$\delta g_R = \frac{\delta\mu}{\mu}\left( g_R^2 \cdot \frac{3}{4\pi^2}\right)\left(1 + 3\cancel{g_R \frac{1}{(4\pi)^2}} \log\frac{\Lambda^2}{\mu^2}\right)^{-1}$$

↑ 1 loop diagram ↑ 2 loops term



$\Lambda \to \infty$ in continuum limit

COEFF = 1 FIELD REN. COND.

Choose a renormalization scale

momentum scale $\mu$ Define the renormalized coupling $\boxed{g_R = \Gamma^{(4)}_{REN}(p_1^{(0)} \cdots}$

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differential form of this variation

$\mu$, $g_R$

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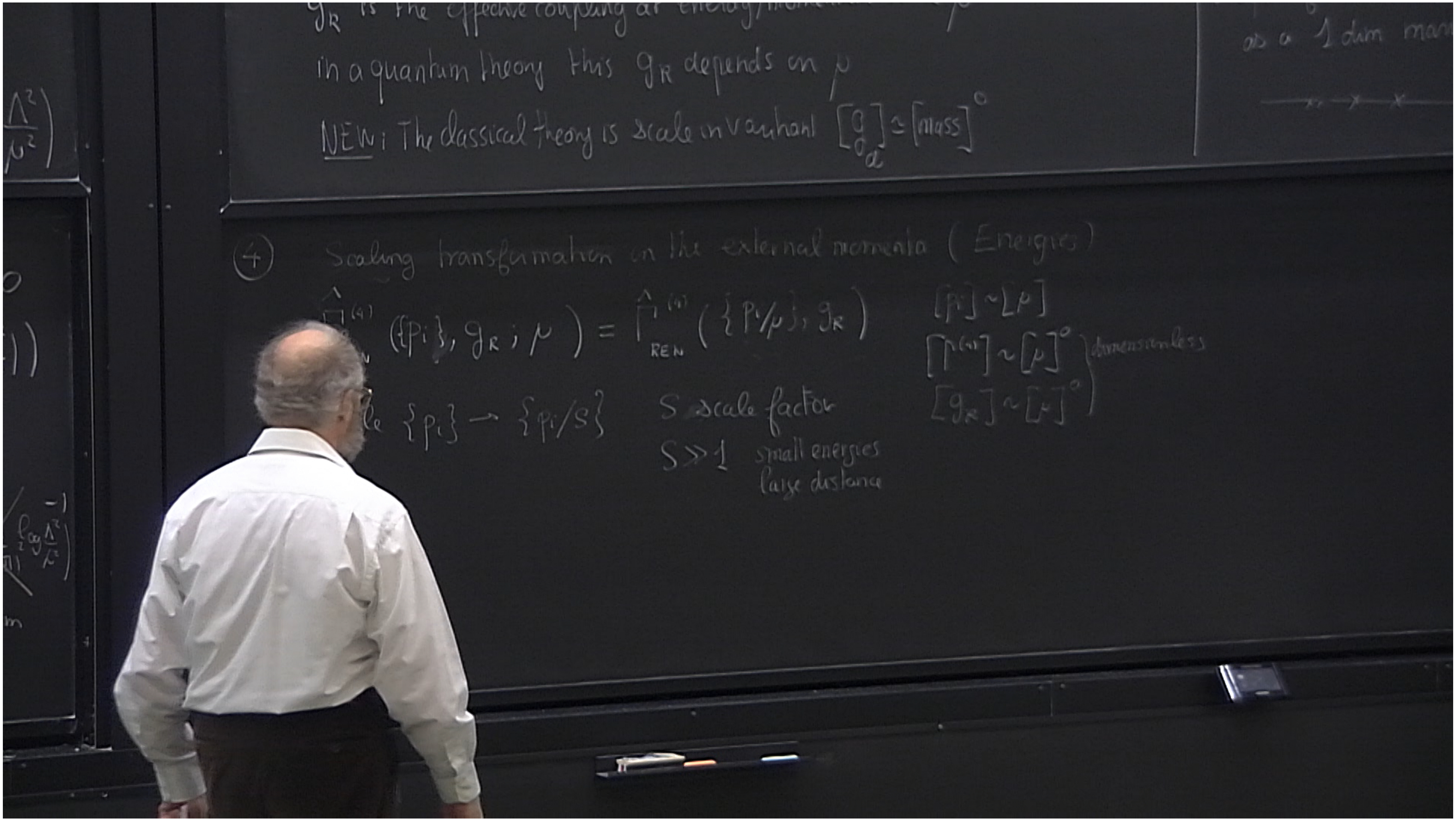
$$\boxed{\mu\frac{dg_R(\mu)}{d\mu} = \frac{3}{(4\pi)^2}g_R^2(\mu) + \cdots = \beta(g_R(\mu))}$$

RG Flow equation

Beta-Function for the coupling $g_R$ at 1 loop

Group (Semi-Group) of Renormalization transformations

$$\mu_0 \longrightarrow S_1^{-1}\mu_0 = \mu_1 \longrightarrow S_2^{-1}\mu_1 = (S_2 S_1)^{-1}\mu_0$$



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NEW: The classical theory is scale invariant $[g_d] \simeq [\text{mass}]^0$

as a 1 dim man...

