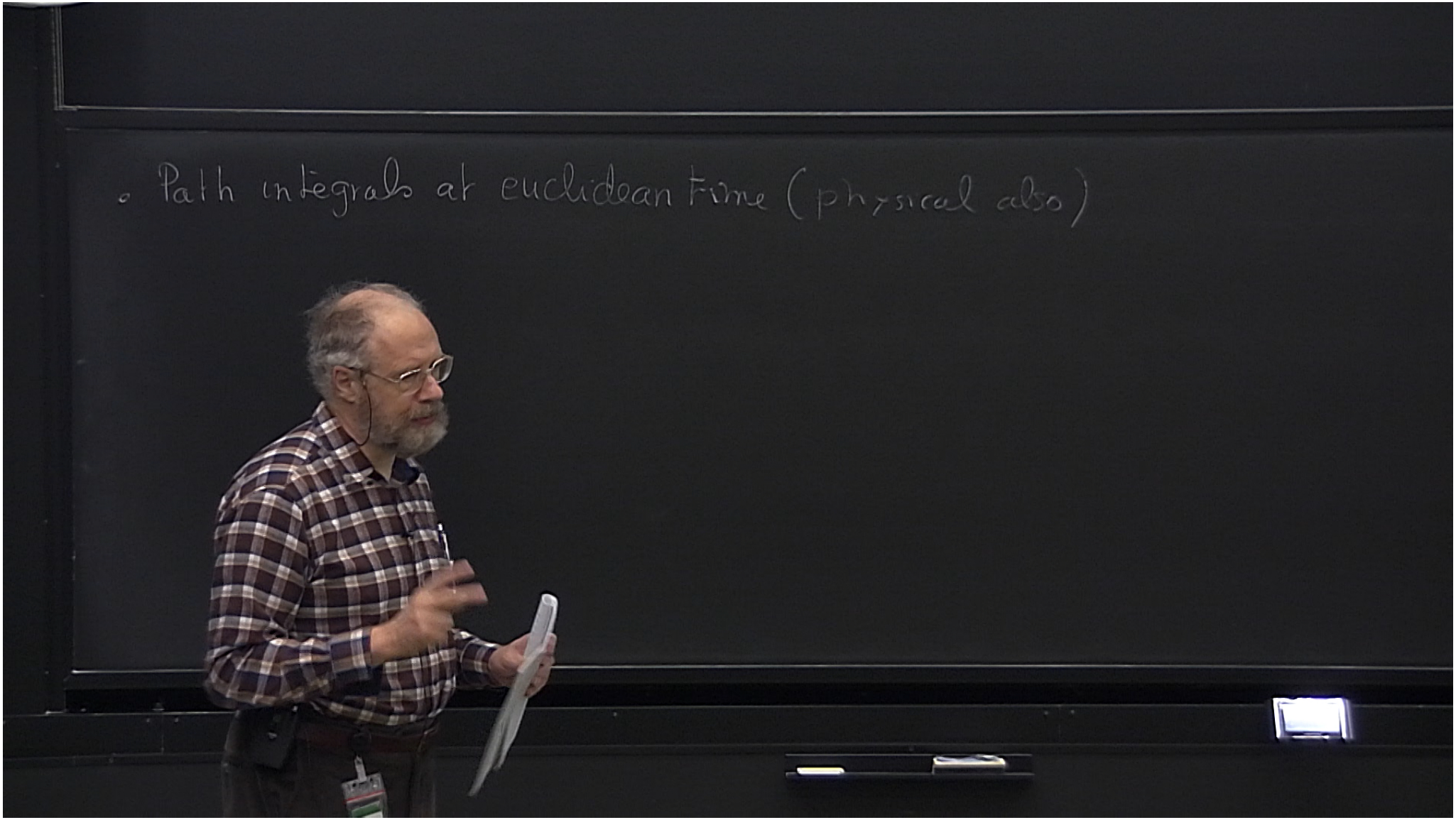
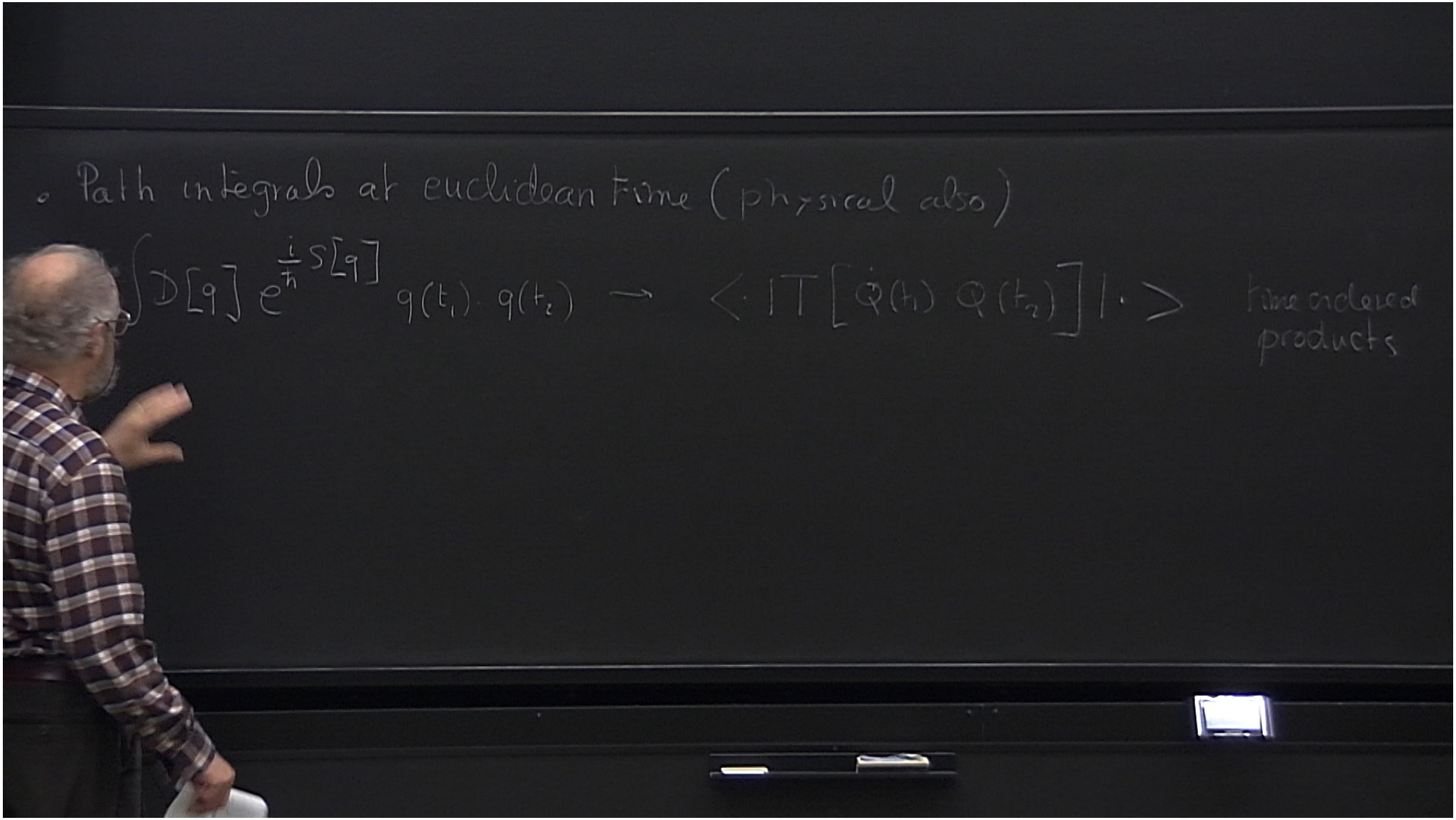
Title: PSI 17/18 - Quantum Field Theory II - Lecture 2

Date: Nov 07, 2017 09:00 AM

URL: http://pirsa.org/17110012

Abstract:


• Path integrals at euclidean time (physical also)


• Path integrals at euclidean time (physical also)
$\int \mathcal{D}[q]\, e^{\frac{i}{\hbar} S[q]}\, q(t_1)\cdot q(t_2) \rightarrow \langle \cdot | T\left[\hat{Q}(t_1)\,\hat{Q}(t_2)\right] | \cdot \rangle$
time ordered products

- Path integrals at euclidean time (physical also)

$$\int \mathcal{D}[q]\, e^{\frac{i}{\hbar} S[q]}\, q(t_1)\cdot q(t_2) \longrightarrow \langle \cdot | T\left[\hat{Q}(t_1)\,\hat{Q}(t_2)\right] | \cdot \rangle$$

time ordered products

$$\int \mathcal{D}[q_E]\, e^{-\frac{1}{\hbar} S_E[q_E]}$$

periodic trajectories

$$q_E[0] = q_E(\tau)$$

period $\tau = \frac{\hbar}{k_B T}$ ← temperature

- Path integrals at euclidean time (physical also)

$$\int \mathcal{D}[q]\, e^{\frac{i}{\hbar} S[q]}\, q(t_1)\cdot q(t_2) \longrightarrow \langle \cdot | T\left[\hat{Q}(t_1)\,\hat{Q}(t_2)\right] | \cdot \rangle$$

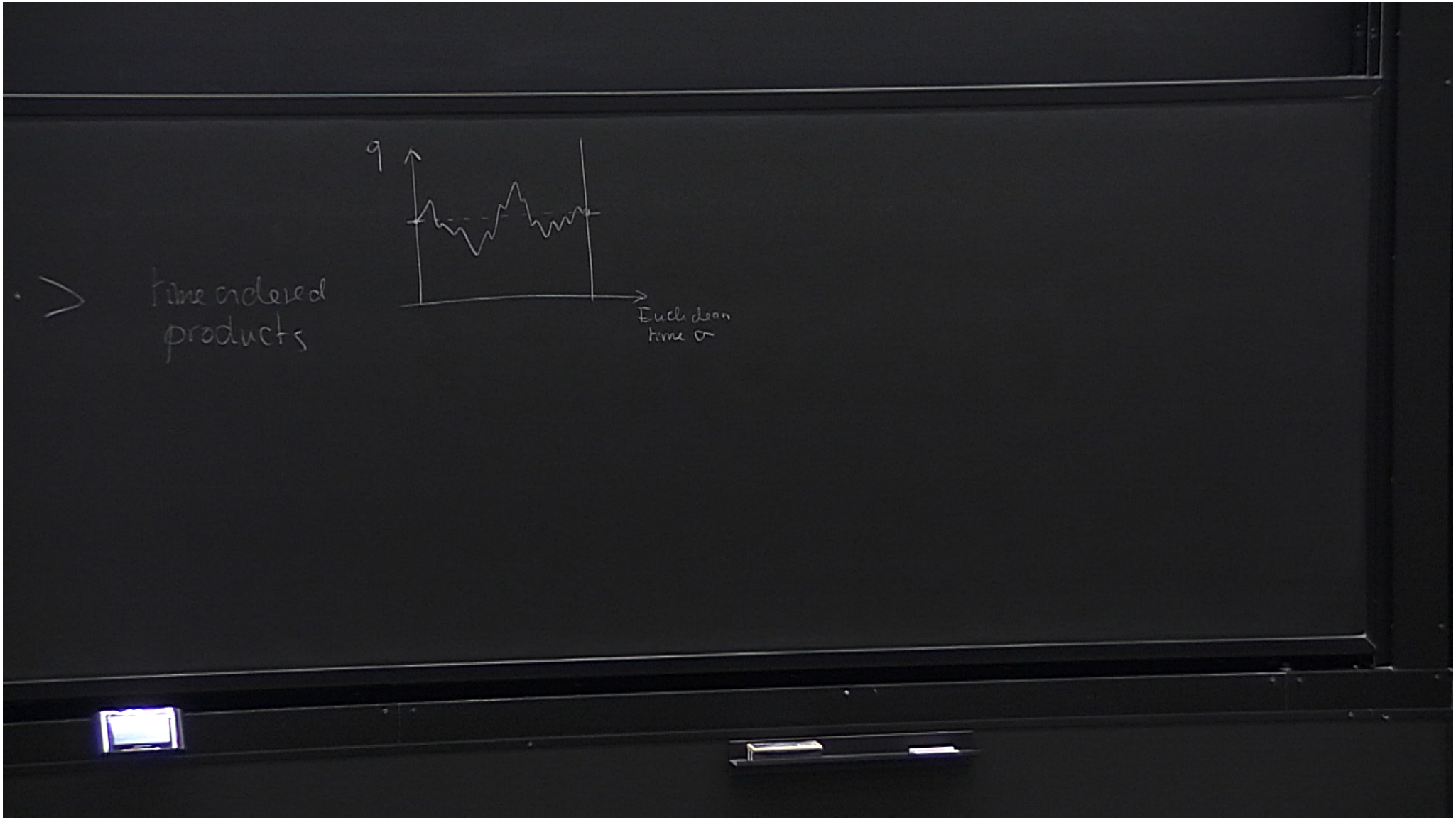
time ordered products

$$\int \mathcal{D}[q_E]\, e^{-\frac{1}{\hbar} S_E[q_E]}\, q(0)$$

periodic trajectories

$$q_E[0] = q_E(\tau)$$

period $\tau = \frac{\hbar}{k_B T}$ ← temperature


time ordered products
Euclidean time σ

- Path integrals at euclidean time (physical also)

$$\int \mathcal{D}[q]\, e^{\frac{i}{\hbar} S[q]}\, q(t_1)\, q(t_2) \longrightarrow \langle 0 | T\left[\hat{Q}(t_1)\, \hat{Q}(t_2)\right] | 0 \rangle$$

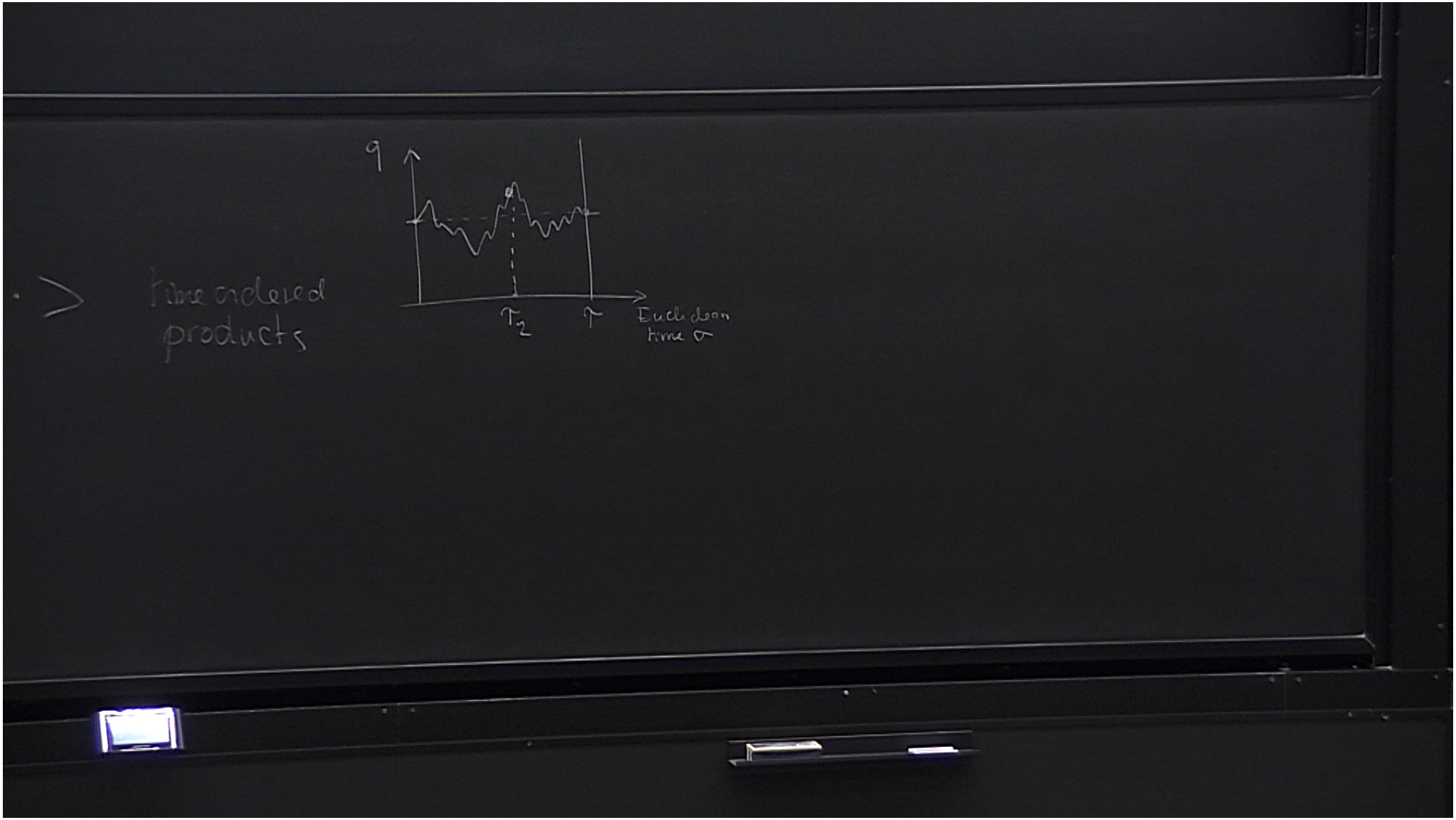
time ordered product

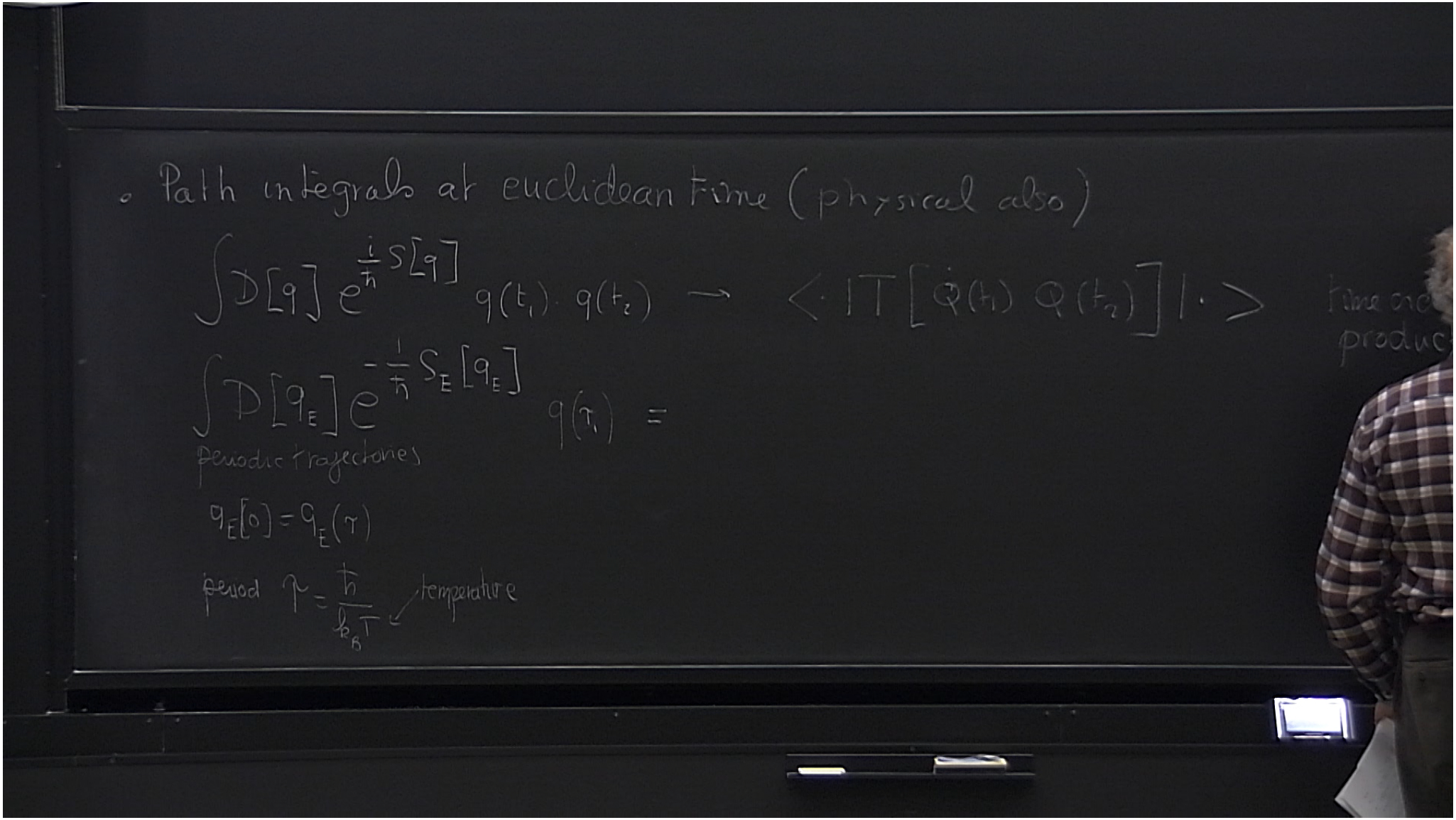
$$\int \mathcal{D}[q_E]\, e^{-\frac{1}{\hbar} S_E[q_E]}\, q(\tau_1)$$

