

Title: PSI 17/18 - Quantum Field Theory II - Lecture 4

Date: Nov 09, 2017 09:00 AM

URL: <http://pirsa.org/17110014>

Abstract:

Today: Free Field. Continued ;  $\hbar=1$   
 Interacting  $\phi^4$  Intro.

Euclidian  $\phi(x)$  real scalar  $x \in \mathbb{R}^d$

$$\int \mathcal{D}[\phi(x)] \exp(-S_E[\phi])$$

$$S_E[\phi] = \int d^d x \frac{1}{2} (\partial_\mu \phi)^2 + \frac{m^2}{2} \phi^2$$

$$= \int d^d x \frac{1}{2} \phi \cdot (-\Delta_x + m^2) \phi$$

Gaussian Funct. Integral

Propagator or Correlation Funct.

$$\langle \phi(x_1) \phi(x_2) \rangle = G(x_1, x_2)$$

$$K \cdot X = K_\mu X^\mu$$

$$K^\mu = \sum_\nu K_\nu \eta^{\mu\nu}$$

$$\eta^E = \begin{pmatrix} 1 & & \\ & -1 & \\ & & \dots \end{pmatrix}$$

Continued

$\hbar = 1$

$G(x)$

$\phi^4$  Intro.

Gaussian Fund. Integral

$x \in \mathbb{R}^d$

Propagator or Correlation Function

$-S_E[\phi]$

$$\langle \phi(x_1) \phi(x_2) \rangle = G(x_1 - x_2) = \int \frac{d^d k}{(2\pi)^d} \frac{e^{i k \cdot x}}{k^2 + m^2}$$

$(\partial_\mu \phi)^2 + \frac{m^2}{2} \phi^2$

$$k \cdot x = k_\mu x^\mu$$

$\eta_{\mu\nu}^E = \delta_{\mu\nu}$  Euclidean  
Metric tensor

$$k^2 = \sum_{\mu, \nu} k_\mu \eta_{\mu\nu}^E k_\nu$$

$$\eta_{\mu\nu}^E = \delta_{\mu\nu}$$

$(-\Delta_x + m^2) \phi^2$

$$G(X) \underset{X \rightarrow 0}{\approx} |X|^{2-d}$$

Short Distance Singularity  
(U.V)  $d \geq 2$

Check: Euclidean  $\rightarrow$  Minkowski Free Field Canonical Quant.

Euclidean Propagator Wick Rotation

$$X_E = (\uparrow, \vec{x})$$

Euclidean time  $\uparrow$  space

$$G(X) \underset{X \rightarrow 0}{\approx} |X|^{2-d}$$

Short Distance Singularity  
(U.V)  $d \geq 2$

Check: Euclidean  $\rightarrow$  Minkowski Free Field Canonical Quant.

Euclidean Propagator Wick Rotation

$$X_E = (\tau, \vec{x})$$

Euclidean time  $\uparrow$   $\uparrow$  space

$$X = (t, \vec{x})$$

physical time  $\uparrow$   $\uparrow$  space

$$G(X) \underset{X \rightarrow 0}{\approx} |X|^{2-d}$$

Short Distance Singularity  
(U.V)  $d \geq 2$

Check: Euclidean  $\rightarrow$  Minkowski Free Field Canonical Quant.

Euclidean Propagator

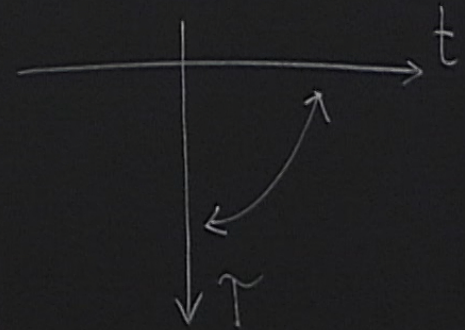
Wick Rotation

$$X_E = (\tau, \vec{x})$$

$$X = (t, \vec{x})$$

Euclidean  
time  $\uparrow$   $\downarrow$  space

physical  
time  $\uparrow$   $\downarrow$  space



$$\tau = -it$$

$$G(x) \underset{x \rightarrow 0}{\approx} |x|^{2-d}$$

Short Distance Singularity  
(U.V)  $d \geq 2$

Check: Euclidean  $\rightarrow$  Minkowski Free Field Canonical Quant.

