Title: PSI 17/18 - Quantum Field Theory II - Lecture 5

Date: Nov 10, 2017 09:00 AM

URL: http://pirsa.org/17110015

Abstract:

Interacting ϕ^4 theory

Simple.
relevant: stat. mechanics
Higgs physics

Scalar field $\phi(x)$ $x \in \mathbb{R}^d$
or $\mathbb{M}^{1,d-1}$

Free 

Interaction  contact (local)

Action (in Euclidean Space)

omit the "E" subscripts

$$S[\phi] = \int_{\mathbb{R}^d} d^d x \left[\frac{1}{2} (\partial_\nu \phi)^2 + \frac{m^2}{2} \phi^2 \right. \qquad \phi^4$$

Action (in Euclidean Space) omit the "E" subscripts

$$S[\phi] = \int_{\mathbb{R}^d} d^d x \left[\frac{1}{2} (\partial_\nu \phi(x))^2 + \frac{m^2}{2} \phi^2(x) + \frac{g}{4!} \phi^4(x) \right]$$

g coupling constant

$g > 0$ repulsion

no boson condensate

eory

$x \in \mathbb{R}^d$

or $\mathbb{M}^{1,d-1}$

ical)

simple:

relevant: stat. mechanics
Higgs physics

Classical Action (in Euclidean Spac

$$S[\phi] = \int_{\mathbb{R}^d} d^d x \left[\frac{1}{2} (\partial_\mu \phi(x)^2 \right.$$

Simple: Classical Action (in Euclidean S

relevant: stat. mechanics
Higgs physics

$$S[\phi] = \int_{\mathbb{R}^d} d^d x \left[\frac{1}{2} (\partial$$

$d-1$

$\langle \phi^2(x) \rangle \approx$ density of particle at point x

true in non-relativistic QFT

so $g\phi^4 \approx$ cost energy g to have two bosons at the same place

lassical Action (in Euclidean Space) omit the "E" subsc

$$S[\phi] = \int_{\mathbb{R}^d} d^d x \left[\frac{1}{2} (\partial_\nu \phi(x))^2 + \frac{m^2}{2} \phi^2(x) + \frac{g}{4!} \phi^4(x) \right]$$

g co

g >

no bos

$$S_0[\phi] = \int d^d x \left[\frac{1}{2} (\partial_\nu \phi)^2 + \frac{m^2}{2} \phi^2 \right]$$

Free action g

nucs

ics

anish at point x

QFT

to have two

same place

Quantization = define a functional integral

$$Z = \frac{1}{Z_0} \int D_0[\phi] \exp\left(-\frac{1}{\hbar} S[\phi]\right)$$

normalization of the functional measure

↑ measure for the free field

$$Z_0 = \int D_0[\phi] \exp\left(-\frac{1}{\hbar} S_0[\phi]\right)$$

interacting theory: expectation values of fields

$$\langle \phi(x_1) \cdots \phi(x_N) \rangle = \frac{\int D_0[\phi] \exp\left(-\frac{1}{\hbar} S\right) \phi \cdots \phi}{\int D_0[\phi] \exp\left(-\frac{1}{\hbar} S\right)}$$

correlation functions

$\exp\left(-\frac{1}{\hbar} S[\phi]\right)$ normalization of the functional measure

...e for the free field

$\exp\left(-\frac{1}{\hbar} S_0[\phi]\right)$ $|\Omega\rangle_{g>0} \neq |0\rangle_{Free}$

...ctation values of fields

$$\frac{\int D_0[\phi] \exp\left(-\frac{1}{\hbar} S\right) \phi \cdots \phi}{\int D_0[\phi] \exp\left(-\frac{1}{\hbar} S\right)} \xrightarrow[\text{Wick}]{} \langle \Omega | T\left[\hat{\Phi}(x_1) \cdots \hat{\Phi}(x_N)\right] | \Omega \rangle$$

$|\Omega\rangle$: vacuum state, $\hat{\Phi}$: Fields operator for interacting theory

$$\frac{1}{Z_0}\int D_0[\phi]\exp\left(-\frac{1}{\hbar}S[\phi]\right)$$

normalization of the functional measure

$\int$ measure for the free field

$$Z_0 = \int D_0[\phi]\exp\left(-\frac{1}{\hbar}S_0[\phi]\right)$$

$$|\Omega\rangle_{g>0} =$$

interacting theory : expectation values of fields

$$\langle \phi(x_1)\cdots\phi(x_N)\rangle = \frac{\int D_0[\phi]\exp\left(-\frac{1}{\hbar}S\right)\phi\cdots\phi}{\int D_0[\phi]\exp\left(-\frac{1}{\hbar}S\right)}$$

correlation functions

$\longrightarrow$ Wick $\longrightarrow$ $\langle\Omega|T[\Phi$

Ω vacuum

omit the "E" subscripts (1)

$\left[\frac{1}{2}\phi^2(x) + \frac{g}{4!}\phi^4(x)\right]$ g coupling constant

$g > 0$ repulsion

no boson condensate

$\left[\frac{1}{2}\phi^2\right]$ Free action $g = 0$

integral

$$\overline{Z} = \frac{1}{Z_0}\int D_0[\phi]\ ex$$

↑ measure fa

$$Z_0 = \int D_0[\phi]\ \exp(-$$

interacting theory : expectatio

$$\langle \phi(x_1)\cdots\phi(x_N)\rangle = \frac{\int D_0}{\int D_0}$$

correlation functions

but perturbation theory : g small : series expansion in g around $g = 0$

Some notations

$$\overline{Z}(x_1, \dots, x_N) = \frac{1}{Z_0} \int \mathcal{D}_0[\phi] \exp\left(-\frac{1}{\hbar} S[\phi]\right) \phi(x_1) \cdots \phi(x_N)$$

N-points

$$\overline{Z} = \frac{1}{Z_0} \int \mathcal{D}_0[\phi] \exp\left(-\frac{1}{\hbar} S[\phi]\right)$$

