Title: PSI 17/18 - Quantum Field Theory II - Lecture 4

Date: Nov 09, 2017 09:00 AM

URL: http://pirsa.org/17110014

Abstract:

Today: Free Field. Continued ; $\hbar = 1$

Interacting ϕ^4 Intro.

Gaussian Fund. Integral

Euclidean $\phi(x)$ real scalar $x \in \mathbb{R}^d$

Propagator or Correlation Fun

$$\begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots 1 \end{pmatrix} = \eta_E \qquad \int D[\phi(x)] \exp\left(-S_E[\phi]\right)$$

$$\langle \phi(x_1)\phi(x_2) \rangle = G(x_1 -$$

$$S_E[\phi] = \int d^d x \, \frac{1}{2}(\partial_\nu \phi)^2 + \frac{m^2}{2}\phi^2$$

$$= \int d^d x \, \frac{1}{2} \cdot \phi \cdot (-\Delta_x + m^2)\phi^2$$

$$K \cdot X = K_\mu X^\mu$$

$$K^2 = \sum_\mu K_\mu \cdot \eta_E^{\mu\nu} \cdot K_\nu$$

$$\eta^E_{\mu\nu} =$$

$$\eta_E^{\mu\nu} =$$

Continued ; $\hbar = 1$

ϕ^4 Intro. Gaussian Fund. Integral

$x \in \mathbb{R}^d$

Propagator or Correlation Function

$-S_E[\phi])$

$$\langle \phi(x_1)\phi(x_2) \rangle = G(x_1 - x_2) = \int \frac{d^d k}{(2\pi)^d} \frac{e^{i k \cdot x}}{k^2 + m^2}$$

$(\partial_\mu \phi)^2 + \frac{m^2}{2}\phi^2$

$k \cdot x = k_\mu x^\mu$

$k^2 = \sum_\mu k_\mu \eta_E^{\mu\nu} k_\nu$

$\eta^E_{\mu\nu} = \delta_{\mu\nu}$ Euclidean Metric tensor

$\eta_E^{\mu\nu} = \delta_{\mu\nu}$

$\phi(-\Delta_x + m^2)\phi^2$

$G(x)$

$$G(x) \simeq_{x \to 0} |x|^{2-d}$$

Short Distance Singularity

(U.V) $\quad d \geq 2$

Check: Euclidean $\to$ Minkowski: FreeField Canonical Quant.

Euclidean Propagator Wick Rotation

$$X_E = (\tau, \vec{x})$$

Euclidean time

space

$$G(X) \simeq_{X \to 0} |X|^{2-d}$$

Short Distance Singularity
(U.V) $d \geq 2$

Check: Euclidean → Minkowski Free Field Canonical Quant.

Euclidean Propagator Wick Rotation

$$X_E = (\tau, \vec{x})$$

$$X = (t, \vec{x})$$

τ: Euclidean time, $\vec{x}$: space

t: physical time, $\vec{x}$: space

$$G(X) \underset{X \to 0}{\simeq} |X|^{2-d}$$

Short Distance Singularity

(U.V) $d \geq 2$

Check: Euclidean $\longrightarrow$ Minkowski

Free Field Canonical Quant.

Euclidean Propagator

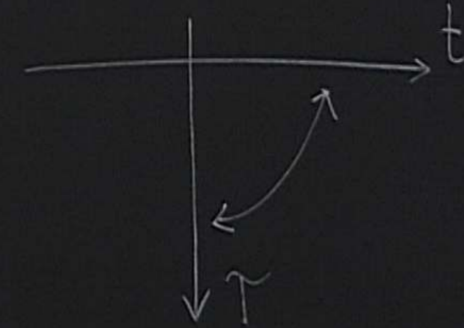
$$X_E = (\tau, \vec{x})$$

Euclidean time, space

Wick Rotation

$$X = (t, \vec{x})$$

physical time, space



$$\tau = -it$$

$$G(X) \underset{X \to 0}{\simeq} |X|^{2-d}$$

Short Distance Singularity
(U.V) $d \geq 2$

Check: Euclidean $\to$ Minkowski   Free Field Canonical Quant.