Euclidean Propagator

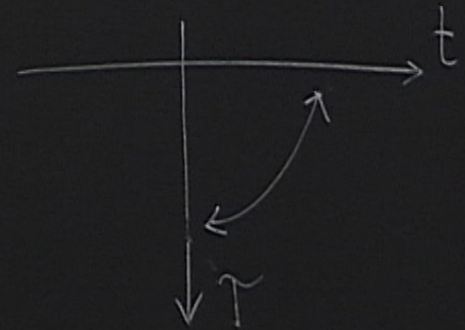
Wick Rotation

$$X_E = (\tau, \vec{x})$$

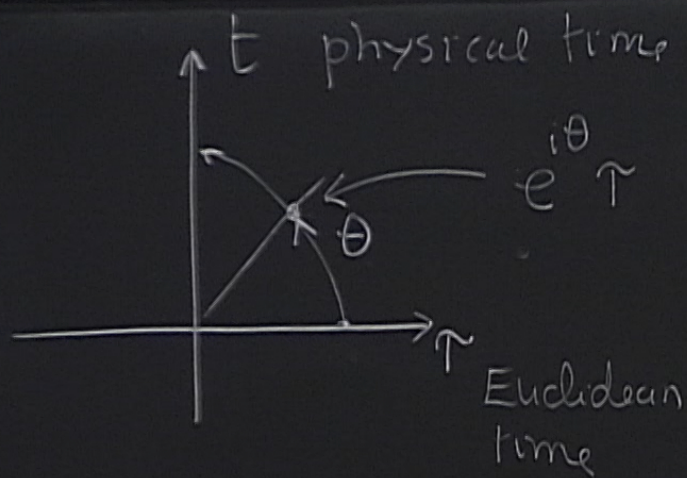
Euclidean time  $\uparrow$   $\uparrow$  space

$$X = (t, \vec{x})$$

physical time  $\uparrow$   $\uparrow$  space



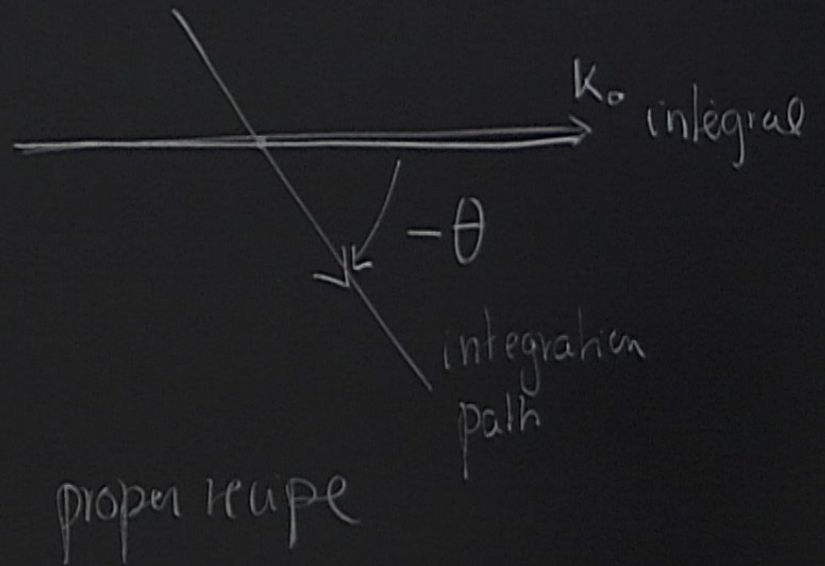
$$\tau = it$$



Wick rotation  
on time

$$\tau = X^0$$

"anti-Wick" rotation  
on  $k_0$   
so that  $k_0 \cdot X^0$  stays real



Momentum Euclidean

$$K_E = (k_0, \vec{k})$$

"Euclidean  
momentum"

↑  
space  
momentum



physical time

$$e^{i\theta}$$

Euclidean time

Euclidean

$$\vec{k}$$

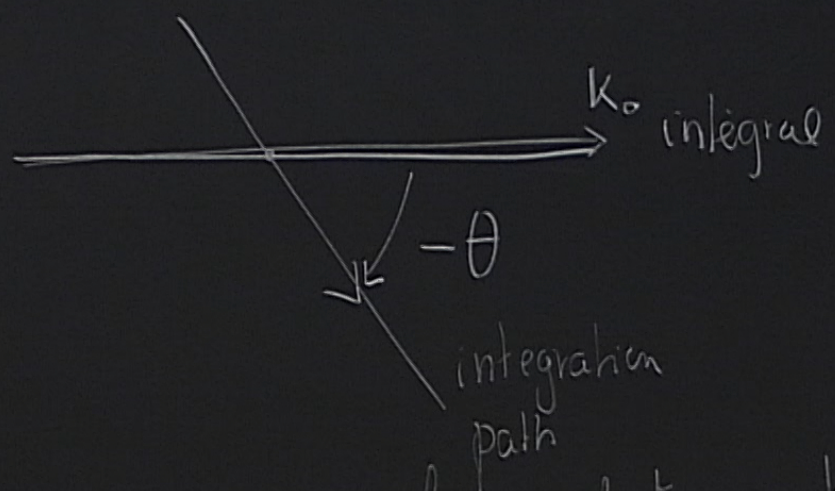
space momentum

Wick rotation on time

$$\tilde{T} = X^0$$

"anti-Wick" rotation on  $k_0$

so that  $k_0 \cdot X^0$  stays real



proper recipe for analytic continuation in time

$$\tau = i t$$

$$k_0 \rightarrow \frac{1}{i} \omega$$

so that the momentum,

$$K = (\omega, \vec{k})$$

energy ↗  
or pulsation ↘      ↖ space momentum

s real  
integral  
analytic continuation

$$G_E = \int_{-\infty}^{+\infty} \frac{dK_0}{2\pi} \int \frac{d^{\vec{k}}}{(2\pi)^{d-1}} \frac{e^{i(K_0 \cdot t + \vec{k} \cdot \vec{x})}}{K_0^2 + \vec{k}^2 + m^2}$$

change of variable (Wick Rot.)

