Title: PSI 17/18 - Quantum Field Theory II - Lecture 7

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Abstract:

Perturbation theory : the quantum effective action

$$\Gamma[\varphi] = S[\varphi] + \hbar \frac{1}{2} \mathrm{Tr}\left[\mathrm{Log}\left(S''[\varphi]\right)\right] + o(\hbar^2)$$

- φ: background field (classical) generating functional for the 1PI functions
- $S[\varphi]$: classical action
- $S''[\varphi]$: Hessian operator, 2nd functional derivative matrix of classical action
- $\hbar$: sum of 1 loop 1PI diagrams

$$S[\varphi] = \int d^d x \, \frac{1}{2} \varphi(x) \left(-\Delta_x + m^2\right) \varphi(x) + \frac{g}{4!} \varphi(x)^4$$

$$S''[\varphi](x,y) = \frac{\delta^2 S[\varphi]}{\delta\varphi(x)\,\delta\varphi(y)} = \left(-\Delta_x + m^2\right)\delta(x-y) + \frac{g}{2}\varphi^2(x)\,\delta(x-y)$$

Kernel of S''

↕ Dirac distribution

$$S''[\varphi] = \left(-\Delta + m^2\right) + \frac{g}{2}\varphi^2$$

diff operator
it is local

$$= \left(-\Delta + m^2\right) \times \left(\mathbb{1} + \frac{1}{-\Delta + m^2} \times \frac{g}{2}\varphi^2\right)$$

$$\left(-\Delta_x + m^2\right)\varphi(x) + \frac{g}{4!}\varphi(x)^4$$

$$= \left(-\Delta_x + m^2\right)\delta(x-y) + \frac{g}{2}\varphi^2(x)\,\delta(x-y)$$

↑ Dirac distribution

φ^2 diff operator it is local

$$\left(1 + \frac{1}{-\Delta + m^2} \times \frac{g}{2}\varphi^2\right)$$

$$\mathrm{Tr}\left(\log(A \cdot B)\right) = \log \mathrm{Det}(A \cdot B)$$
$$= \log\left(\det(A) \times \det B\right)$$
$$= \log \det(A) + \log(\det B)$$

$$\mathrm{Tr}\left[\log\left(S''[\varphi]\right)\right] = T$$

$$\mathrm{Tr}\left[\mathrm{Log}\left(S''[\varphi]\right)\right] = \mathrm{Tr}\left[\mathrm{Log}\left(-\Delta + m^2\right)\right] + \mathrm{Tr}\left[\mathrm{Log}\left(\mathbb{1} + \frac{g}{2}\cdot\frac{1}{-\Delta+m^2}\cdot\varphi^2\right)\right]$$

↑ Free Field

↑ interacting theory

Series expansion in g

$$\square = \sum_{K=1}^{\infty} \frac{(-1)^{K-1}}{K}\left(\frac{g}{2}\right)^K \mathrm{Tr}\left[\underbrace{\frac{1}{-\Delta+m^2}\cdot\varphi^2 \cdot \frac{1}{-\Delta+m^2}\cdot\varphi^2 \cdots \frac{1}{-\Delta+m^2}\cdot\varphi^2}_{K \text{ times}}\right]$$

Kernels of trace class operators

$A(x,y)$ $\quad \mathrm{Tr}\, A = \int d^d x\, A(x,x)$

$B(x,y)$ $\quad (A\cdot B)(x,z) = \int d^d y\, A(x,y)\, B(y,z)$

$$\varphi^2(x,y) = \varphi^2(x)\,\delta(x-y)$$

$$\left(\frac{1}{-\Delta+m^2}\right)(x,y) = G_0(x-y) = \int \frac{d^d k}{(2\pi)^d} \frac{e^{ik(x-y)}}{k^2+m^2}$$

Free Field Propagator



$$= \mathrm{Tr}\left[\mathrm{Log}\left(-\Delta + m^2\right)\right] + \mathrm{Tr}\left[\mathrm{Log}\left(\mathbb{I} + \frac{g}{2}\cdot\frac{1}{-\Delta+m^2}\cdot\varphi^2\right)\right]$$