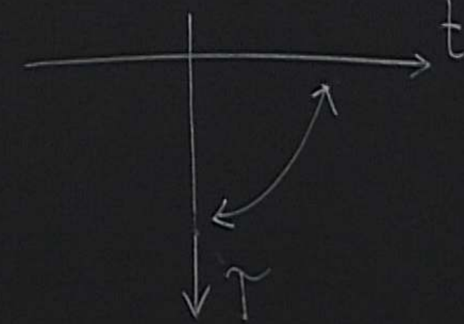
Euclidean Propagator   Wick Rotation

$$X_E = (\tau, \vec{x})$$

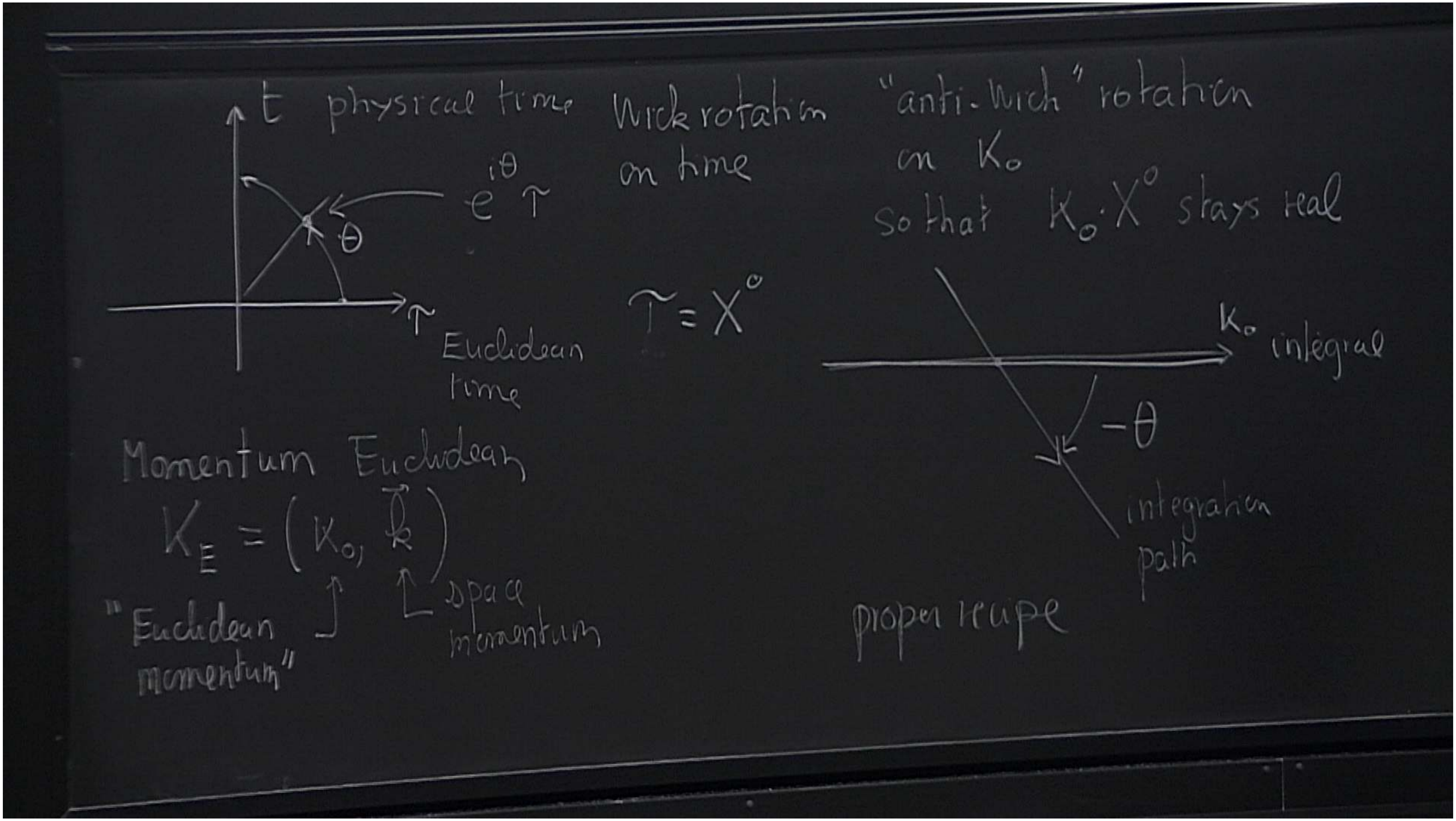
Euclidean time   space

$$X = (t, \vec{x})$$

physical time   space



$$\tau = it$$


t physical time
Wick rotation on time
"anti-Wick" rotation on K_0
so that $K_0 \cdot X^0$ stays real
$e^{i\theta}\tau$
θ
τ Euclidean time
$\tau = X^0$
K_0 integral
$-\theta$
integration path
proper recipe
Momentum Euclidean
$K_E = (K_0, \vec{k})$
"Euclidean momentum"
space momentum

physical time

$e^{i\theta}\,\tau$

$\longrightarrow \tau$ Euclidean time

Euclidean

$(\vec{k})$

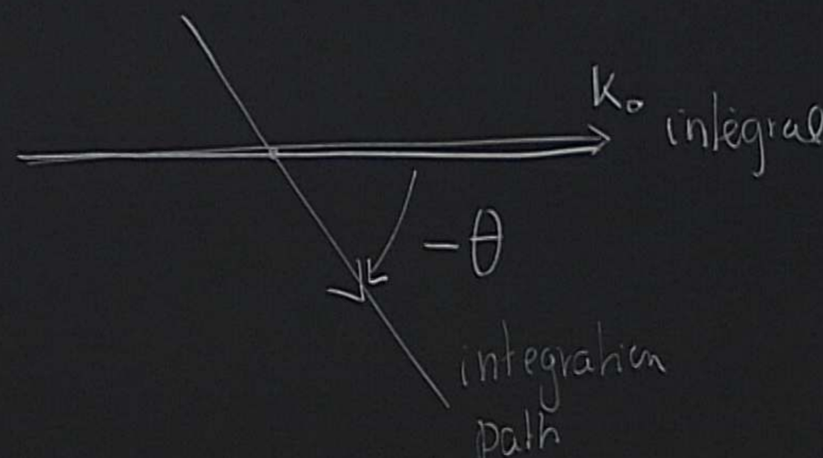
space momentum

Wick rotation on time

$\tau = X^0$

"anti-Wick" rotation on K_0

so that $K_0 \cdot X^0$ stays real



proper recipe for analytic continuation in time s real

integral

ytic continuation

$$\tau = i\,t$$

$$K_0 \to \frac{1}{i}\,\omega$$

so that the momentum

$$K = (\omega, \vec{k})$$

energy or pulsation

space momentum

ntum

e momentum

$$G_E \equiv \int_{-\infty}^{+\infty} \frac{dk_0}{2\pi} \int \frac{d^{d-1}\vec{k}}{(2\pi)^{d-1}} \frac{e^{i(k_0 \cdot \tau + \vec{k}\cdot\vec{x})}}{k_0^2 + \vec{k}^2 + m^2}$$

change of variable (Wick Rot.)

$$\boxed{\int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} (-i) \int \frac{d^{d-1}\vec{k}}{(2\pi)^{d-1}} \frac{e^{i(\omega \cdot t + \vec{k}\cdot\vec{x})}}{-\omega^2 + \vec{k}^2 + m^2}}$$

…ysical time

$e^{i\theta} \tau$

$\rightarrow \tau$ Euclidean time

Euclidean

$(\ldots \vec{k})$

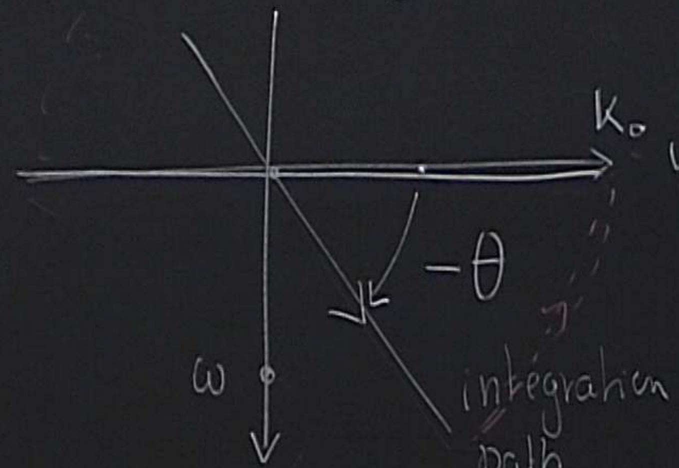
space momentum

Wick rotation on time

$\tau = X^0$

"anti-Wick" rotation on K_0

so that $K_0 \cdot X^0$ stays real



proper recipe for analytic continuation in time

$$\boxed{\begin{aligned} \tau &= i\,t \\ K_0 &\rightarrow \frac{1}{i}\,\omega \end{aligned}}$$