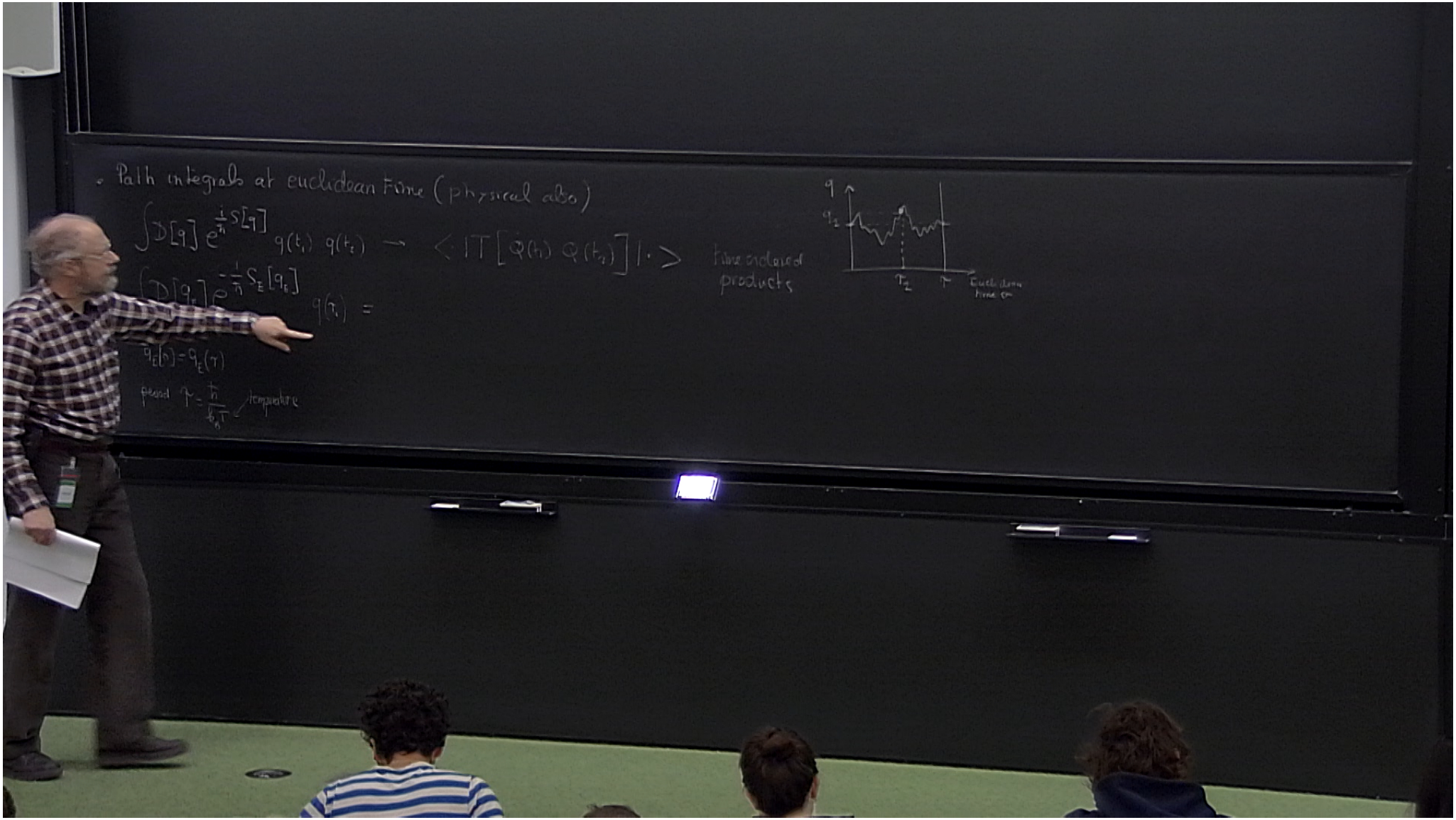
periodic trajectories

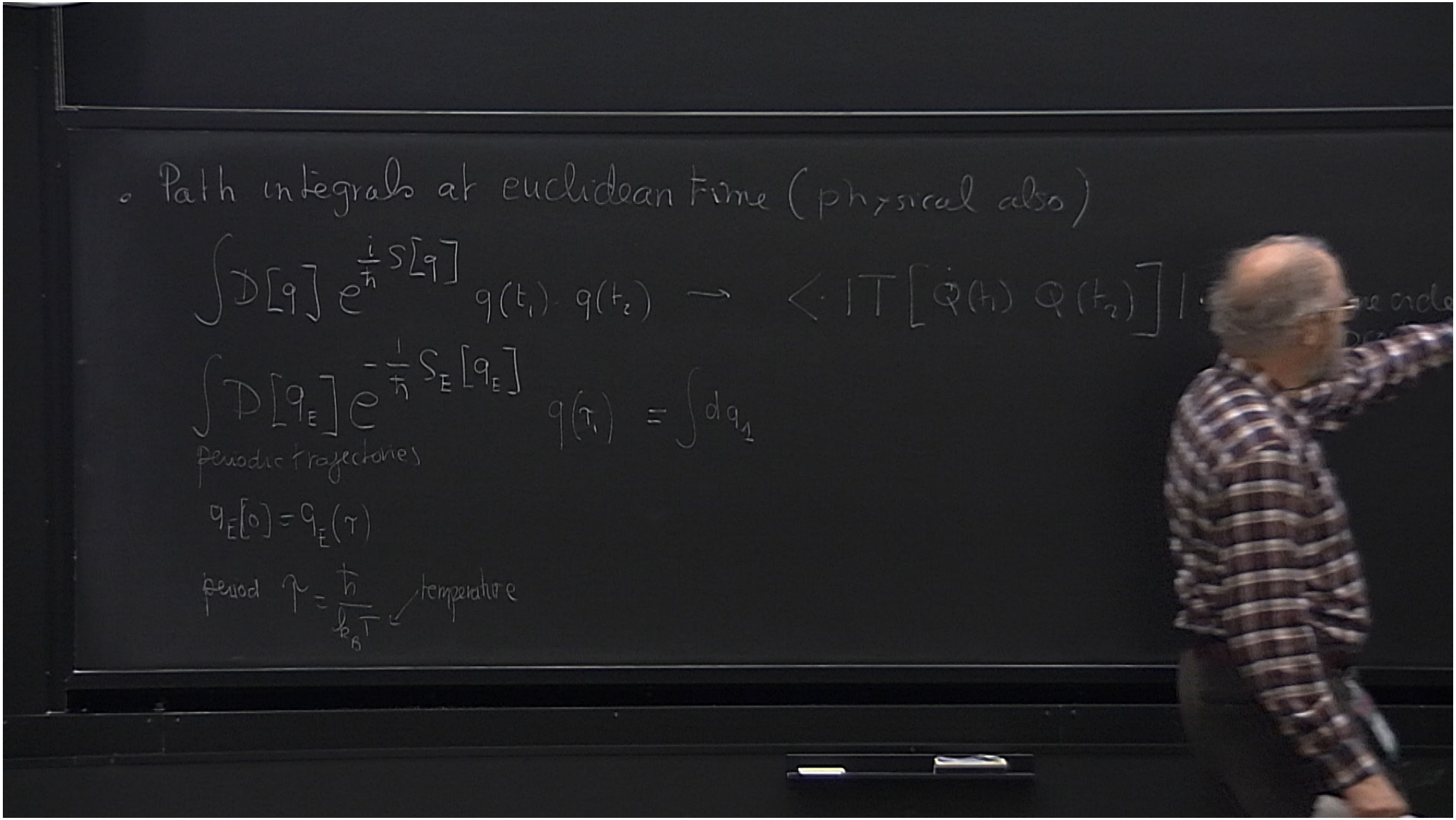
$$q_E[0] = q_E(\tau)$$

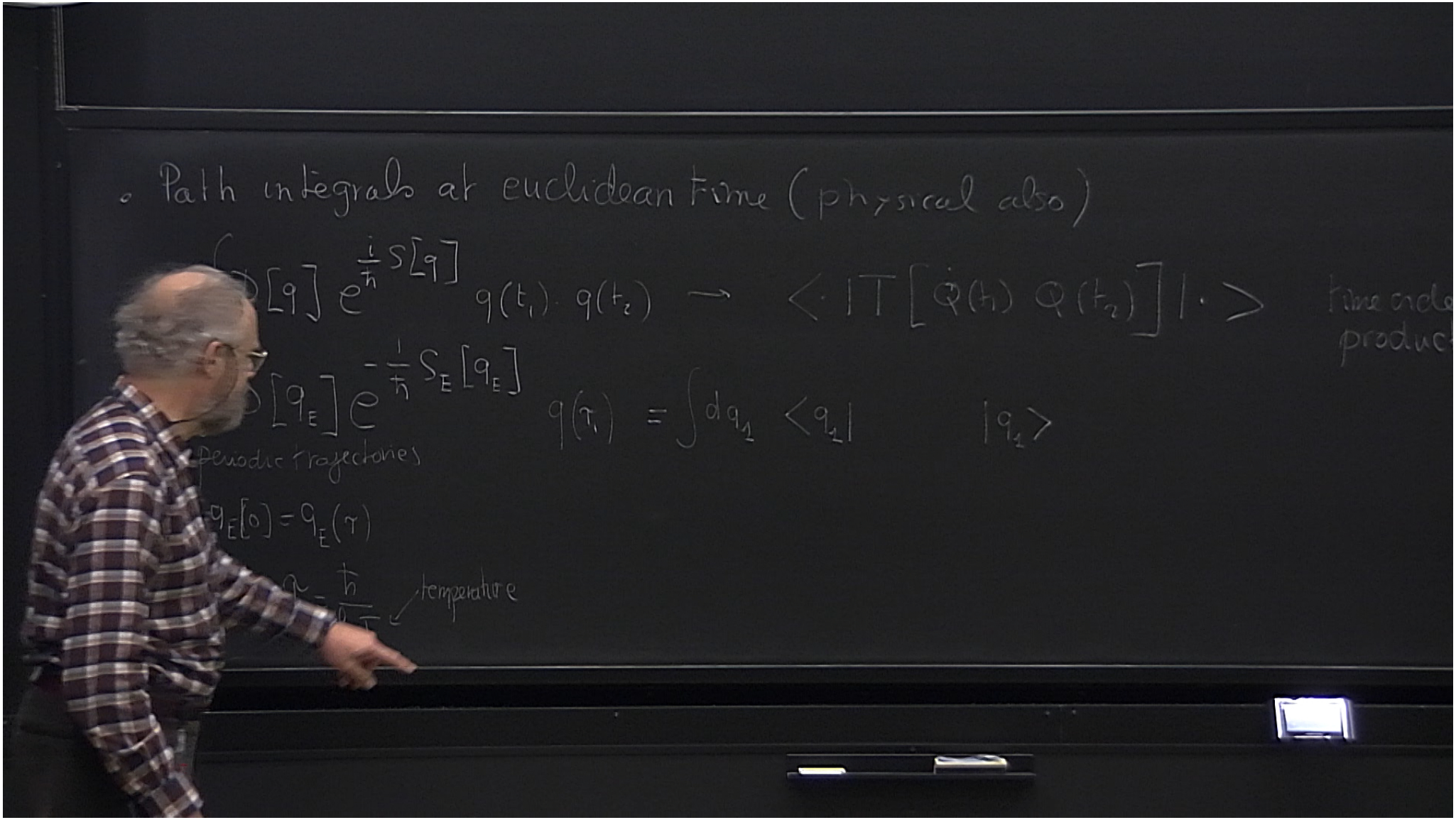
period $\tau = \frac{\hbar}{k_B T}$ ← temperature

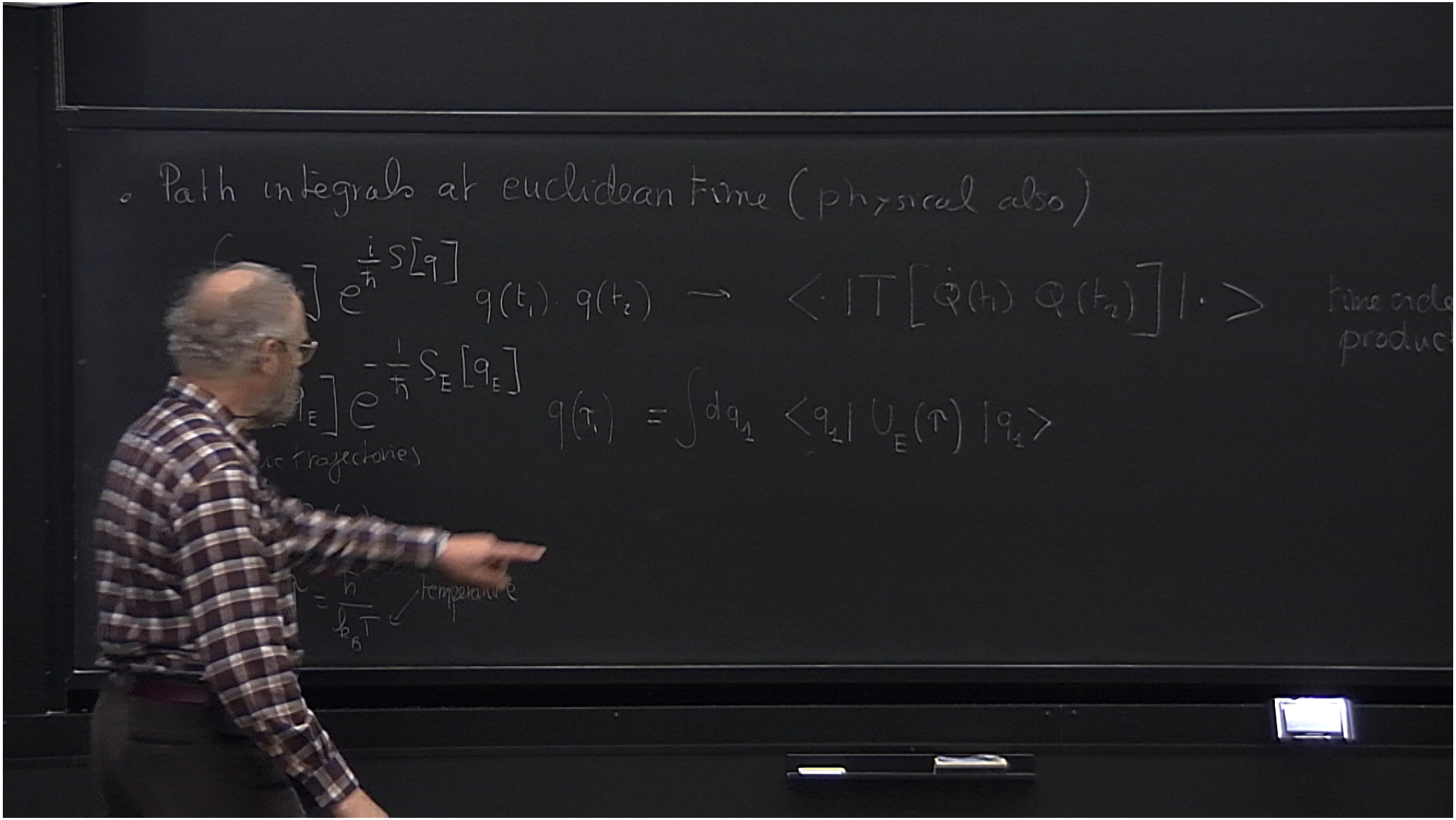

q
τ_2
τ
Euclidean time σ
> time ordered products

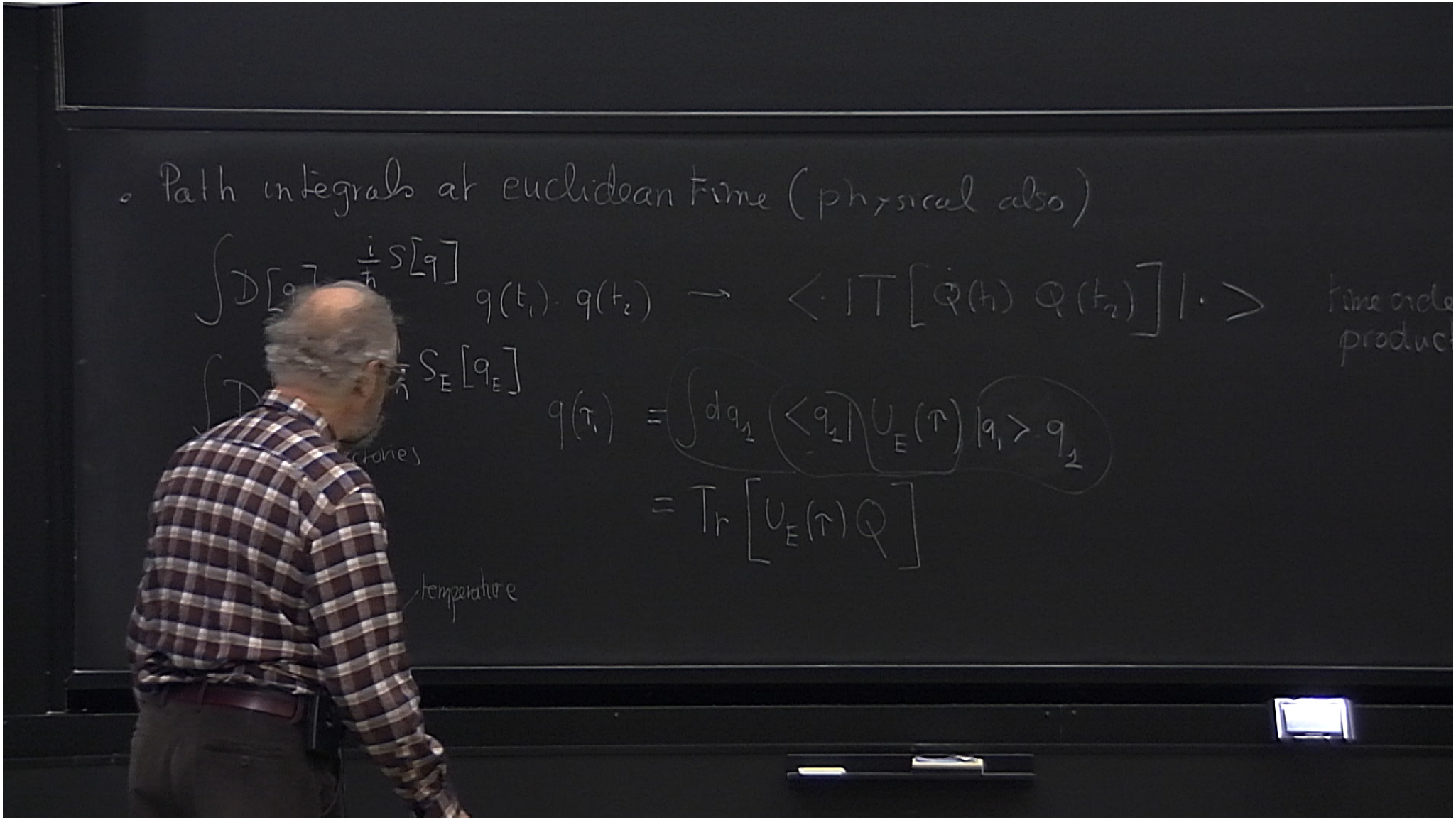

• Path integrals at euclidean time (physical also)
$\int \mathcal{D}[q]\, e^{\frac{i}{\hbar} S[q]}\, q(t_1)\cdot q(t_2) \rightarrow \langle \cdot | T\left[\hat{Q}(t_1)\, Q(t_2)\right] | \cdot \rangle$
time ord
produc
$\int \mathcal{D}[q_E]\, e^{-\frac{1}{\hbar} S_E[q_E]}\, q(\tau_1) =$
periodic trajectories
$q_E[0] = q_E(\tau)$
period $\tau = \frac{\hbar}{k_B T}$
temperature

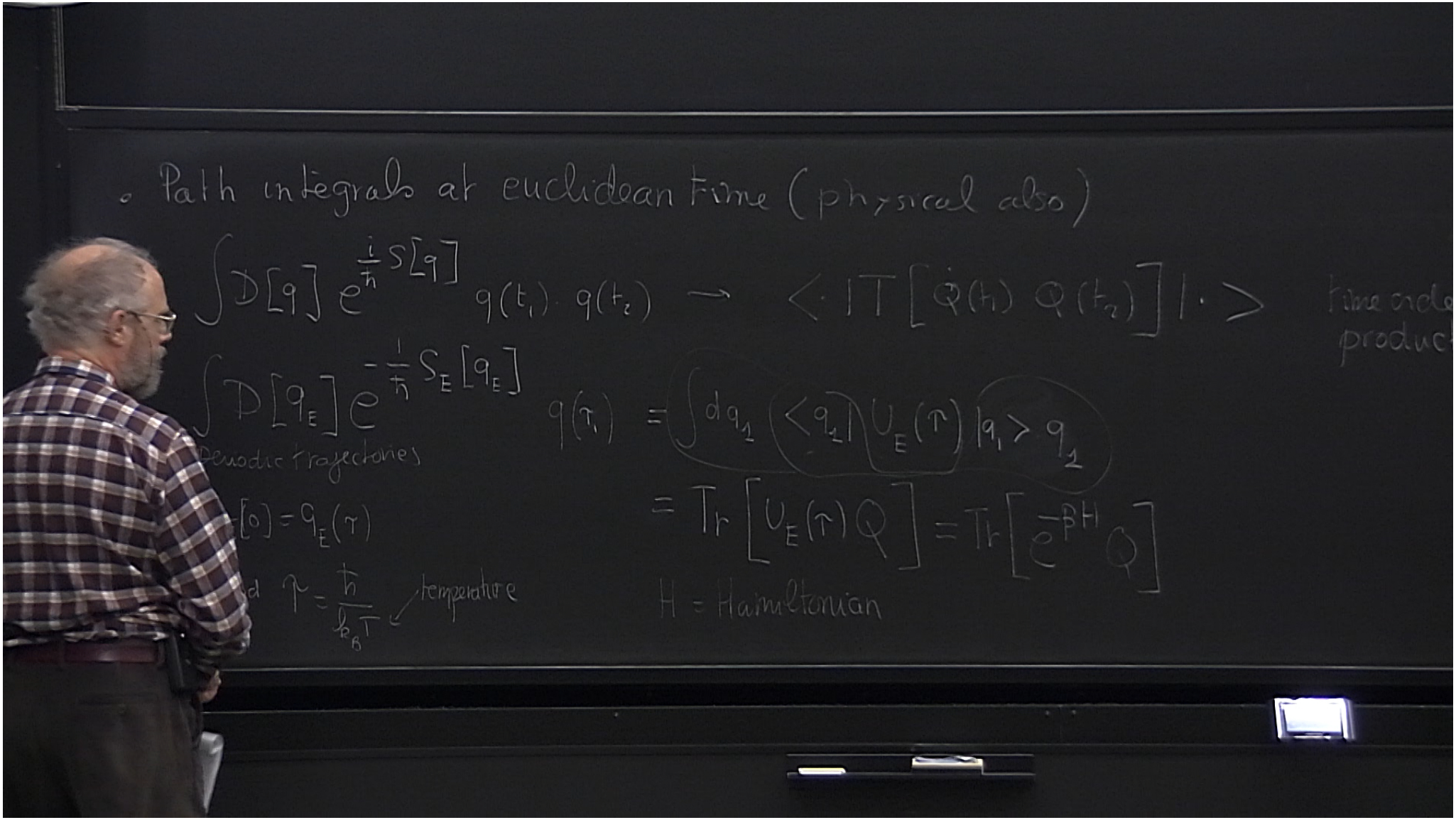

• Path integrals at euclidean time (physical also)
$\int \mathcal{D}[q]\, e^{\frac{i}{\hbar} S[q]}\, q(t_1)\, q(t_2) \rightarrow \langle 0 | T[\hat{q}(t_1)\, \hat{q}(t_2)] | 0 \rangle$
time ordered products
$\int \mathcal{D}[q_E]\, e^{-\frac{1}{\hbar} S_E[q_E]}\, q(\tau_1) =$
$q_E(\tau) = q_E(\tau)$
period $\gamma = \frac{\hbar}{k_B T}$ temperature
q
q_2
τ_2
γ
Euclidean time τ

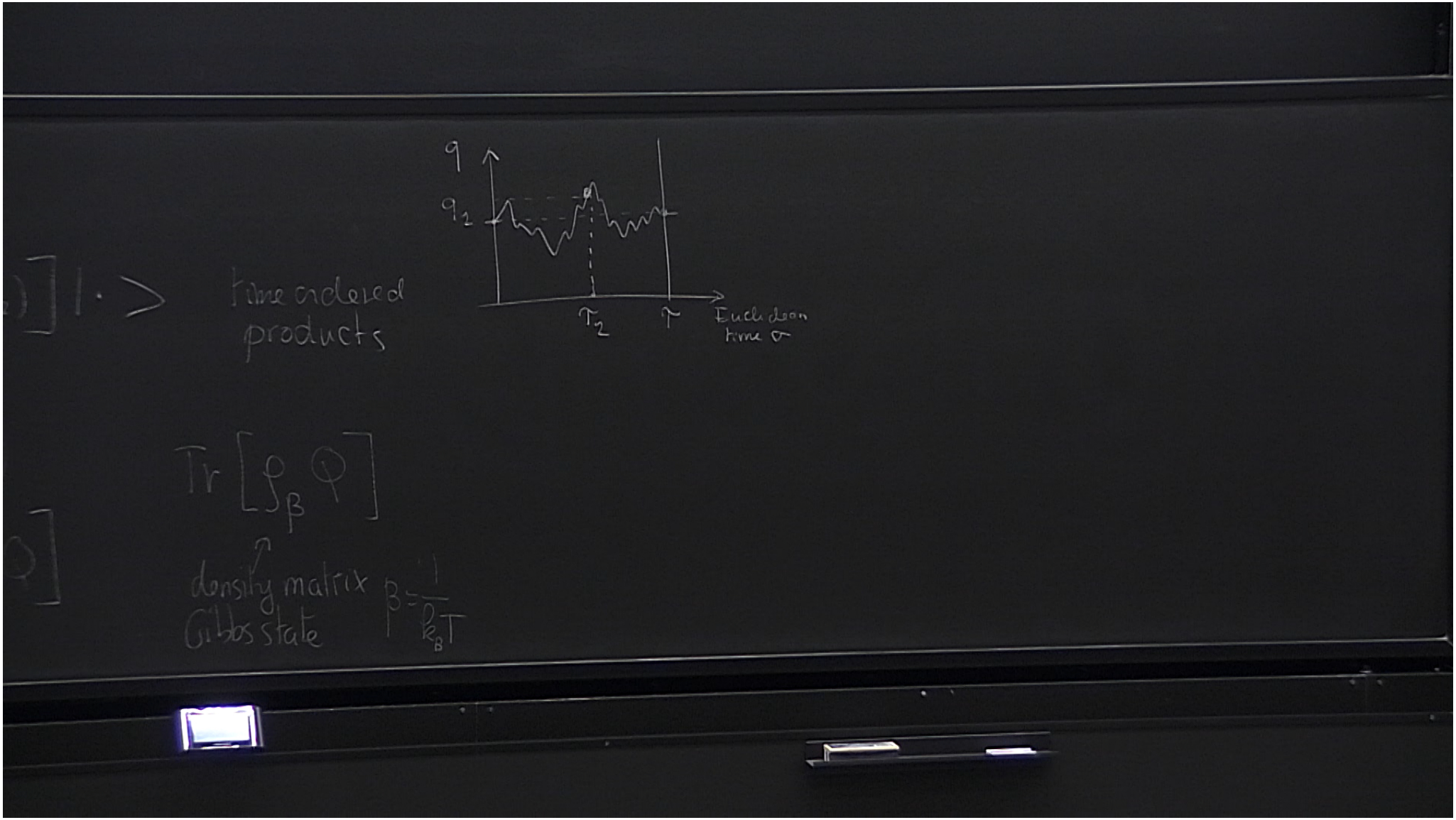

Path integrals at euclidean time (physical also)
$\int \mathcal{D}[q]\, e^{\frac{i}{\hbar} S[q]}\, q(t_1)\, q(t_2) \rightarrow \langle |T[\hat{Q}(t_1)\, Q(t_2)]|$
$\int \mathcal{D}[q_E]\, e^{-\frac{1}{\hbar} S_E[q_E]}$
$q(T_1) = \int dq_1$
periodic trajectories
$q_E[0] = q_E(T)$
period $T = \frac{\hbar}{k_B T}$
temperature

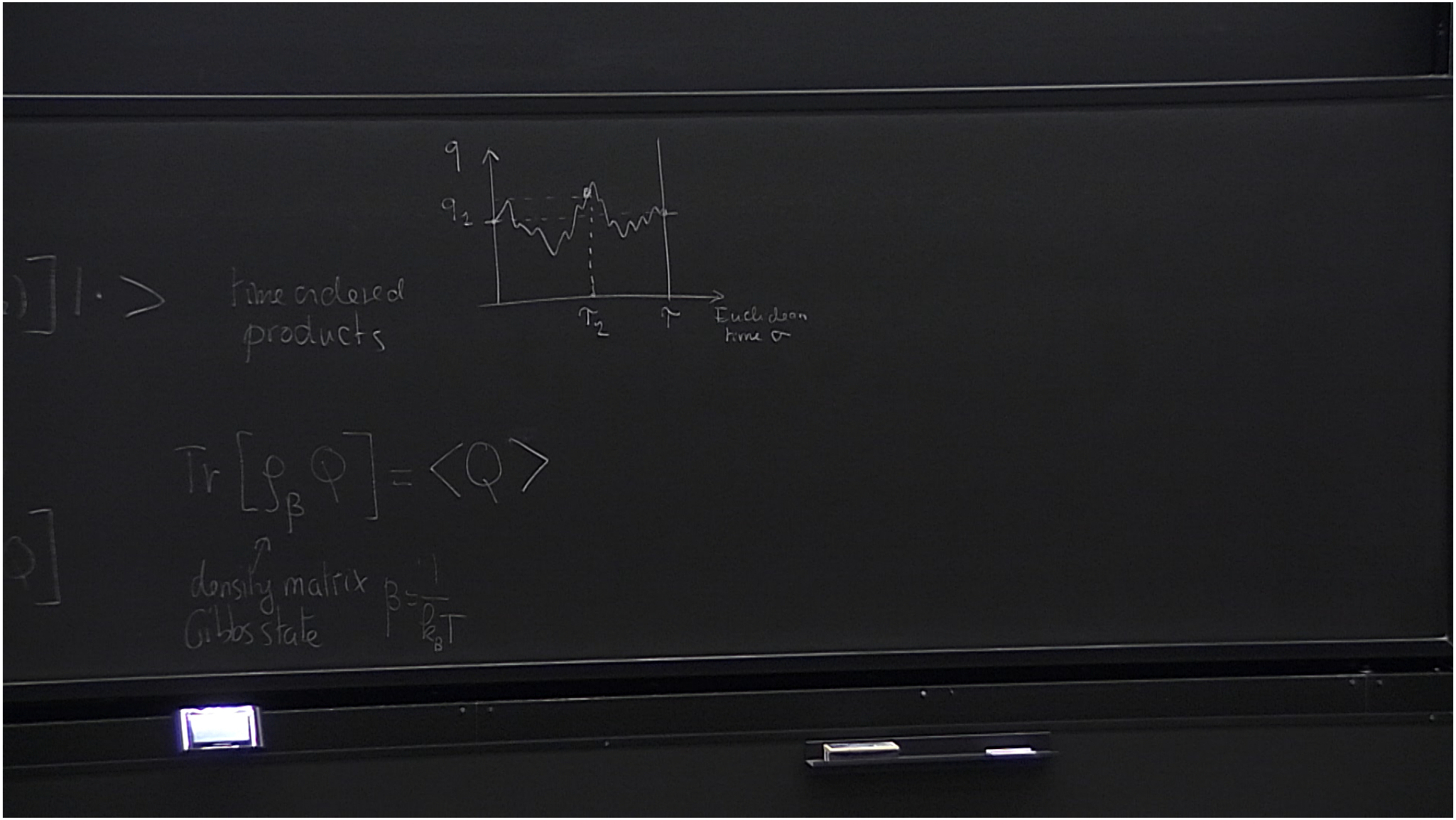

Path integrals at euclidean time (physical also)
$\int \mathcal{D}[q]\, e^{\frac{i}{\hbar} S[q]}\; q(t_1)\cdot q(t_2) \rightarrow \langle \cdot | T\left[\hat{Q}(t_1)\, Q(t_2)\right] | \cdot \rangle$
time ordered product
$\int \mathcal{D}[q_E]\, e^{-\frac{1}{\hbar} S_E[q_E]}$
$q(\tau_1) = \int dq_1 \langle q_2 |$
$| q_1 \rangle$
periodic trajectories
$q_E[0] = q_E(\tau)$
$\beta = \frac{\hbar}{k_B T}$
temperature

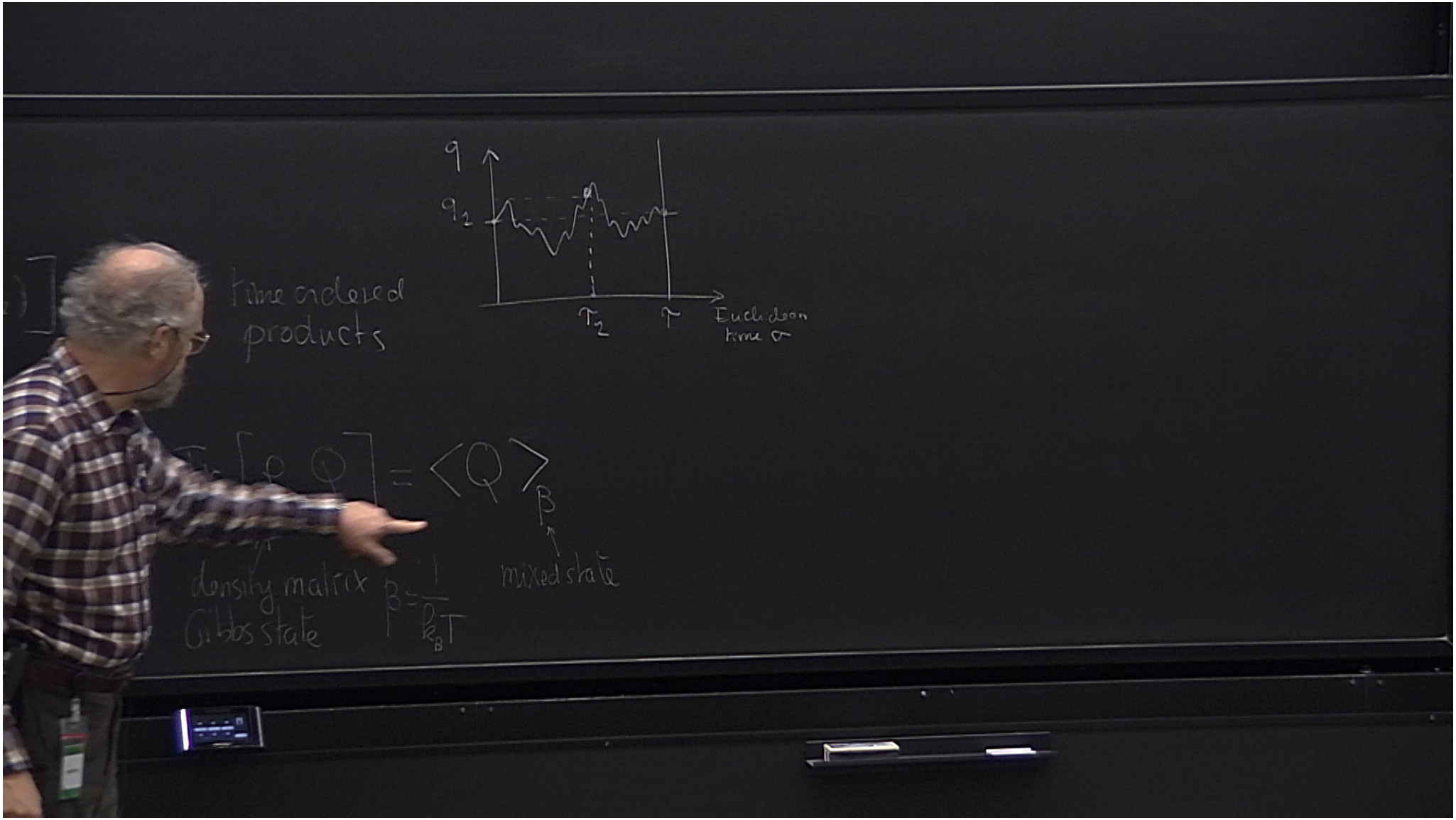

Path integrals at euclidean time (physical also)
$e^{\frac{i}{\hbar}S[q]}$
$q(t_1)\cdot q(t_2) \rightarrow \langle \cdot | T\left[\hat{Q}(t_1)\,\hat{Q}(t_2)\right] | \cdot \rangle$
time ordered product
$e^{-\frac{1}{\hbar}S_E[q_E]}$
$q(\tau_1) = \int dq_1 \langle q_1 | U_E(\tau) | q_1 \rangle$
trajectories
$= \frac{\hbar}{k_B T}$
temperature