④ Scaling transformation in the external momenta ( Energies )

$$\hat{\Gamma}^{(4)}(\{p_i\}, g_R; \mu) = \hat{\Gamma}^{(4)}_{REN}(\{p_i/\mu\}, g_R)$$

$[p_i] \sim [\mu]$

$[\Gamma^{(4)}] \sim [\mu]^0$ dimensionless

$[g_R] \sim [\mu]^0$

...le $\{p_i\} \to \{p_i/S\}$

S scale factor

$S \gg 1$ small energies large distance

**NEW**: The classical theory is scale invariant $[g_d] \simeq [\text{mass}]$

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rescale $\{p_i\} \to \{p_i/S\}$ $\quad$ $S$ scale factor

$S \gg 1$ small energies
large distance

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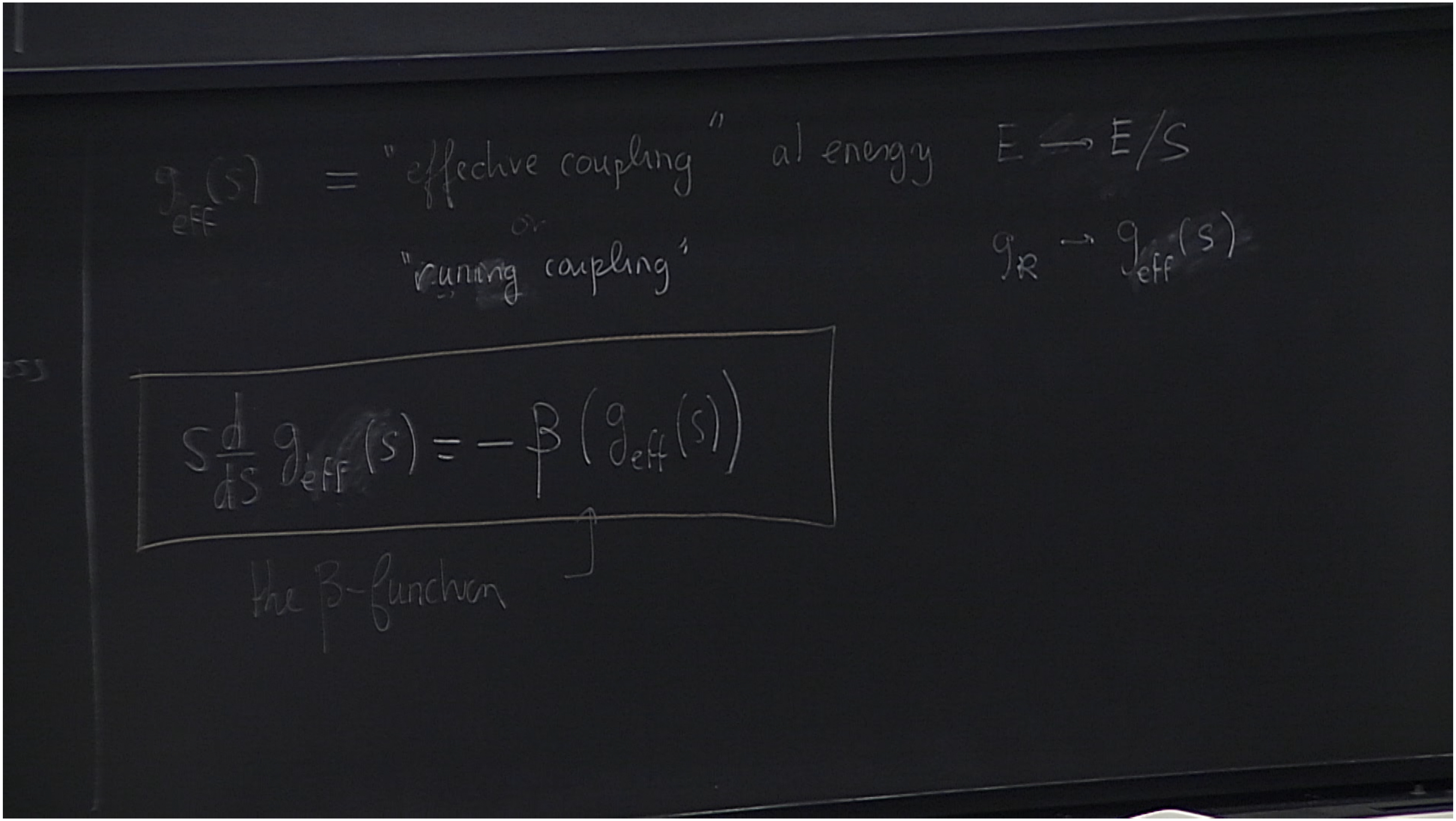
rescale $\{p_i\} \to \{p_i/s\}$   $s$ scale factor

$s \gg 1$ small energies
large distance

$$\hat{\Gamma}^{(4)}_{REN}(\{p_i/s\}, g_R; \mu) \underset{\substack{\uparrow \\ \text{dimensional analysis}}}{=} \hat{\Gamma}^{(4)}_{REN}(\{p_i\}, g_R; \mu s) \underset{\substack{\uparrow \\ \text{non-trivial} \\ \text{RG analysis}}}{=} \hat{\Gamma}^{(4)}_{REN}(\{p_i\}, g_R(s); \mu)$$