↑ Free Field

↑ interacting theory

Kernels of trace class operators $A(x,$ $B(x,$ $\varphi^2(x$ $\left(\frac{1}{-\Delta+m^2}\right.$

$$\frac{(-1)^{k-1}}{k}\left(\frac{g}{2}\right)^k \mathrm{Tr}\left[\underbrace{\frac{1}{-\Delta+m^2}\cdot\varphi^2}_{k\text{ times}}\cdot\underbrace{\frac{1}{-\Delta+m^2}\cdot\varphi^2}\cdots\underbrace{\frac{1}{-\Delta+m^2}\cdot\varphi^2}\right]$$

$$= \int dx_1 dy_1 \cdots G_0(x_1-y_1)\,\varphi^2(y_1)\,\delta(y_1-x_2)\,G_0(x_2-y_2)\,\varphi^2(y_2)\,\delta(y_2-x_3)\cdots\varphi^2(y_k)\,\delta(y_k-x_1)$$

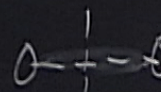
$$(x,y) = \varphi^2(x)\,\delta(x-y)$$

$$(x,y) = G_0(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik(x-y)}}{k^2+m^2}$$

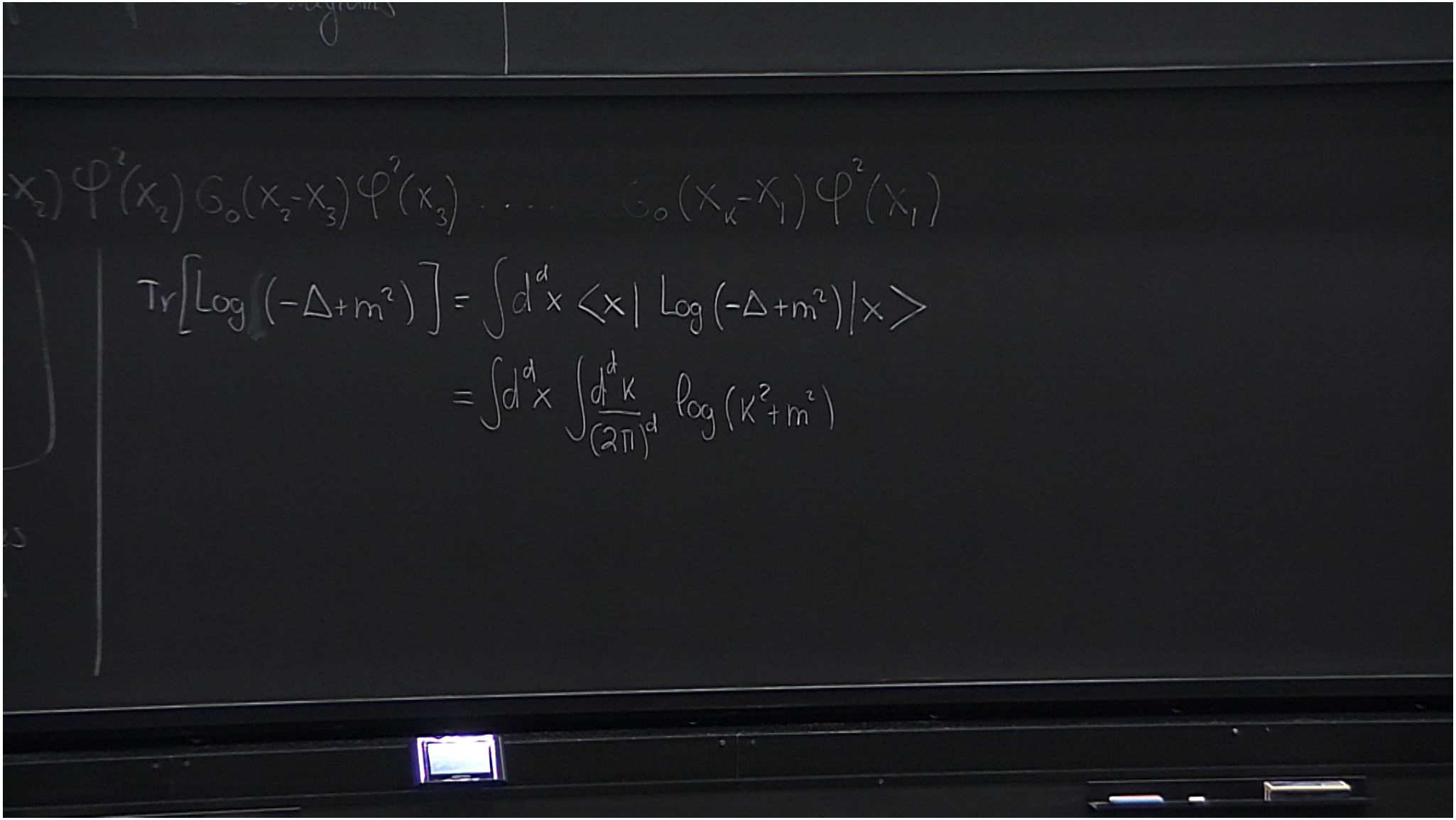
Free Field Propagator



$$\delta(x-y) =$$

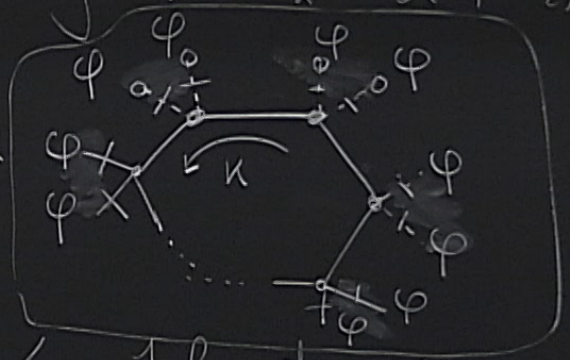


amputated or truncated line


$\varphi^2(x_2)\, G_0(x_2-x_3)\, \varphi^2(x_3) \cdots G_0(x_K-x_1)\, \varphi^2(x_1)$
$\mathrm{Tr}\left[\mathrm{Log}\,(-\Delta+m^2)\right] = \int d^d x \,\langle x|\, \mathrm{Log}\,(-\Delta+m^2)\,|x\rangle$