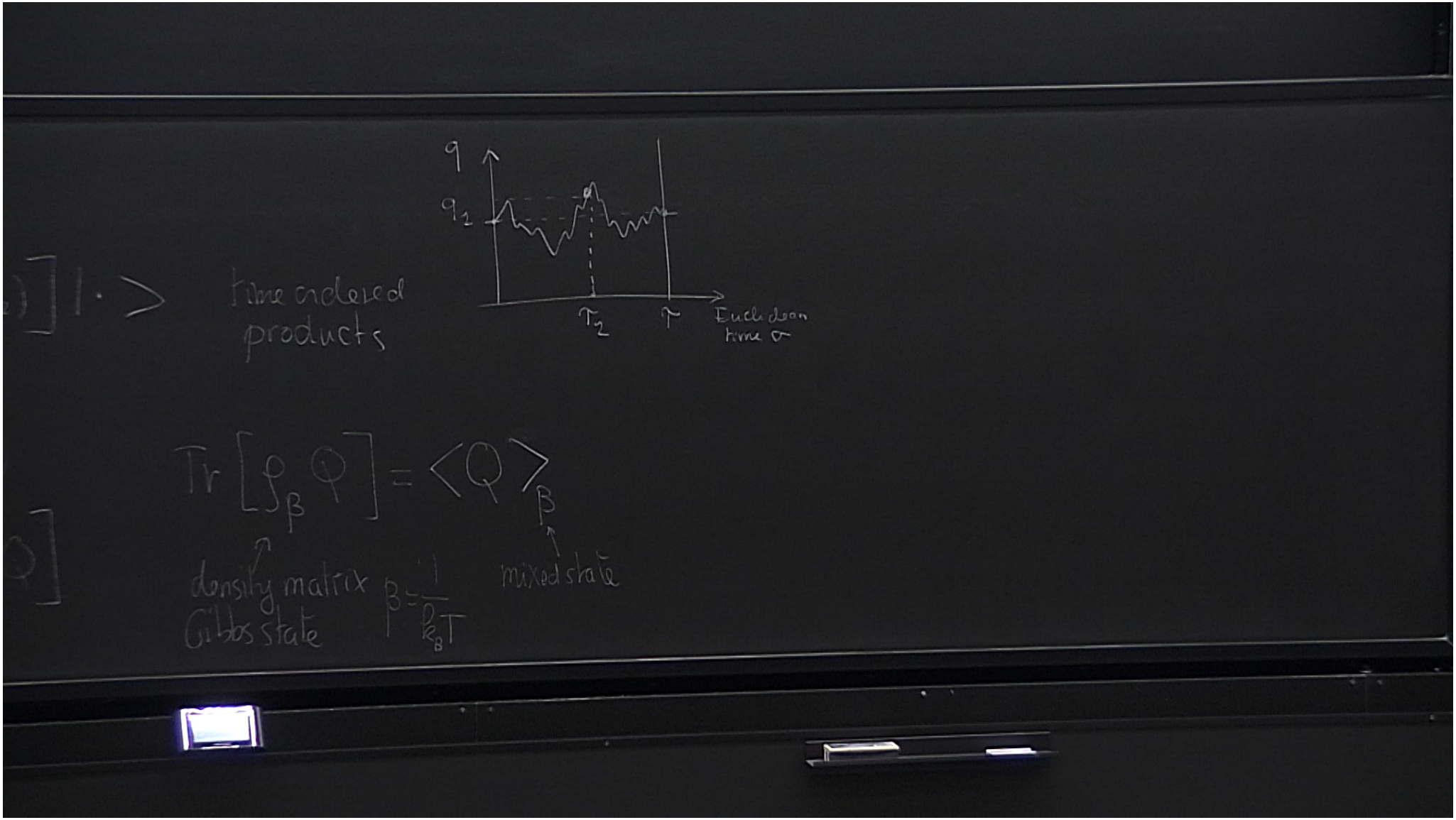

Path integrals at euclidean time (physical also)
$\int \mathcal{D}[q]\, e^{\frac{i}{\hbar} S[q]}\, q(t_1)\, q(t_2) \rightarrow \langle \cdot | T\left[\hat{Q}(t_1)\, \hat{Q}(t_2)\right] | \cdot \rangle$
time ordered product
$\int \mathcal{D}[q]\, e^{-\frac{1}{\hbar} S_E[q_E]}$
trajectories
$q(\tau_1) = \int dq_1 \langle q_1 | U_E(\tau) | q_1 \rangle q_1$
$= Tr\left[U_E(\tau)\, Q\right]$
temperature

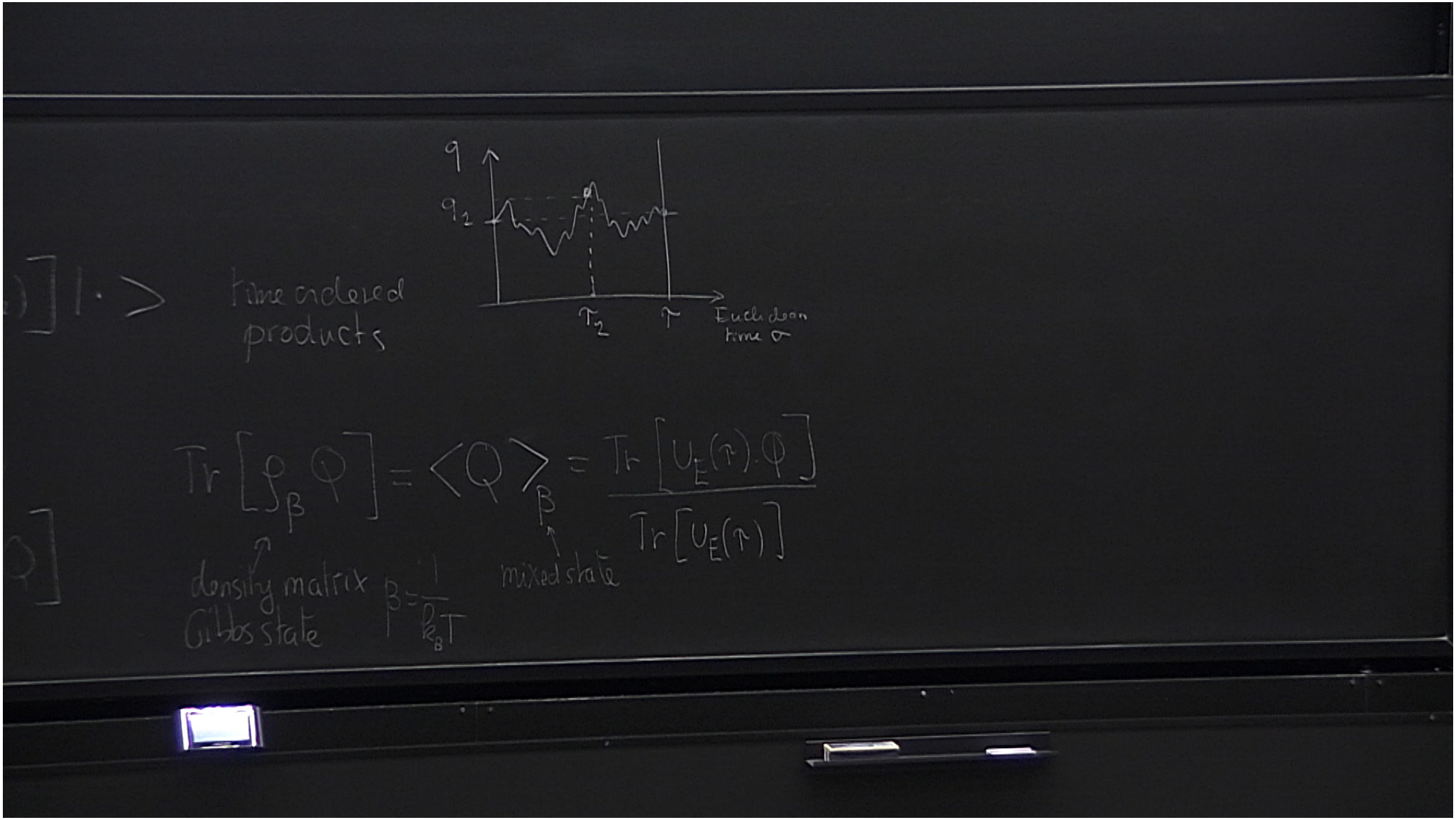

Path integrals at euclidean time (physical also)
$\int D[q]\, e^{\frac{i}{\hbar}S[q]}\, q(t_1)\cdot q(t_2) \rightarrow \langle \cdot | T\left[\hat{Q}(t_1)\,\hat{Q}(t_2)\right] | \cdot \rangle$
time ordered product
$\int D[q_E]\, e^{-\frac{1}{\hbar}S_E[q_E]}$
periodic trajectories
$q(\tau_1) = \int dq_1 \langle q_1 | U_E(\tau) | q_1 \rangle q_1$
$= Tr\left[U_E(\tau)\,Q\right] = Tr\left[e^{-\beta H}\,Q\right]$
$[0] = q_E(\tau)$
$\tau = \frac{\hbar}{k_B T}$
temperature
H = Hamiltonian

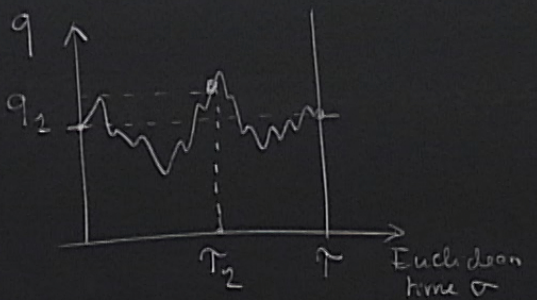

time ordered products
q
q_2
τ_2
τ
Euclidean time σ
$\mathrm{Tr}[\rho_\beta \varphi]$
density matrix
Gibbs state
$\beta = \frac{1}{k_B T}$


q
q_2
τ_2
τ
Euclidean time σ
time ordered products
$\mathrm{Tr}[\rho_\beta Q] = \langle Q \rangle$
density matrix
Gibbs state
$\beta = \frac{1}{k_B T}$


q
q_2
τ_2
τ
Euclidean time σ
time ordered products
ρ
density matrix
Gibbs state
$\beta = \frac{1}{k_B T}$
$\langle Q \rangle_\beta$
mixed state


time ordered products
q
q_2
τ_2
τ
Euclidean time σ
$\mathrm{Tr}[\rho_\beta \mathcal{O}] = \langle \mathcal{O} \rangle_\beta$
density matrix
Gibbs state
$\beta = \frac{1}{k_B T}$
mixed state


time ordered products
q
q_2
τ_2
τ
Euclidean time σ
$\mathrm{Tr}\left[\rho_\beta \mathcal{O}\right] = \langle \mathcal{O} \rangle_\beta = \frac{\mathrm{Tr}\left[U_E(\tau)\cdot\mathcal{O}\right]}{\mathrm{Tr}\left[U_E(\tau)\right]}$
density matrix
Gibbs state
$\beta = \frac{1}{k_B T}$
mixed state



time ordered products

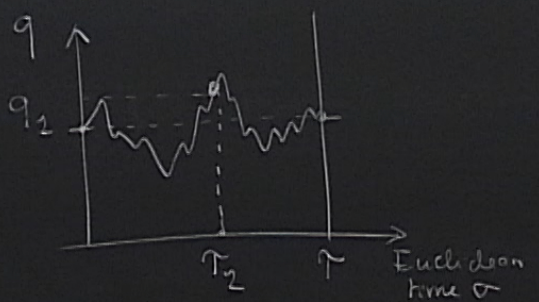
$$\mathrm{Tr}\left[\rho_\beta Q\right] = \langle Q \rangle_\beta = \frac{\mathrm{Tr}\left[U_E(\tau)\cdot Q\right]}{\mathrm{Tr}\left[U_E(\tau)\right]} = \frac{\int \mathcal{D}[q]\, e^{-\frac{1}{\hbar}S_E[q]}\, q(\tau_1)}{\int \mathcal{D}[q]\, e^{-\frac{1}{\hbar}S_E[q]}}$$

density matrix Gibbs state $\beta = \frac{1}{k_B T}$

mixed state

periodic trajectories $\tau = \frac{i}{\hbar}\beta$

time ordered products



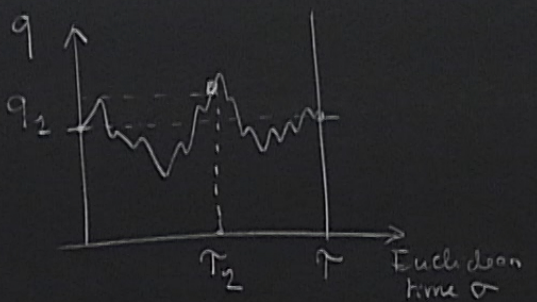
$$\mathrm{Tr}\left[\rho_\beta Q\right] = \langle Q \rangle_\beta = \frac{\mathrm{Tr}\left[U_E(\tau)\cdot Q\right]}{\mathrm{Tr}\left[U_E(\tau)\right]} = \frac{\int \mathcal{D}[q_E]\, e^{-\frac{1}{\hbar}S_E[q_E]}\, q(\tau_1)}{\int \mathcal{D}[q_E]\, e^{-\frac{1}{\hbar}S_E[q_E]}}$$

density matrix
Gibbs state

$\beta = \frac{1}{k_B T}$

mixed state

periodic trajectories $\tau = \frac{i}{\hbar}\beta$



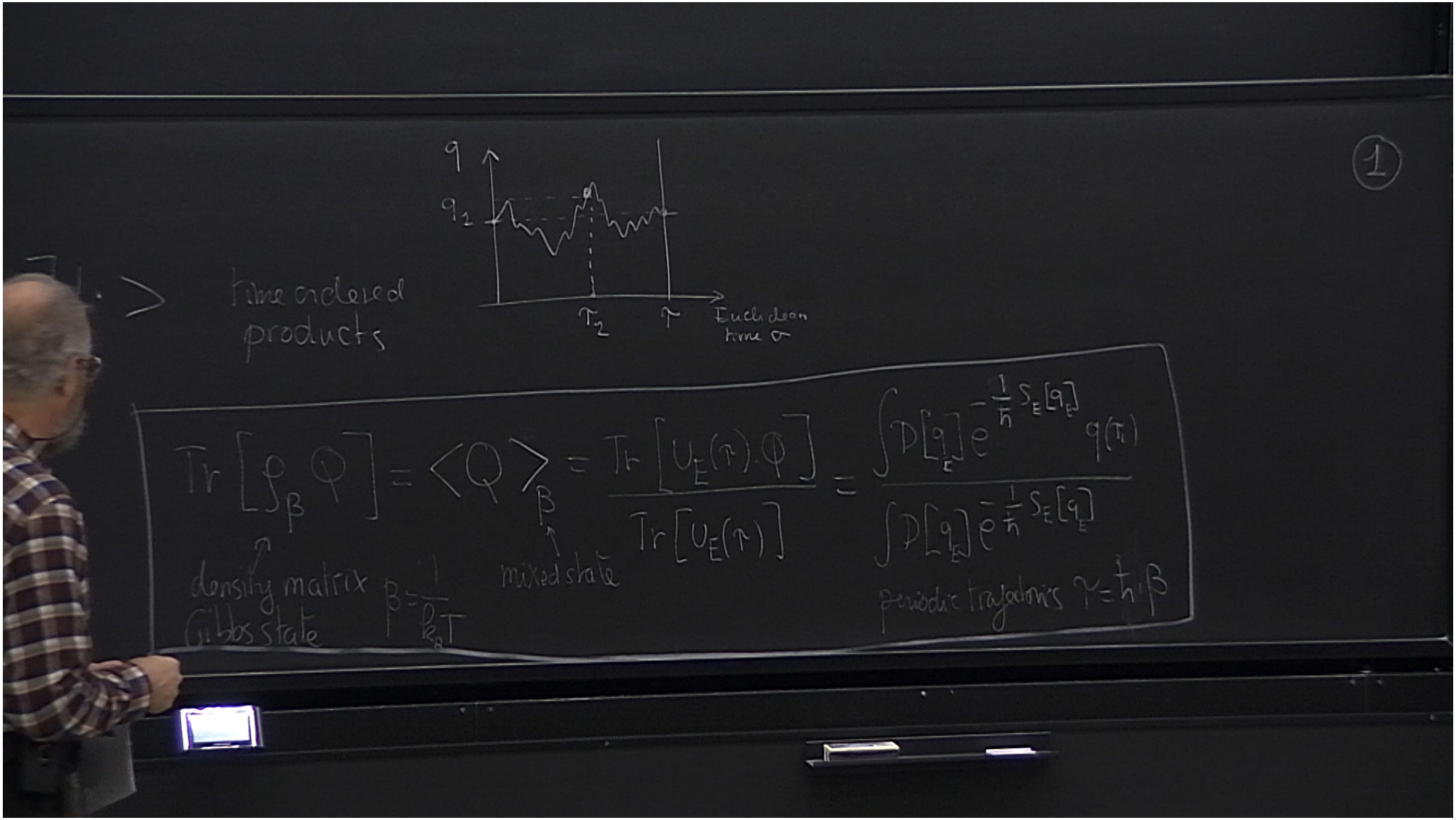
time ordered products

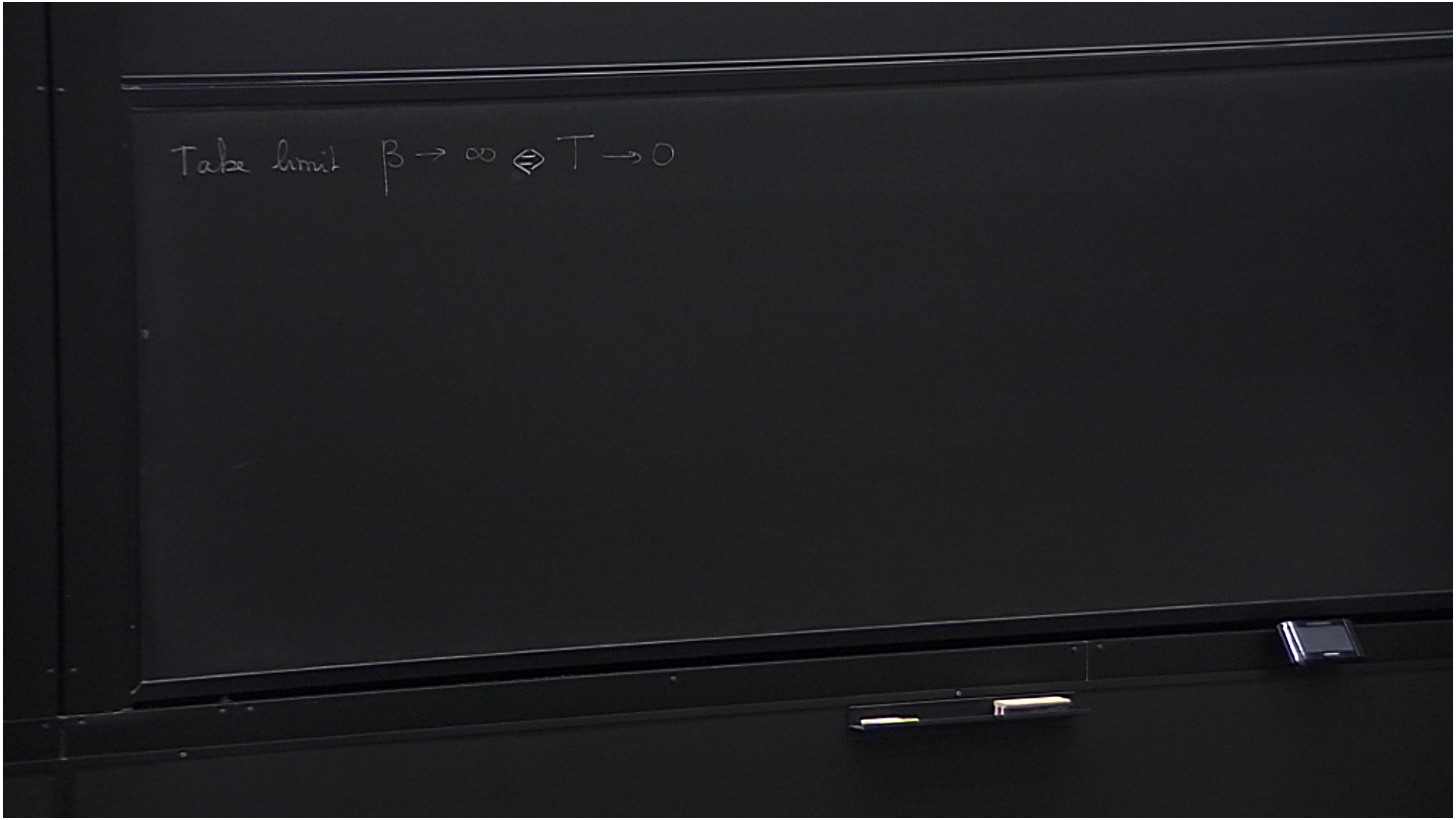
$$\mathrm{Tr}\left[\rho_\beta Q\right] = \langle Q \rangle_\beta = \frac{\mathrm{Tr}\left[U_E(\tau)\, Q\right]}{\mathrm{Tr}\left[U_E(\tau)\right]} = \frac{\int \mathcal{D}[q_E]\, e^{-\frac{1}{\hbar} S_E[q_E]}\, q(\tau_1)}{\int \mathcal{D}[q_E]\, e^{-\frac{1}{\hbar} S_E[q_E]}}$$

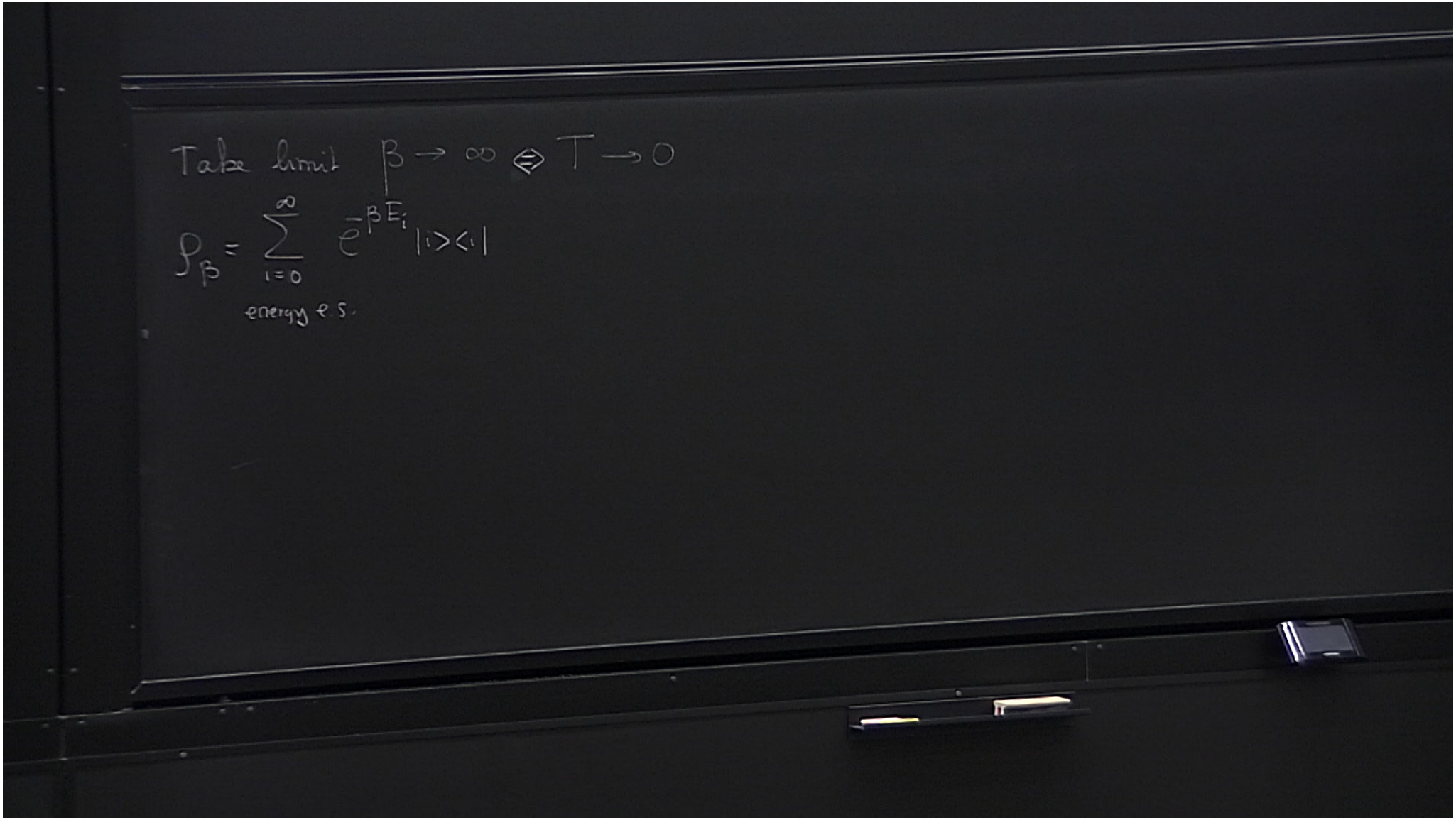
density matrix
Gibbs state $\beta = \frac{1}{k_B T}$