rescaling momenta ⟺ change the coupling



$g_{\text{eff}}(s)$ = "effective coupling" at energy $E \longrightarrow E/s$

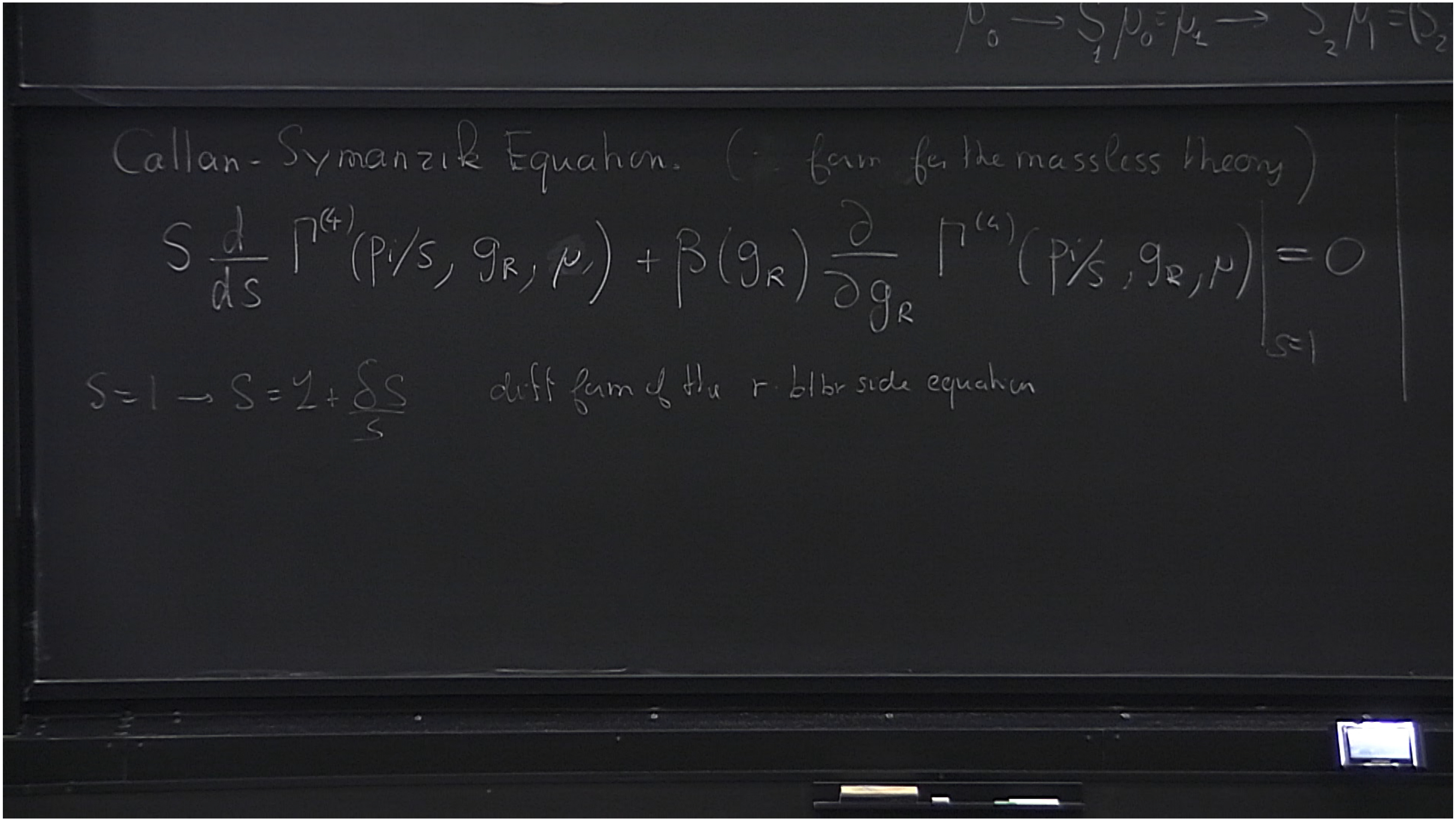
or

"running coupling"

$g_R \to g_{\text{eff}}(s)$

$$\boxed{s\frac{d}{ds}\, g_{\text{eff}}(s) = -\beta\left(g_{\text{eff}}(s)\right)}$$

the $\beta$-function

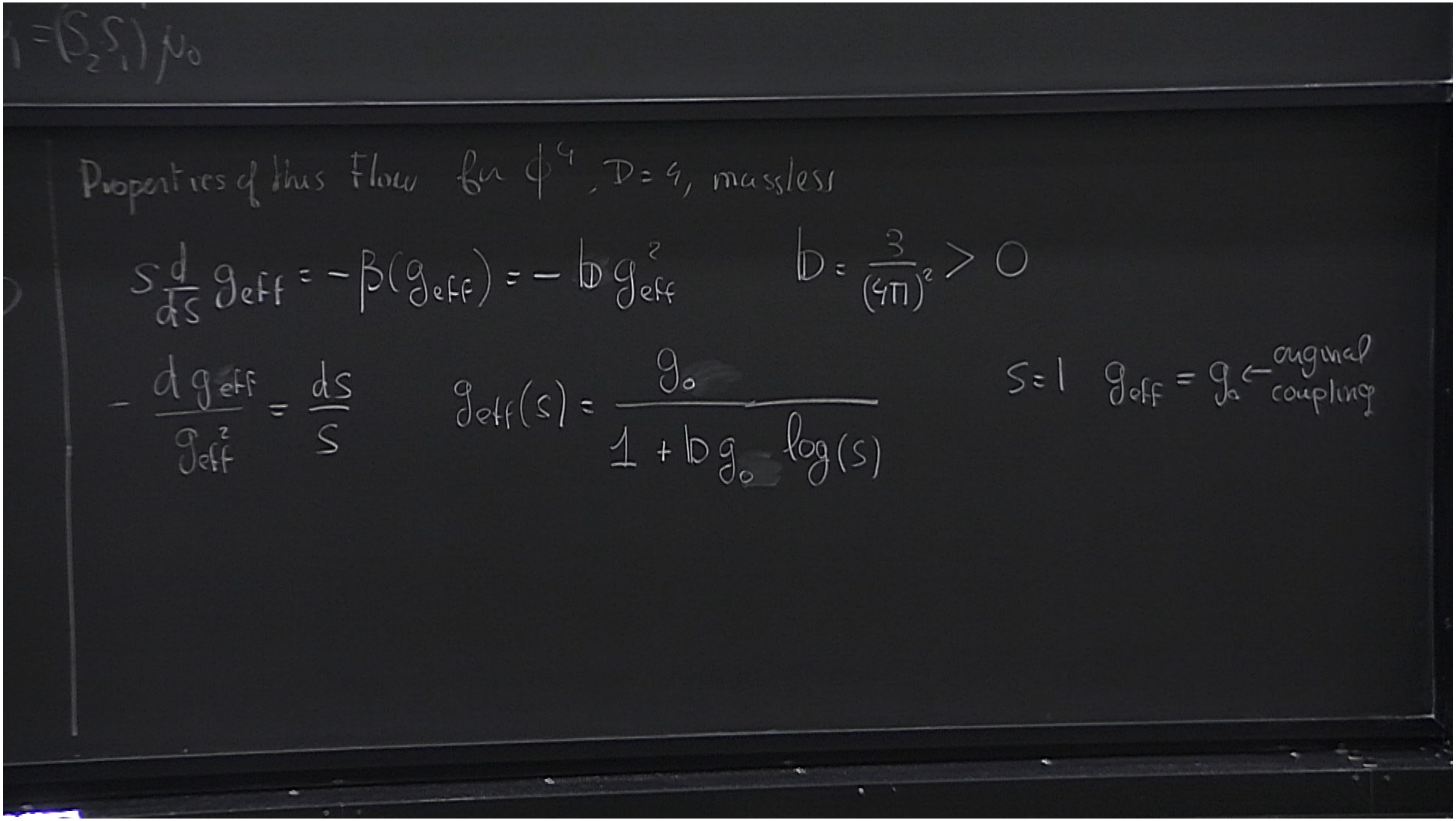


$$\mu_0 \longrightarrow S_1 \mu_0 = \mu_1 \longrightarrow S_2 \mu_1 = \mu_2$$

Callan-Symanzik Equation. ( form for the massless theory)

$$S \frac{d}{ds} \Gamma^{(4)}(p_i/s, g_R, \mu) + \beta(g_R) \frac{\partial}{\partial g_R} \Gamma^{(4)}(p_i/s, g_R, \mu) \Big|_{s=1} = 0$$