So that the mo…

$K = (\omega, \vec{k}$ …

energy or pulsation

$$G_E = \int_{-\infty}^{+\infty} \frac{dk_0}{2\pi} \int \frac{d^{d-1}\vec{k}}{(2\pi)^{d-1}} \frac{e^{i(k_0 \cdot \tau + \vec{k}\cdot\vec{x})}}{k_0^2 + \vec{k}^2 + m^2}$$

change of variable (Wick Rot.)

$$\boxed{\int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} (-i) \int \frac{d^{d-1}\vec{k}}{(2\pi)^{d-1}} \frac{e^{i(\omega \cdot t + \vec{k}\cdot\vec{x})}}{-\omega^2 + \vec{k}^2 + m^2}}$$

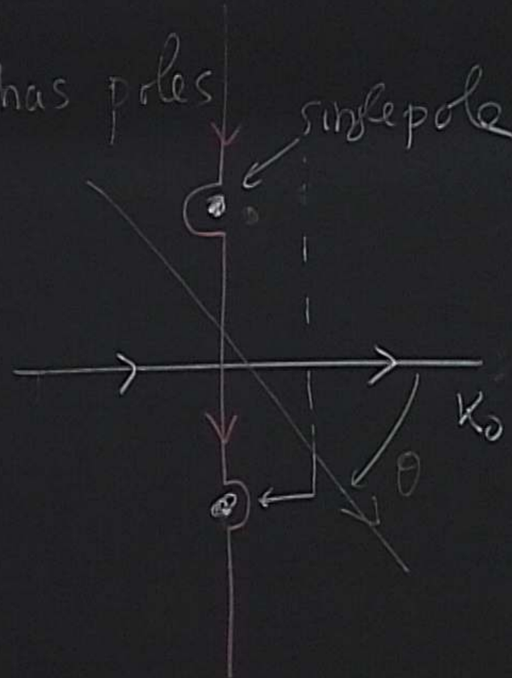
momentum

space momentum

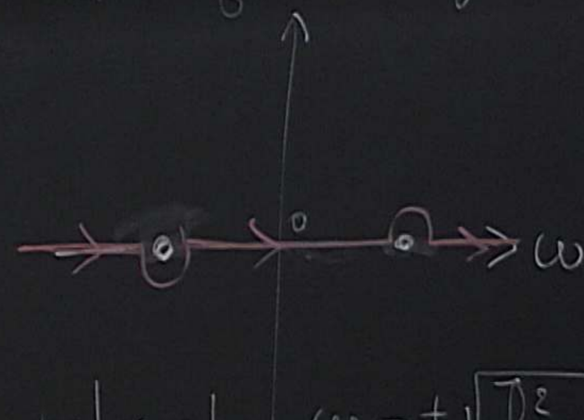
Denominator has poles

$$\frac{1}{k_0^2 + \vec{k}^2 + m^2}$$

$$k_0 = \pm i \sqrt{\vec{k}^2 + m^2}$$



In the final integral over ω



poles at $\omega = \pm \sqrt{\vec{k}^2 + m^2}$

l over ω

prescription of integration
for the Feynman propagator

ω

$\overline{\quad k^2+m^2}$

$$\int \frac{d\omega}{2\pi}\int\frac{d^{d-1}\vec{k}}{(2\pi)^{d-1}}\; i\; \frac{e^{i(\omega t+\vec{k}\cdot\vec{x})}}{\omega^2-\vec{k}^2-m^2+i0_+}$$

with $X=(t,\vec{x})$

$$= G_F(t,\vec{x})$$

$$= \langle 0|\,T[\Phi(0)\Phi(x)\,|0\rangle$$

time ordered product

Field operators

integration
man pragator

$$\int \frac{d\omega}{2\pi} \int \frac{d^{d-1}\vec{k}}{(2\pi)^{d-1}} \; i \; \frac{e^{i(\omega t + \vec{k}\cdot\vec{x})}}{\omega^2 - \vec{k}^2 - m^2 + i\,0_+}$$

$$= G_F(t, \vec{x}) \qquad \text{with } X = (t, \vec{x})$$

$$= \langle 0 | T [\Phi(0)\Phi(X) | 0 \rangle$$

time ordered product

Field operators

prescription of integration
for the $\boxed{\text{Feynman propagator}}$

$$\int \frac{d\omega}{2\pi} \int \frac{d\vec{k}}{(2\pi)^{d-1}} \cdot \frac{e}{\omega^2 - \vec{k}^2 - m^2 +}$$

$$= G_F(t, \vec{x})$$

$$= \langle 0 | T[\Phi(0)\Phi(x)] | 0 \rangle$$

time ordered product

Field operators

with X

$$K = (K^\mu) = (E, \vec{P})$$

energy momentum

Proof: finite number of r.v ϕ^a $a = 1, \mathcal{N}$

propability distrib $\exp\left(-\frac{1}{2}\phi^a K_{ab}\phi^b\right)$

$K = (K_{ab})$ symmetric positive real matrix

$$\langle \phi^a \phi^b \rangle = (K^{-1})_{ab} = G^{ab}$$ 2 points correlator

$$\langle \phi^a \rangle = 0$$

$$\langle \phi^{a_1} \cdots \phi^{a_N} \rangle = \frac{\int \prod_a d\phi^a \, \exp\left(-\frac{1}{2}\phi \cdot K \cdot \phi\right) \phi^{a_1} \cdots \phi^{a_N}}{\int \prod_a d\phi^a \, \exp\left(-\frac{1}{2}\phi \cdot K \phi\right)}$$