$= \int d^d x \int \frac{d^d k}{(2\pi)^d}\, \mathrm{Log}\,(k^2+m^2)$

$$\mathrm{Tr}\left[\mathrm{Log}\left(\mathbb{1}+\dots\right)\right] = \sum_K \frac{(-1)^{K-1}}{K}\left(\frac{g}{2}\right)^K \int d^d x_1 \, d^d x_K \; G_0(x_1-x_2)\,\varphi^2(x_2)\,G_0(x_2-x_3)\,\varphi^2(x_3)\,\dots$$

$$= \sum_K \frac{(-1)^{K-1}}{K}\left(\frac{g}{2}\right)^K$$



ϕ^4 theory

1 loop diagram with 2 truncated lines at each vertex, with 2 $\varphi(x)$ attached

$$\mathrm{Tr}\left[\mathrm{Log}\left(-\Delta+m^2\right)\right] =$$

$$=$$

Final result for the 1 loop term

$$+\hbar\left[-\frac{1}{2}\;\bigcirc\;+\frac{g}{4}\;\varphi\;\bigcirc\;\varphi\;-\frac{g^2}{16}\;\varphi\;\bigcirc\;\varphi\;\varphi\;\varphi\right.$$

$$\left.+\frac{g^3}{48}\;\varphi\;\bigcirc\;\varphi\;\varphi\;\varphi\;\varphi\;+\;\cdots\right]$$

↑ correct symmetry factors for the 1PI diagrams

How to derive this ? Saddle point approximation.

$$\Gamma[\varphi] = \varphi \cdot j - W[j] \qquad \varphi = \frac{\delta W[j]}{\delta j}$$