$S=1 \rightarrow S = 1 + \frac{\delta S}{S}$ diff form of the r.h.s side equation



$= (S_2 S_1)\mu_0$

Properties of this Flow for $\phi^4$, $D = 4$, massless

$$s\frac{d}{ds} g_{eff} = -\beta(g_{eff}) = -b\, g_{eff}^2 \qquad b = \frac{3}{(4\pi)^2} > 0$$

$$-\frac{d g_{eff}}{g_{eff}^2} = \frac{ds}{s} \qquad g_{eff}(s) = \frac{g_0}{1 + b g_0 \log(s)}$$

$s = 1 \quad g_{eff} = g_0 \leftarrow$ original coupling

$= (S_2 S_1)\mu_0$

Properties of this Flow for $\phi^4$, $D = 4$, massless

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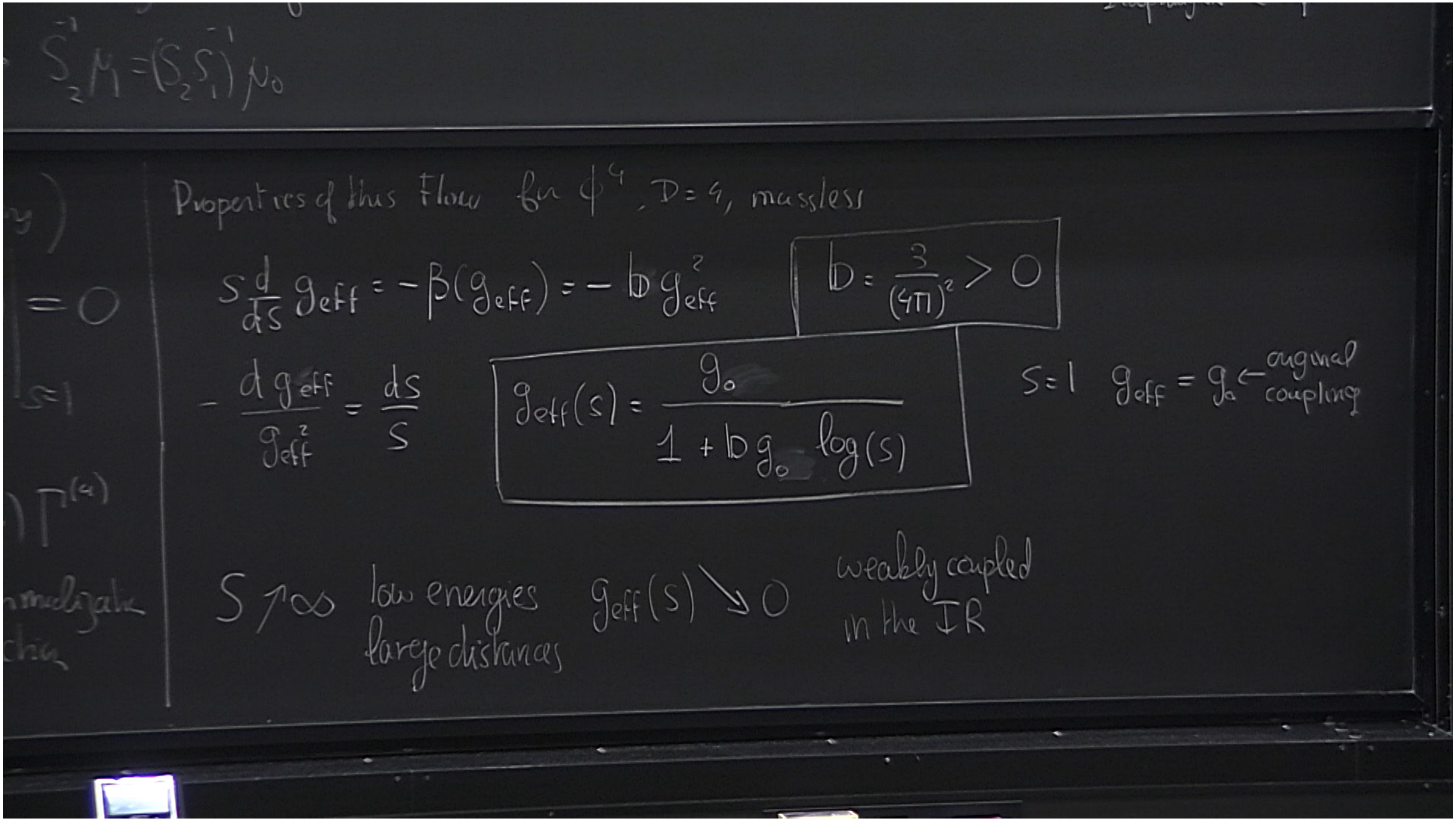