$$\int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} (-i) \int \frac{d^{\vec{k}}}{(2\pi)^{d-1}} \frac{e^{i(\omega \cdot t + \vec{k} \cdot \vec{x})}}{-\omega^2 + \vec{k}^2 + m^2}$$

physical time

$$e^{i\theta}$$

Euclidean time

Euclidean

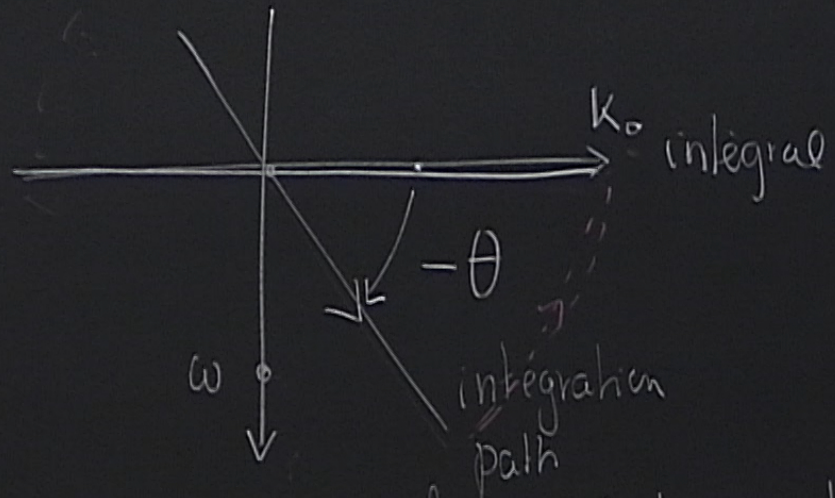
space momentum

Wick rotation on time

$$\tau = X^0$$

"anti-Wick" rotation on  $K_0$

so that  $K_0 \cdot X^0$  stays real



proper recipe for analytic continuation in time

$$\tau = i t$$

$$K_0 \rightarrow \frac{1}{i} \omega$$

so that the m...

$$K = (\omega, \vec{k})$$

energy or pulsation

$$G_E = \int_{-\infty}^{+\infty} \frac{dK_0}{2\pi} \int \frac{d^{d-1} \vec{k}}{(2\pi)^{d-1}} \frac{e^{i(K_0 \cdot t + \vec{k} \cdot \vec{x})}}{K_0^2 + \vec{k}^2 + m^2}$$

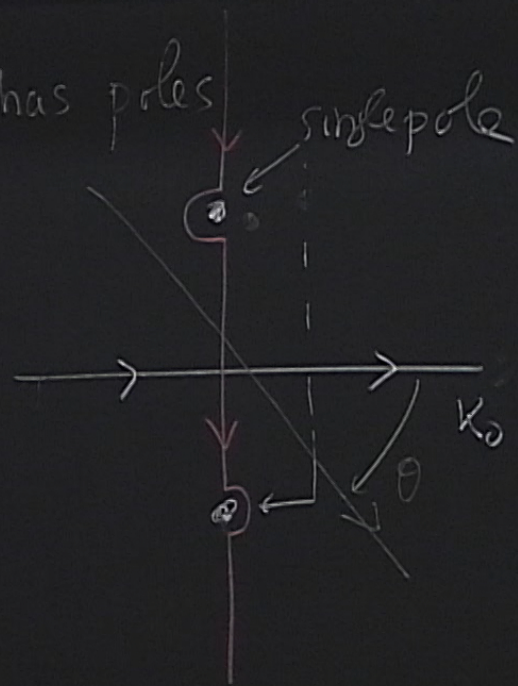
change of variable ( Wick Rot. )

$$\int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} (-i) \int \frac{d^{d-1} \vec{k}}{(2\pi)^{d-1}} \frac{e^{i(\omega \cdot t + \vec{k} \cdot \vec{x})}}{-\omega^2 + \vec{k}^2 + m^2}$$