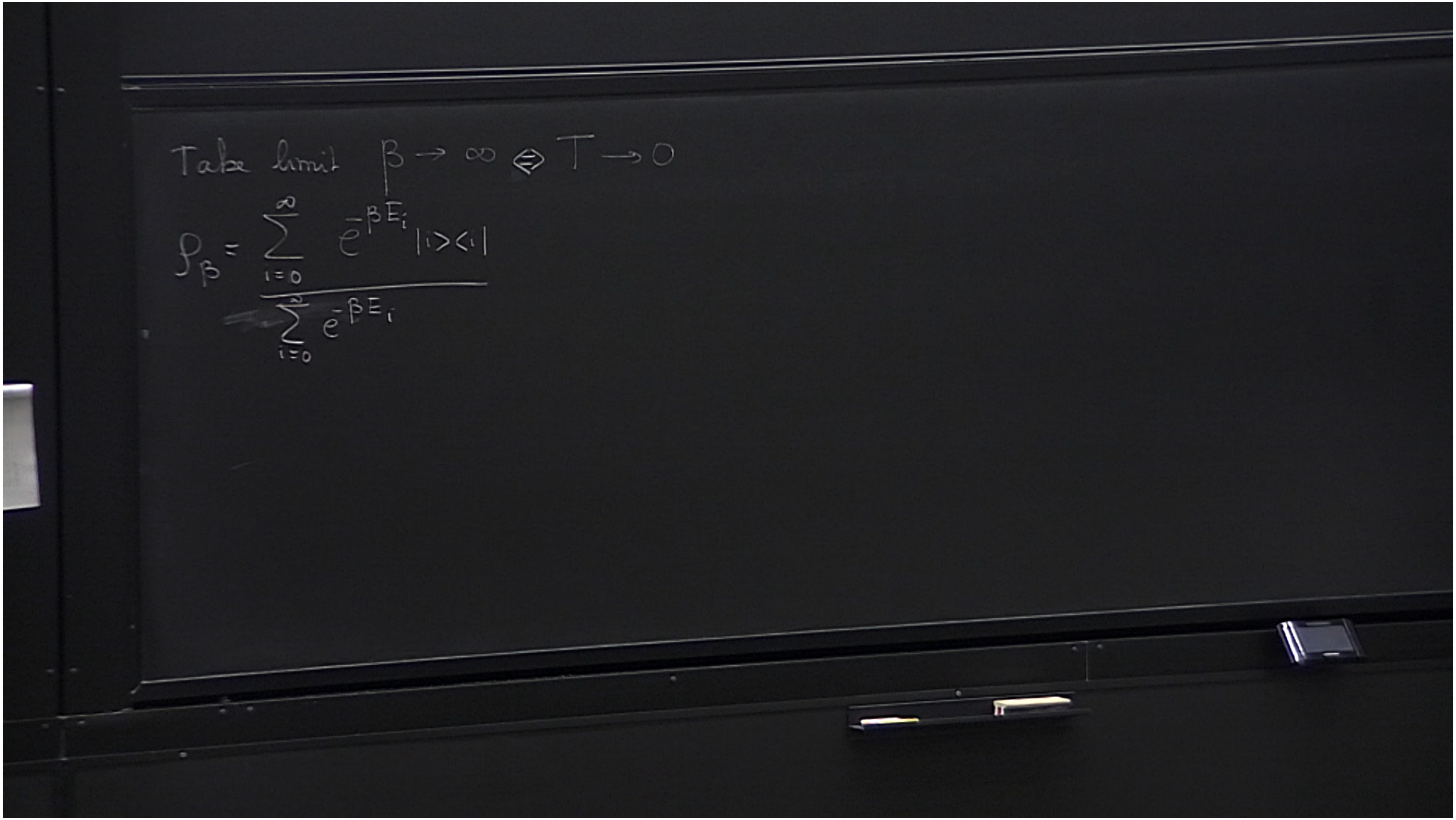
mixed state

periodic trajectories $\tau = \frac{i}{\hbar}\beta$


q
q_2
τ_2
τ
Euclidean time σ
time ordered products
$\mathrm{Tr}\left[\rho_\beta Q\right] = \langle Q \rangle_\beta = \frac{\mathrm{Tr}\left[U_E(\tau)\cdot Q\right]}{\mathrm{Tr}\left[U_E(\tau)\right]} = \frac{\int \mathcal{D}[q_E]\, e^{-\frac{1}{\hbar}S_E[q_E]}\, q(\tau_1)}{\int \mathcal{D}[q_E]\, e^{-\frac{1}{\hbar}S_E[q_E]}}$
density matrix
Gibbs state
$\beta = \frac{1}{k_B T}$
mixed state
periodic trajectories $\tau = \frac{i}{\hbar}\beta$
1


Take limit $\beta \to \infty \Leftrightarrow T \to 0$

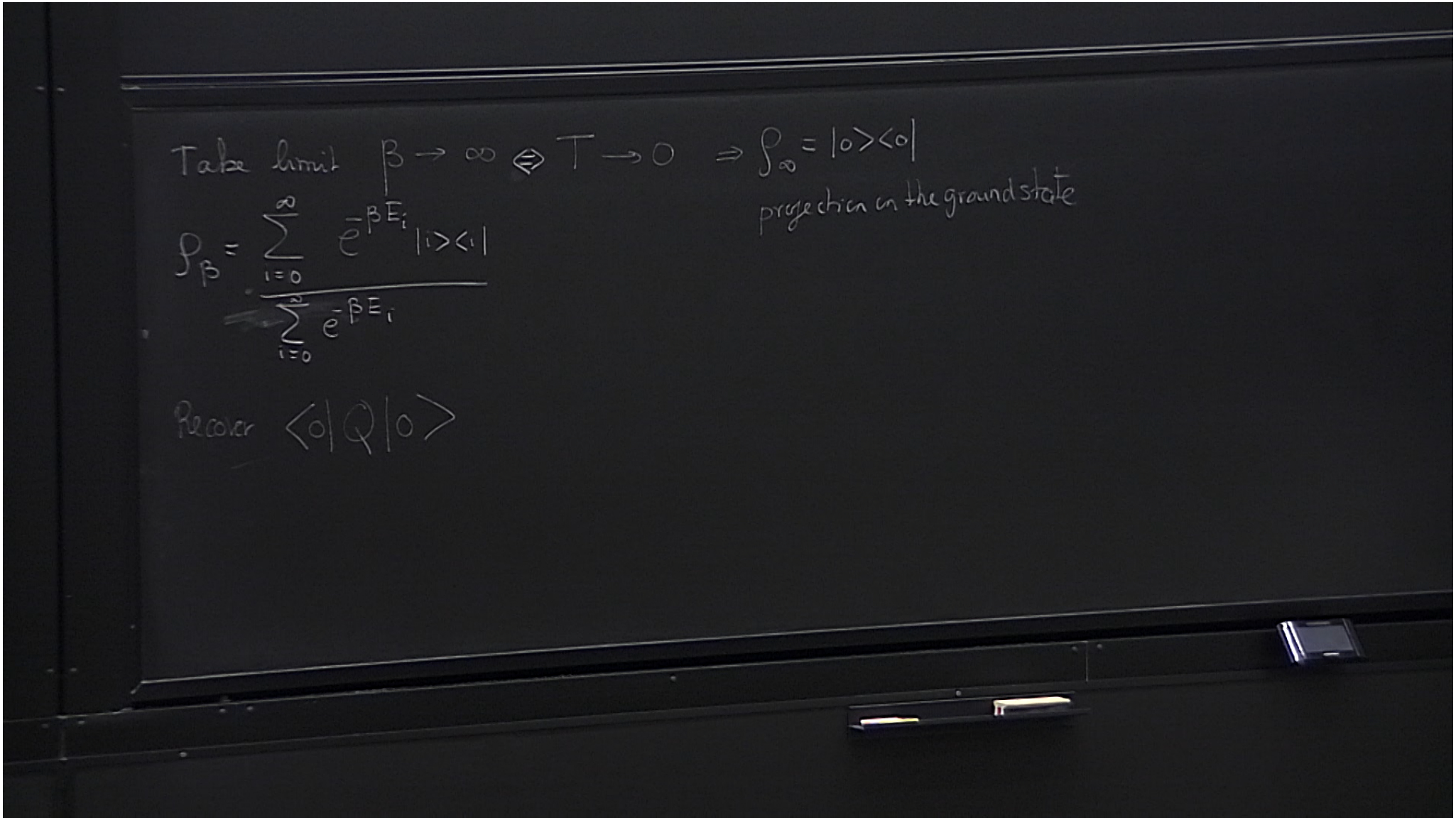

Take limit $\beta \to \infty \Leftrightarrow T \to 0$
$\rho_\beta = \sum_{i=0}^{\infty} e^{-\beta E_i} |i\rangle\langle i|$
energy e.s.


Take limit $\beta \to \infty \Leftrightarrow T \to 0$
$\rho_\beta = \frac{\sum_{i=0}^{\infty} e^{-\beta E_i} |i\rangle\langle i|}{\sum_{i=0}^{\infty} e^{-\beta E_i}}$

Take limit $\beta \to \infty \Leftrightarrow T \to 0 \Rightarrow \rho_\infty = |0\rangle\langle 0|$

projection on the ground state

$$\rho_\beta = \frac{\sum_{i=0}^{\infty} e^{-\beta E_i} |i\rangle\langle i|}{\sum_{i=0}^{\infty} e^{-\beta E_i}}$$


Take limit $\beta \to \infty \Leftrightarrow T \to 0 \Rightarrow \rho_\infty = |0\rangle\langle 0|$
projection on the ground state
$\rho_\beta = \frac{\sum_{i=0}^{\infty} e^{-\beta E_i} |i\rangle\langle i|}{\sum_{i=0}^{\infty} e^{-\beta E_i}}$
Recover $\langle 0|Q|0\rangle$

Take limit $\beta \to \infty \Leftrightarrow T \to 0 \Rightarrow \rho_\infty = |0\rangle\langle 0|$

projection on the ground state

$$\rho_\beta = \frac{\sum_{i=0}^{\infty} e^{-\beta E_i} |i\rangle\langle i|}{\sum_{i=0}^{\infty} e^{-\beta E_i}}$$

Recover $\langle 0|Q|0\rangle$ = limit

proud $\tau \to \infty$

Take limit $\beta \to \infty \Leftrightarrow T \to 0 \Rightarrow \rho_\infty = |0\rangle\langle 0|$

projection on the ground state

$$\rho_\beta = \frac{\sum_{i=0}^{\infty} e^{-\beta E_i} |i\rangle\langle i|}{\sum_{i=0}^{\infty} e^{-\beta E_i}}$$

Recover $\langle 0|Q|0\rangle = \lim_{\text{Wick } \tau \to \infty} \frac{\int \mathcal{D}[q] e^{-\frac{1}{\hbar} S_E} q}{\int \mathcal{D}[q] e^{-\frac{1}{\hbar} S_E}}$

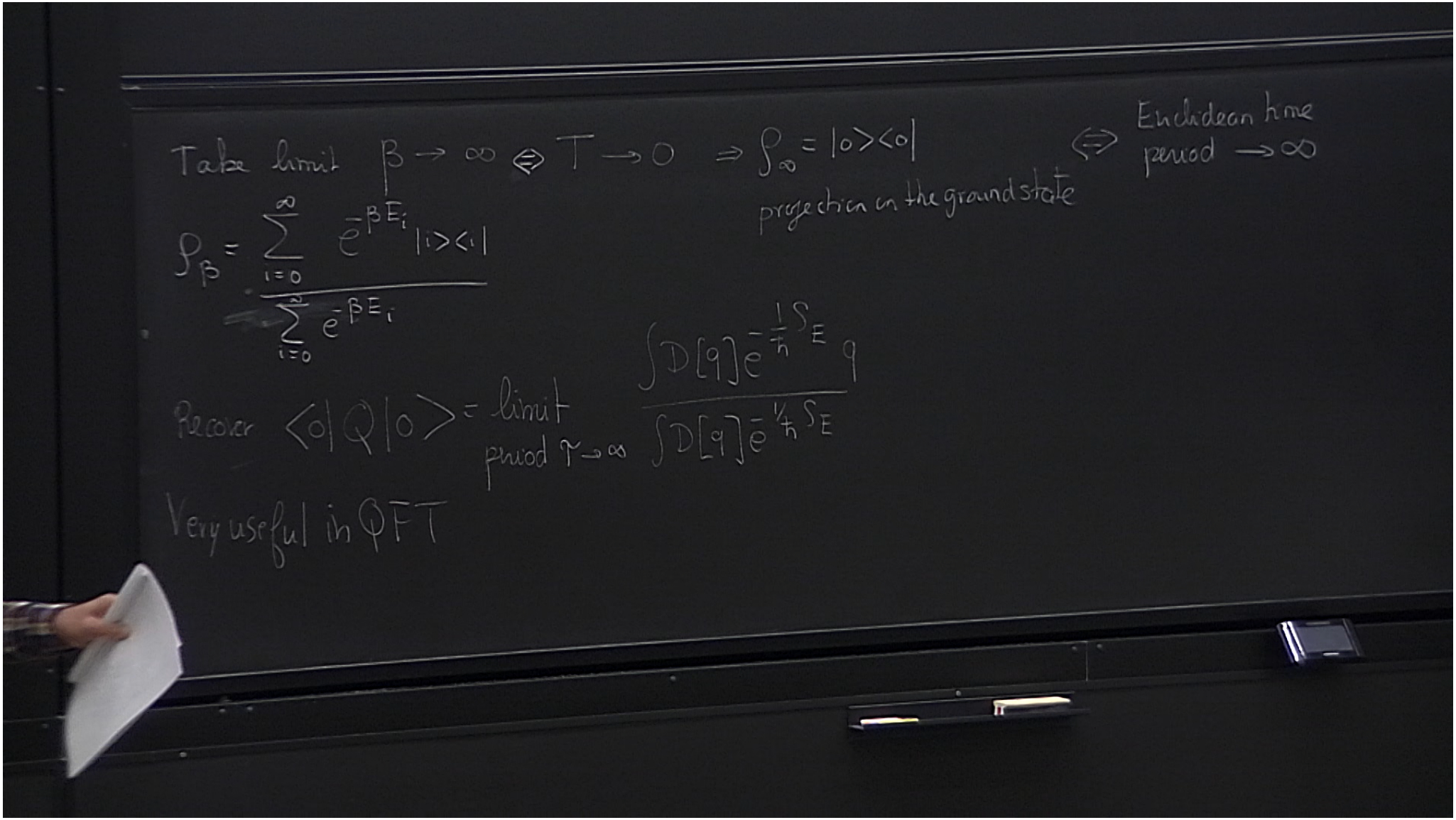
Take limit $\beta \to \infty \Leftrightarrow T \to 0 \Rightarrow \rho_\infty = |0\rangle\langle 0|$

projection on the ground state

$$\rho_\beta = \frac{\sum_{i=0}^{\infty} e^{-\beta E_i} |i\rangle\langle i|}{\sum_{i=0}^{\infty} e^{-\beta E_i}}$$

Recover $\langle 0|Q|0\rangle = \lim_{\text{fmod } \tau \to \infty} \frac{\int \mathcal{D}[q] e^{-\frac{1}{\hbar} S_E} q}{\int \mathcal{D}[q] e^{-\frac{1}{\hbar} S_E}}$

Very useful in QFT

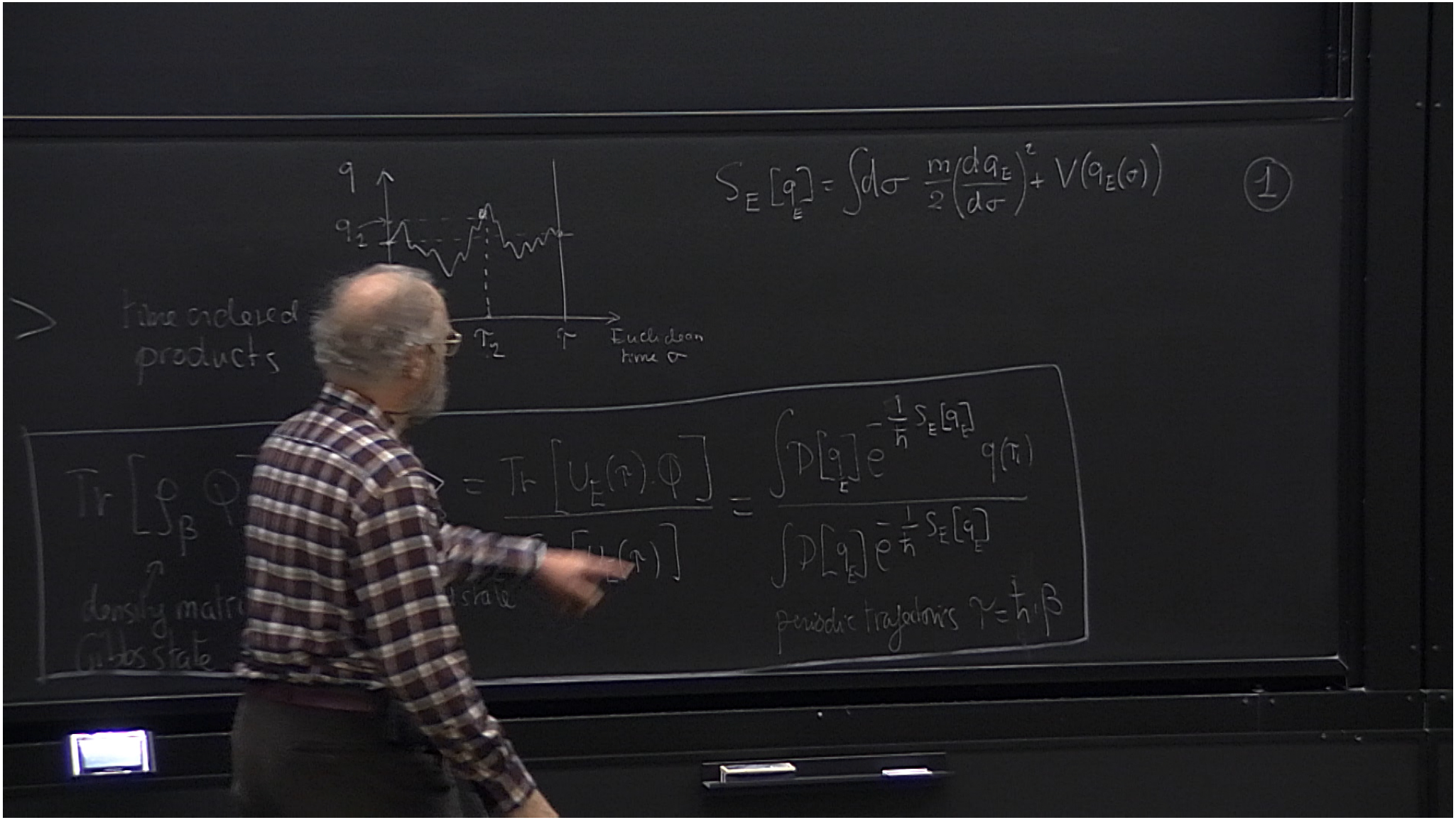

Take limit $\beta \to \infty \Leftrightarrow T \to 0 \Rightarrow \rho_\infty = |0\rangle\langle 0|$
projection on the ground state
$\Leftrightarrow$ Euclidean time period $\to \infty$
$\rho_\beta = \frac{\sum_{i=0}^{\infty} e^{-\beta E_i} |i\rangle\langle i|}{\sum_{i=0}^{\infty} e^{-\beta E_i}}$
Recover $\langle 0|Q|0\rangle$ = limit period $\tau \to \infty$ $\frac{\int D[q] e^{-\frac{1}{\hbar} S_E} q}{\int D[q] e^{-\frac{1}{\hbar} S_E}}$
Very useful in QFT

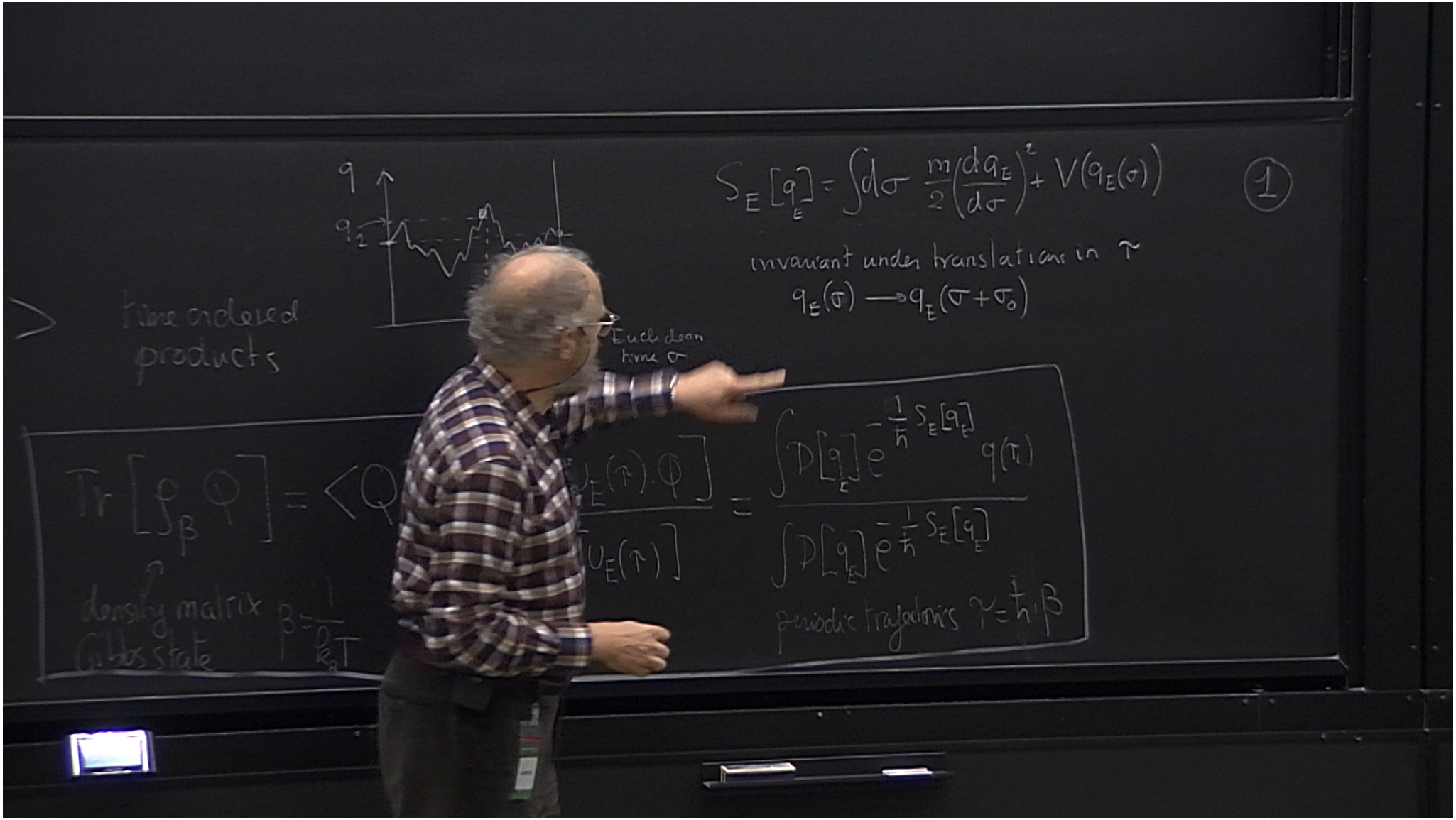
Take limit $\beta \to \infty \Leftrightarrow T \to 0 \Rightarrow \rho_\infty = |0\rangle\langle 0|$ projection on the ground state $\Leftrightarrow$ Euclidean time period $\to \infty$, $\tau \in \mathbb{R}$

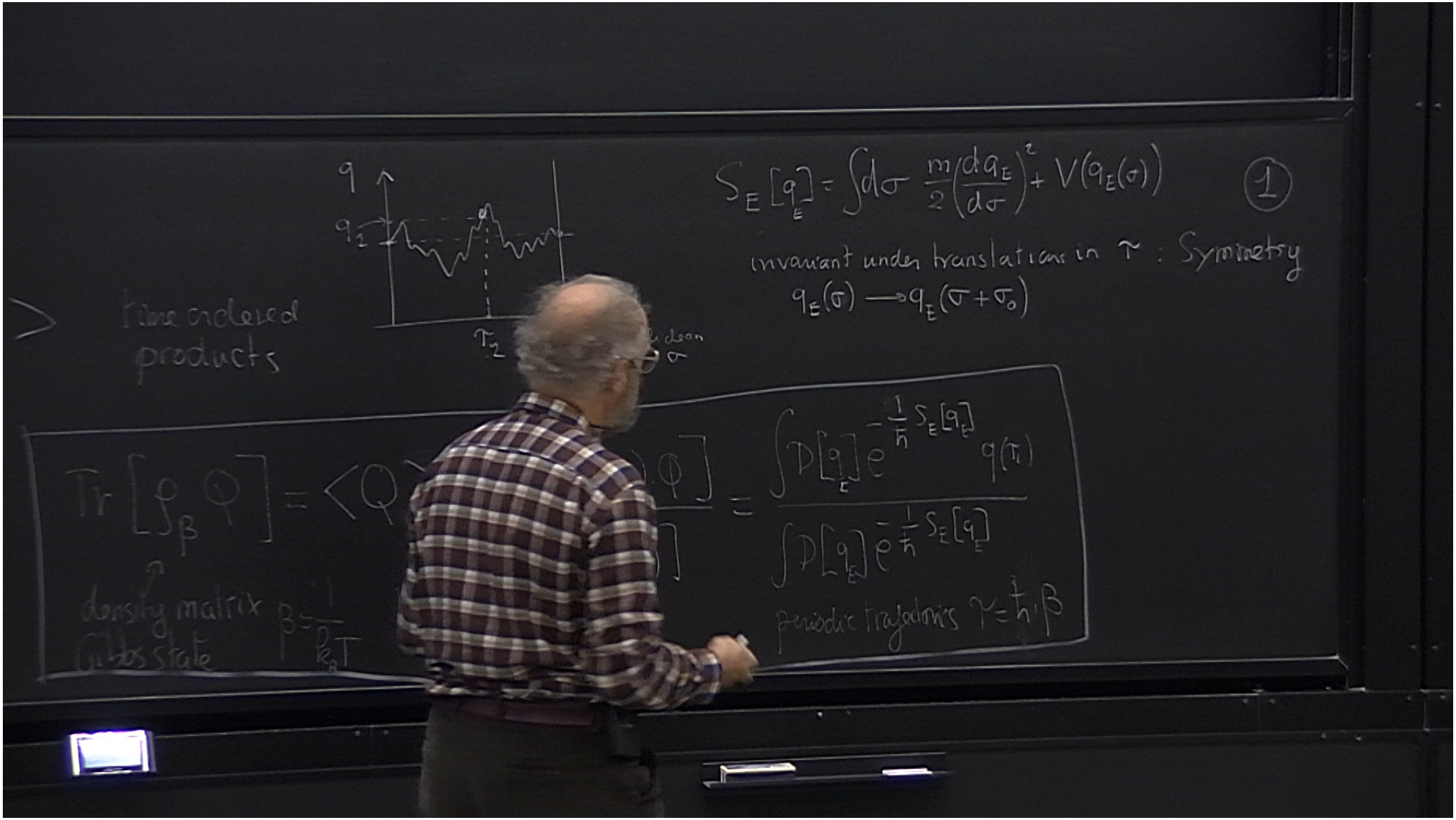
$$\rho_\beta = \frac{\sum_{i=0}^{\infty} e^{-\beta E_i} |i\rangle\langle i|}{\sum_{i=0}^{\infty} e^{-\beta E_i}}$$