Generating Function

vector J_a not a random variable

source term

$$Z(j) = \int \prod_a d\phi^a \, \exp\left(-\frac{1}{2}\phi^a K_{ab}\phi^b + J_c \phi^c\right)$$

$$\phi\cdot K\cdot\phi\Big)\ \phi^{a_1}\cdots\phi^{a_N}$$

$$\phi\cdot K\cdot\phi\Big)$$

change of variables

$$\phi^a = \phi'^a + G^{ab} J_b$$

so $\boxed{Z(J) = Z\cdot\exp\left(\frac{1}{2} J_a G^{ab} J_b\right)}$

$$_{ab}\phi^b + J_c\phi^c\Big) = \int \prod d\phi'^a \exp\left(-\frac{1}{2}\phi'^a K_{ab}\phi'^b\right)\cdot\exp\left(\frac{1}{2}J_a G^{ab} J_b\right)$$

$$\langle \phi^{a_1} \rangle = \frac{1}{Z(j)} \frac{\partial}{\partial J_{a_1}} Z(j) \Big|_{j=0} = G^{a_1 \ldots}$$

$$\langle \phi^{a_1} \phi^{a_2} \rangle = \frac{1}{Z(j)} \frac{\partial}{\partial J_{a_1}} \frac{\partial}{\partial J_{a_2}} \Big|_{j=0} \qquad \frac{\partial}{\partial J_{a_1}} \ldots$$

$$\vdots$$

$$\langle \phi^{a_1} \cdots \phi^{a_N} \rangle = \frac{1}{Z(j)} \frac{\partial}{\partial J_{a_1}} \cdots \frac{\partial}{\partial J_{a_N}} Z(j) \Big|_{j=0}$$

$$G^{a_1 b} j_b \Big|_{j=0} = 0$$

$$\frac{\partial}{\partial J_{a_1}} \frac{\partial}{\partial J_{a_2}} \exp\left(\frac{1}{2} J \cdot G J\right) = \frac{\partial}{\partial J_{a_1}} \left[G^{a_2 b} j_b \cdot \exp\left(\frac{1}{2} J \cdot G \cdot J\right) \right] = \underbrace{G^{a_2 a_1}} + \overbrace{G^{a_2 b} j_b \cdot G^{a_1 c} j_c}^{=0 \text{ when } j=0}$$

$$\frac{\partial}{\partial j} \frac{\partial}{\partial j} \frac{\partial}{\partial j} \frac{\partial}{\partial j} \exp\left(\frac{1}{2} j G j\right)$$

$$\left. \cdots J\cdot G J\right)\Big] = \Big(G^{a_2 a_1} + \overbrace{G^{a_2 b} j_b \cdot G^{a_1 c} j_c}^{=0 \text{ when } j=0} \Big) \exp\left(\tfrac{1}{2} j\cdot G j\right)$$

$$= G^{a_2 a_1} G^{a_4 a_3} + G^{a_2 a_3} G^{a_1 a_4} + G^{a_2 a_4} G^{a_1 a_3}$$

$$1\ 2\ \ 3\ 4 \quad + \quad 1\ 2\ \ 3\ 4 \quad + \quad 1\ 2\ 3\ 4$$

Wick Theorem holds for Functional Integral Quantization

Euclidean $* = \langle \phi(x_1)\phi(x_2) \cdots \phi(x_N) \rangle = \dfrac{\int D[\phi]\, \exp(-S}{\int D[\phi]\, \exp(-S}$

E - omitted

N odd $N = 2M+1$ this is zero

N even $N = 2M$ non-zero

$\langle \phi(x_1) \rangle = 0$

Gaussian random variables

$$* = \sum_{\text{pairings in M pairs}} \prod_{\text{pairs}} \langle \phi(x_\cdot)\phi(x_\cdot) \rangle$$

Wick Theorem

Quantization

$$= \frac{\int D[\phi] \exp(-S[\phi]) \, \phi(x_1) \cdots \phi(x_N)}{\int D[\phi] \exp(-S[\phi])}$$

$$\langle \phi(x_1) \rangle = 0$$

Wick Theorem

Draw $\underset{x_1}{\circ}\!\!-\!\!-\!\!-\!\!\underset{x_2}{\circ} = \langle \phi(x_1) \phi(x_2) \rangle$

$$\langle \phi(x_1) \phi(x_2$$

$$= \langle \phi(x_1) \phi(x_2) \rangle$$

$$+ \langle \phi(x_1) \phi(x_3$$

$$+ \langle \phi(x_1) \phi(x_4$$

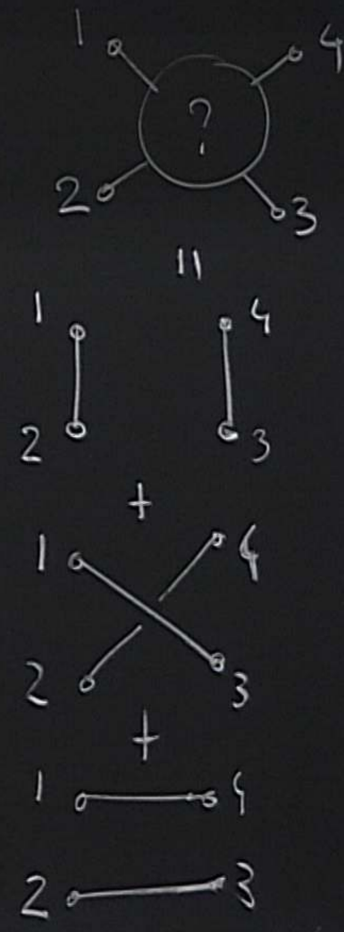
$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle$$

$$= \langle \phi(x_1)\phi(x_2)\rangle\langle \phi(x_3)\phi(x_4)\rangle$$

$$+ \langle \phi(x_1)\phi(x_3)\rangle\langle \phi(x_2)\phi(x_4)\rangle$$

$$+ \langle \phi(x_1)\phi(x_4)\rangle\langle \phi(x_2)\phi(x_3)\rangle$$

Famous Feynman diagram representation



In a QFT

$$|\psi\rangle = \phi\,\phi \cdots \phi\,|0\rangle$$

$$\langle\Phi|\psi\rangle = \langle 0|\,\phi\,\phi \cdots \phi \times \phi \cdots \phi\,|0\rangle$$

Fock Space $\leftarrow$ Correlation Functions $\leftarrow$ Wick Th & Functional Integral

In a QFT

$$|\psi\rangle = \phi\,\phi\cdots\phi\,|0\rangle$$

$$\langle\phi|\psi\rangle = \langle 0|\,\phi\,\phi\cdots\phi \times \phi\cdots\phi\,|0\rangle$$

Fock Space $\longleftarrow$ Correlation Functions $\longleftarrow$ Wick Th & Functional Integral