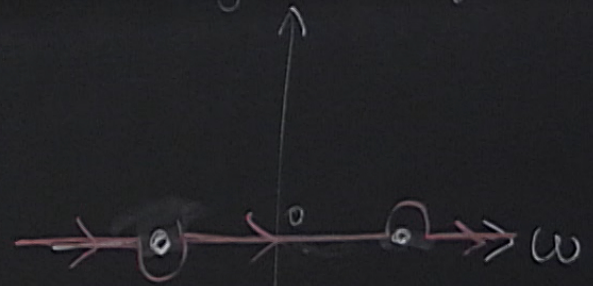
Denominator has poles

$$\frac{1}{k_0^2 + \vec{k}^2 + m^2}$$

$$k_0 = \pm i \sqrt{\vec{k}^2 + m^2}$$



In the final integral over  $\omega$



poles at  $\omega = \pm \sqrt{\vec{k}^2 + m^2}$

prescription of integration  
for the Feynman propagator

$$\int \frac{d\omega}{2\pi} \int \frac{d^{d-1}\vec{k}}{(2\pi)^{d-1}} i \frac{e^{i(\omega t + \vec{k} \cdot \vec{x})}}{\omega^2 - \vec{k}^2 - m^2 + i0_+}$$

$$= G_F(t, \vec{x}) \quad \text{with } X = (t, \vec{x})$$

$$= \langle 0 | T [\Phi(t) \Phi(x)] | 0 \rangle$$

$\nearrow$  time ordered product       $\nwarrow$  Field operators

Integration  
Feynman propagator

$$\int \frac{d\omega}{2\pi} \int \frac{d\vec{k}}{(2\pi)^{d-1}} e^{i(\omega t + \vec{k} \cdot \vec{x})} \frac{1}{\omega^2 - \vec{k}^2 - m^2 + i0_+}$$

$$= G_F(t, \vec{x}) \quad \text{with } X = (t, \vec{x})$$

$$= \langle 0 | T [\Phi(t) \Phi(x)] | 0 \rangle$$

time ordered product      Field operators





prescription of integration  
for the Feynman propagator

$$\int \frac{d\omega}{2\pi} \int \frac{d^d k}{(2\pi)^{d-1}} \frac{1}{\omega^2 - \vec{k}^2 - m^2 + i\epsilon}$$

$$K = (K^\mu) = (E, \vec{p})$$

energy momentum

$$= G_F(t, \vec{x})$$

with  $X$

$$= \langle 0 | T [\Phi(0) \Phi(x)] | 0 \rangle$$

time ordered product      Field operators

Proof: finite number of r.v  $\phi^a$   $a=1, \dots, d$

probability distrib  $\exp(-\frac{1}{2} \phi^a K_{ab} \phi^b)$

$K = (K_{ab})$  symmetric positive  
real matrix

$$\langle \phi^a \phi^b \rangle = (K^{-1})_{ab} = G^{ab} \quad \text{2 points correlator}$$

$$\langle \phi^a \rangle = 0$$

$$\langle \phi^{a_1} \dots \phi^{a_N} \rangle = \frac{\int \prod_a d\phi^a \exp(-\frac{1}{2} \phi \cdot K \cdot \phi) \phi^{a_1} \dots \phi^{a_N}}{\int \prod_a d\phi^a \exp(-\frac{1}{2} \phi \cdot K \cdot \phi)}$$

Generating Function

vector  $J_a$  not a random variable

Source term

$$Z(j) = \int \prod_a d\phi^a \exp\left(-\frac{1}{2} \phi^a K_{ab} \phi^b + J_c \phi^c\right)$$

$$\psi(\mathbf{k}, \phi) \phi^{a_1} \dots \phi^{a_N}$$

change of variables

$$\psi(\mathbf{k}, \phi)$$

$$\phi^a = \phi'^a + G^{ab} J_b$$

$$\text{so } Z(\mathbf{J}) = Z \cdot \exp\left(\frac{1}{2} J_a G^{ab} J_b\right)$$

$$\left( \phi^a K_{ab} \phi^b + J_c \phi^c \right) = \int \prod d\phi'^a \exp\left(-\frac{1}{2} \phi'^a K_{ab} \phi'^b\right) \cdot \exp\left(\frac{1}{2} J_a G^{ab} J_b\right)$$

$$\langle \phi^{a_1} \rangle = \frac{1}{Z(j)} \frac{\partial}{\partial J_{a_1}} Z(j) \Big|_{j=0} = G^{a_1}$$

$$\langle \phi^{a_1} \phi^{a_2} \rangle = \frac{1}{Z(j)} \frac{\partial}{\partial J_{a_1}} \frac{\partial}{\partial J_{a_2}} Z(j) \Big|_{j=0} = \frac{\partial}{\partial J_{a_1}} G^{a_2}$$

$$\langle \phi^{a_1} \dots \phi^{a_N} \rangle = \frac{1}{Z(j)} \frac{\partial}{\partial J_{a_1}} \dots \frac{\partial}{\partial J_{a_N}} Z(j) \Big|_{j=0}$$

$$G^{a_2 b} J_b \Big|_{j=0} = 0$$

= 0 when  $j=0$

$$\frac{\partial}{\partial J_{a_1}} \frac{\partial}{\partial J_{a_2}} \exp\left(\frac{1}{2} J \cdot G \cdot J\right) = \frac{\partial}{\partial J_{a_1}} \left[ G^{a_2 b} J_b \cdot \exp\left(\frac{1}{2} J \cdot G \cdot J\right) \right] = \underbrace{G^{a_2 a_1}} + \overbrace{G^{a_2 b} J_b \cdot G^{a_1 c} J_c}$$

$$\frac{\partial}{\partial J} \frac{\partial}{\partial J} \frac{\partial}{\partial J} \frac{\partial}{\partial J} \exp\left(\frac{1}{2} J \cdot G \cdot J\right)$$