...tor for interacting theory

No more Gaussian Integral ; difficult non-perturbative

but perturbation theory : g small : series expansion in g around $g = 0$

Some notations

$$\bar{Z}(X_1, \cdots, X_N) = \frac{1}{Z_0} \int D_0[\phi] \exp\left(-\frac{1}{\hbar} S[\phi]\right) \phi(X_1) \cdots \phi(X_N)$$

N-points

$$\bar{Z} = \frac{1}{Z_0} \int D_0[\phi] \exp\left(-\frac{1}{\hbar} S[\phi]\right)$$

obviously $\langle \phi(X_1) \cdots \phi(X_N) \rangle = G(X_1 \cdots X_N) = \dfrac{\bar{Z}(X_1, \cdots X_N)}{\bar{Z}}$ ← depends on g coupling

Taylor series of

$$-\frac{1}{\hbar}\frac{g}{4!}\int d^d x\,\phi^4(x)\Big)\,\phi(z_1)\cdots\phi(z_N) \qquad \exp\Big(-\frac{g}{\hbar\cdot 4!}\int d^d$$

Interversion $\int \mathcal{D}_0[\phi]\cdot\sum_K$

$$\int \frac{\mathcal{D}_0[\phi]}{Z_0}\exp\Big(-\frac{1}{\hbar}S_0[\phi]\Big)\,\phi(z_1)\cdots\phi(z_N)\,\phi^4(x_1)\cdots\phi^4(x_K)\Big]$$

$$\langle\phi(z_1)\cdots\phi(z_N)\,\phi^4(x_1)\cdots\phi^4(x_K)\rangle_0$$ v.e.v in the free theory

Perturbation theory

$$\overline{Z}(z_1 \dots z_N) = \frac{1}{Z_0} \int D_0[\phi] \exp\left(-\frac{1}{\hbar} S_0[\phi]\right) \cdot \exp\left(-\frac{1}{\hbar} \frac{g}{4!} \int d^d x\, \phi^4(x)\right) \phi(z_1) \dots$$

↑ "external points"

$$= \sum_{K=0}^{\infty} \frac{1}{K!} \left(\frac{-g}{4! \hbar}\right)^K \int \dots \int d^d x_1 \dots d^d x_K \left[\int \frac{D_0[\phi]}{Z_0} \exp\left(-\frac{1}{\hbar} S_0[\phi]\right) \phi \right.$$

$$\left. \langle \phi(z_1) \dots \phi(z_N)\, \phi^4(x_1) \dots \phi^4 \right.$$

$$\boxed{\overline{Z}(z_1 \dots z_N) = \sum_{K=0}^{\infty} \frac{1}{K!} \left(\frac{-g}{4! \hbar}\right)^K \int d^d x_1 \dots d^d x_K \langle \phi(z_1) \dots \phi(z_N)\, \phi^4(x_1) \dots \phi^4(x_K) \rangle_0}$$

Taylor series of

$$\exp\left(-\frac{g}{\hbar \cdot 4!}\int d^d x\, \phi^4(x)\right) = \sum_{K=0}^{\infty} \frac{1}{K!}\left(\frac{-g}{4!\,\hbar}\right)^K \left[\int d^d x\, \phi^4(x)\right]^K$$

Interversion

$$\int D_0[\phi] \cdot \sum_K = \sum_K \int D_0[\phi]$$

⚠ Dangerous

$\phi^4(x_1) \cdots \phi^4(x_K)$

v.e.v in the free theory

↑ Entire Function convergent series

not a convergent series, radius cv is zero!
asymptotic series when $g \to 0$

Exercise:

$d = 0$

$$\exp\left(-\frac{g}{\hbar \cdot 4!}\int d^d x\, \phi^4(x)\right) = \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{-g}{4!\,\hbar}\right)^k \left[\int d^d x\, \phi^4(x)\right]^k$$

$$\boxed{\int D_0[\phi] \cdot \sum_k = \sum_k \int D_0[\phi]}$$

Dangerous

entire function

convergent series

not a convergent series. radius cv is zero!

asymptotic series when $g \to 0$

Exercise:

$d = 0$

$$\int_{-\infty}^{+\infty} d\phi\, \exp\left(-\frac{1}{2}\phi^2 - \frac{g}{4!}\phi^4\right) = \sum_k \frac{(-1)^k g^k}{k!\,(4!)^k} \langle \phi^{4k} \rangle_{\text{GAUSS}} \simeq k!$$

How to compute. use Wick theorem

$$\langle \phi(z_1) \cdots \phi(z_N)\, \phi^4(x_1) \cdots \phi^4(x_K) \rangle_0 = \sum_{\text{pairings}} \prod_{\text{pairs}} \langle \phi(\cdot)\, \phi(\cdot) \rangle_0$$