Canonical Quant. $=$ Functional Integral ick Th & Functional Integral

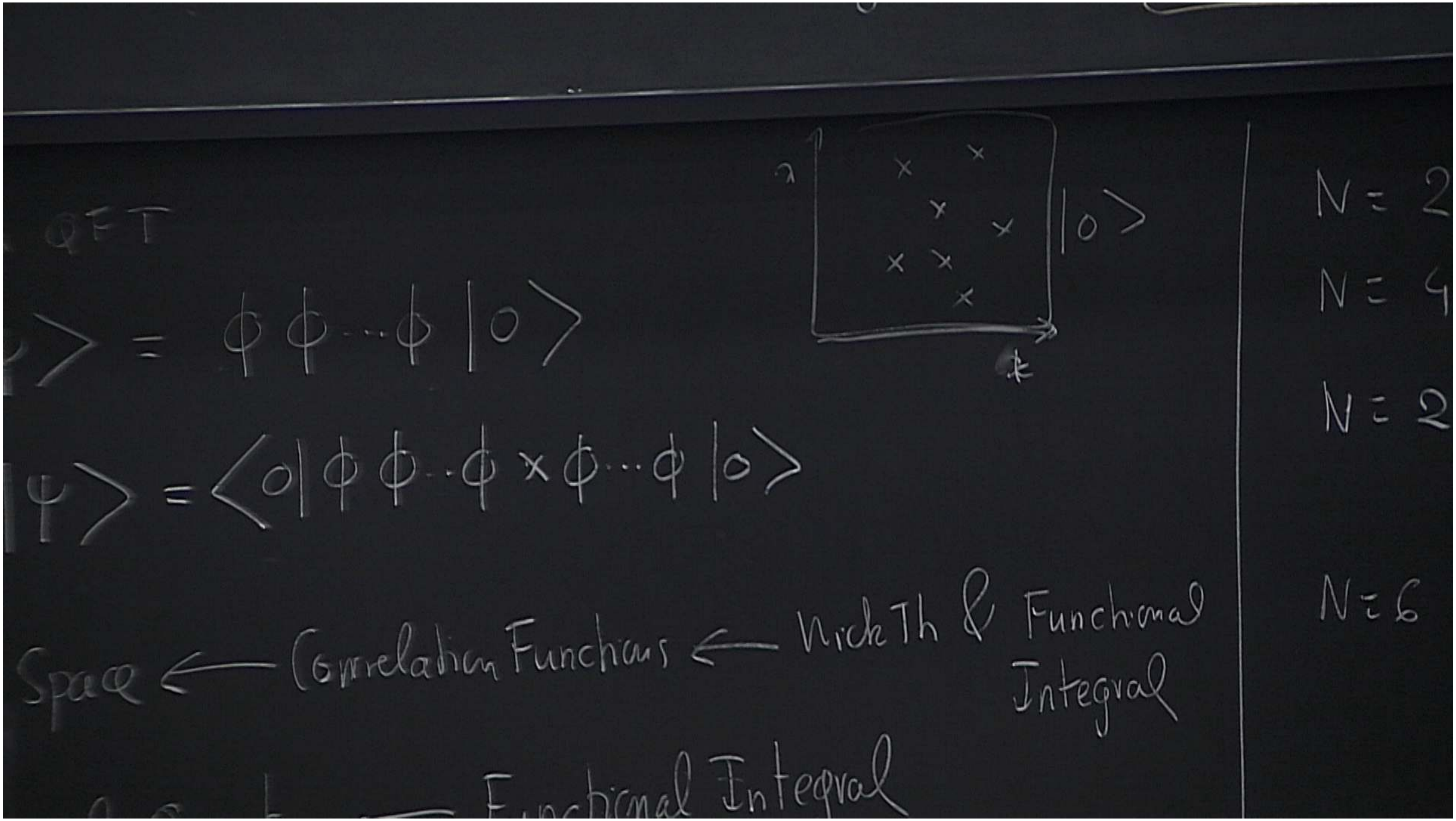
gral

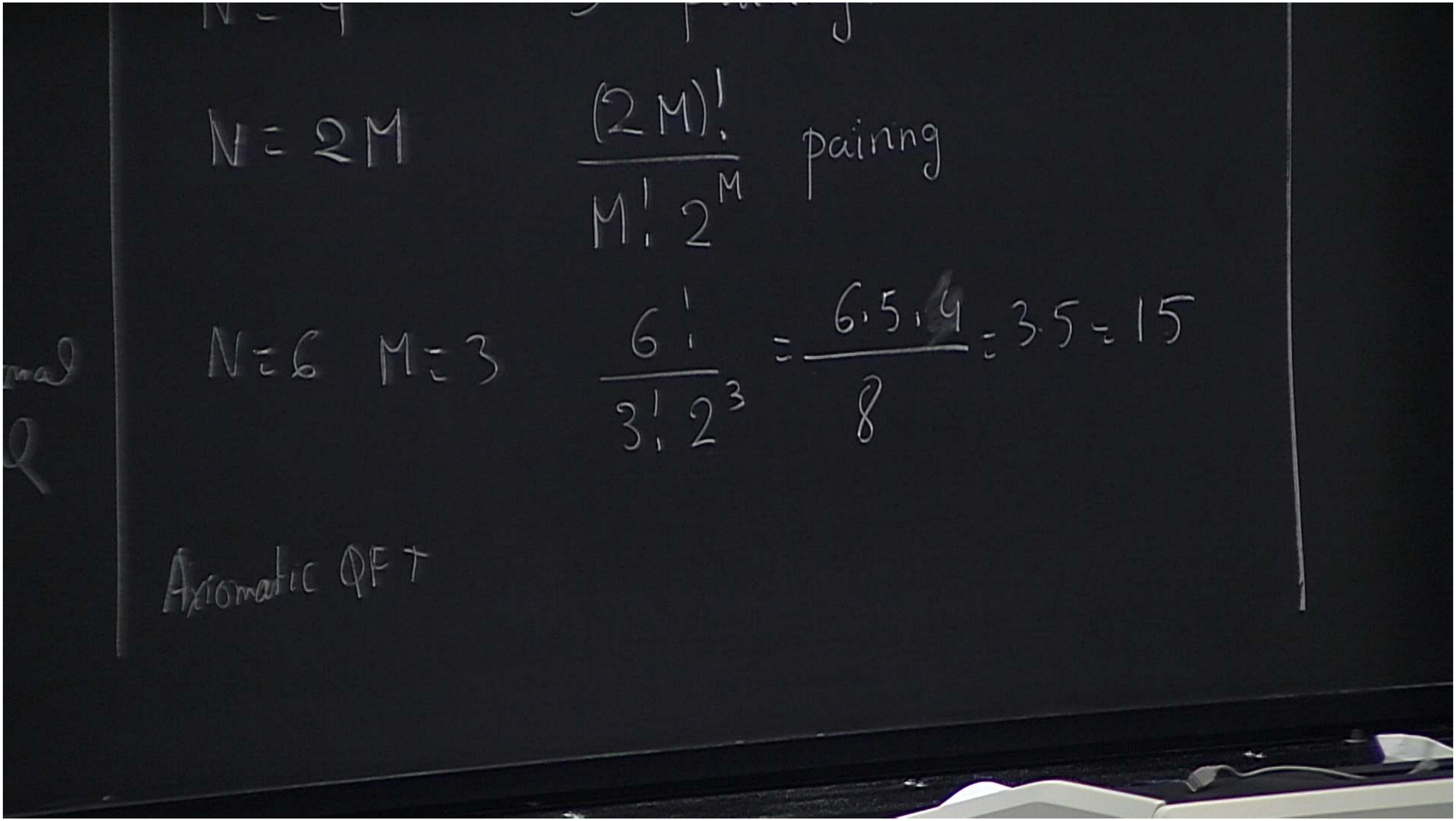
$N = 2$ 1 pairing

$N = 4$ 3 pairings

$N = 2M$ $\dfrac{(2M)!}{M!\,2^M}$ pairing

$N = 6 \quad M = 3 \qquad \dfrac{6!}{3!\,2^3} = \dfrac{6.5.4}{8} = 3.5 = 15$


QFT
$|\psi\rangle = \phi\phi\cdots\phi|0\rangle$
$\langle\psi|\psi\rangle = \langle 0|\phi\phi\cdots\phi \times \phi\cdots\phi|0\rangle$
$|0\rangle$
Space ← Correlation Functions ← Wick Th & Functional Integral
Functional Integral
N = 2
N = 4
N = 2
N = 6


$N = 2M$ $\frac{(2M)!}{M!\,2^M}$ pairing
$N = 6$ $M = 3$ $\frac{6!}{3!\,2^3} = \frac{6 \cdot 5 \cdot 4}{8} = 3 \cdot 5 = 15$
Axiomatic QFT

Composite operators and Normal products: $\langle \phi(x)^2 \rangle = \infty$

$$\phi(x) \longrightarrow \Phi(x)$$

random variable — Field operator

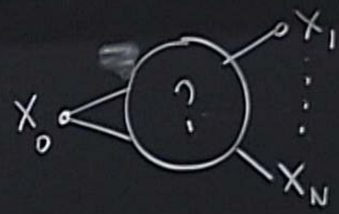
properties of $\phi^2(x)$ in Functional Integral

* average of $\phi^2(x) = +\infty$ 2-d

* $\langle \phi(x)\phi(y) \rangle \underset{y \to x}{\simeq} |x-y| \underset{y=x}{\to} \infty$