= 0 when  $j=0$

$$\begin{aligned}
 \left. \dots \right] &= \left( G^{a_2 a_2} + G^{a_2 b} j_b \cdot G^{a_1 c} j_c \right) \exp\left(\frac{1}{2} j \cdot G j\right) \\
 &= G^{a_2 a_1} G^{a_4 a_3} + G^{a_2 a_3} G^{a_1 a_4} + G^{a_2 a_4} G^{a_1 a_3} \\
 &\quad \begin{array}{ccc} 1 & 2 & 3 & 4 \\ \text{└─} & \text{└─} & & \end{array} + \begin{array}{ccc} 1 & 2 & 3 & 4 \\ & \text{└─} & \text{└─} & \end{array} + \begin{array}{ccc} 1 & 2 & 3 & 4 \\ & & \text{└─} & \text{└─} \\ & & & \text{└─} \end{array}
 \end{aligned}$$

Wick Theorem holds for Functional Integral Quantization

Euclidean  $*$  =  $\langle \phi(x_1) \phi(x_2) \dots \phi(x_N) \rangle = \frac{\int \mathcal{D}[\phi] \exp(-S)}  
E-omitted$

N odd  $N=2M+1$  this is zero

N even  $N=2M$  non-zero

$$\langle \phi(x_i) \rangle = 0$$

Gaussian random variables

$$* = \sum_{\text{pairings in } M \text{ pairs}} \prod \langle \phi(x_i) \phi(x_j) \rangle$$

Wick Theorem

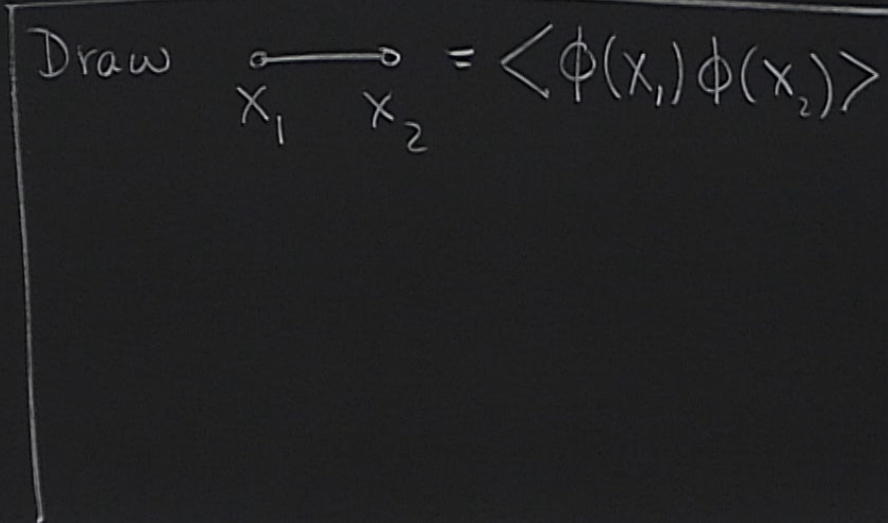


Quantization

$$\frac{\int \mathcal{D}[\phi] \exp(-S[\phi]) \phi(x_1) \dots \phi(x_n)}{\int \mathcal{D}[\phi] \exp(-S[\phi])}$$

$$\langle \phi(x) \rangle = 0$$

Wick Theorem



$$\begin{aligned} &\langle \phi(x_1) \phi(x_2) \rangle \\ &= \langle \phi(x_1) \phi(x_2) \rangle \\ &+ \langle \phi(x_1) \phi(x_3) \rangle \\ &+ \langle \phi(x_1) \phi(x_4) \rangle \end{aligned}$$

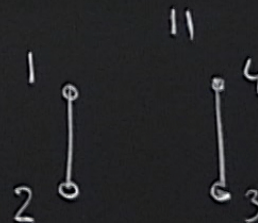
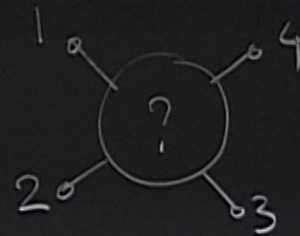
$$\langle \phi(x_1) \phi(x_2) \phi(x_3) \phi(x_4) \rangle$$

$$= \langle \phi(x_1) \phi(x_2) \rangle \langle \phi(x_3) \phi(x_4) \rangle$$

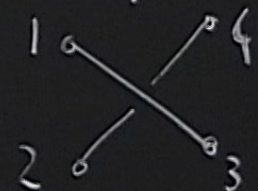
$$+ \langle \phi(x_1) \phi(x_3) \rangle \langle \phi(x_2) \phi(x_4) \rangle$$

$$+ \langle \phi(x_1) \phi(x_4) \rangle \langle \phi(x_2) \phi(x_3) \rangle$$

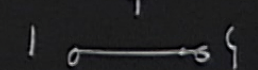
Famous Feynman diagram representation



+



+



+



In a QFT

$$|\psi\rangle = \phi \phi \dots \phi |0\rangle$$

$$\langle \phi | \psi \rangle = \langle 0 | \phi \phi \dots \phi \times \phi \dots \phi | 0 \rangle$$