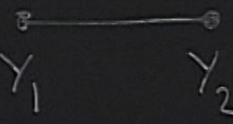
external points

internal points to be integrated over

z's or x's

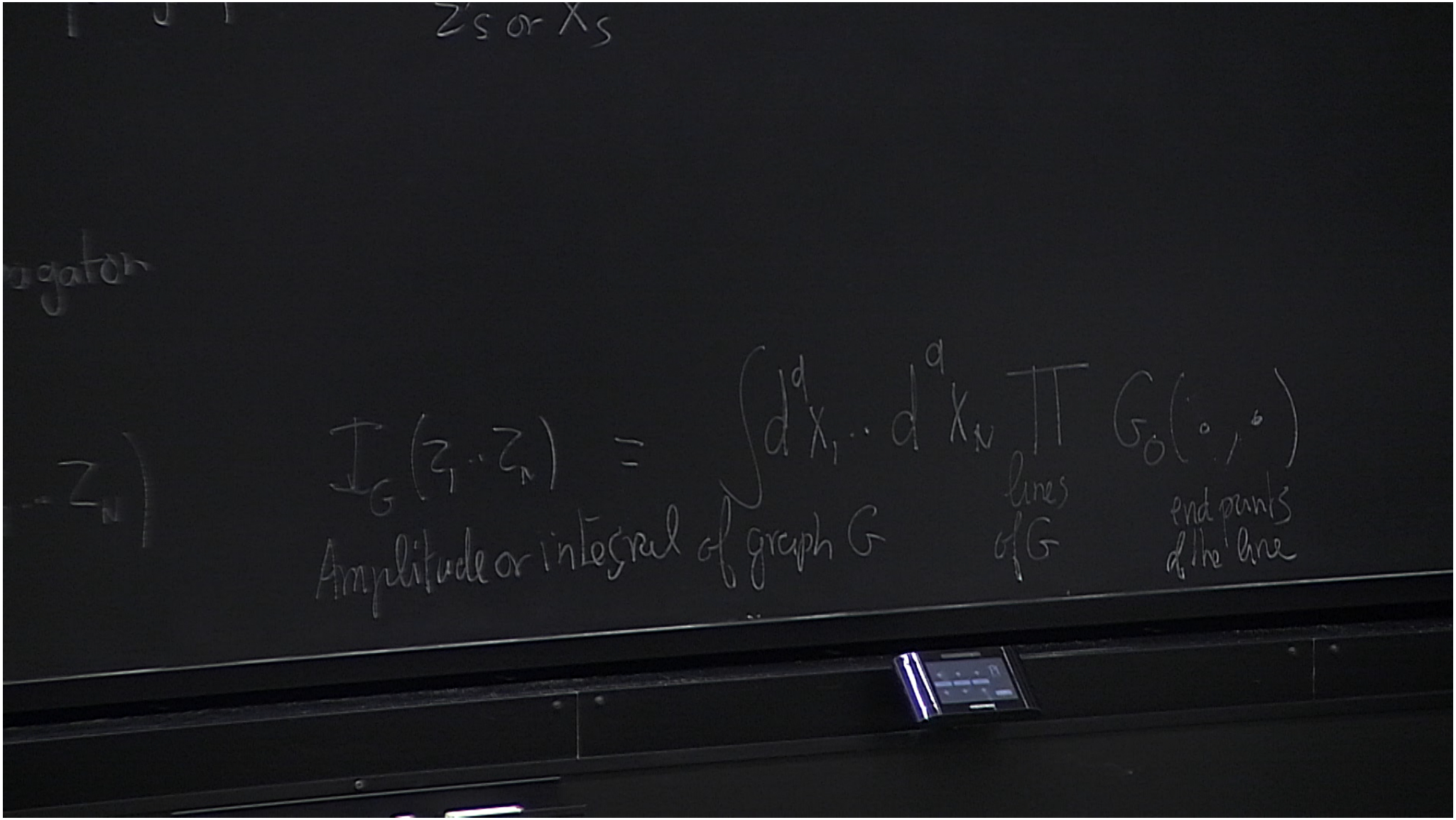
Feynman diagrams

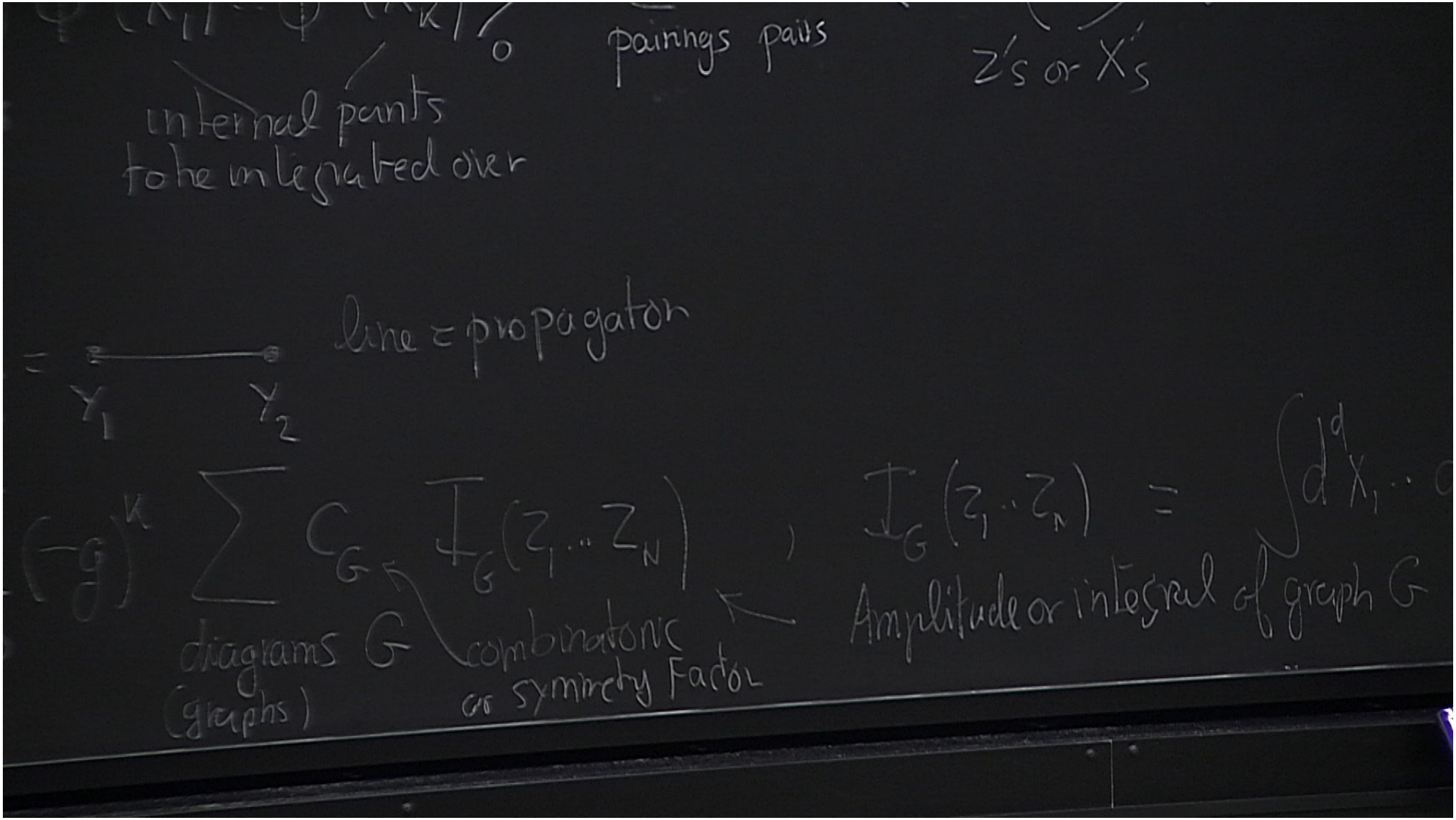
$$\langle \phi(y_1)\, \phi(y_2) \rangle_0 =$$



line = propagator

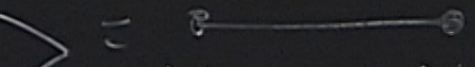
$$Z(z_1, \ldots z_N) = \sum_{K=0}^{\infty} (-g)^K \sum_{\substack{\text{diagrams } G \\ \text{(graphs)}}}$$