1-P-I bkgr field source connected

$$W[j] = \hbar \log Z[j],$$

with

$$Z[j] = \int \mathcal{D}[\phi] \exp\left(\frac{-1}{\hbar}\left(S[\phi] - j\cdot\phi\right)\right)$$

random variable (quantum)

ximation $\hbar$ small:

minimum of $S[\phi] - j\cdot\phi$

ϕ_c : $\frac{\delta S}{\delta \phi(x)}[\phi] - j(x) = 0$

$\phi_c = \{\phi_c(x)\}$ functional of j

$S'[\phi_c] = j$ saddle point

pand $\phi = \phi_c + \tilde{\phi}$

classical J solution

$\tilde{\phi}$ random variable Q. fluctuation

$$j\cdot\phi = \int d^d x\, j(x)\phi(x)$$

For ϕ^4 theory

$$(-\Delta_x + m^2)\phi_c(x) + \frac{g}{6}\phi_c^3(x) = j(x)$$

$$j \cdot \phi = \int d^d x \, j(x) \phi(x)$$

For ϕ^4 theory

$$\left(-\Delta_x + m^2\right) \phi_c(x) + \frac{g}{6} \phi_c^3(x) = j(x)$$

$$Z[j] = \exp\left(-\frac{1}{\hbar}\left(S[\phi_c] - j \cdot \phi_c\right)\right) \int \mathcal{D}[\tilde{\phi}] \exp\left(-\frac{1}{\hbar}\left(\frac{1}{2}\tilde{\phi} \cdot S''[\phi_c] \cdot \tilde{\phi} + \text{high order terms}\right)\right)$$

$\tilde{\phi}$ are $\sim \hbar^{1/2}$

$$\text{Det}\left[S''[\phi_c]\right]^{-1/2} \left(1 + O(\hbar)\right)$$

$$W[j] = j \cdot \phi_c - S[\phi_c] - \frac{\hbar}{2} \mathrm{Tr}\left[\mathrm{Log}\left(S''[\phi_c]\right)\right]$$

with $\phi_c \leftarrow S'[\phi_c] = j$

Legendre transform

$$\varphi = \frac{\delta W}{\delta j} = \phi_c + \frac{\delta \phi_c}{\delta j} \cdot \left(j - S'[\phi_c]\right) + O(\hbar)$$

φ ↑ bkgr field

$j - S'[\phi_c] = 0$

$$\boxed{\varphi = \phi_c + O(\hbar)}$$

for the 1PI diagrams

Invert. j functional of φ & Legendre transform ⑤

$$\Gamma[\varphi] = j\cdot\varphi - W[J] = \underbrace{j\cdot(\varphi-\phi_c) + S[\phi_c]}_{= S[\varphi] + O(\hbar^2)} + \frac{\hbar}{2}\,\mathrm{Tr}\underbrace{\left(\mathrm{Log}\left(S''[\phi_c]\right)\right)}_{\mathrm{Tr}\left[\mathrm{Log}\left(S''[\varphi]\right)\right]} + O(\hbar^2)$$

Exercise: check that using $\varphi = \phi_c + O(\hbar)$ — a bit of work; easy

Invert J functional of φ & Legendre transform

$$\Gamma[\varphi] = J\cdot\varphi - W[J] = \underbrace{J\cdot(\varphi-\phi_c) + S[\phi_c]}_{= S[\varphi]+O(\hbar^2)} + \frac{\hbar}{2}\,\mathrm{Tr}\underbrace{\left(\mathrm{Log}\left(S''[\phi\right.\right.}_{\mathrm{Tr}\left[\mathrm{Log}\left(S''[\varphi\right.\right.}$$

Exercise: check that using $\varphi = \phi_c + O(\hbar)$ a bit of work

easy

$$\boxed{\Gamma[\varphi] = S[\varphi] + \frac{\hbar}{2}\,\mathrm{Tr}\left[\mathrm{Log}\left(S''[\varphi]\right)\right] + O(\hbar^2)}$$

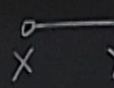
the explicit form of $S[\phi]$ never plays a role

random variable (quantum) ... classical solution ... Q. fluctuation

• Structure of Feynman integrals & their UV singularities



$= \langle \phi^2(0) \rangle_0 =$



$= \int \frac{d^d k}{(2\pi)^d} e^{iK(x-y)} = \langle \phi(x)\phi(x) \rangle_0$

↓ FT

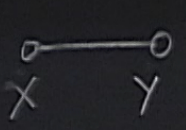
$= \langle \hat{\phi}(K_1)\hat{\phi}(K_2) \rangle$