Fock Space  $\leftarrow$  Correlation Functions  $\leftarrow$  WickTh & Functional Integral

In a QFT

$$|\psi\rangle = \phi \phi \dots \phi |0\rangle$$

$$\langle \phi | \psi \rangle = \langle 0 | \phi \phi \dots \phi \times \phi \dots \phi | 0 \rangle$$

Fock Space  $\leftarrow$  Correlation Functions  $\leftarrow$  Wick Th & Functional Integral

Canonical Quant:  $\equiv$  Functional Integral

$$N = 2 \quad 1 \text{ pairing}$$

$$N = 4 \quad 3 \text{ pairings}$$

$$N = 2M \quad \frac{(2M)!}{M! 2^M} \text{ pairing}$$

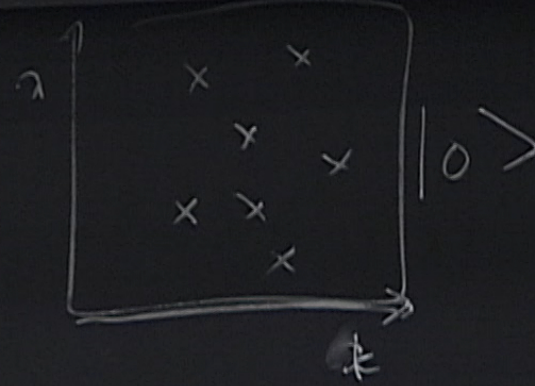
$$N = 6 \quad M = 3 \quad \frac{6!}{3! 2^3} = \frac{6 \cdot 5 \cdot 4}{8} = 3 \cdot 5 = 15$$

Math & Functional  
Integral

goal

QFT

$$|n\rangle = \phi \phi \dots \phi |0\rangle$$



$$\langle \Psi | = \langle 0 | \phi \phi \dots \phi x \phi \dots \phi | 0 \rangle$$

Space  $\leftarrow$  Correlation Functions  $\leftarrow$  Wick Th & Functional Integral

Functional Integral

$$N = 2$$

$$N = 4$$

$$N = 2$$

$$N = 6$$

$$N = 2M$$

$$\frac{(2M)!}{M! 2^M} \text{ pairing}$$

$$N = 6 \quad M = 3$$

$$\frac{6!}{3! 2^3} = \frac{6 \cdot 5 \cdot 4}{8} = 3 \cdot 5 = 15$$

Axiomatic QFT

Composite operators and Normal products:  $\langle \phi(x)^2 \rangle = \infty$  ☹️

$\phi(x)$   $\longrightarrow$   $\bar{\Phi}(x)$   
Random variable      Field Operator

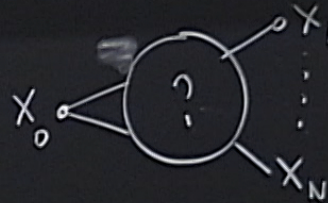
properties of  $\phi^2(x)$   
in Functional Integral

\* average of  $\phi^2(x) = +\infty$  2-d

\*  $\langle \phi(x)\phi(y) \rangle \underset{y \rightarrow x}{\simeq} |x-y| \underset{y=x}{\rightarrow} \infty$



pts:  $\langle \phi(x)^2 \rangle = \infty$  ☹️

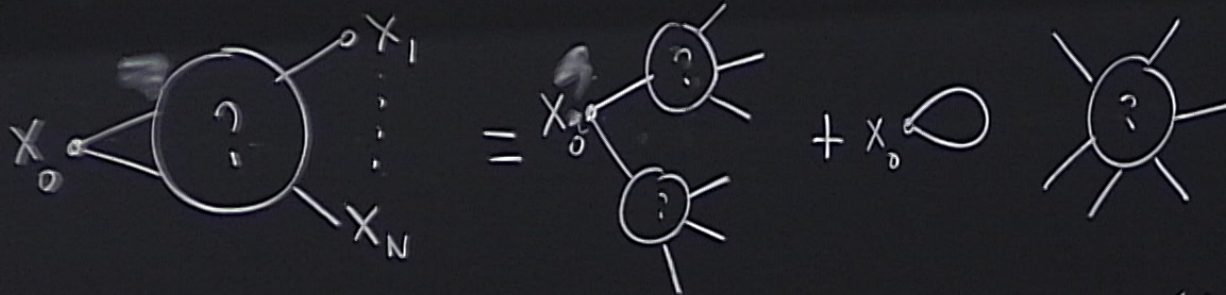


all matrix elements are  $\infty$

$$\underbrace{\langle \phi(x_1) \dots \phi(x_N) \phi^2(x_0) \rangle}_{N+2 \text{ points}} =$$