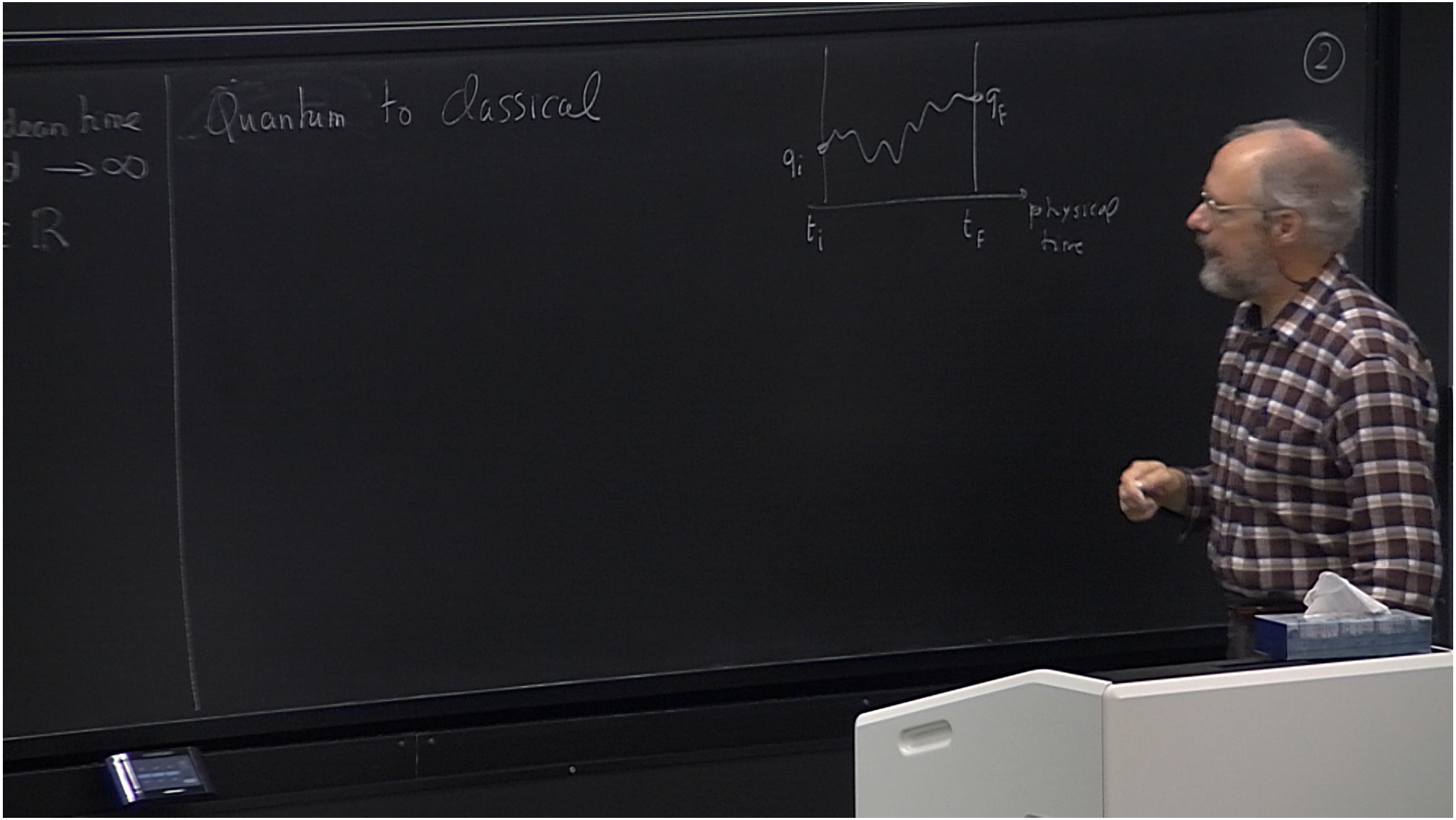
Recover $\langle 0|Q|0\rangle$ = limit period $\tau \to \infty$ $\dfrac{\int \mathcal{D}[q] e^{-\frac{1}{\hbar} S_E} q}{\int \mathcal{D}[q] e^{-\frac{1}{\hbar} S_E}}$

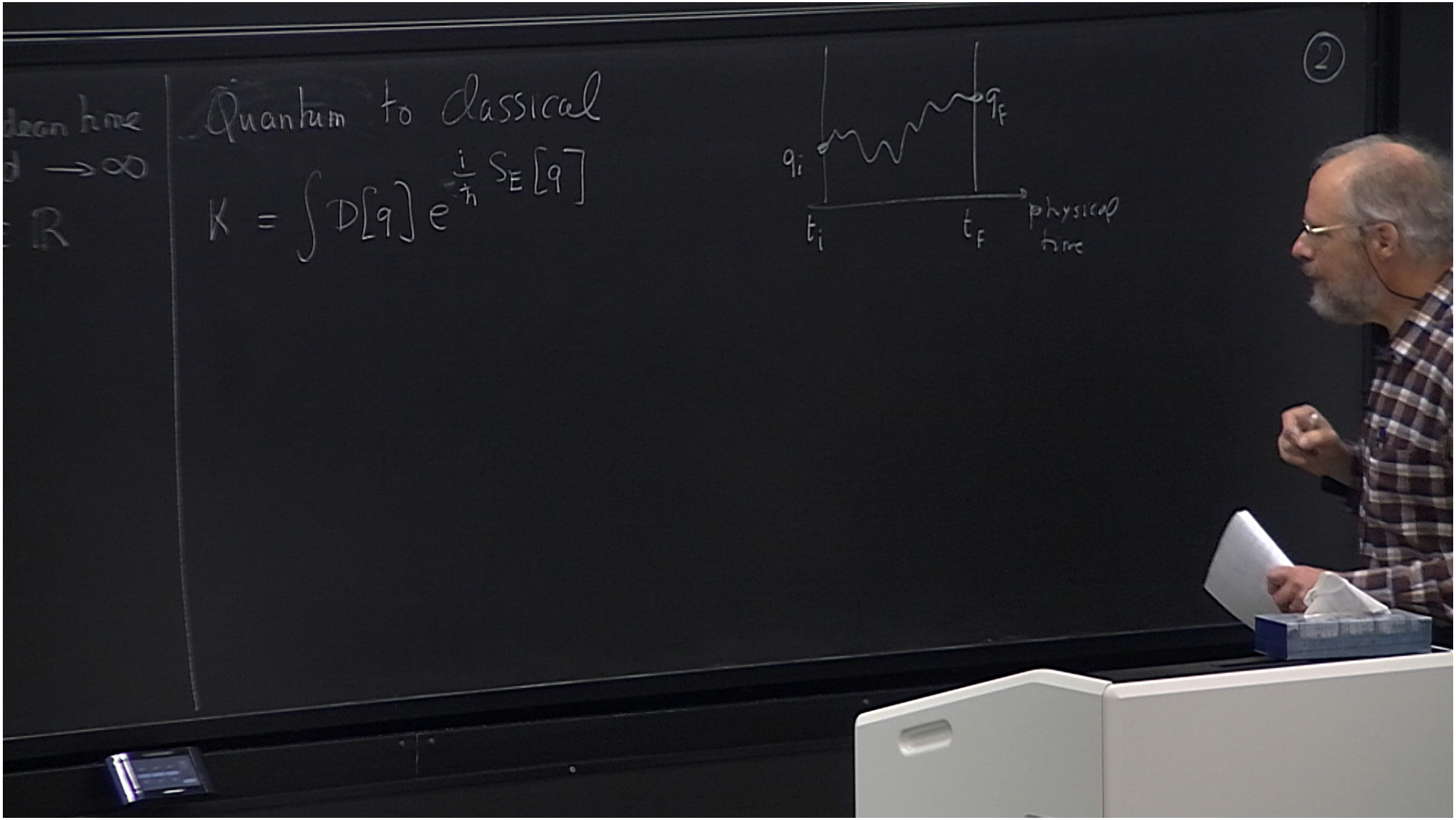
Very useful in QFT

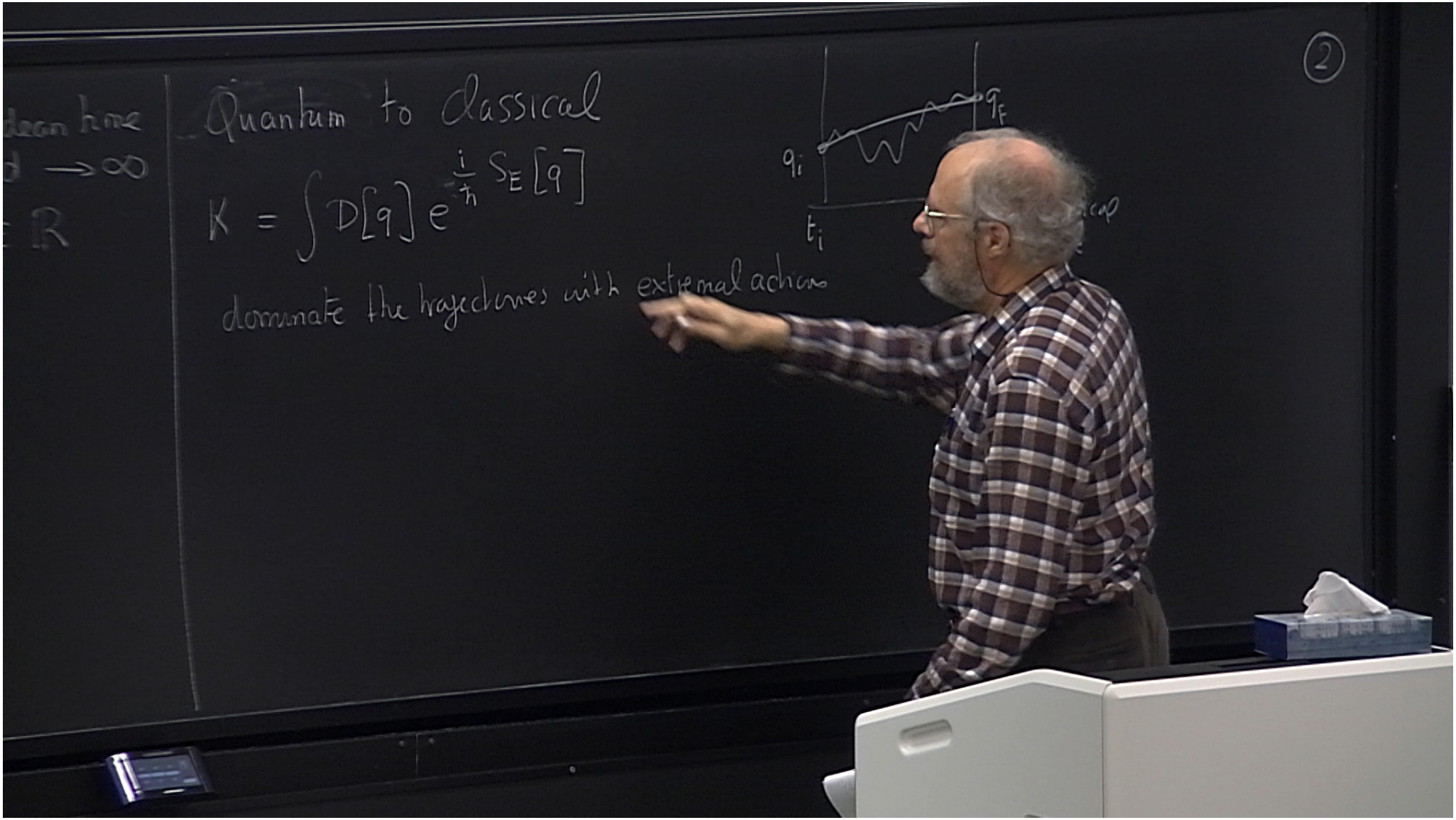

$S_E[q_E] = \int d\sigma \; \frac{m}{2}\left(\frac{dq_E}{d\sigma}\right)^2 + V(q_E(\sigma))$ (1)
q
q_2
τ_2
τ
Euclidean time σ
time ordered products
$\mathrm{Tr}\left[\rho_\beta \Phi\right] = \mathrm{Tr}\left[U_E(\tau)\,\Phi\right] = \frac{\int \mathcal{D}[q_E]\, e^{-\frac{1}{\hbar} S_E[q_E]}\, q(\tau_1)}{\int \mathcal{D}[q_E]\, e^{-\frac{1}{\hbar} S_E[q_E]}}$
density matrix
Gibbs state
periodic trajectories $\tau = \frac{1}{\hbar}\beta$

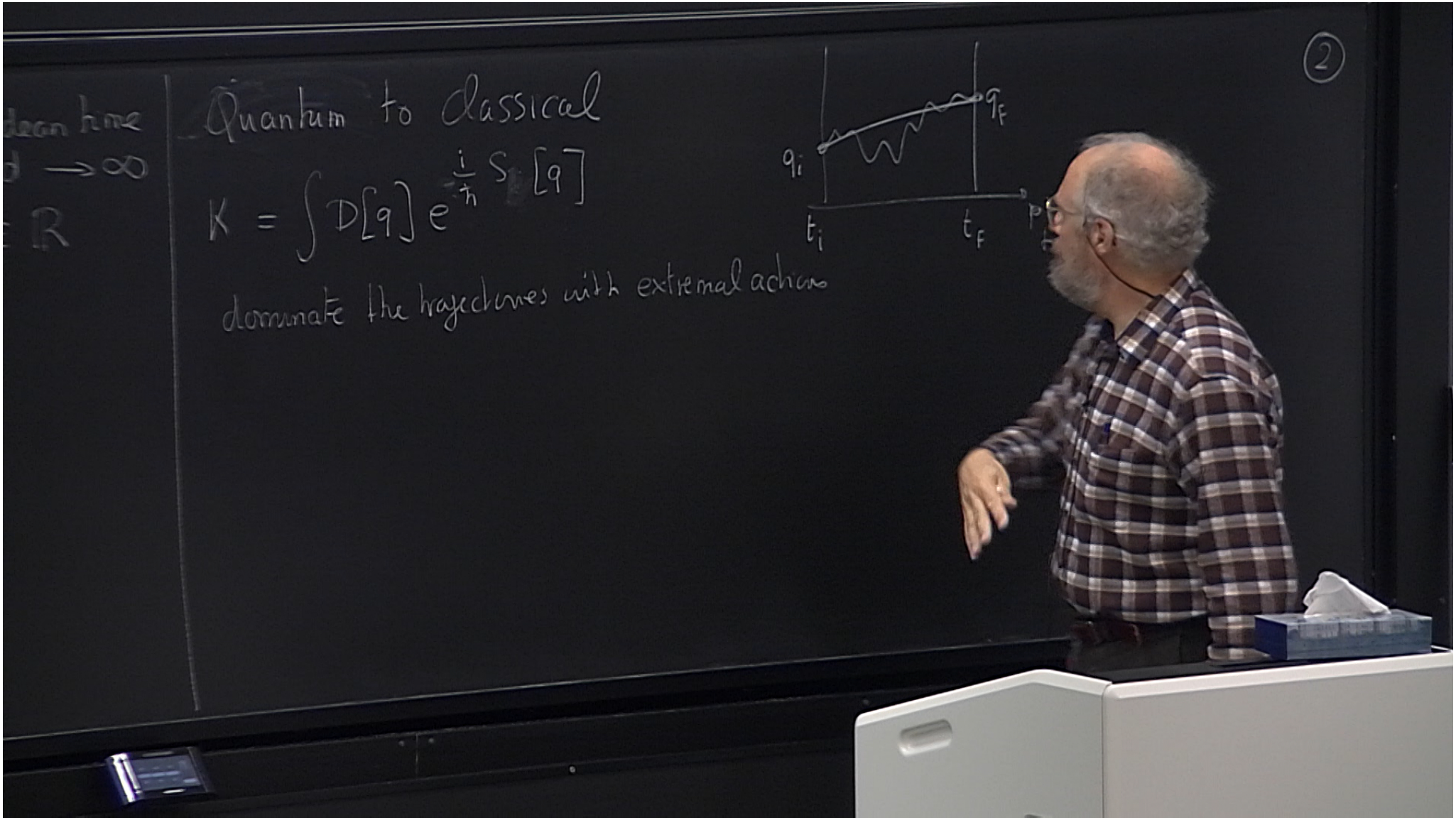

$S_E[q_E] = \int d\sigma \, \frac{m}{2}\left(\frac{dq_E}{d\sigma}\right)^2 + V(q_E(\sigma))$ (1)
invariant under translations in τ
$q_E(\sigma) \longrightarrow q_E(\sigma + \sigma_0)$
time ordered products
Euclidean time σ
$\mathrm{Tr}\left[\rho_\beta \, \varphi\right] = \frac{\int \mathcal{D}[q_E] \, e^{-\frac{1}{\hbar} S_E[q_E]} \, q(\tau_i)}{\int \mathcal{D}[q_E] \, e^{-\frac{1}{\hbar} S_E[q_E]}}$
density matrix
Gibbs state
$\beta = \frac{1}{k_B T}$
periodic trajectories $\tau = \frac{i}{\hbar}\beta$

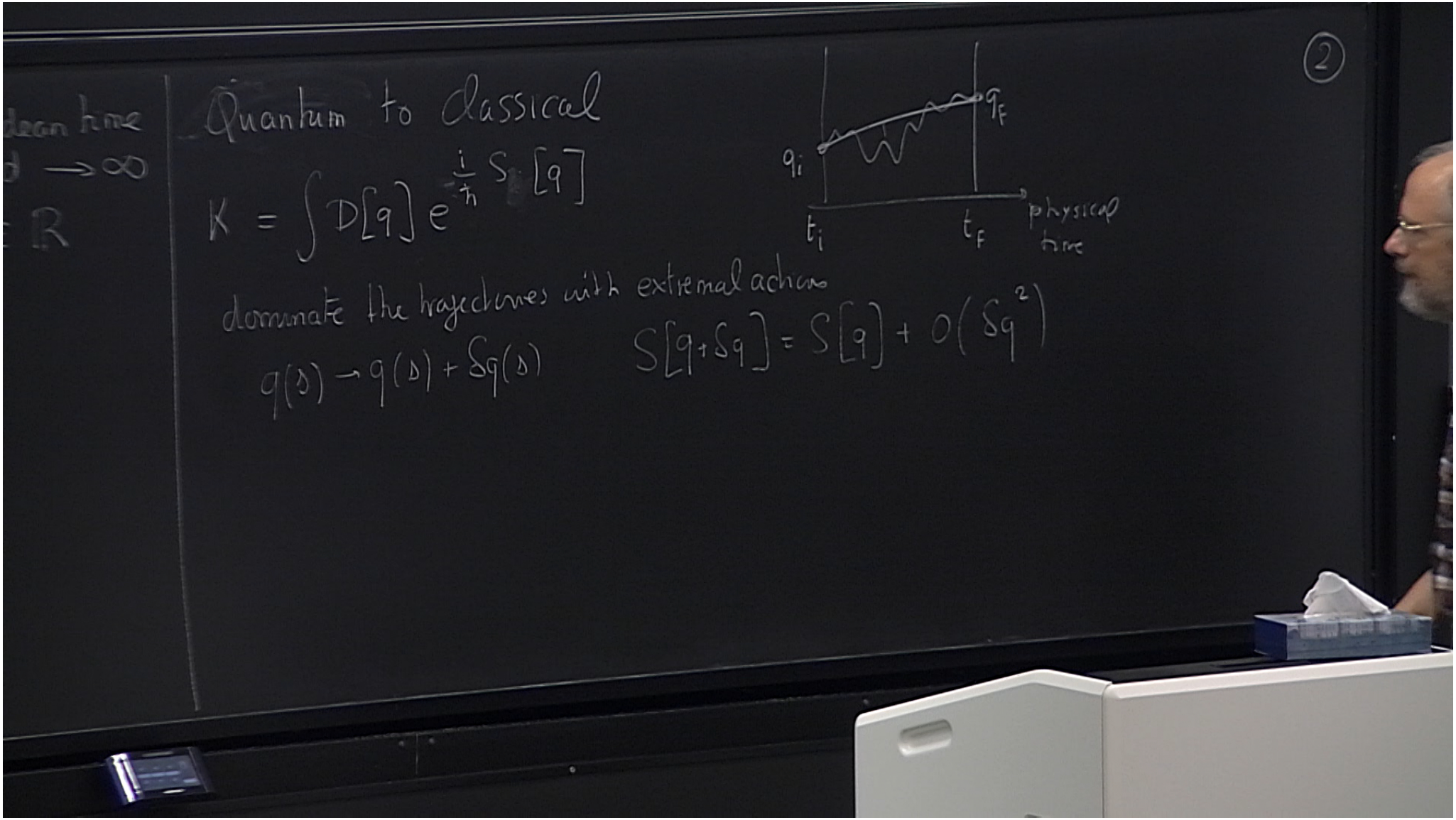

$S_E[q_E] = \int d\sigma \; \frac{m}{2}\left(\frac{dq_E}{d\sigma}\right)^2 + V(q_E(\sigma))$ (1)
invariant under translations in τ : Symmetry
$q_E(\sigma) \longrightarrow q_E(\sigma + \sigma_0)$
time ordered products
$\mathrm{Tr}\left[\rho_\beta Q\right] = \langle Q \rangle = \frac{\int \mathcal{D}[q_E] e^{-\frac{1}{\hbar} S_E[q_E]} q(\tau_1)}{\int \mathcal{D}[q_E] e^{-\frac{1}{\hbar} S_E[q_E]}}$
density matrix
Gibbs state
$\beta = \frac{1}{k_B T}$
periodic trajectories $\tau = \frac{i}{\hbar}\beta$

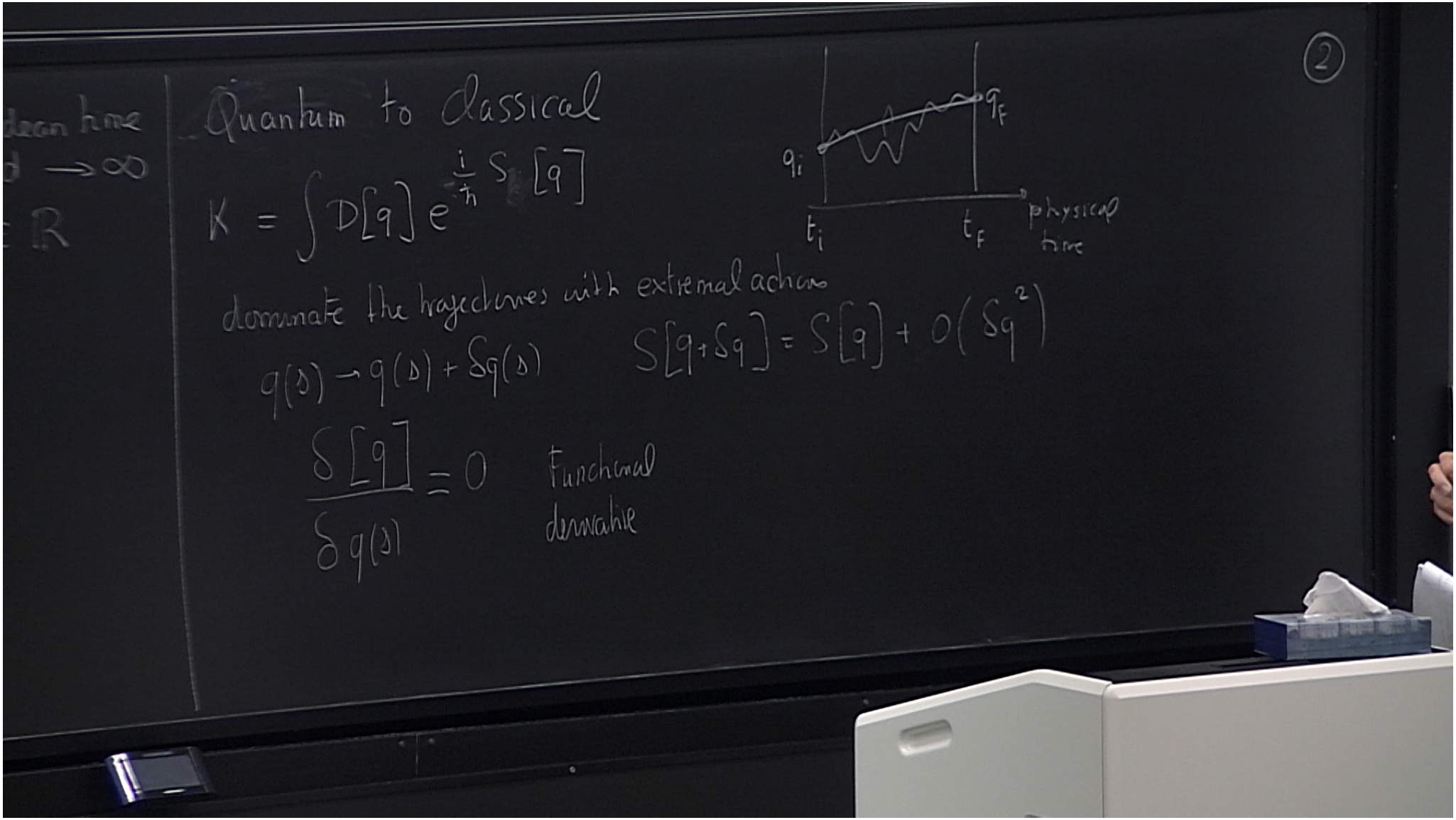

Quantum to classical
q_i
q_F
t_i
t_F
physical time

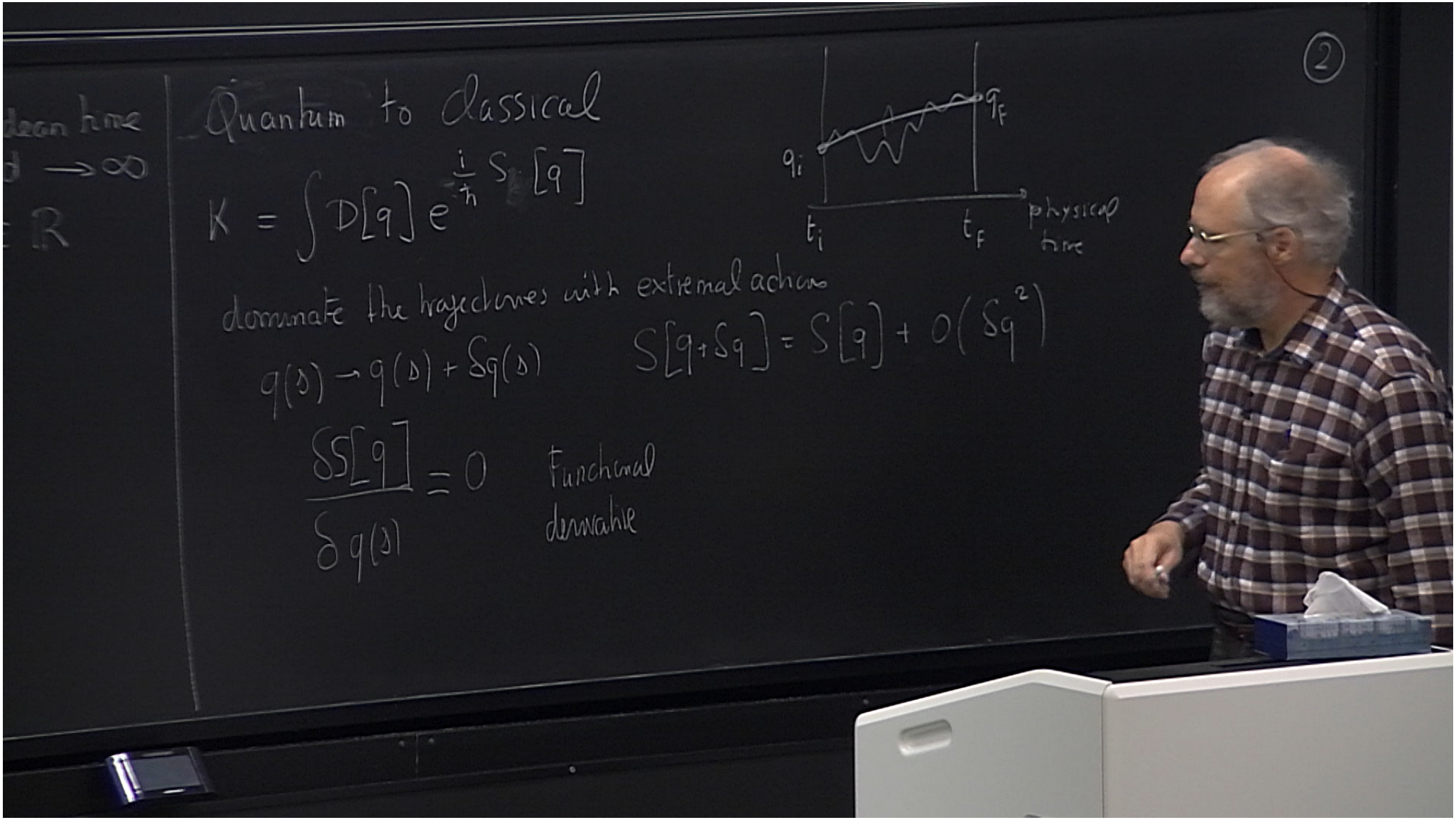

Quantum to classical
$K = \int D[q]\, e^{\frac{i}{\hbar} S_E[q]}$
q_i
q_F
t_i
t_F
physical time
②