Tadpole diagram



...and their UV singularities

$$\hat{\phi}(k) = \int d^d x \, e^{-ikx} \phi(x)$$

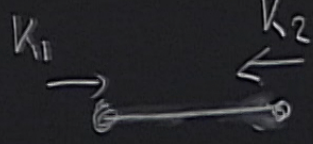


$$= \int \frac{d^d k}{(2\pi)^d} e^{ik(x-y)} = \langle \phi(x)\phi(y) \rangle_0$$

$\downarrow$ FT

$$(2\pi)^d \delta(k_1+k_2) \frac{1}{k_1^2+m^2} = \langle \hat{\phi}(k_1)\hat{\phi}(k_2) \rangle$$

conservation of momenta





$$\frac{1}{k^2+m^2}$$

Feynman propagator in Euclidean space

$$k^2 = \sum_\mu k_\mu^2$$

$e^{-ikx}\,\phi(x)$

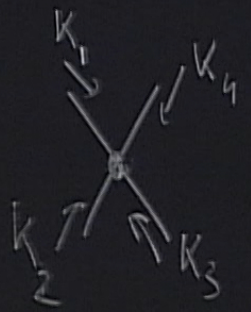
Still in Euclidean theory (UV singularities is common to Euclidean & Minkowski space-time

$\times \rightarrow \int dx_1 \phi^4(x_1) = \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\,\delta(x_2-x_1)\,\delta(x_3-x_1)\,\delta(x_4-x_1)$

vertex at point x

$\downarrow FT$

$$\hat{\phi}(k_1)\hat{\phi}(k_2)\hat{\phi}(k_3)\hat{\phi}(k_4)\,\boxed{(2\pi)^d \delta(k_1+k_2+k_3+k_4)}$$



momentum conservation

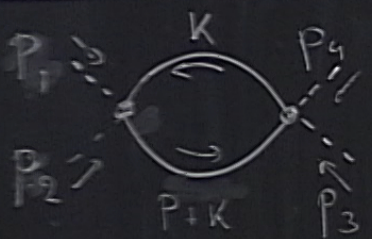
$$\sum_1^4 k_i = 0$$



Tadpole

$$T(m) = \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 + m^2}$$

UV divergent when $d \geq 2$ at $k \to \infty$



$P = P_1 + P_2$ incoming momentum

$$B(P, m) = \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 + m^2} \frac{1}{(P+k)^2 + m^2}$$

$G_0(X_1 - X_2)^2 \xrightarrow{FT}$ UV divergent sometimes

Bubble diagram

UV divergent when $k \to \infty$ $d \geq 4$

Effective action at order φ^2

2 point 1PI function,

$$\Gamma_{(2)} = \left.\frac{\delta^2 \Gamma[\varphi]}{\delta\varphi\,\delta\varphi}\right|_{\varphi=0} = -\Delta + m^2 + \hbar \frac{g}{2}$$ 

$$\Gamma_{(2)}(z_1, z_2) = (-\Delta_{z_1} + m^2)\,\delta(z_1 - z_2) + \hbar \frac{g}{2} T(m)\,\delta(z_1 - z_2)$$

$\downarrow$ FT

$$\hat{\Gamma}_{(2)}(p_1, p_2) = (2\pi)^d \delta(p_1 + p_2)\left(p_1^2 + m^2 + \hbar \frac{g}{2} T(m) + O(\hbar^2)\right)$$