$S_2^{-1}\mu_1 = (S_2 S_1)^{-1}\mu_0$

$=0$

$s=1$

$\Gamma^{(4)}$

Properties of this Flow for $\phi^4$, $D=4$, massless

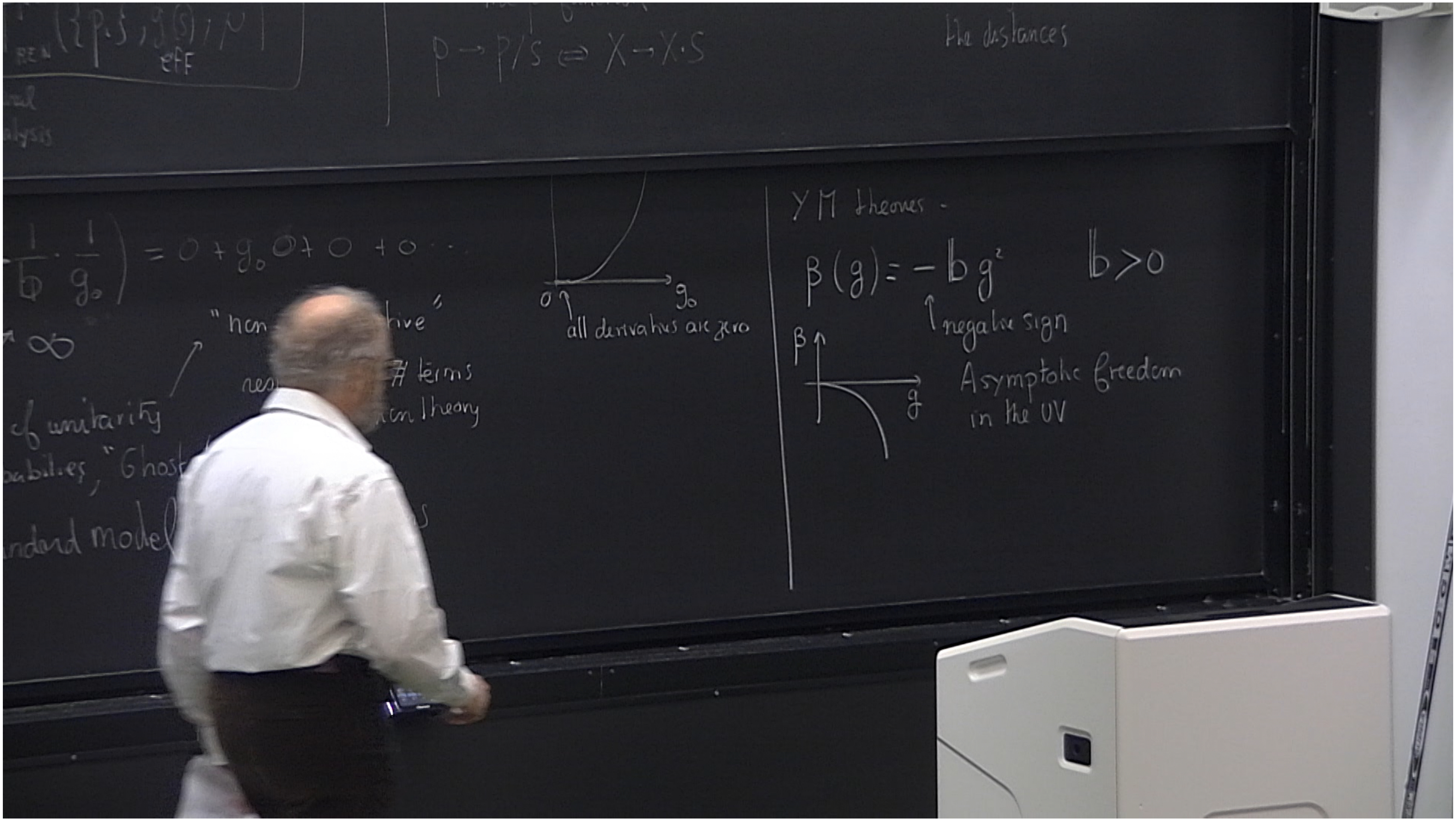
$$s\frac{d}{ds}g_{eff} = -\beta(g_{eff}) = -b\,g_{eff}^2 \qquad \boxed{b = \frac{3}{(4\pi)^2} > 0}$$

$$-\frac{dg_{eff}}{g_{eff}^2} = \frac{ds}{s}$$

$$\boxed{g_{eff}(s) = \frac{g_0}{1 + b\,g_0 \log(s)}}$$

$s=1 \quad g_{eff} = g_0 \leftarrow$ original coupling

$S \nearrow \infty$ low energies large distances $g_{eff}(s) \searrow 0$ weakly coupled in the IR



$p \to p/s \Leftrightarrow X \to X \cdot s$

the distances

$= 0 + g_0 0 + 0 + 0 \cdots$

"non tive"