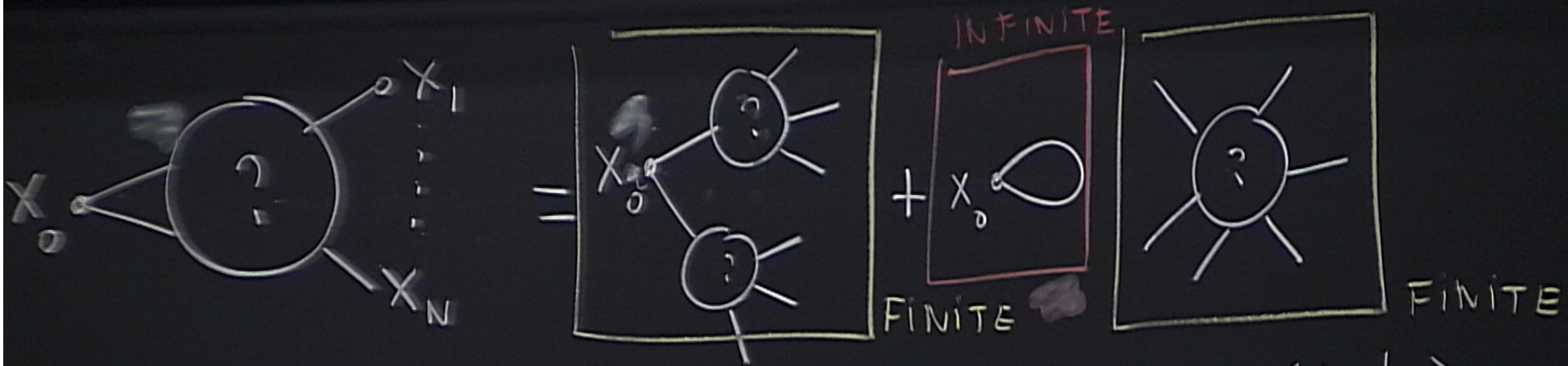
Wick Theorem pairings

$$\sum \prod \langle \phi \phi \rangle = \langle \phi(x_0) \phi(x_1) \rangle \langle \phi(x_0) \phi(x_1) \rangle \langle \phi \phi \rangle + \langle \phi(x_0) \phi(x_0) \rangle \langle \phi \phi \rangle \dots$$



$\infty$

$$\sum_{\text{Wick Theorem pairings}} \prod \langle \phi \phi \rangle = \langle \phi(x_0) \phi(x_1) \rangle \langle \phi(x_0) \phi(x_1) \rangle \langle \phi \phi \rangle + \langle \phi(x_0) \phi(x_0) \rangle \langle \phi \phi \rangle \dots$$



ainings

$$\sum \prod \langle \phi \phi \rangle = \langle \phi(x_0) \phi(x_1) \rangle \langle \phi(x_0) \phi(x_1) \rangle \langle \phi \phi \rangle + \langle \phi(x_0) \phi(x_0) \rangle \langle \phi \phi \rangle \dots$$

all matrix elements are  $\infty$

$$\underbrace{\langle \phi(x_1) \dots \phi(x_N) \phi^2(x_0) \rangle}_{N+2 \text{ points}} =$$

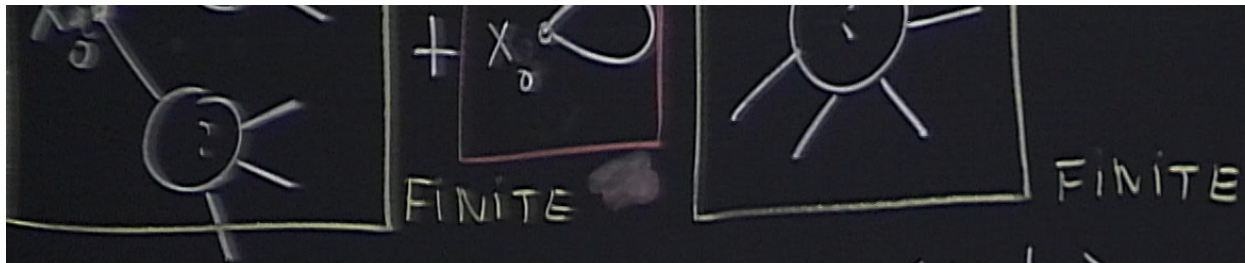
Wick Theorem pairings

$$\sum \prod \langle \phi \phi \rangle = \langle \phi(x_0) \phi(x_0) \rangle + \dots$$