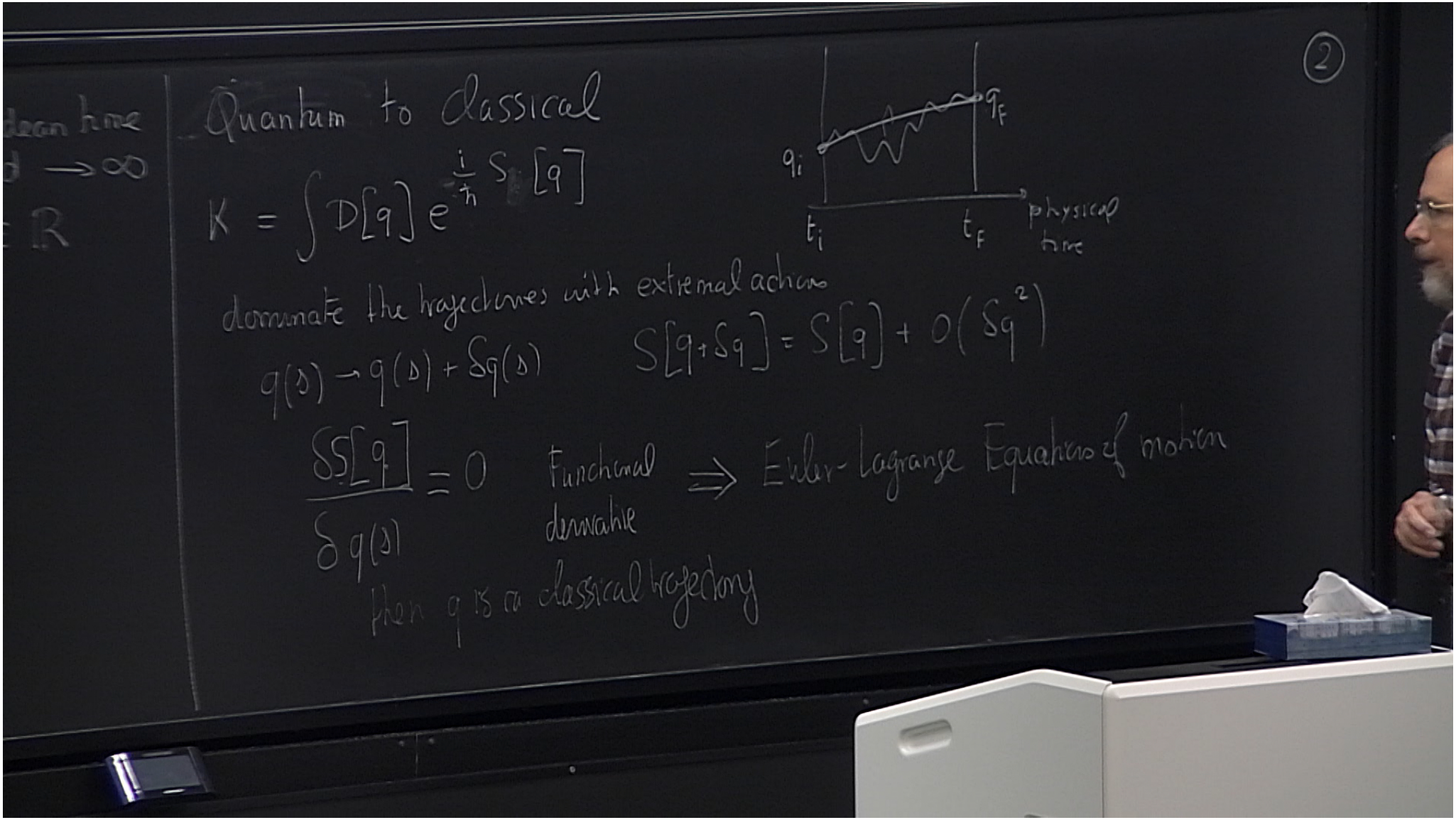

Quantum to classical
$K = \int D[q]\, e^{\frac{i}{\hbar} S_E[q]}$
dominate the trajectories with extremal action
q_i
q_F
t_i
②

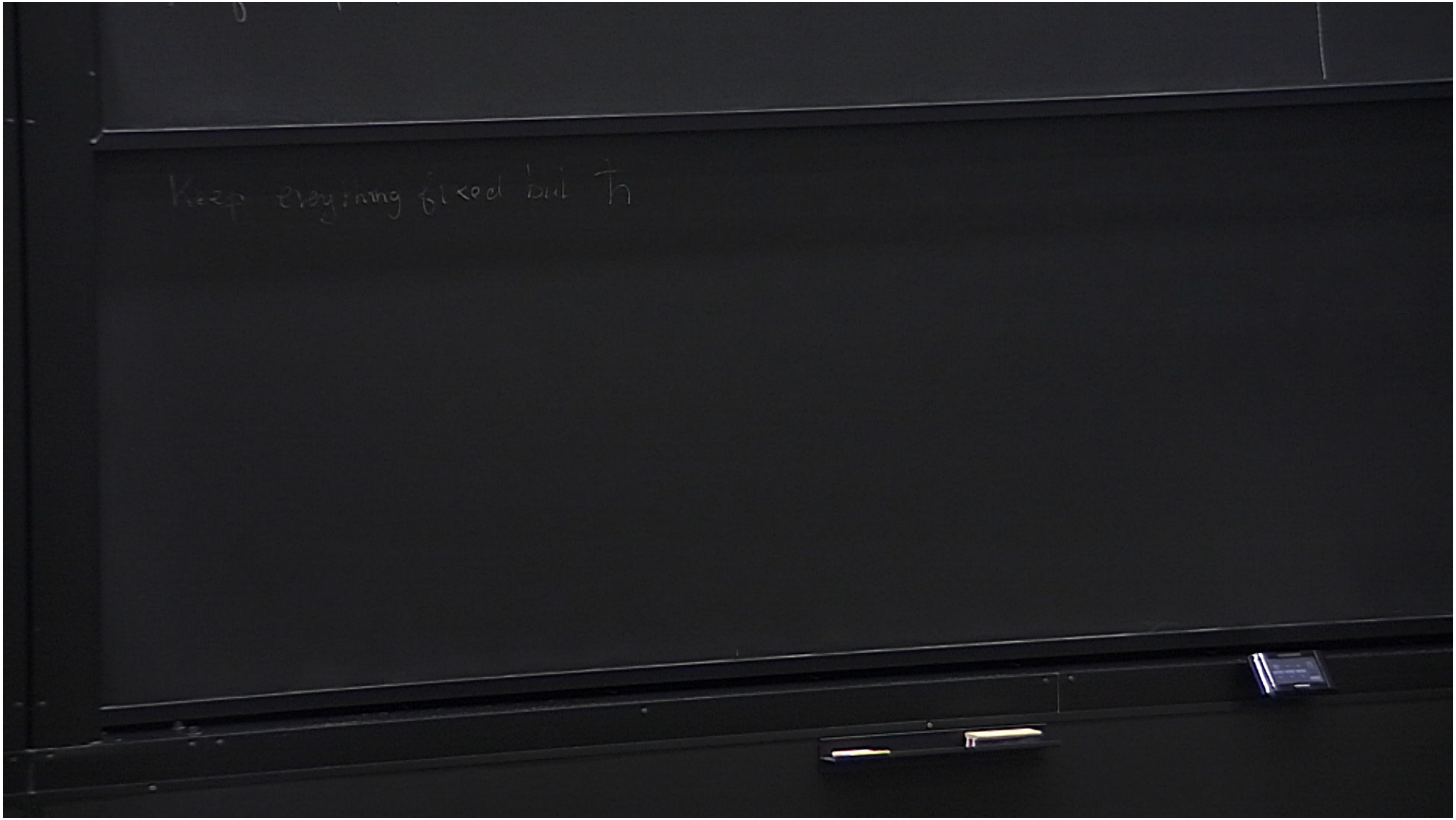

Quantum to classical
$K = \int D[q]\, e^{\frac{i}{\hbar} S[q]}$
dominate the trajectories with extremal action
q_i
q_F
t_i
t_F
2

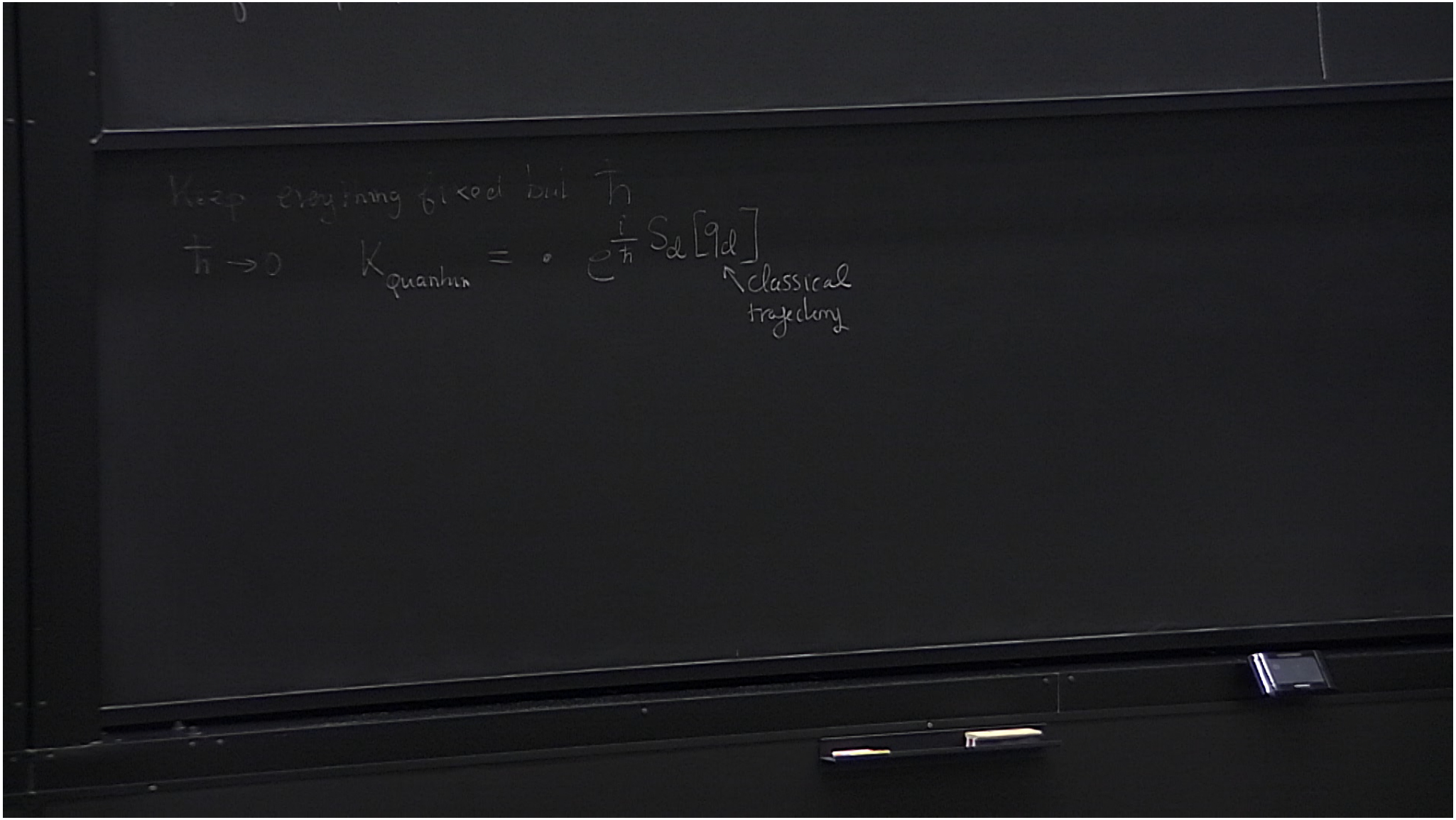

Quantum to classical
$K = \int D[q]\, e^{\frac{i}{\hbar} S[q]}$
q_i
q_F
t_i
t_F
physical time
dominate the trajectories with extremal action
$q(s) \to q(s) + \delta q(s)$
$S[q+\delta q] = S[q] + O(\delta q^2)$
②

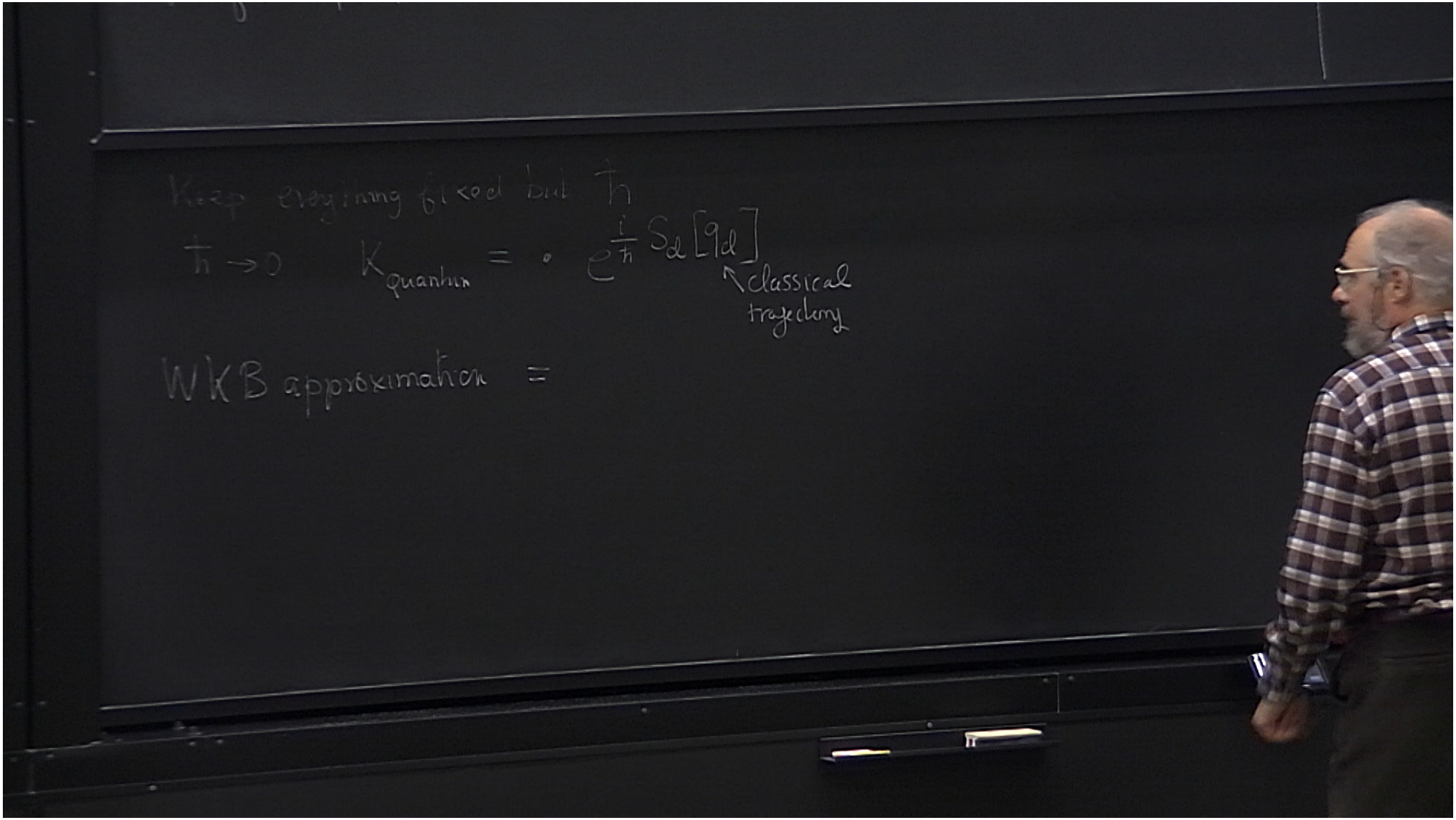

Quantum to classical
$K = \int D[q]\, e^{\frac{i}{\hbar} S[q]}$
q_i
q_F
t_i
t_F
physical time
dominate the trajectories with extremal action
$q(s) \to q(s) + \delta q(s)$
$S[q+\delta q] = S[q] + O(\delta q^2)$
$\frac{\delta S[q]}{\delta q(s)} = 0$
Functional derivative
2

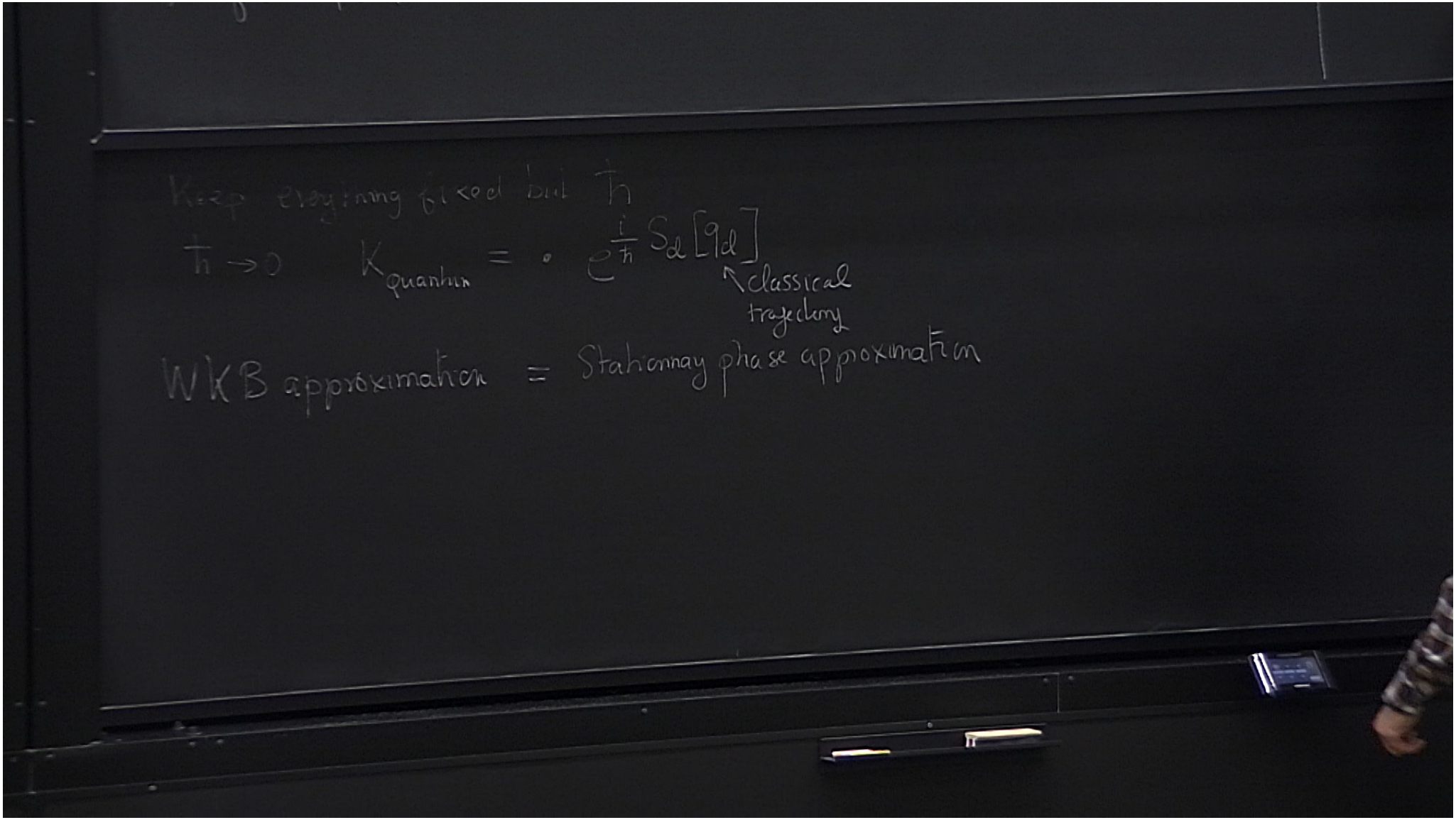

Quantum to classical
$K = \int D[q]\, e^{\frac{i}{\hbar} S[q]}$
q_i
q_F
t_i
t_F
physical time
dominate the trajectories with extremal action
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Functional derivative
②

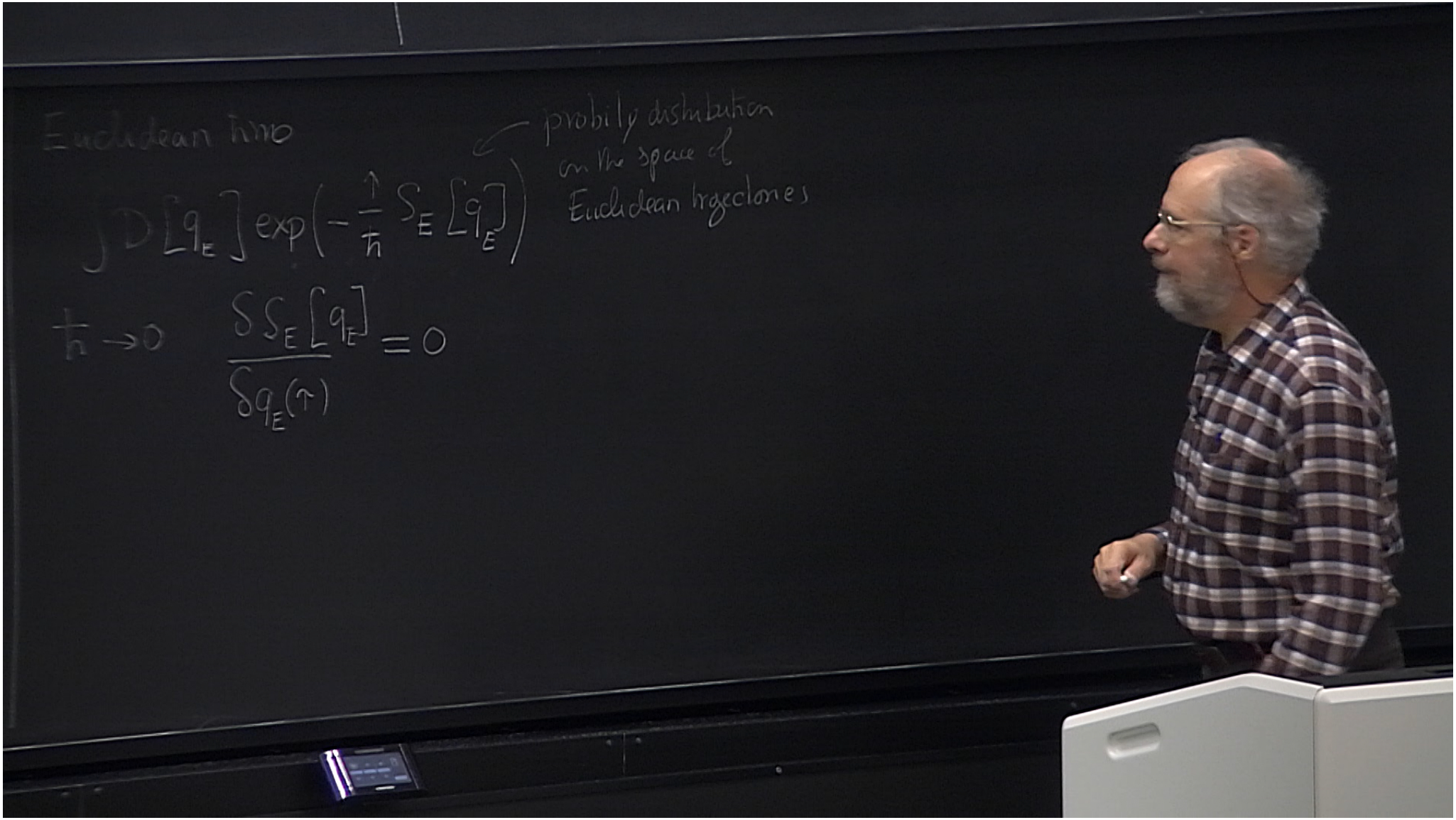

Quantum to classical
$K = \int D[q]\, e^{\frac{i}{\hbar} S[q]}$
q_i
q_F
t_i
t_F
physical time
dominate the trajectories with extremal action
$q(s) \to q(s) + \delta q(s)$
$S[q+\delta q] = S[q] + O(\delta q^2)$
$\frac{\delta S[q]}{\delta q(s)} = 0$
Functional derivative
$\Rightarrow$ Euler-Lagrange Equations of motion
then q is a classical trajectory
2


Keep beginning fixed but Th


Keep beginning fixed but $\hbar$
$\hbar \to 0$ $K_{quantum} = \cdot e^{\frac{i}{\hbar} S_{cl}[q_{cl}]}$
classical trajectory

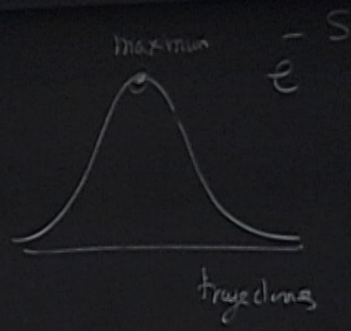

Keep everything fixed but $\hbar$
$\hbar \to 0$
$K_{quantum} = \cdot e^{\frac{i}{\hbar} S_{cl}[q_{cl}]}$
↖classical trajectory
WKB approximation =


Keep beginning fixed but $\hbar$
$\hbar \to 0$ $K_{quantum} = \cdot e^{\frac{i}{\hbar} S_{cl}[q_{cl}]}$
classical trajectory
WKB approximation = Stationary phase approximation


Euclidean time
$\int \mathcal{D}[q_E] \exp\left(-\frac{1}{\hbar} S_E[q_E]\right)$
probability distribution on the space of Euclidean trajectories
$\hbar \to 0 \quad \frac{\delta S_E[q_E]}{\delta q_E(\tau)} = 0$

Euclidean time

$\int \mathcal{D}[q_E] \exp\left(-\frac{1}{\hbar} S_E[q_E]\right)$ ← probability distribution on the space of Euclidean trajectories



$\frac{1}{\hbar} \to 0 \qquad \frac{\delta S_E[q_E]}{\delta q_E(\tau)} = 0$

maximize probability
minimize the action

Keep everything fixed but $\hbar$

$\hbar \to 0$ $\quad K_{quantum} = \cdot\, e^{\frac{i}{\hbar} S_{cl}[q_{cl}]}$ ← classical trajectory

Physical time

WKB approximation = Stationary phase approximation

Euclidean time = Saddle point approximation

Euclidean time

$$\int D[q_E] \exp\left(-\frac{1}{\hbar} S_E[q_E]\right)$$

$\hbar \to 0$ $\quad \frac{\delta S_E[q_E]}{\delta q_E(\tau)} = 0$

Keep everything fixed but $\hbar$

$\hbar \to 0$ $\quad K_{quantum} = \cdot\, e^{\frac{i}{\hbar} S_{cl}[q_{cl}]}$

↑ classical trajectory

Physical time

WKB approximation $\left(\begin{array}{l} = \text{Stationnay phase approximation} \\ = \text{Saddle punt approximation} \end{array}\right.$