Z_S or X_S
gator
$I_G(z_1 \cdots z_N) = \int d^q x_1 \cdots d^q x_N \prod G_0(\cdot, \cdot)$
Amplitude or integral of graph G
lines of G
end points of the line


pairings pairs
Z's or X's
internal points
to be integrated over
line = propagator
Y_1
Y_2
$(-g)^K \sum_{\text{diagrams } G} C_G \, I_G(Z_1 \cdots Z_N)$
$I_G(Z_1 \cdots Z_N) = \int d^d X_1 \cdots$
diagrams G
(graphs)
combinatoric
or symmetry Factor
Amplitude or integral of graph G

to be integrated over

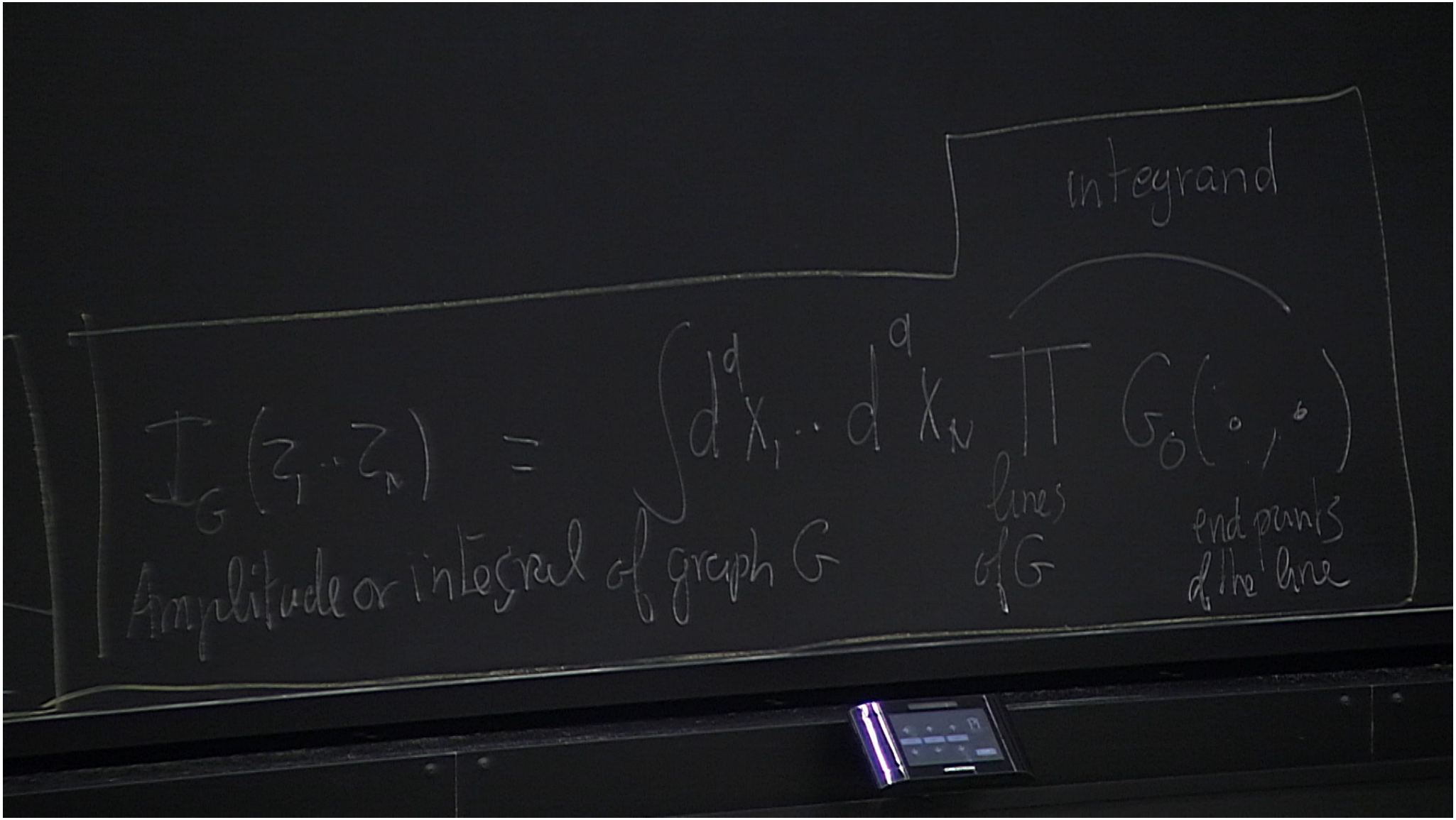
Feynman diagrams

$\langle \phi(Y_1) \phi(Y_2) \rangle_0 =$ 

line = propagator

$$Z(Z_1, \dots Z_N) = \sum_{K=0}^{\infty} (-g)^K \sum_{\text{diagrams } G \text{ (graphs)}} C_G \, I_G(Z_1 \dots Z_N)$$

C_G: combinatoric or symmetry factor


integrand
$I_G(z_1 \ldots z_N) = \int d^d x_1 \ldots d^d x_V \prod_{\text{lines of } G} G_0(\ldots)$
Amplitude or integral of graph G
lines of G
end points of the line