cts: $\langle \phi(x)^2 \rangle = \infty$

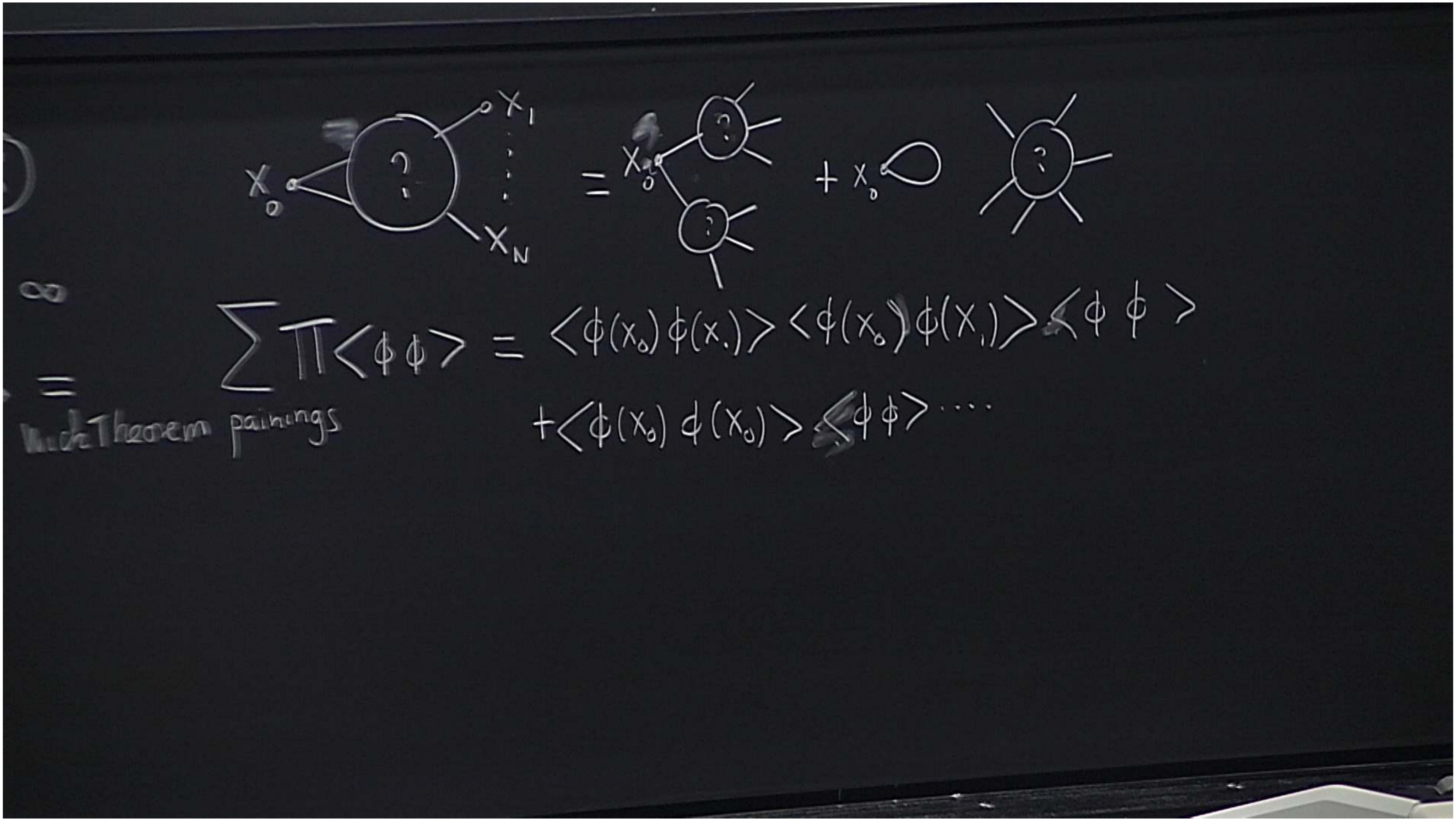
all matrix elements are ∞

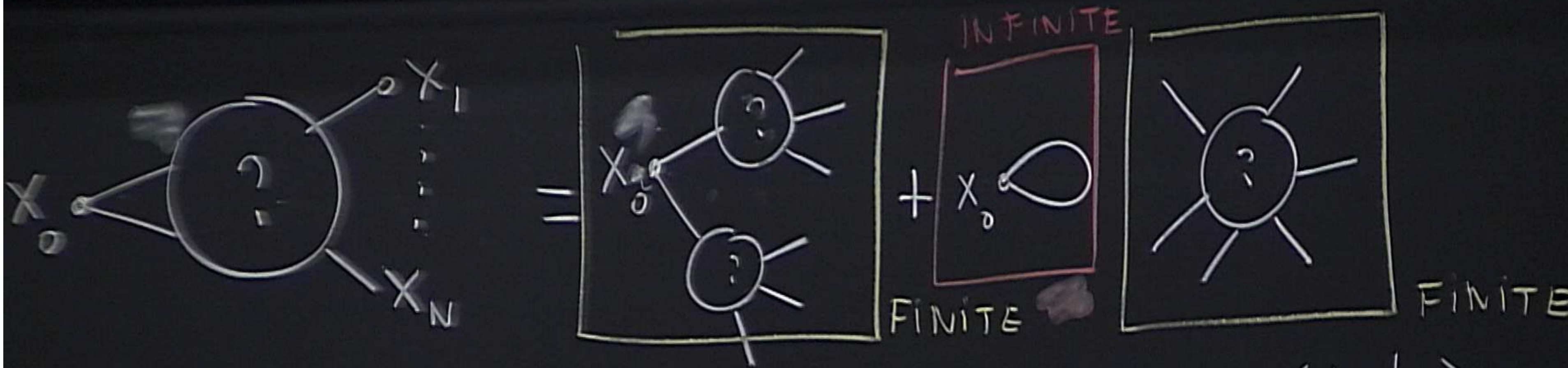


$$\underbrace{\langle \phi(x_1)\cdots\phi(x_N)\,\phi^2(x_0)\rangle}_{N+2 \text{ points}} = \sum\prod\langle\phi\phi\rangle = \langle\phi(x_0)\phi(x_1)\rangle\langle\phi(x_0)\phi(x_1)\rangle\langle\phi\,\phi\rangle$$

Wick Theorem pairings

$$+\langle\phi(x_0)\,\phi(x_0)\rangle\langle\phi\phi\rangle\cdots$$


x_0
x_1
x_N
Theorem pairings
$\sum \prod \langle \phi \phi \rangle = \langle \phi(x_0) \phi(x_1) \rangle \langle \phi(x_0) \phi(x_1) \rangle \langle \phi \phi \rangle$
$+ \langle \phi(x_0) \phi(x_0) \rangle \langle \phi \phi \rangle \cdots$



$$\sum_{\text{airings}} \prod \langle \phi \phi \rangle = \langle \phi(x_0)\phi(x_1)\rangle \langle \phi(x_0)\phi(x_1)\rangle \langle \phi \phi \rangle$$

$$+ \langle \phi(x_0)\phi(x_0)\rangle \langle \phi \phi \rangle \cdots$$

all matrix elements are ∞

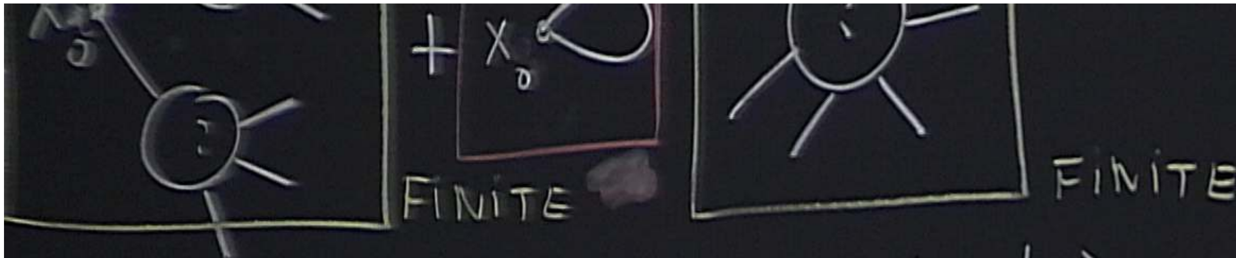
$$\langle \underbrace{\phi(x_1)\cdots\phi(x_N)\,\phi^2(x_0)}_{N+2 \text{ points}} \rangle = \sum_{\text{Wick Theorem pairings}} \prod \langle \phi\,\phi \rangle = \langle \phi(x_0)\,\phi(x \cdots$$