# terms

n theory

of unitarity

babilities, "Ghost

ndard model

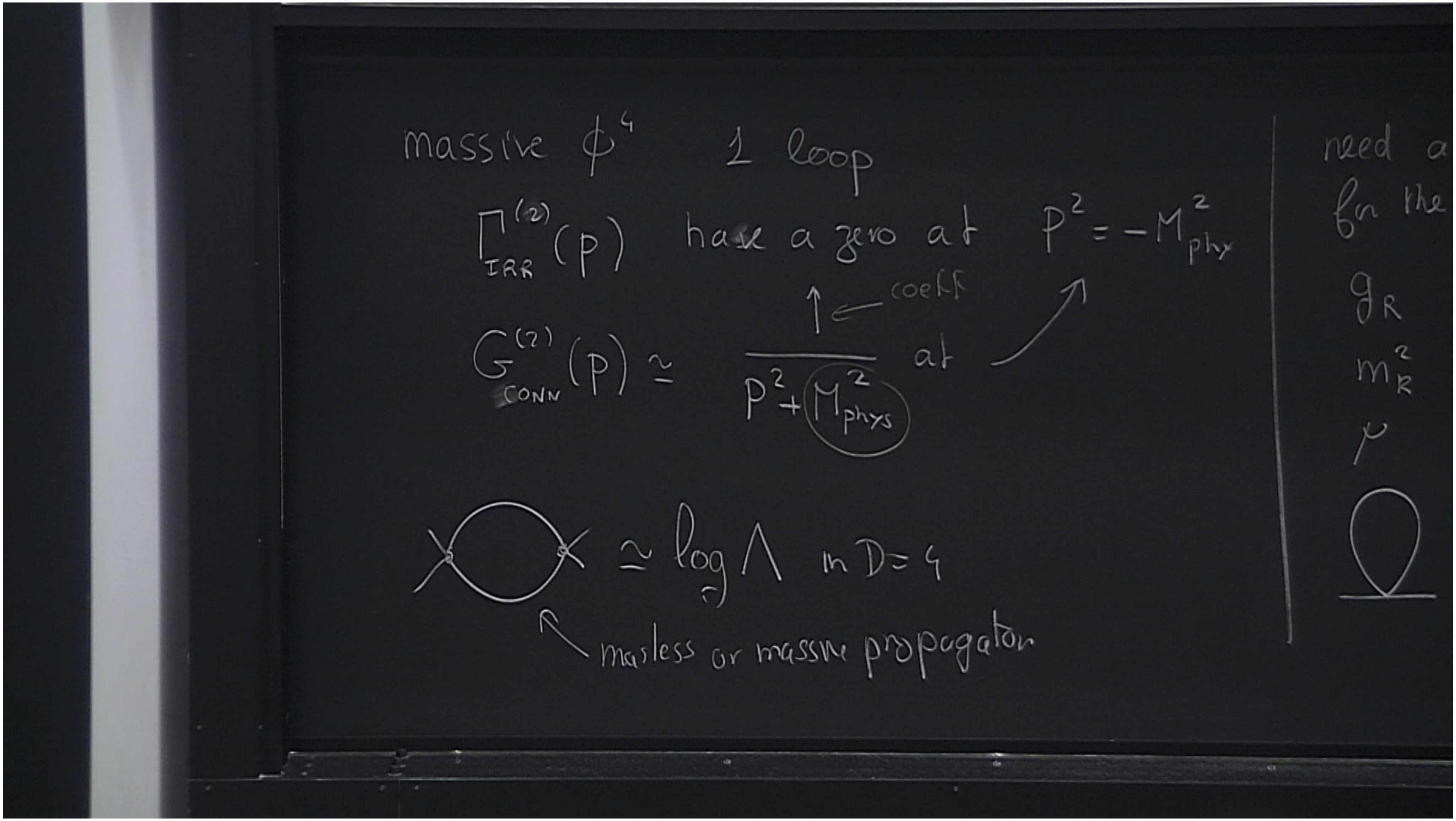
all derivatives are zero

YM theories -

$\beta(g) = -b g^2$   $b > 0$

↑ negative sign

Asymptotic freedom in the UV



massive $\phi^4$   1 loop

$\Gamma^{(2)}_{IRR}(p)$ has a zero at $p^2 = -M^2_{phys}$

$\uparrow \leftarrow$ coeff

$$G^{(2)}_{CONN}(p) \simeq \frac{\uparrow}{p^2 + \boxed{M^2_{phys}}} \text{ at } \nearrow$$

$\simeq \log \Lambda$ in $D=4$

$\nwarrow$ massless or massive propagator

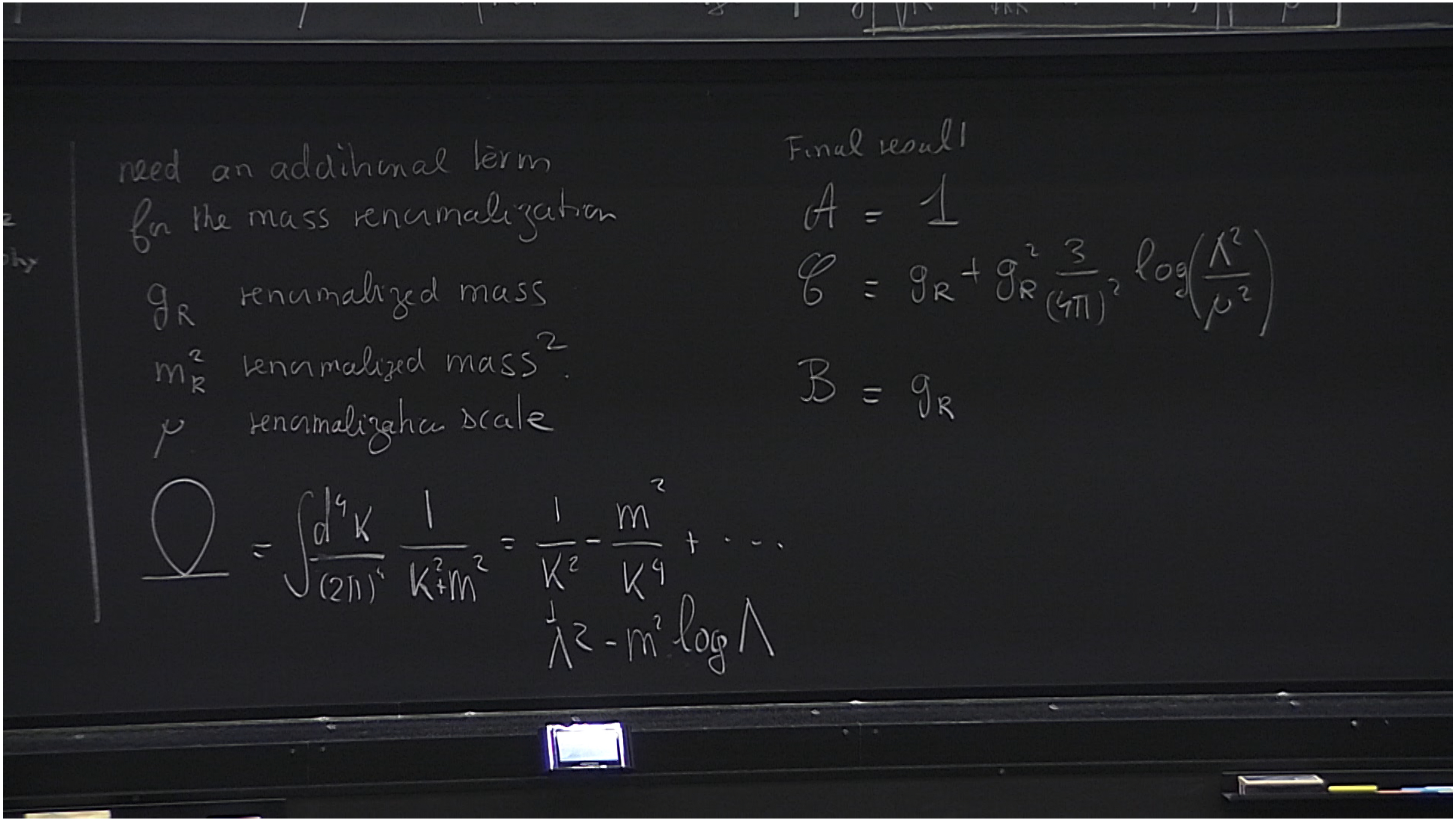
need a

for the

$g_R$

$m_R^2$

$\mu$



need an additional term
for the mass renormalization

$g_R$ renormalized mass

$m_R^2$ renormalized mass$^2$.