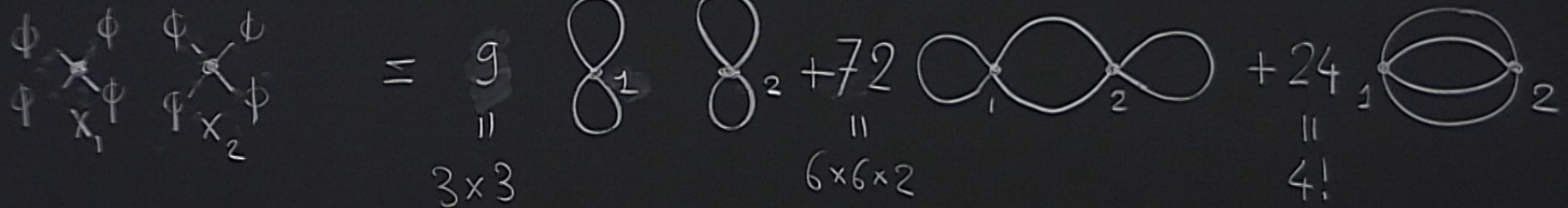
Examples

$\hbar = 1$

expansion of $\bar{Z}$, $N=0$ external pts

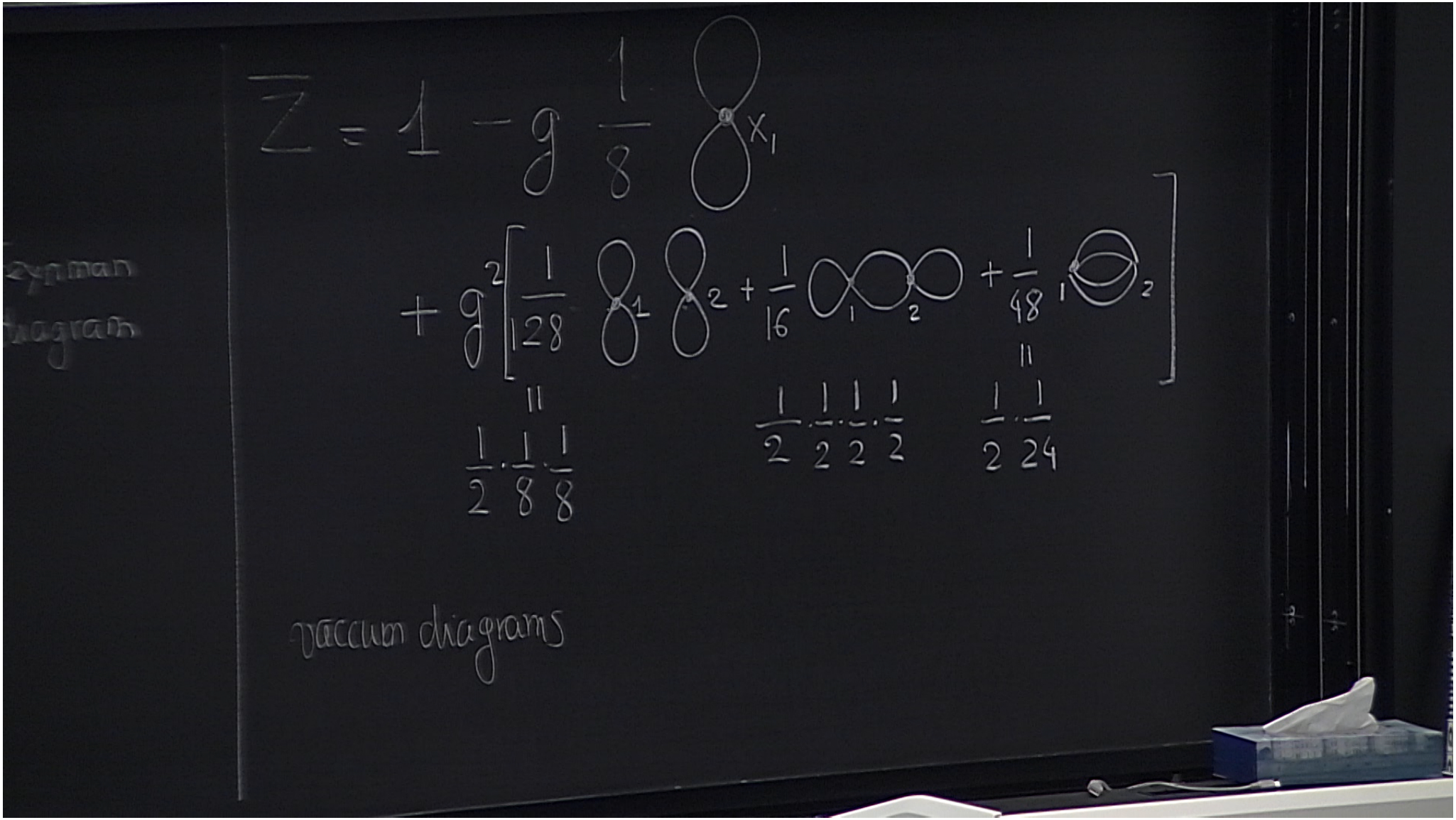
$$\bar{Z} = 1 + \left(\frac{-g}{4!}\right)\int d^d x_1 \langle \phi^4(x_1)\rangle_0 + \frac{1}{2!}\left(\frac{-g}{4!}\right)^2 \int d^d x_1 d^d x_2 \langle \phi^4(x_1)\phi^4(x_2)\rangle_0 + O(g^3)$$

$$\langle \phi^4(x_1)\,\phi^4(x_2)\rangle_0 = 9\,\langle\phi^2(x_1)\rangle_0^2\,\langle\phi^2(x_2)\rangle_0^2 + 72\,\langle\phi^2(x_1)\rangle_0\,\langle\phi(x_1)\phi(x_2)\rangle_0^2\,\langle\phi$$