$$+ \langle \phi(x_0)\,\phi(\cdots$$

INFINITE

random variable

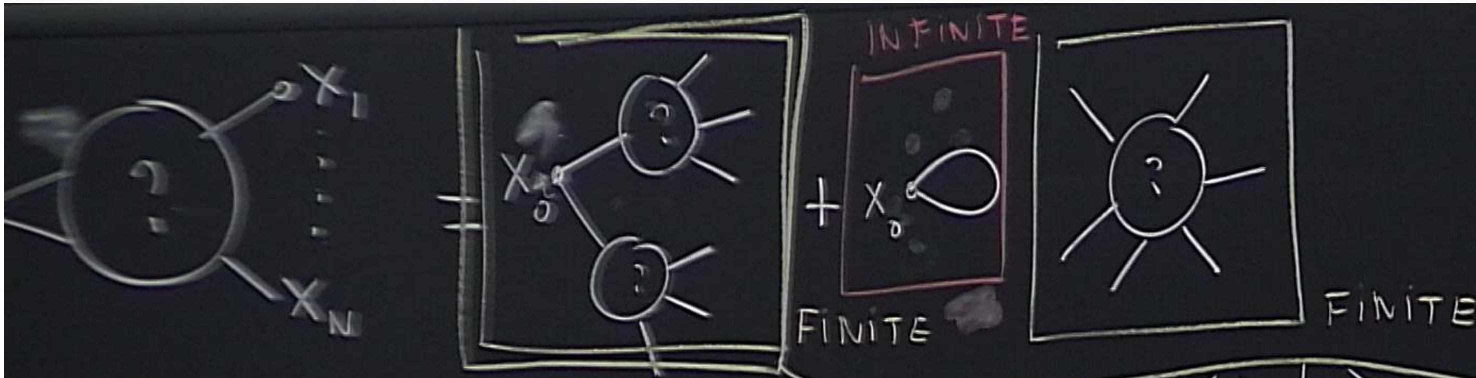
$$\lim_{y \to x_0} \Big[\underbrace{\phi(x_0)\,\phi(y)}_{\text{random variable}} - \underbrace{\langle \phi(x_0)\,\phi(y) \rangle}_{G(x_0 - y)} \cdot \underbrace{1}_{\substack{1 \text{ random variable} \\ \text{not fluctuating}}} \Big] = \underbrace{:\phi^2(x_0):}_{\text{random variable}}$$



$$(x_0)\,\phi(x_\cdot)\rangle\,\langle\phi(x_0)\,\phi(x_1)\rangle\,\langle\phi\,\phi\rangle$$

$$(x_0)\,\phi(x_0)\rangle\,\langle\phi\,\phi\rangle\cdots$$

$$\langle\phi(x_1)\cdots\phi(x_N)\,:\phi^2(x_0):\,\rangle = \text{Finite} =$$



$$\ldots\langle\phi\phi\rangle = \langle\phi(x_0)\phi(x_1)\rangle\langle\phi(x_0)\phi(x_1)\rangle\langle\phi\ \phi\rangle$$

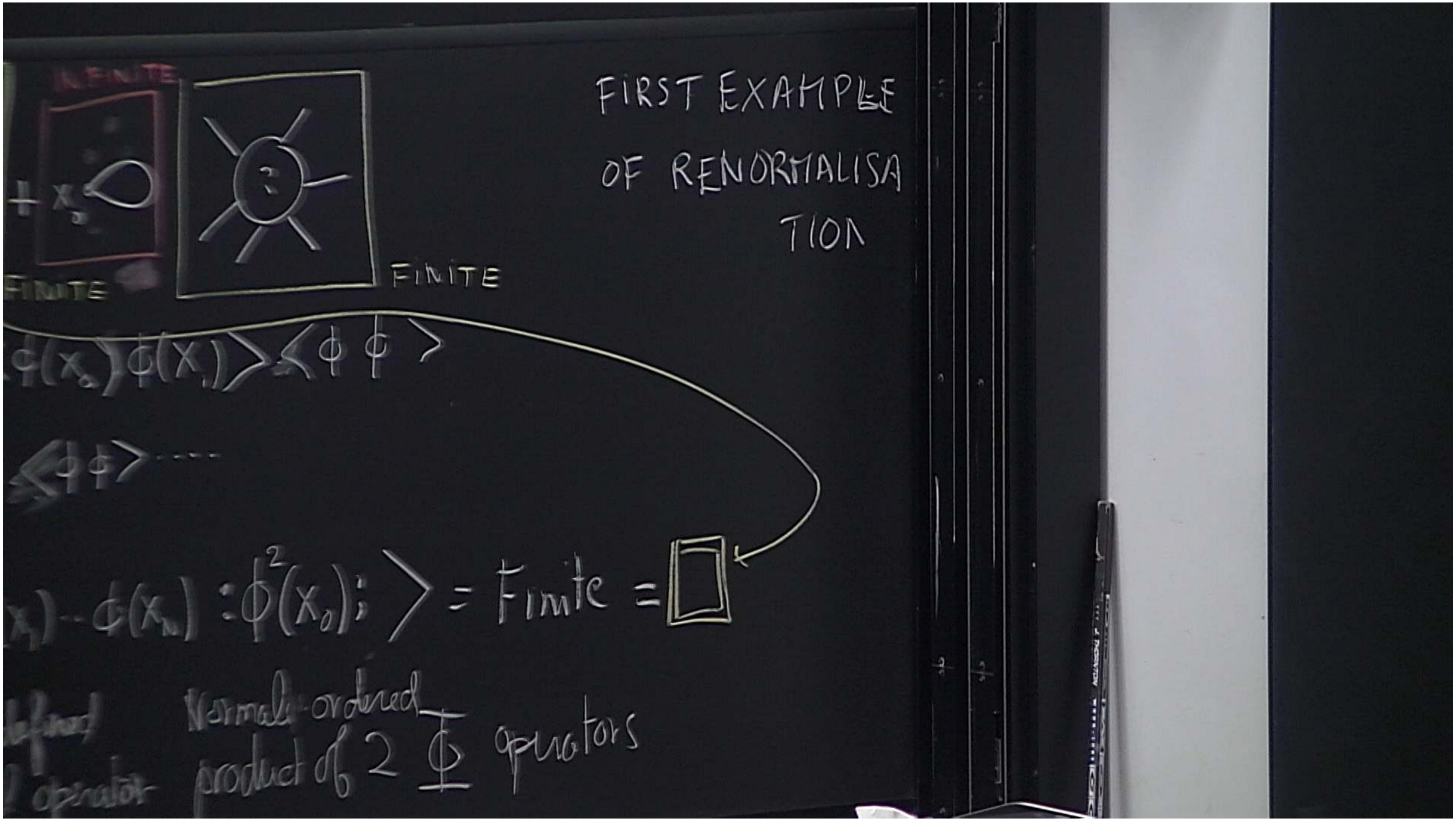
$$+\langle\phi(x_0)\ \phi(x_0)\rangle\langle\phi\ \phi\rangle\cdots$$

$$\langle\phi(x_1)\cdots\phi(x_N)\ :\phi^2(x_0):\ \rangle = \text{Finite} = \square$$

$$= :\phi^2(x_0):$$

random variable

$\longleftrightarrow$ Well defined local operator   Normal ordered product of 2 Φ operators


FIRST EXAMPLE
OF RENORMALISA
TION
FINITE
= Finite =
Normally-ordered
product of 2 Φ operators