INFINITE

random variable

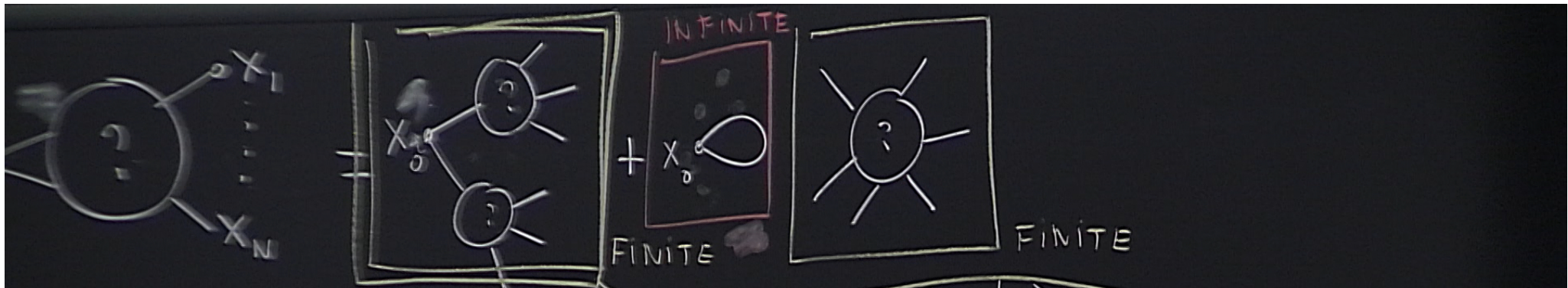
$$\lim_{Y \rightarrow X_0} \left[ \underbrace{\phi(x_0) \phi(Y)}_{\text{random variable}} - \underbrace{\langle \phi(x_0) \phi(Y) \rangle}_{G(x_0 - Y)} \cdot \underbrace{1}_{\substack{\uparrow \text{random variable} \\ \text{not fluctuating}}} \right] = \underbrace{:\phi^2(x_0):}_{\text{random variable}}$$



$$\langle x_0 | \phi(x_1) \rangle \langle \phi(x_0) | \phi(x_1) \rangle \langle \phi | \phi \rangle$$

$$\langle x_0 | \phi(x_0) \rangle \langle \phi | \phi \rangle \dots$$

$$\langle \phi(x_1) \dots \phi(x_N) : \phi^2(x_0) \rangle = \text{Finite} =$$



$$\langle \phi \phi \rangle = \langle \phi(x_0) \phi(x_1) \rangle \langle \phi(x_0) \phi(x_2) \rangle \langle \phi \phi \rangle + \langle \phi(x_0) \phi(x_0) \rangle \langle \phi \phi \rangle + \dots$$

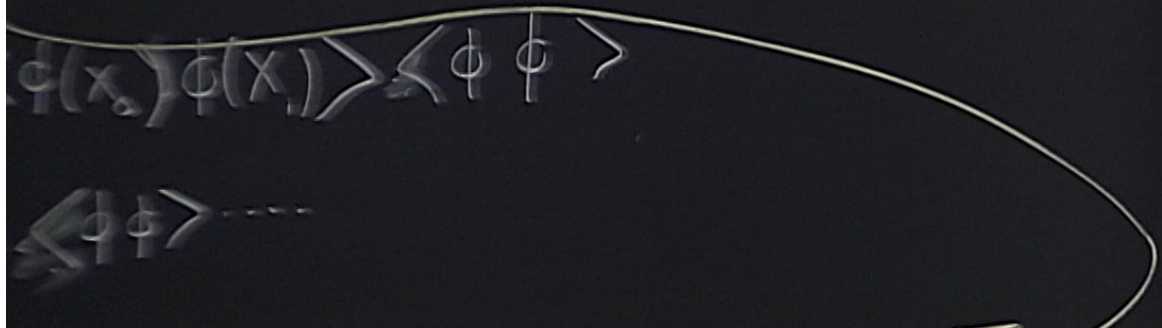
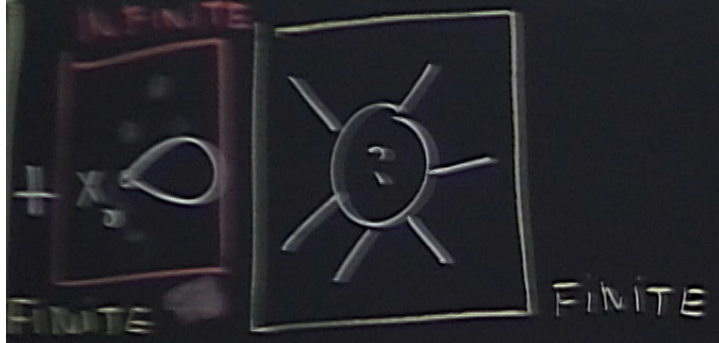
$$\langle \phi(x_1) \dots \phi(x_N) : \phi^2(x_0) : \rangle = \text{Finite} = \boxed{\phantom{0}}$$

$$= : \phi^2(x_0) :$$

random variable

Well defined local operator      Normally ordered product of 2  $\phi$  operators

# FIRST EXAMPLE OF RENORMALISA TION



$$\langle \phi(x_0) \phi(x_0) \rangle = \text{Finite} = \boxed{\phantom{0}}$$

defined  
operator

Normal-ordered  
product of 2  $\phi$  operators