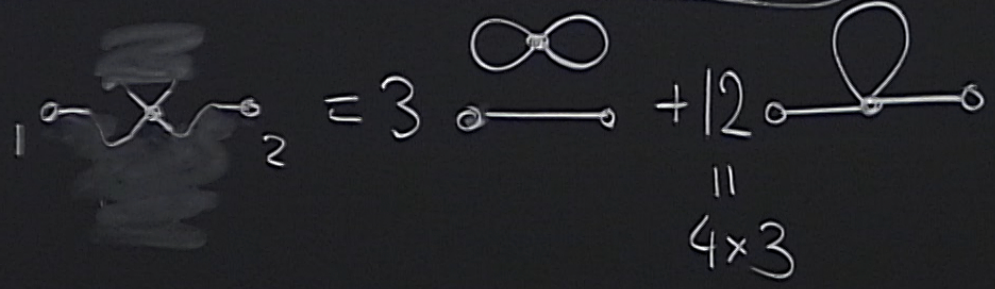
$$= \underset{\parallel}{9} \quad + \underset{\parallel}{72} \quad + \underset{\parallel}{24}$$

$$3\times 3 \qquad 6\times 6\times 2 \qquad 4!$$


$Z = 1 - g \frac{1}{8}$ x_1
$+ g^2 \left[\frac{1}{128} \; x_1 \; x_2 + \frac{1}{16} \; 1 \; 2 + \frac{1}{48} \; 1 \; 2 \right]$
$\frac{1}{2} \cdot \frac{1}{8} \cdot \frac{1}{8}$
$\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}$
$\frac{1}{2} \cdot \frac{1}{24}$
vaccum diagrams

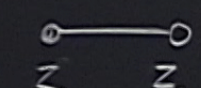
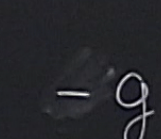
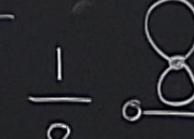
$$\phi^4(x_1)\rangle_0 + \frac{1}{2}\left(\frac{-g}{4!}\right)^2 \int d^d x_1\, d^d x_2 \langle \phi(z_1)\phi(z_2)\phi^4(x_1)\phi^4(x_2)\rangle_0 + O(g^3)$$

$$\langle \phi(z_1)\phi(z_2)\phi^4(x_1)\phi^4(x_2)\rangle_0 + O(g^3)$$



$$= 4\times 3$$

$N=2$ $\quad \Sigma(z_1, z_2) = \langle \phi(z_1)\phi(z_2) \rangle_0 + \left(\frac{-g}{4!}\right) \int d^d x_1 \langle \phi(z_1)\phi(z_2)\phi^4(x_1) \rangle_0$

$=$  $- g \left[\frac{1}{8} \right.$  $+ \frac{1}{2}$  $\left. \right] + g^2 \left[\frac{1}{128} \right.$ 

external points fixed

$G_0(z_1 - z_2) - g \left[\frac{1}{8} G_0(z_1 - z_2) \int d^d x_1 \, G_0(0)^2 + \frac{1}{2} \int d^d x_1 \, G_0(z_1 - x_1) G_0(x_1 - z_2) G_0(0) \right]$

$$\langle \phi(z_1)\phi(z_2)\rangle_0 + \left(\frac{-g}{4!}\right)\int d^d x_1 \langle \phi(z_1)\phi(z_2)\phi^4(x_1)\rangle_0 + \frac{1}{2}\left(\frac{-g}{4!}\right)^2 \int d^d x_1\, d^d x_2 \langle \phi(z_1)\phi($$

$$\frac{1}{8} \quad + \frac{1}{2} \quad \Big] + g^2 \Big[\frac{1}{128} \quad + \frac{1}{16} \quad + \frac{1}{48} \quad + \frac{1}{16} \quad + \frac{1}{4} \quad + \frac{1}{6} \quad \Big]$$

NEW + CONNECTED

NEW + CONNECTED

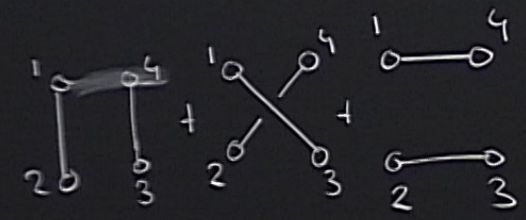
$$G_0(z_1-z_2)\int d^d x_1\, G_0(0)^2 + \frac{1}{2}\int d^d x_1\, G_0(z_1-x_1)\,G_0(x_1-z_2)\,G_0(0)\Big]$$