$\mu$ renormalization scale

$$= \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2+m^2} = \frac{1}{k^2} - \frac{m^2}{k^4} + \dots$$

$$\downarrow$$

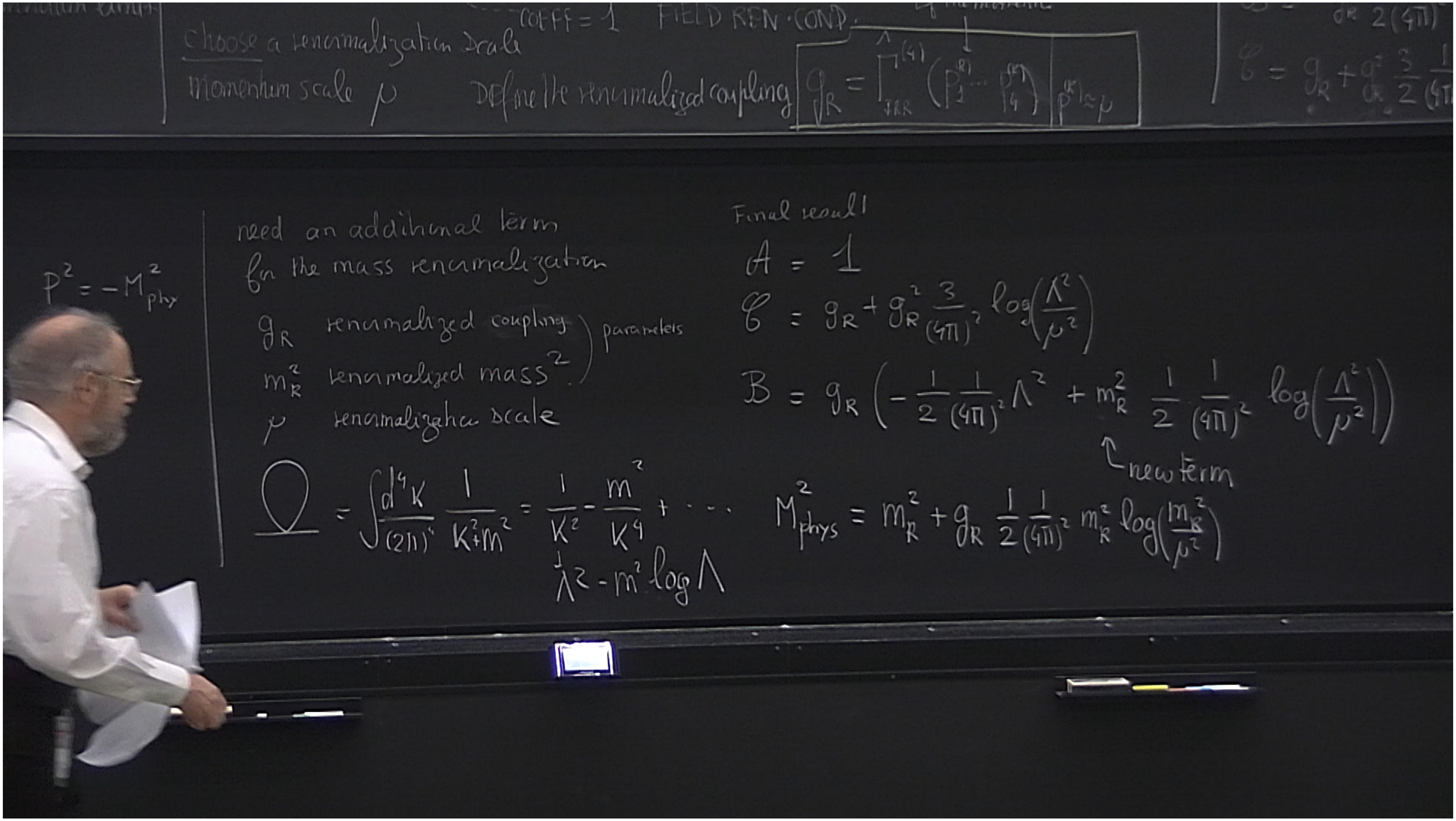
$$\Lambda^2 - m^2 \log \Lambda$$

Final result

$$A = 1$$

$$C = g_R + g_R^2 \frac{3}{(4\pi)^2} \log\left(\frac{\Lambda^2}{\mu^2}\right)$$

$$B = g_R$$



COEFF = 1 FIELD REN · COND.

Choose a renormalization scale

momentum scale $\mu$

Define the renormalized coupling $\boxed{g_R = \hat{\Gamma}^{(4)}_{IRR}\left(p_1^{(R)} \dots p_4^{(R)}\right) \quad p^{(R)} \simeq \mu}$

$\mathcal{C} = g_R + g_R^2 \frac{3}{2} \frac{1}{(4\pi}$

$p^2 = -M^2_{phy}$

need an additional term

for the mass renormalization

$g_R$ renormalized coupling

$m_R^2$ renormalized mass$^2$

$\mu$ renormalization scale

parameters

$$= \int \frac{d^4K}{(2\pi)^4} \frac{1}{K^2+m^2} = \frac{1}{K^2} - \frac{m^2}{K^4} + \dots$$

$$\Lambda^2 - m^2 \log \Lambda$$

Final result!