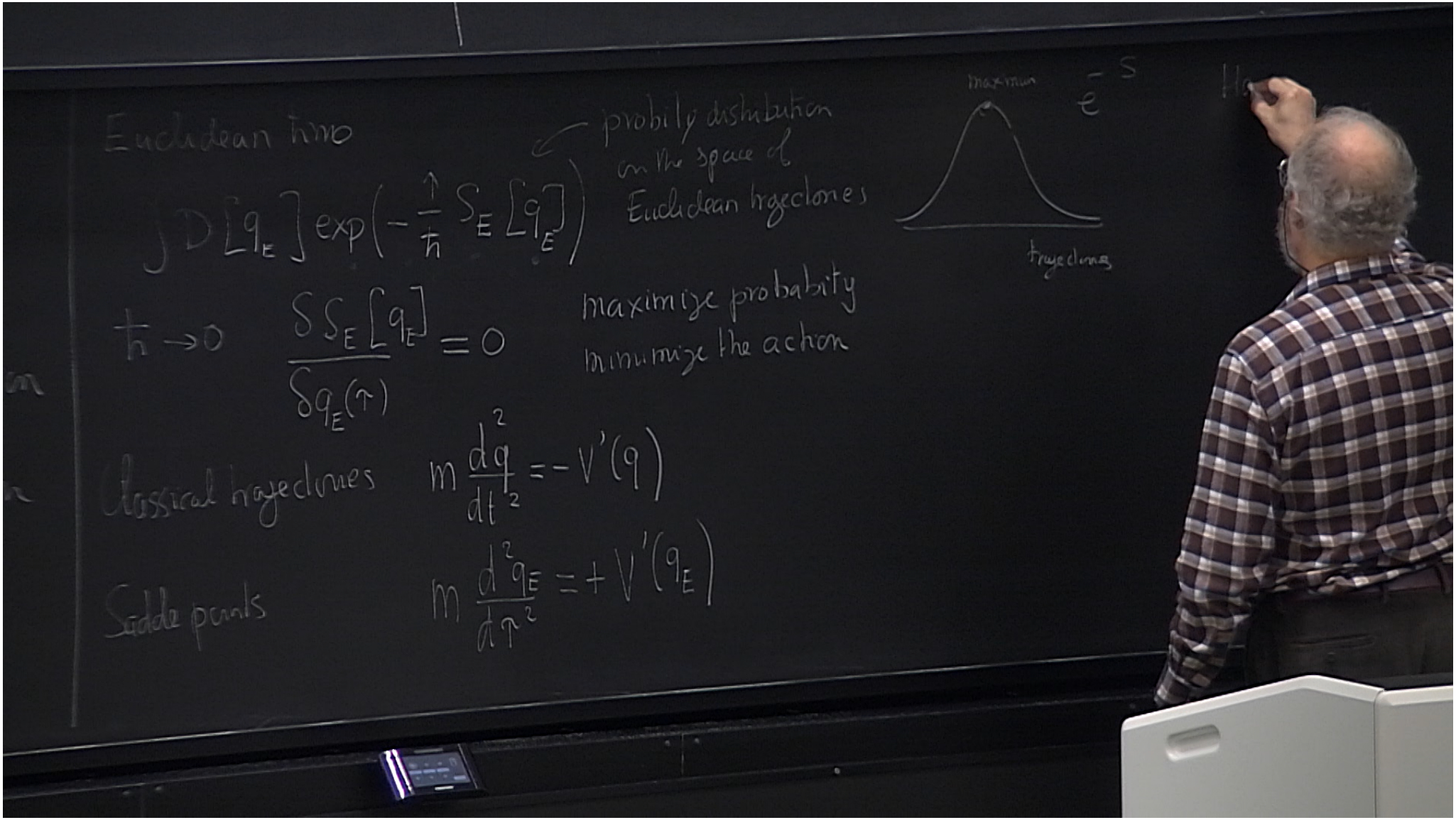
Euclidean time

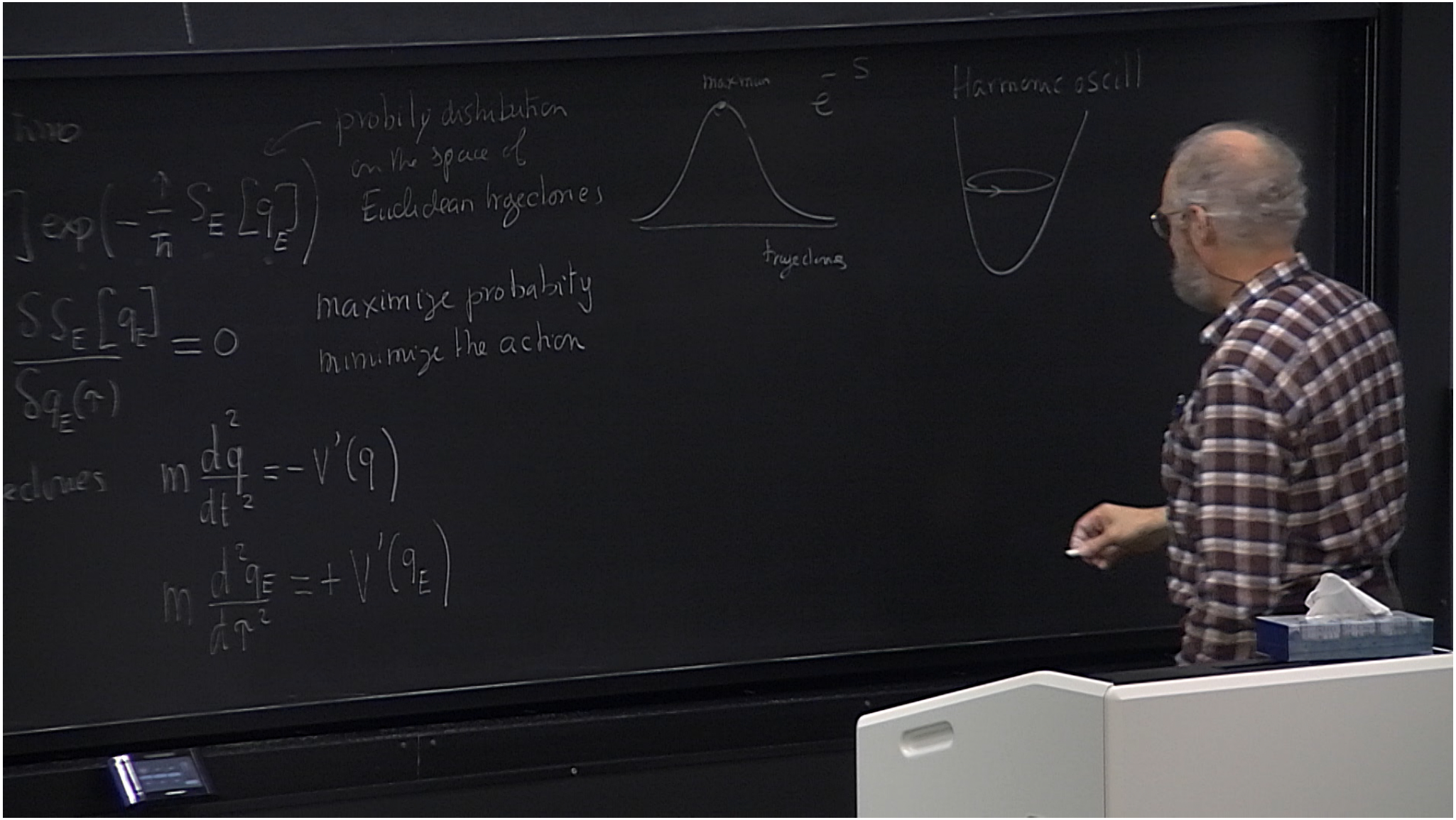
Complex WKP approximation

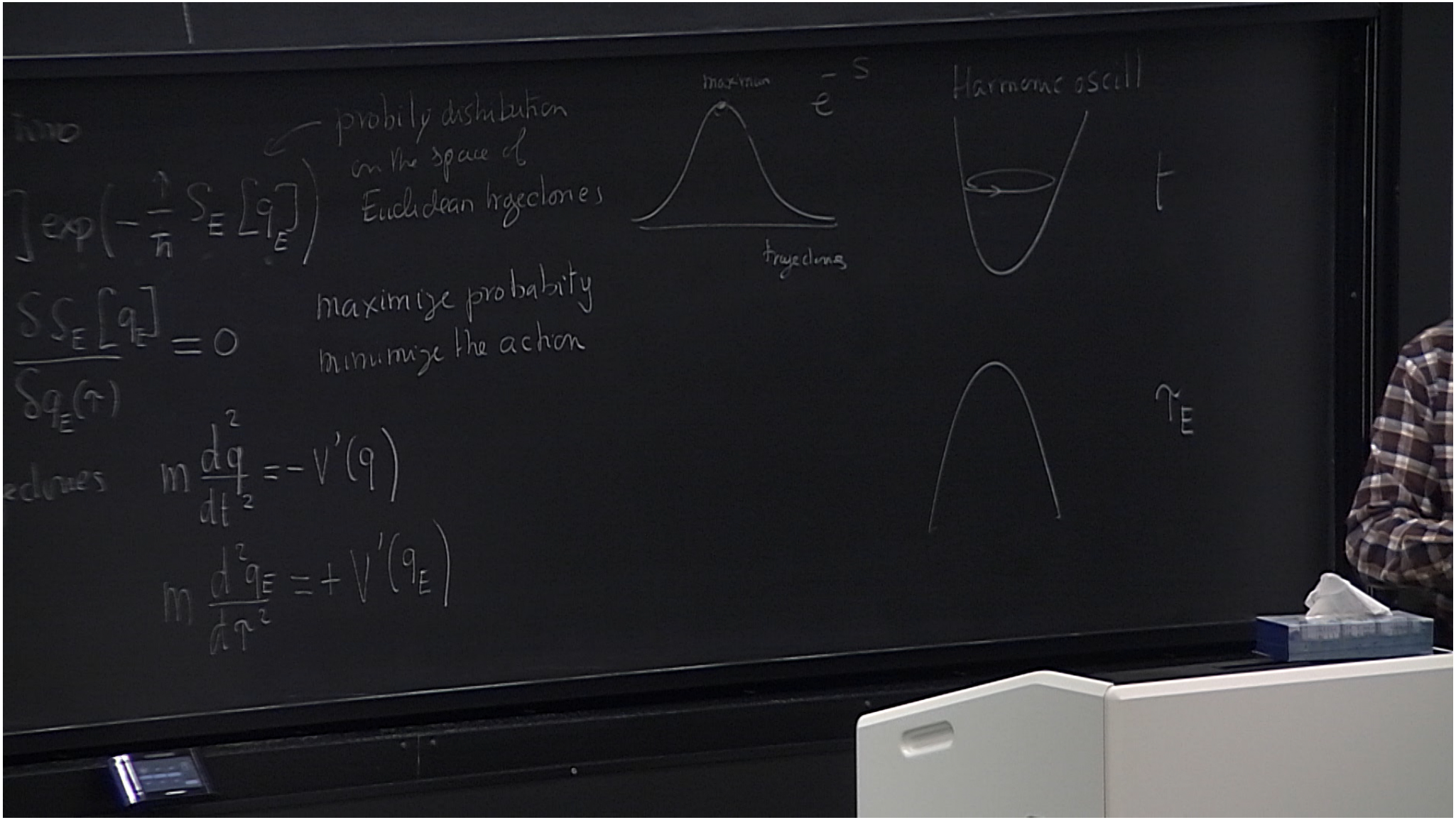
Euclidean time

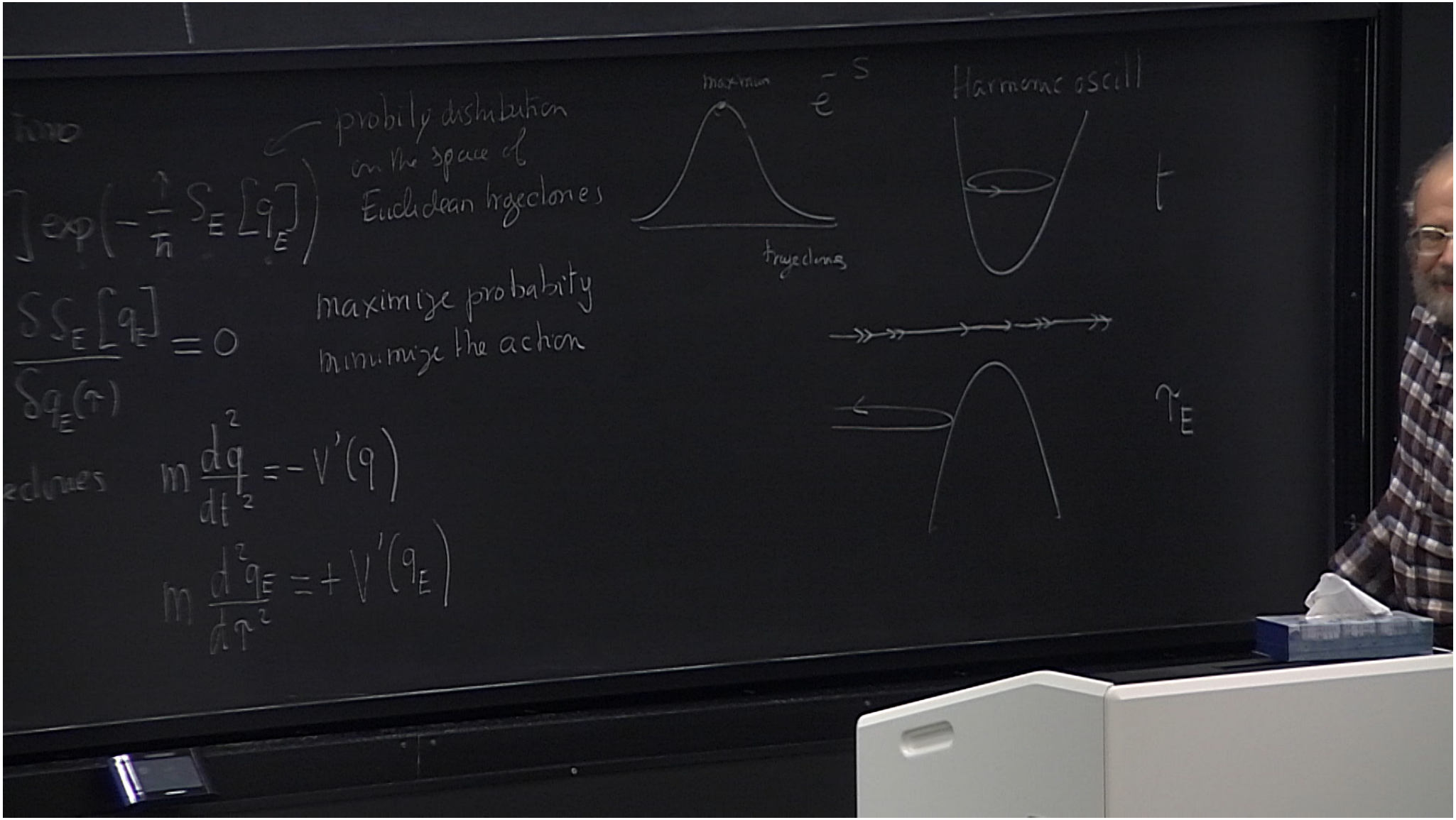
$$\int \mathcal{D}[q_E] \exp\left(-\frac{1}{\hbar} S_E[q_E]\right)$$

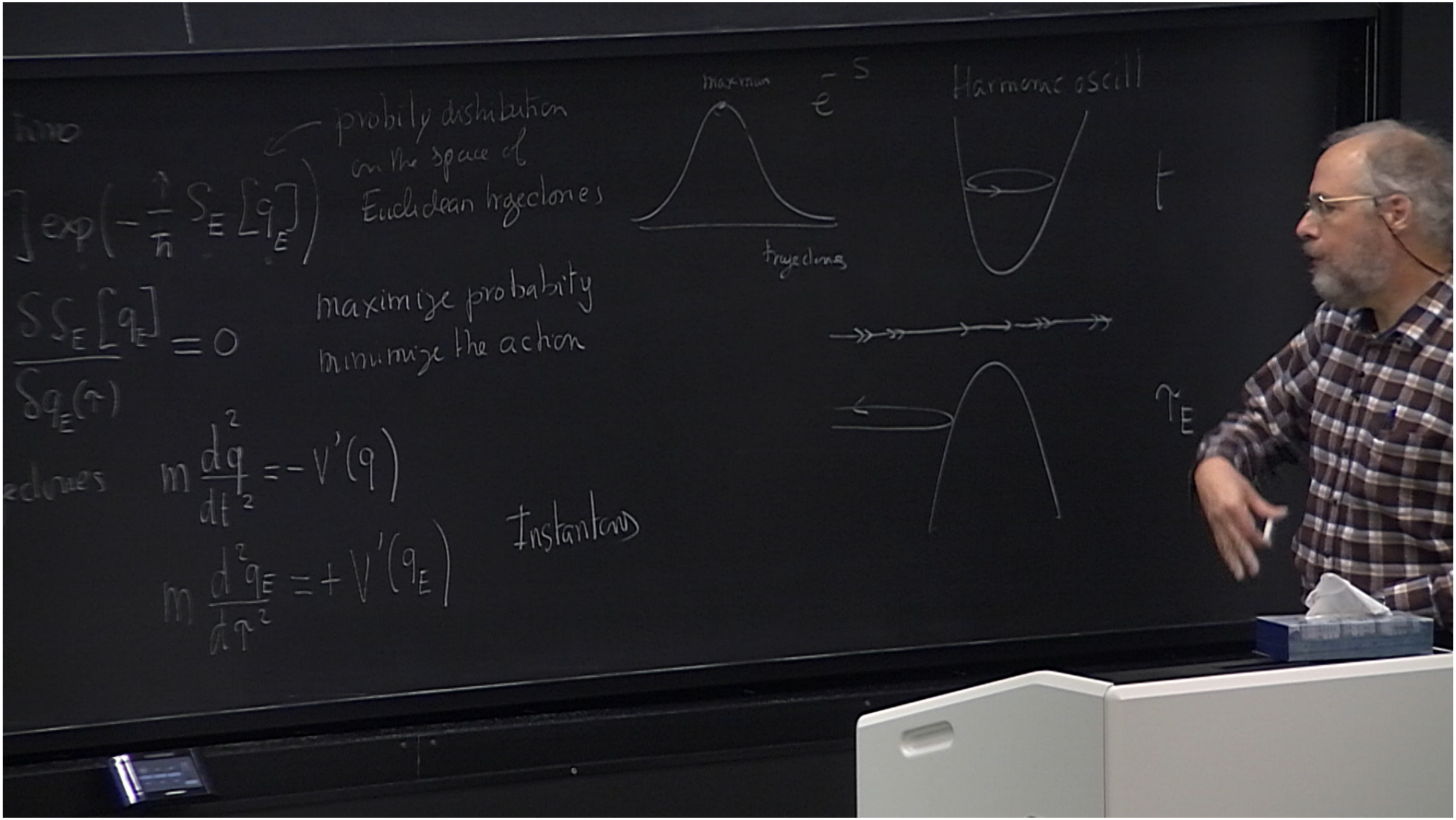
$\hbar \to 0$ $\quad \frac{\delta S_E[q_E]}{\delta q_E(\tau)} = 0$


Euclidean time
$\mathcal{D}[q_E]\exp\left(-\frac{1}{\hbar}S_E[q_E]\right)$
probability distribution on the space of Euclidean trajectories
$\hbar \to 0 \quad \frac{\delta S_E[q_E]}{\delta q_E(\tau)} = 0$
maximize probability
minimize the action
classical trajectories
$m\frac{d^2q}{dt^2} = -V'(q)$
Saddle points
$m\frac{d^2q_E}{d\tau^2} = +V'(q_E)$
maximum
e^{-S}
trajectories


probability distribution
on the space of
Euclidean trajectories
maximize probability
minimize the action
maximum
e^{-S}
trajectories
Harmonic oscill
$\frac{\delta S_E[q_E]}{\delta q_E(\tau)} = 0$
$m\frac{d^2q}{dt^2} = -V'(q)$
$m\frac{d^2q_E}{d\tau^2} = +V'(q_E)$


probily distribution
on the space of
Euclidean trajectories
maximum
e^{-S}
trajectories
Harmonic oscill
t
τ_E
maximize probabity
minimize the action
$\frac{\delta S_E[q_E]}{\delta q_E(\tau)} = 0$
$m\frac{d^2q}{dt^2} = -V'(q)$
$m\frac{d^2q_E}{d\tau^2} = +V'(q_E)$


probability distribution in the space of Euclidean trajectories
maximize probability
minimize the action
maximum
trajectories
Harmonic oscill
t
τ_E


probity distribution
on the space of
Euclidean trajectories
maximize probability
minimize the action
maximum
trajectories
Harmonic oscill
Instantons

period $\tau = \frac{\hbar}{k_B T}$ ← temperature

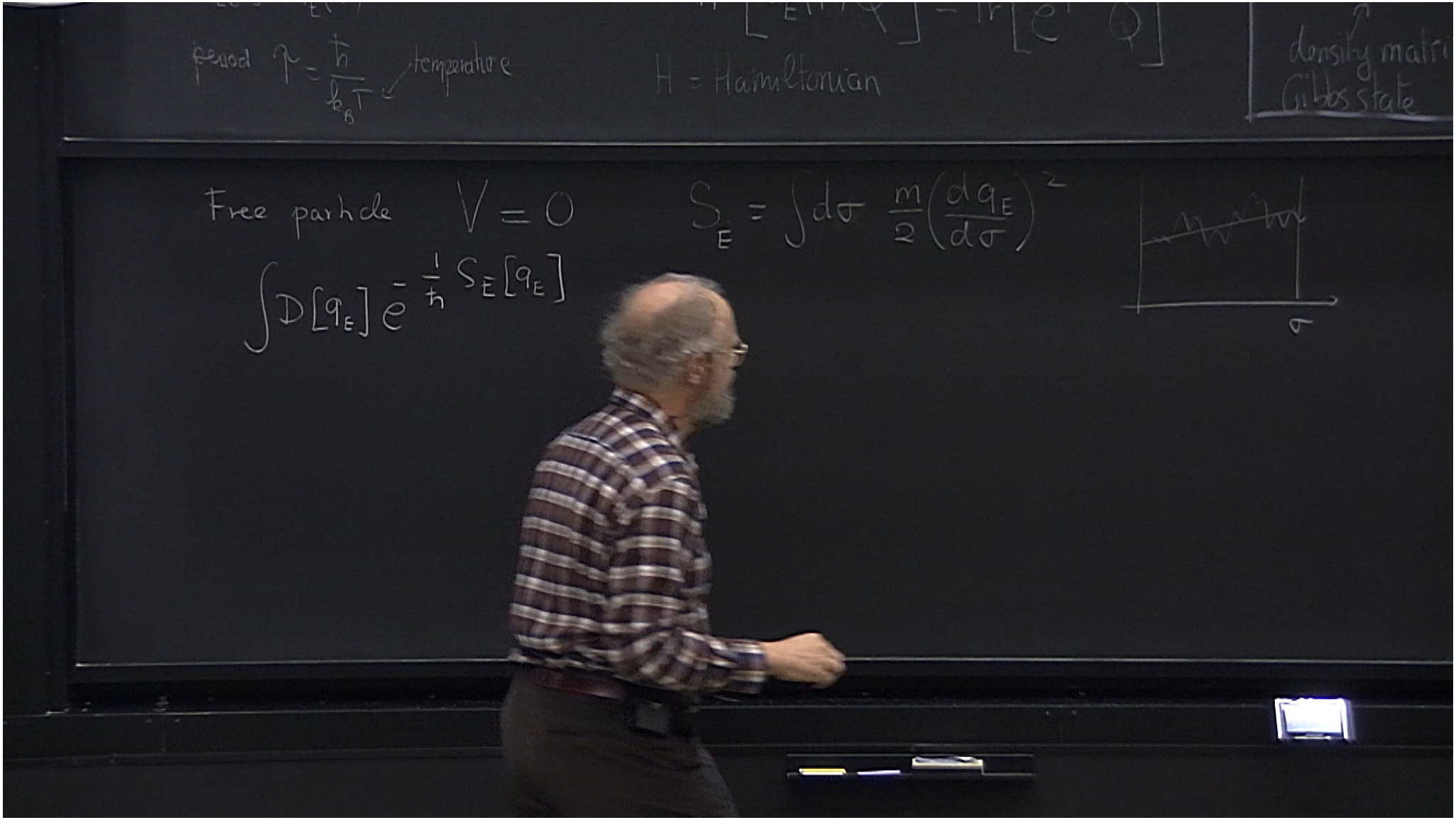
H = Hamiltonian

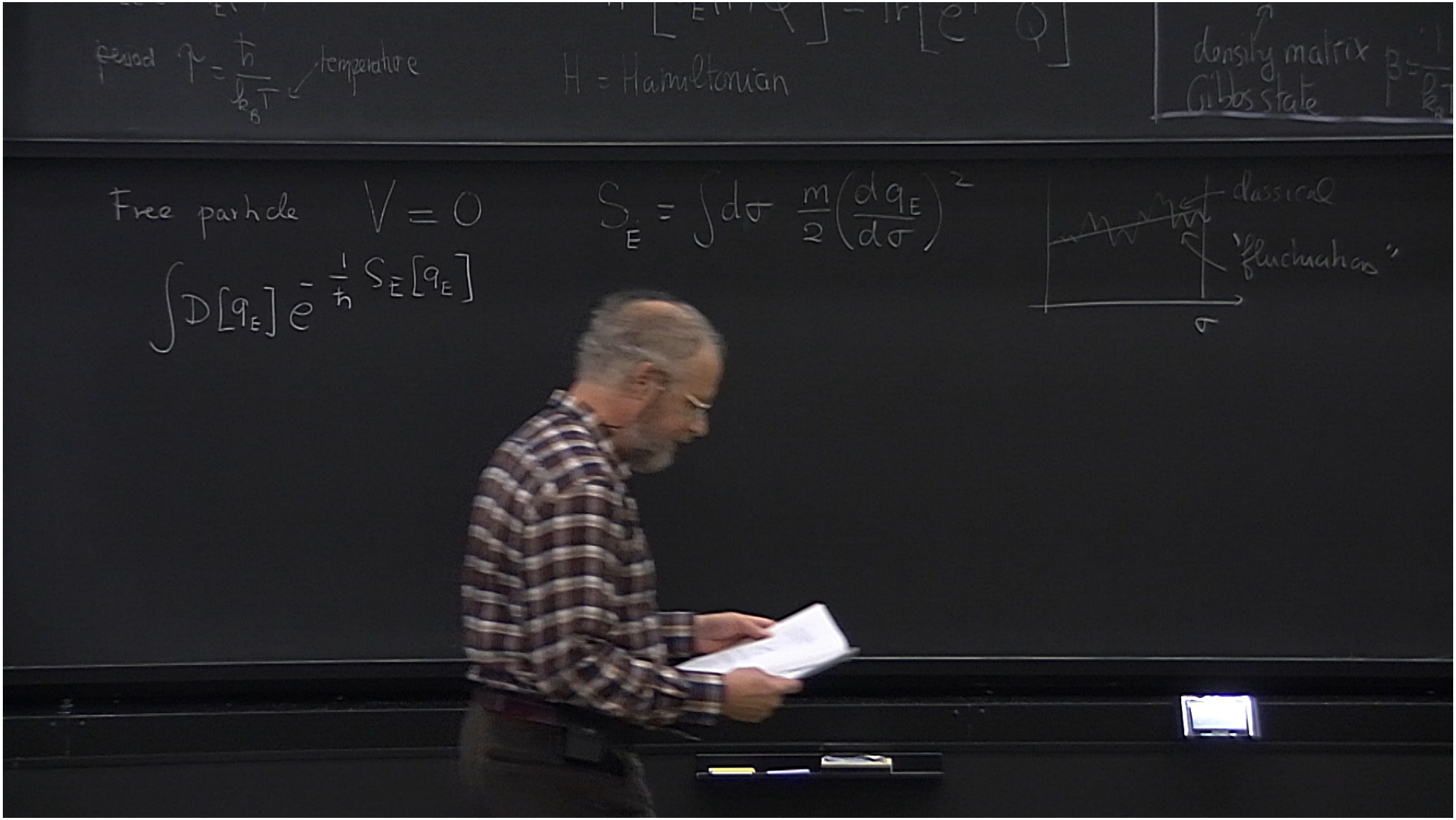
density matrix
Gibbs state

Free particle $V = 0$

$$S_E = \int d\sigma \, \frac{m}{2} \left(\frac{dq_E}{d\sigma} \right)^2$$

$$\int D[q_E] \, e^{-\frac{1}{\hbar} S_E[q_E]}$$


period $\tau = \frac{\hbar}{k_B T}$
temperature
H = Hamiltonian
density matrix
Gibbs state
Free particle $V = 0$
$S_E = \int d\sigma \frac{m}{2}\left(\frac{dq_E}{d\sigma}\right)^2$
$\int D[q_E] e^{-\frac{1}{\hbar} S_E[q_E]}$
σ


period $\tau = \frac{\hbar}{k_B T}$
temperature
H = Hamiltonian
density matrix
Gibbs state
$\beta = \frac{1}{k_B T}$
Free particle $V = 0$
$S_E = \int d\sigma \frac{m}{2}\left(\frac{dq_E}{d\sigma}\right)^2$
$\int D[q_E] e^{-\frac{1}{\hbar} S_E[q_E]}$
classical
"fluctuation"
σ

period $\tau = \frac{\hbar}{k_B T}$ ← temperature

H = Hamiltonian

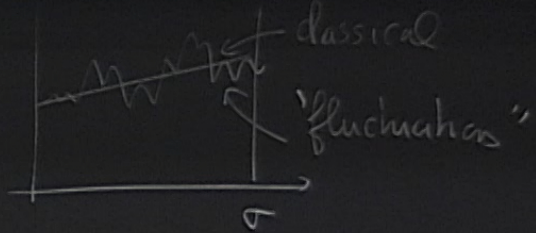
density matrix
Gibbs state $\beta = \frac{1}{k_B T}$