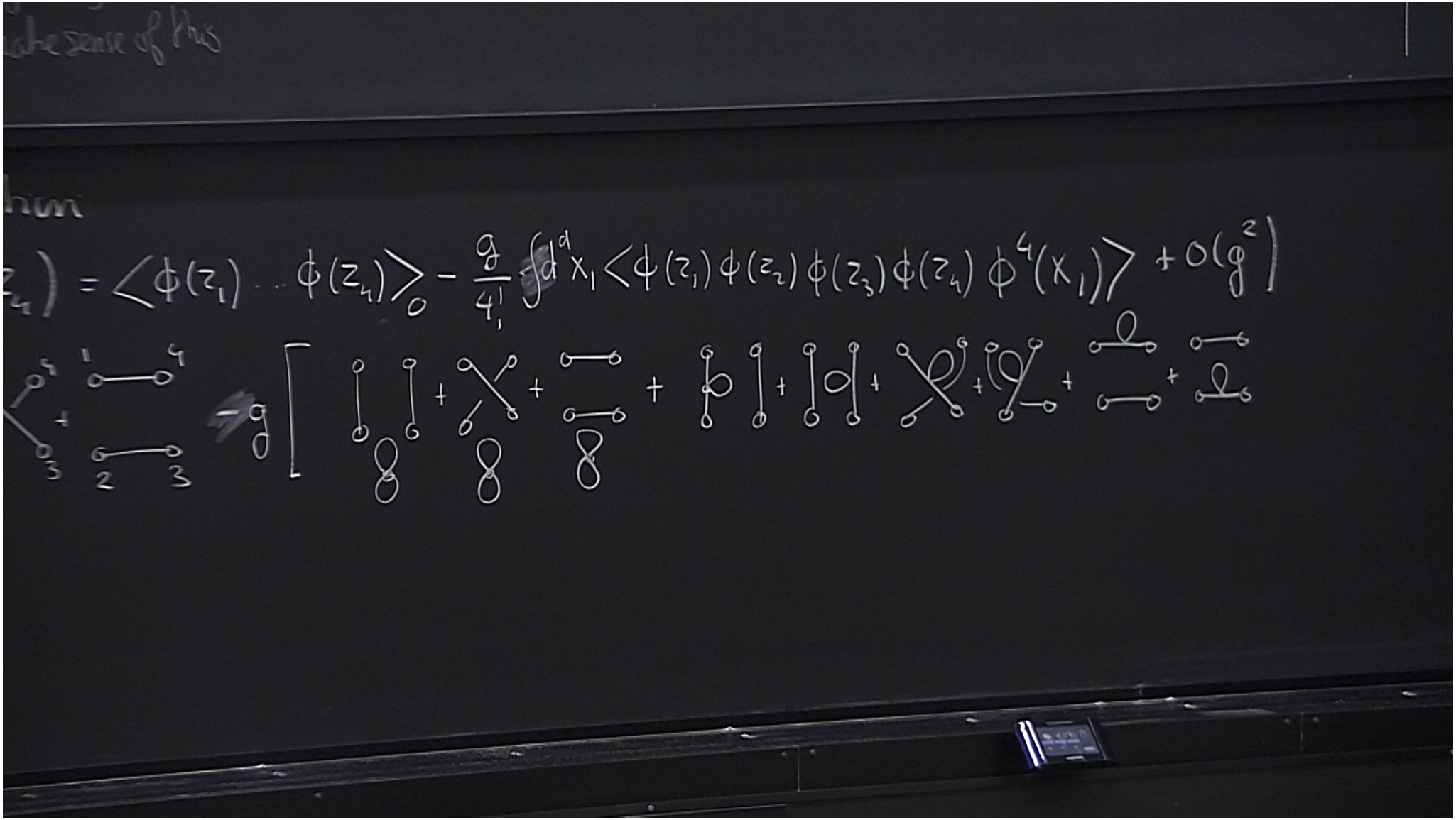
periodic b.c make sense of this

4 point function

$$\overline{Z}(z_1, z_2, z_3, z_4) = \langle \phi(z_1) \cdots \phi(z_4) \rangle_0 - \frac{g}{4!} \int d^d x_1 \langle \phi(z_1) \phi(z_2) \phi(z_3) \phi(z_4) \phi^4(x$$

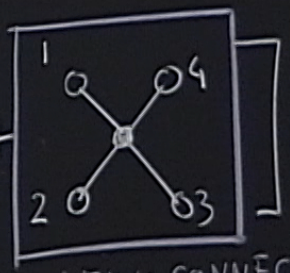
$$= \quad + \quad + \quad = g \Big[$$




the sense of this
hm
$\phi(z_1)\cdots\phi(z_4)\rangle_0 - \frac{g}{4!}\int d^d x_1 \langle \phi(z_1)\phi(z_2)\phi(z_3)\phi(z_4)\phi^4(x_1)\rangle + O(g^2)$
$= -g\Big[$
1
2
3
4

$$\langle z_4 \rangle_0 - \frac{g}{4!} \int d^d x_1 \langle \phi(z_1)\phi(z_2)\phi(z_3)\phi(z_4)\phi^4(x_1) \rangle + O(g^2)$$

$$\left(\quad + \quad + \quad \right) + \frac{1}{2} \left(\quad + \quad + \quad + \quad + \quad + \quad + \quad \right) + \quad$$