Simple property of the 2 point function

connected 2 point function

$$W_{(2)}(z_1, z_2) = \langle \phi(z_1)\phi(z_2) \rangle$$

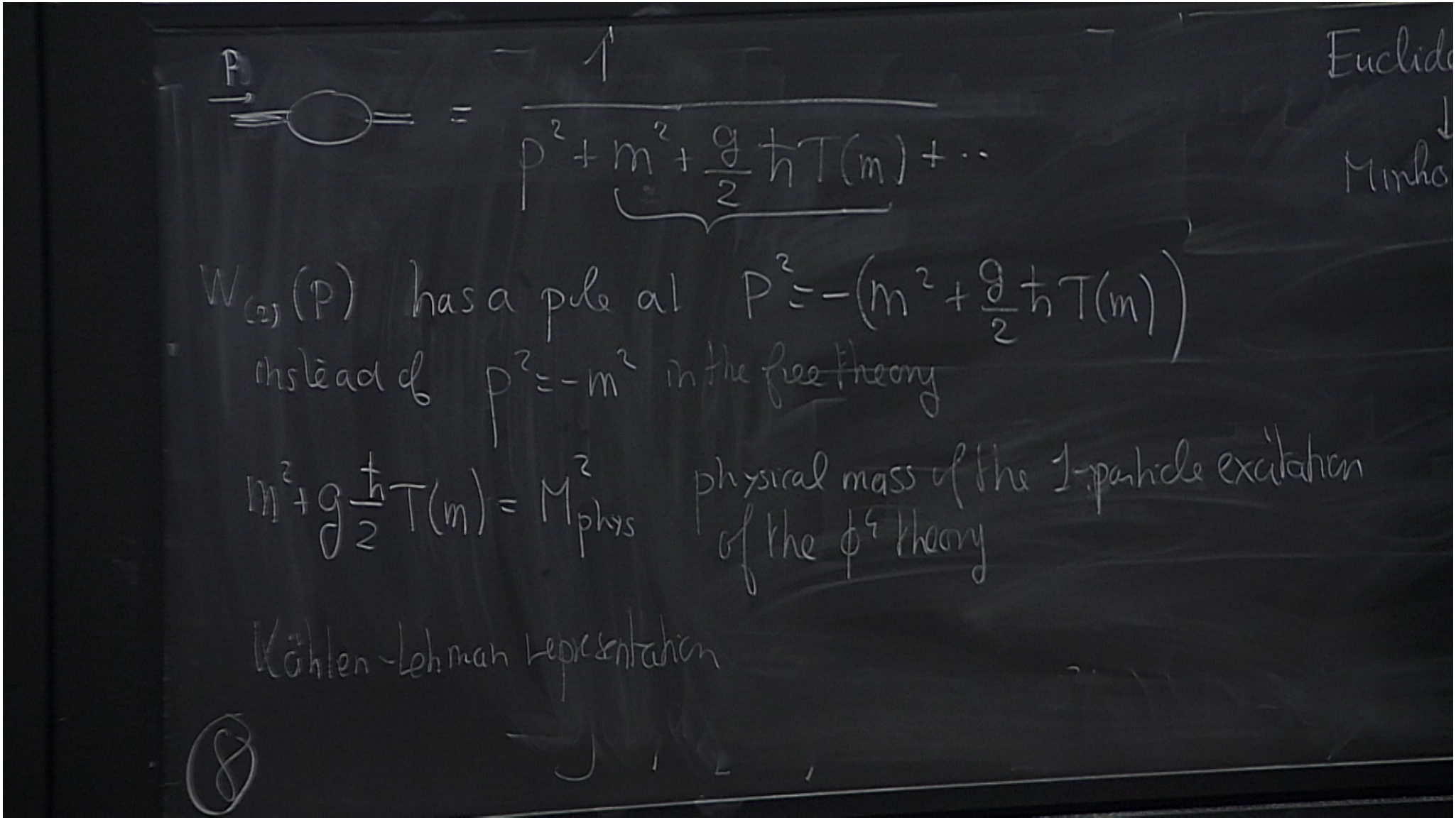
$$\downarrow \text{FT}$$

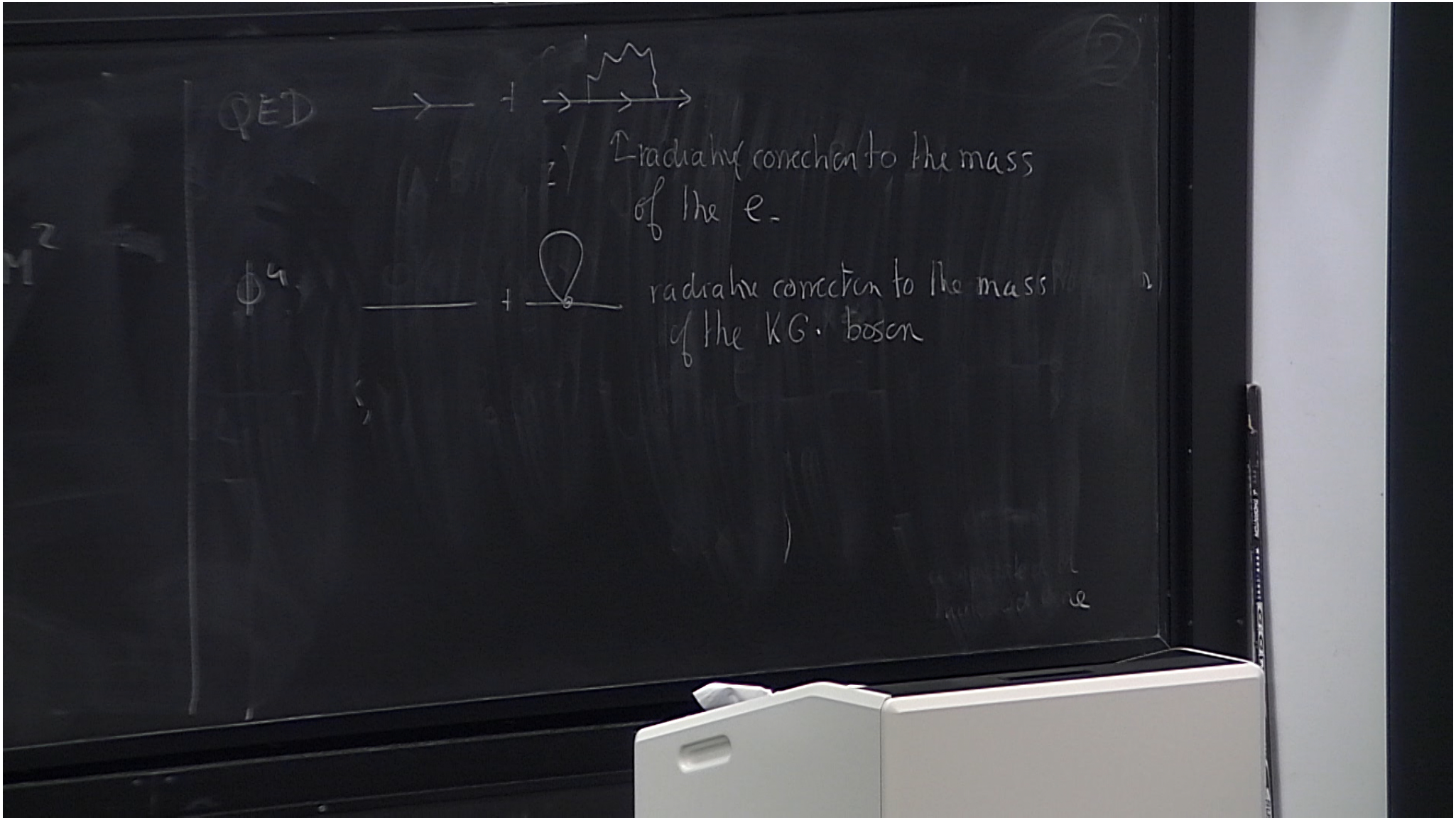
$$W_{(2)}(p_1, p_2) = (2\pi)^d \delta(p_1 + p_2) \frac{1}{\hat{\Gamma}_{(2)}(p_1)}$$

full propagator

$=$

inverse of 1PI 2pt function in momentum space


P
=
1
$p^2 + m^2 + \frac{g}{2}\hbar T(m) + \cdots$
Euclide
Minko
$W_{(2)}(P)$ has a pole at $P^2 = -\left(m^2 + \frac{g}{2}\hbar T(m)\right)$
instead of $p^2 = -m^2$ in the free theory
$m^2 + g\frac{\hbar}{2}T(m) = M^2_{phys}$ physical mass of the 1-particle excitation of the ϕ^4 theory
Källen-Lehmann representation
8


QED
+
radiative correction to the mass of the e-
ϕ^4
+
radiative correction to the mass of the KG. boson