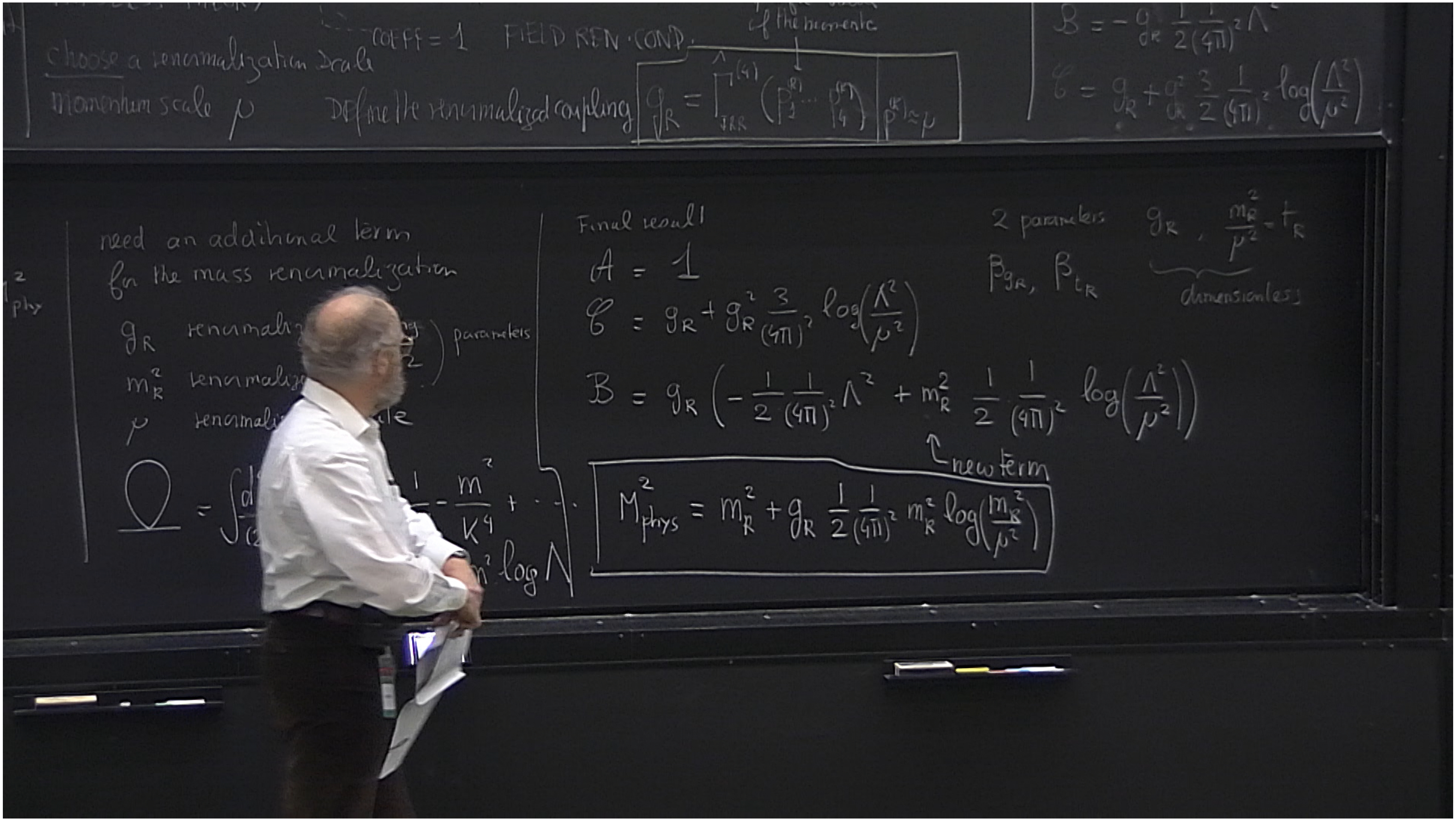
$$A = 1$$

$$\mathcal{C} = g_R + g_R^2 \frac{3}{(4\pi)^2} \log\left(\frac{\Lambda^2}{\mu^2}\right)$$

$$B = g_R \left( -\frac{1}{2}\frac{1}{(4\pi)^2}\Lambda^2 + m_R^2 \frac{1}{2}\frac{1}{(4\pi)^2} \log\left(\frac{\Lambda^2}{\mu^2}\right)\right)$$

new term

$$M^2_{phys} = m_R^2 + g_R \frac{1}{2}\frac{1}{(4\pi)^2} m_R^2 \log\left(\frac{m_R^2}{\mu^2}\right)$$



COEFF = 1 FIELD REN. COND.

of the momenta

choose a renormalization scale

momentum scale $\mu$

Define the renormalized coupling $\boxed{g_R = \hat{\Gamma}^{(4)}_{JRR}(p_1^{(R)} \dots p_4^{(R)}) \quad |p^{(R)}| \approx \mu}$

$$\mathcal{B} = -g_R \frac{1}{2} \frac{1}{(4\pi)^2} \Lambda^2$$

$$\mathcal{C} = g_R + g_R^2 \frac{3}{2} \frac{1}{(4\pi)^2} \log\left(\frac{\Lambda^2}{\mu^2}\right)$$

$^{2}_{phy}$

need an additional term

for the mass renormalization

$g_R$ renormaliz... ) parameters

$m_R^2$ renormaliz...

$\mu$ renormali... ...le

$$= \int \frac{d\dots}{(2\dots} \dots \frac{1}{\dots} - \frac{m^2}{K^4} + \dots \Big] \quad \dots m^2 \log \Lambda$$

Final result!

$$\mathcal{A} = 1$$

$$\mathcal{C} = g_R + g_R^2 \frac{3}{(4\pi)^2} \log\left(\frac{\Lambda^2}{\mu^2}\right)$$

$$\mathcal{B} = g_R \left( -\frac{1}{2} \frac{1}{(4\pi)^2} \Lambda^2 + \underset{\uparrow \text{new term}}{m_R^2} \frac{1}{2} \frac{1}{(4\pi)^2} \log\left(\frac{\Lambda^2}{\mu^2}\right) \right)$$

$$\boxed{M^2_{phys} = m_R^2 + g_R \frac{1}{2} \frac{1}{(4\pi)^2} m_R^2 \log\left(\frac{m_R^2}{\mu^2}\right)}$$

2 parameters $\underbrace{g_R, \quad \frac{m_R^2}{\mu^2} = t_R}_{\text{dimensionless}}$

$\beta_{g_R}, \quad \beta_{t_R}$