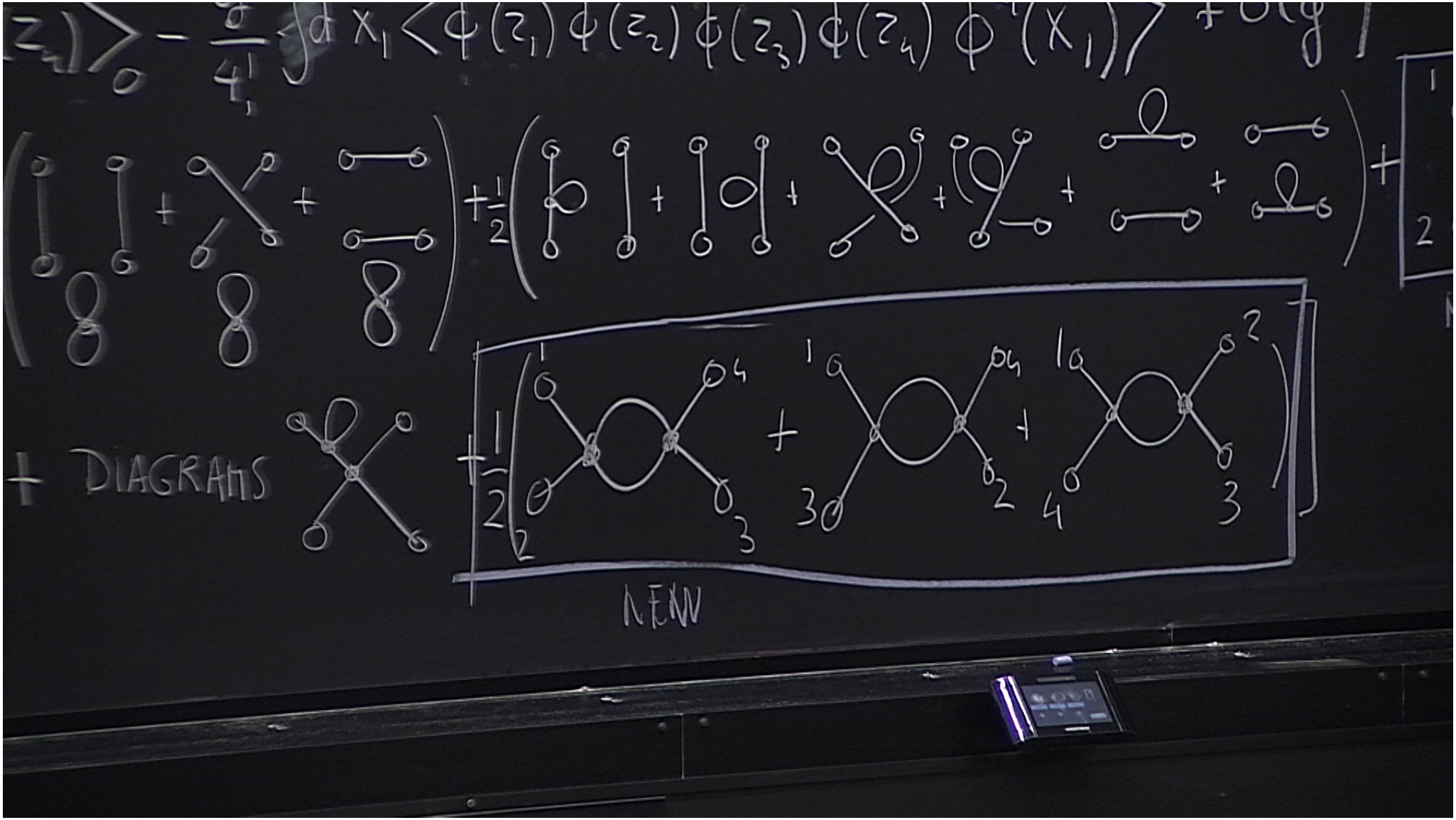
NEW: CONNECTED

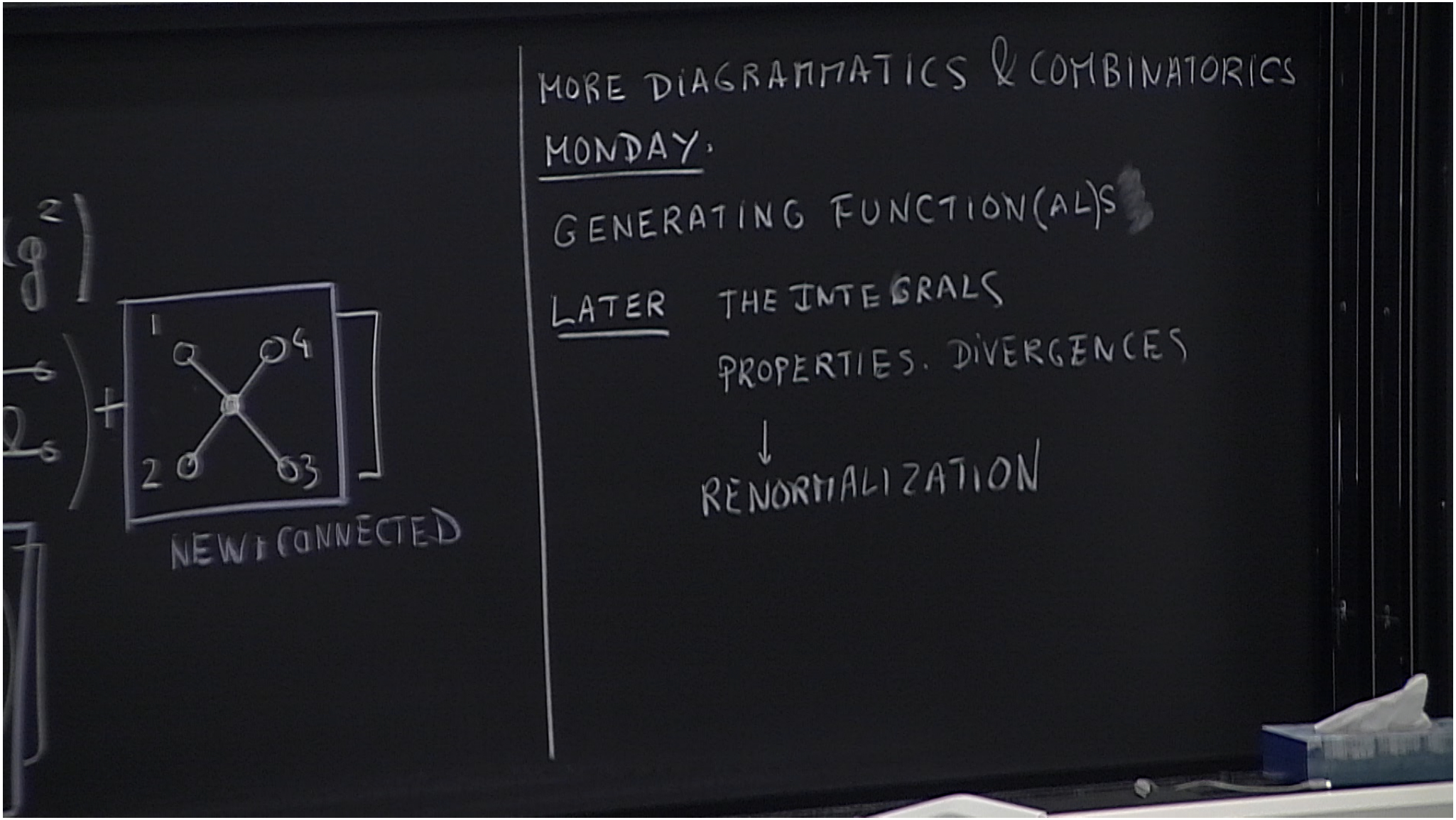
4 point function

$$\overline{Z}(z_1,z_2,z_3,z_4) = \langle \phi(z_1) \dots \phi(z_4) \rangle_0 - \frac{g}{4!}\int d^d x_1 \langle \phi(z_1)\phi(z_2)\phi(z_3)\phi(z_4)\phi^4(x_1)$$

$$= \quad + \quad + \quad - g\left[\frac{1}{8}\left(\quad + \quad + \quad \right) + \frac{1}{2}\left(\quad + \quad + \quad + \quad + \right.\right.$$

$$+ g^2 \Big[\text{DISCONNECTED DIAGRAMS} + \text{DIAGRAMS} \quad +$$


+ DIAGRAMS
NEW


MORE DIAGRAMMATICS & COMBINATORICS
MONDAY:
GENERATING FUNCTION(AL)S
LATER THE INTEGRALS
PROPERTIES. DIVERGENCES
↓
RENORMALIZATION
1
4
2
3
NEW: CONNECTED