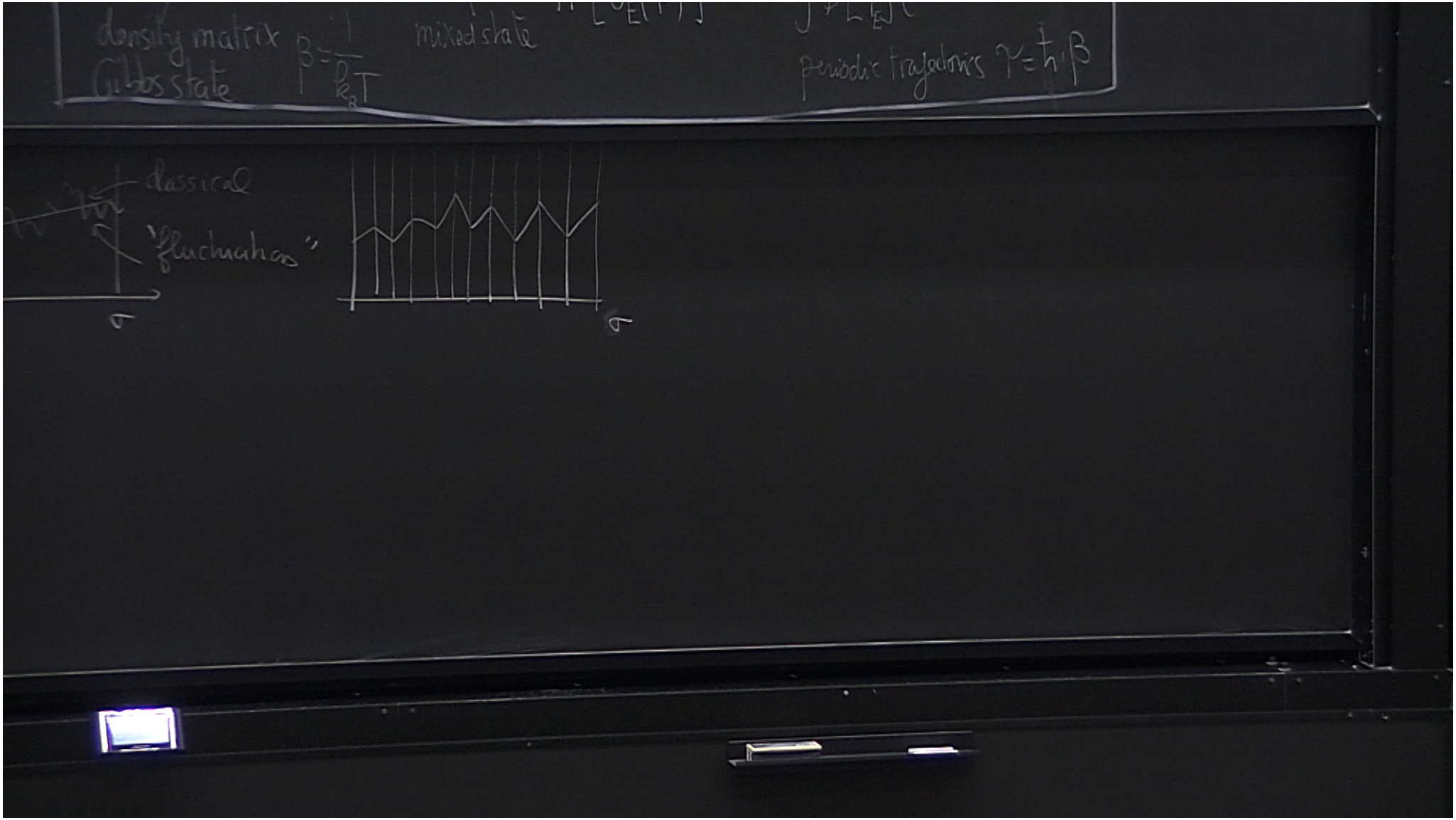
Free particle $V = 0$

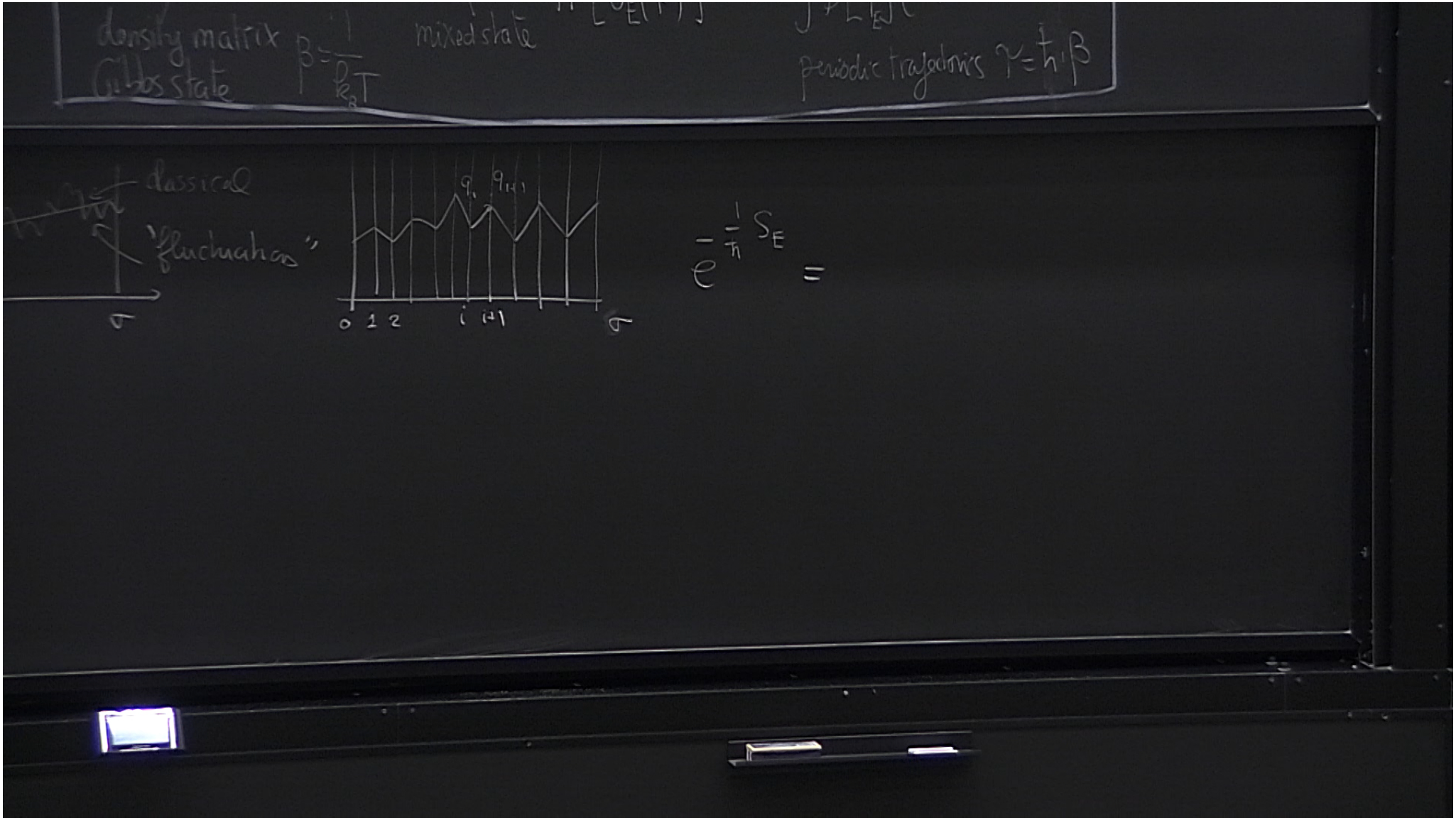
$$S_E = \int d\sigma \, \frac{m}{2}\left(\frac{dq_E}{d\sigma}\right)^2$$

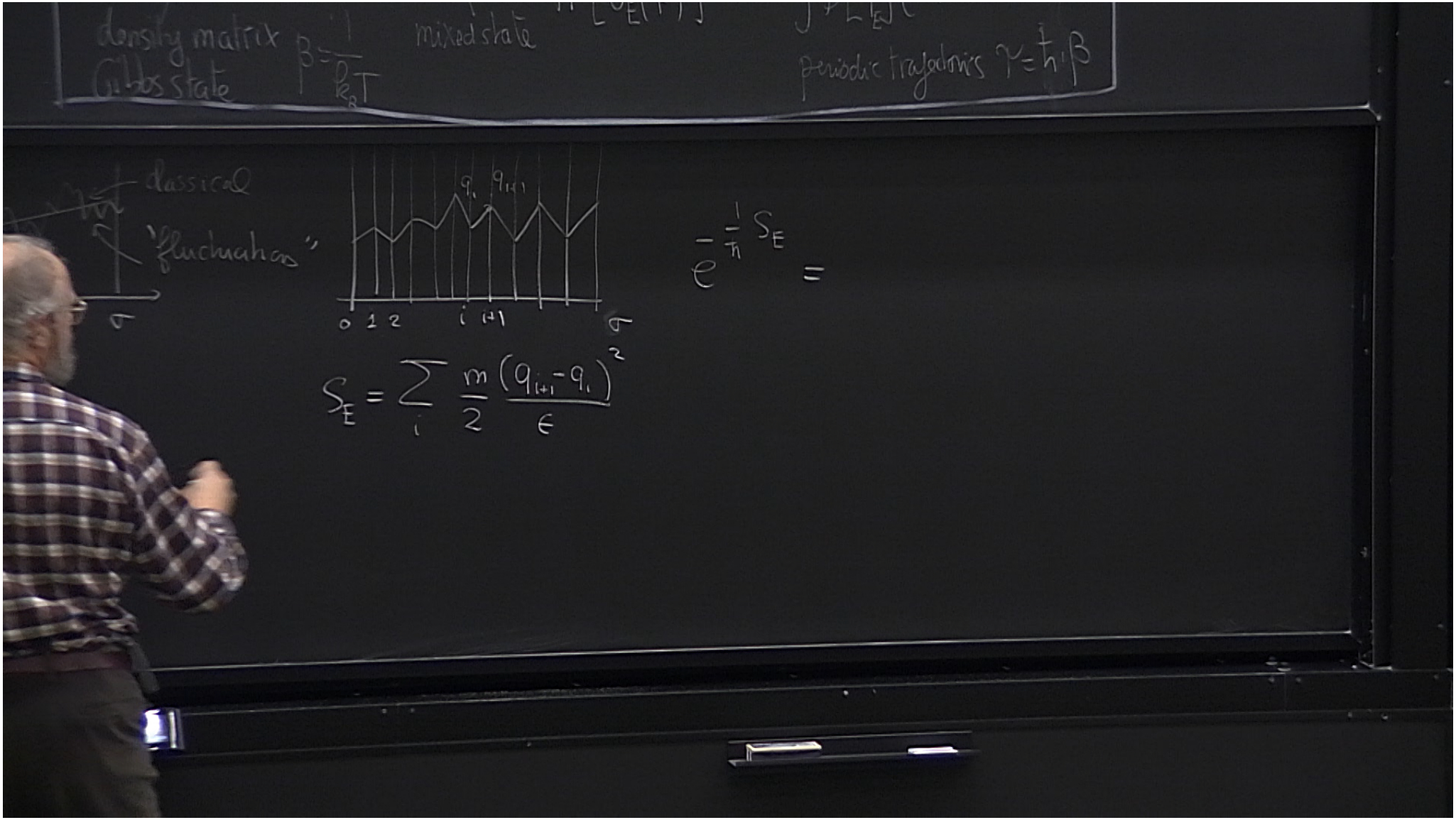
$[q_E]\, e^{-\frac{1}{\hbar} S_E[q_E]}$

$\Rightarrow$ Random Brownian motion



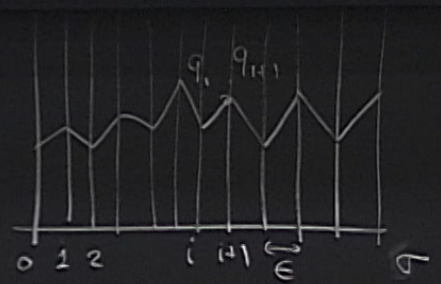

density matrix
Gibbs state
$\beta = \frac{1}{k_B T}$
mixed state
periodic trajectories $\tau = \frac{1}{\hbar} \cdot \beta$
classical
"fluctuations"
σ
σ


density matrix
Gibbs state
mixed state
periodic trajectories
classical
"fluctuations"
0 1 2 i i+1


density matrix
Gibbs state
$\beta = \frac{1}{k_B T}$
mixed state
periodic trajectories $\tau = \frac{1}{\hbar}\beta$
classical
"fluctuations"
q_i q_{i+1}
0 1 2 i i+1
$e^{-\frac{1}{\hbar} S_E} =$
$S_E = \sum_i \frac{m}{2} \frac{(q_{i+1} - q_i)^2}{\epsilon}$

density matrix $\beta = \frac{1}{k_B T}$
Gibbs state
mixed state
periodic trajectories $\tau = \frac{t}{\hbar} \cdot \beta$

classical
"fluctuations"

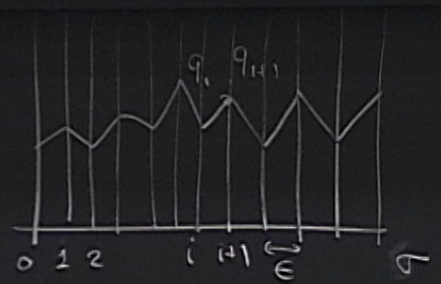


$$e^{-\frac{1}{\hbar} S_E} =$$

$$S_E = \sum_i \frac{m}{2} \frac{(q_{i+1} - q_i)^2}{\epsilon}$$

density matrix $\beta = \frac{1}{k_B T}$
Gibbs state
mixed state
periodic trajectories $\tau = \frac{1}{\hbar} \cdot \beta$

classical
"fluctuations"



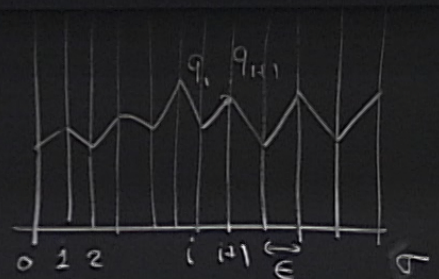
$$S_E = \sum_i \frac{m}{2} \frac{(q_{i+1} - q_i)^2}{\epsilon}$$

$$e^{-\frac{1}{\hbar} S_E} = \prod_i e^{-\frac{m}{2\hbar\epsilon}(q_{i+1} - q_i)^2}$$

density matrix $\beta = \frac{1}{k_B T}$
Gibbs state
mixed state
periodic trajectories $\tau = \frac{1}{\hbar}\beta$

classical

"fluctuations"



$$S_E = \sum_i \frac{m}{2} \frac{(q_{i+1} - q_i)^2}{\epsilon}$$

$$e^{-\frac{1}{\hbar} S_E} = \prod_i e^{-\frac{m}{2\hbar\epsilon}(q_{i+1} - q_i)^2}$$

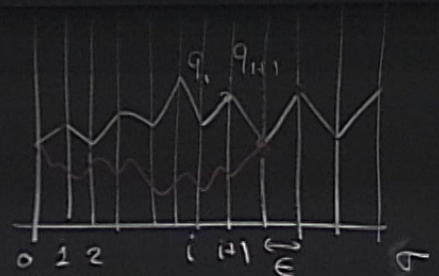
typical trajectory q_i : stochastic process in time

density matrix $\beta = \frac{1}{k_B T}$ mixed state periodic trajectories $\tau = \frac{1}{\hbar}\beta$

Gibbs state

classical

"fluctuations"



$$S_E = \sum_i \frac{m}{2} \frac{(q_{i+1} - q_i)^2}{\epsilon}$$

$$e^{-\frac{1}{\hbar} S_E} = \prod_i e^{-\frac{m}{2\hbar\epsilon}(q_{i+1} - q_i)^2}$$

typical trajectory q_i : stochastic process in time

Markov process

period $\beta = \frac{\hbar}{k_B T}$ ← temperature

H = Hamiltonian

density matrix
Gibbs state

Free particle $V = 0$ $\qquad S_E = \int d\sigma \, \frac{m}{2}\left(\frac{dq_E}{d\sigma}\right)^2$

$$\int \mathcal{D}[q_E] \, e^{-\frac{1}{\hbar} S_E[q_E]}$$

$\Rightarrow$ Random Brownian motion

classical

fluctuations

σ

Path Integrals $\longleftrightarrow$ Stochastic Processes Kac, ...

period $\tau = \frac{\hbar}{k_B T}$ ← temperature

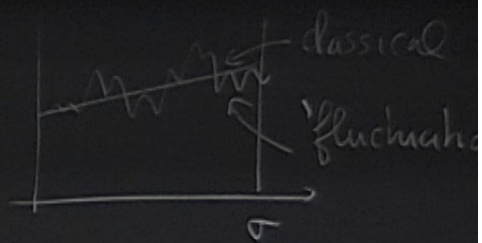
H = Hamiltonian

density matrix
Gibbs state

Free particle $V = 0$ $\qquad S_E = \int d\sigma \, \frac{m}{2}\left(\frac{dq_E}{d\sigma}\right)^2$

$$\int \mathcal{D}[q_E] \, e^{-\frac{1}{\hbar} S_E[q_E]}$$

$\Rightarrow$ Random Brownian motion



Path Integrals $\longleftrightarrow$ Stochastic Processes $\qquad$ Kac, ...

QM at Euclean time $\longleftrightarrow$ Diffusion processes

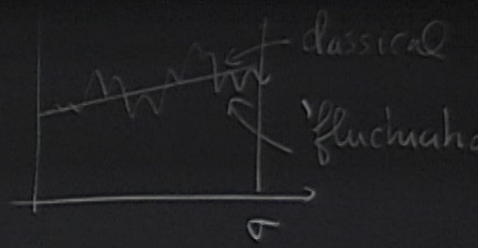
period $\beta = \frac{\hbar}{k_B T}$ ← temperature

H = Hamiltonian

density matrix
Gibbs state

Free particle $V = 0$ $\quad S_E = \int d\sigma \, \frac{m}{2}\left(\frac{dq_E}{d\sigma}\right)^2$

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Path Integrals $\longleftrightarrow$ Stochastic Processes Kac, ...

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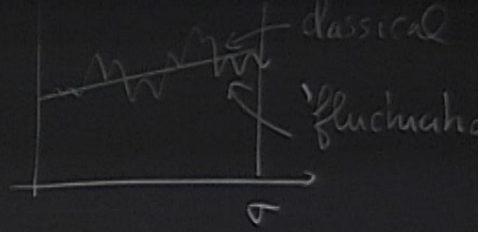
Schrödinger Eq $i\hbar \frac{\partial}{\partial t}\psi = \cdot \Delta \psi$

period $\beta = \frac{\hbar}{k_B T}$ ← temperature

H = Hamiltonian

density matrix
Gibbs state

Free particle $V = 0$ $\quad S_E = \int d\sigma \, \frac{m}{2}\left(\frac{dq_E}{d\sigma}\right)^2$



$$\int D[q_E]\, e^{-\frac{1}{\hbar} S_E[q_E]} \Rightarrow \text{Random Brownian motion}$$

Path Integrals ⟷ Stochastic Processes Kac, ...

QM at Euclidean time ⟷ Diffusion processes

Schrödinger Eq $i\hbar \frac{\partial}{\partial t}\psi = \Delta\psi \xrightarrow[\text{Euclidean}]{} -\hbar \frac{\partial}{\partial \tau}\psi = \Delta\psi$

period $\beta = \frac{\hbar}{k_B T}$ ← temperature

H = Hamiltonian

density matrix
Gibbs state

Free particle $V = 0$ $\quad S_E = \int d\sigma \, \frac{m}{2}\left(\frac{dq_E}{d\sigma}\right)^2$

$\int D[q_E] \, e^{-\frac{1}{\hbar} S_E[q_E]}$ $\Rightarrow$ Random Brownian motion

classical

fluctuations

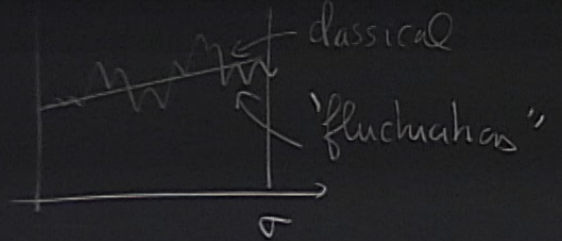
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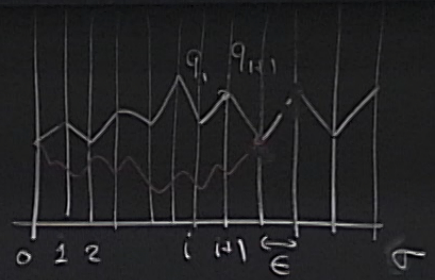
Integrals $\longleftrightarrow$ Stochastic Processes Kac, ...

QM at Euclean time $\longleftrightarrow$ Diffusion processes

Schrödinger Eq $i\hbar \frac{\partial}{\partial t}\psi = \Delta\psi \xrightarrow[\text{Euclidean}]{} -\hbar \frac{\partial}{\partial \tau}\psi = \Delta\psi$ Heat Equation (Fourier)

density matrix $\beta = \frac{1}{k_B T}$ mixed state periodic trajectories $\tau = \frac{i}{\hbar}\beta$



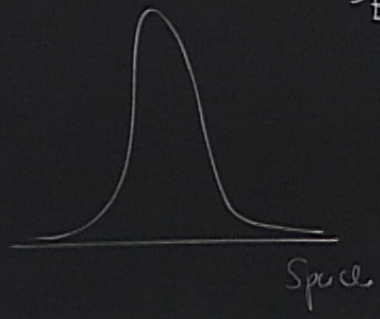


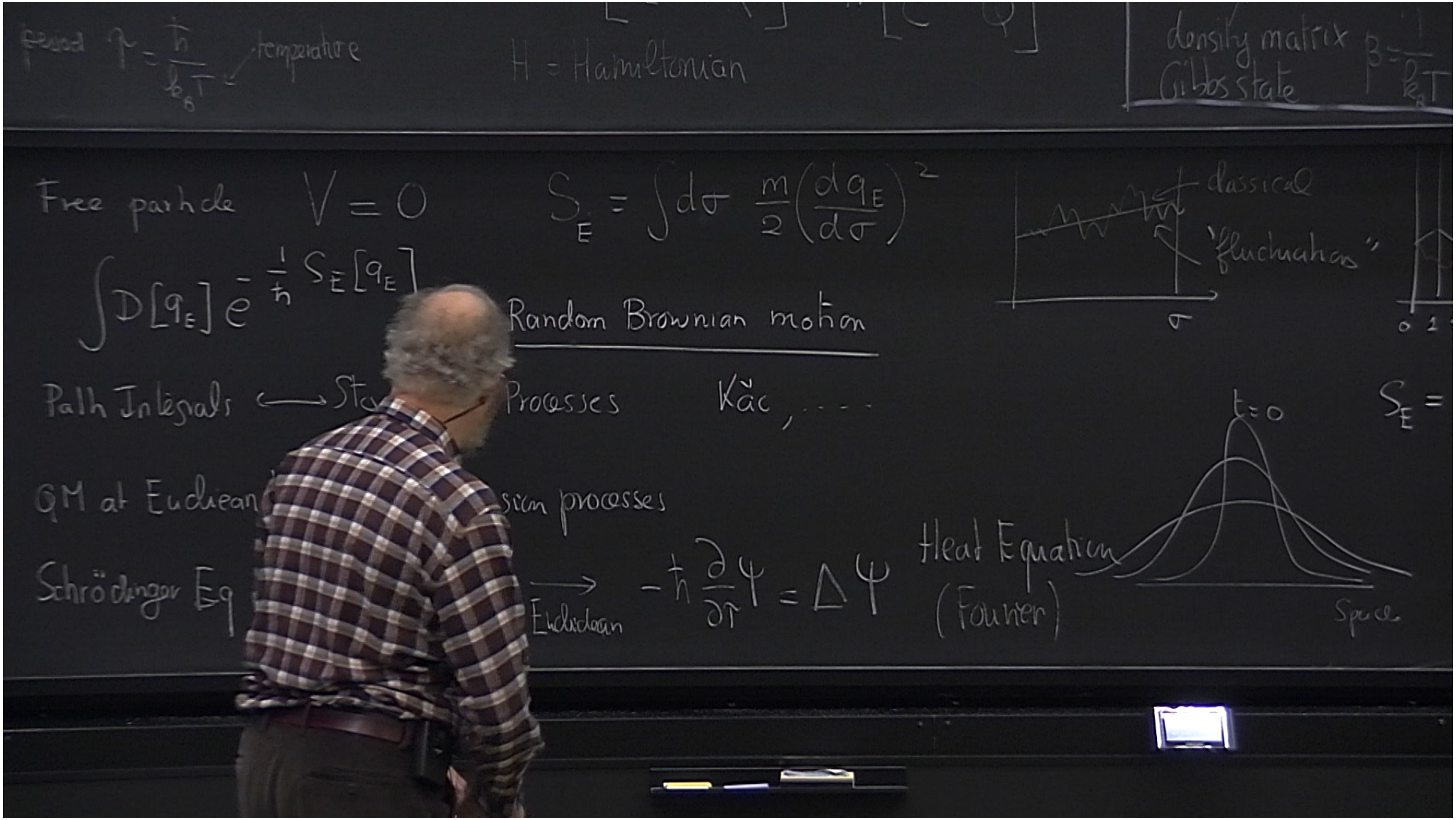
$$S_E = \sum_i \frac{m}{2} \frac{(q_{i+1} - q_i)^2}{\epsilon}$$

$$e^{-\frac{1}{\hbar} S_E} = \prod_i e^{-\left(\frac{m}{2\hbar\epsilon}\right)(q_{i+1} - q_i)^2}$$

typical trajectory q_i : stochastic process in Markov process

Heat Equation (Fourier)

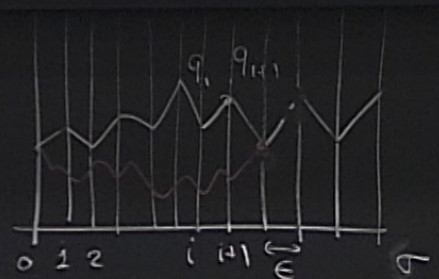



temperature
H = Hamiltonian
density matrix
Gibbs state
Free particle V = 0
classical
"fluctuations"
Random Brownian motion
Path Integrals
Processes
Kac, ...
QM at Euclidean
sion processes
Schrödinger Eq
Euclidean
Heat Equation
(Fourier)
t=0
Space

density matrix $\beta = \frac{1}{k_B T}$
Gibbs state
mixed state
periodic trajectories $\tau = \hbar \cdot \beta$

classical

"fluctuations"

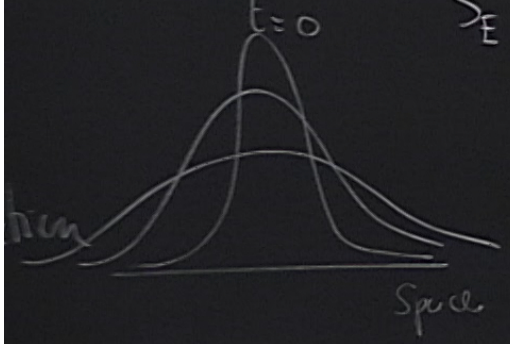


$$e^{-\frac{1}{\hbar} S_E} = \prod_i e^{-\left(\frac{m}{2\hbar\epsilon}(q_{i+1}-q_i)^2\right)}$$

typical trajectory q_i : stochastic process in time
Markov process

$$S_E = \sum_i \frac{m}{2} \frac{(q_{i+1}-q_i)^2}{\epsilon}$$

Brownian Motion

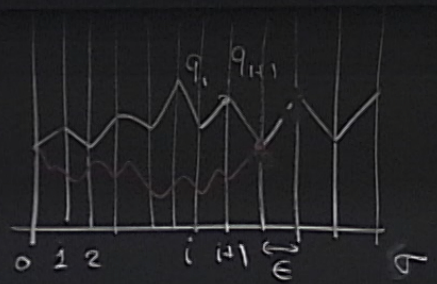


density matrix $\beta = \frac{1}{k_B T}$ mixed state periodic trajectories $\tau = \frac{1}{\hbar}\beta$

Gibbs state

classical

"fluctuations"

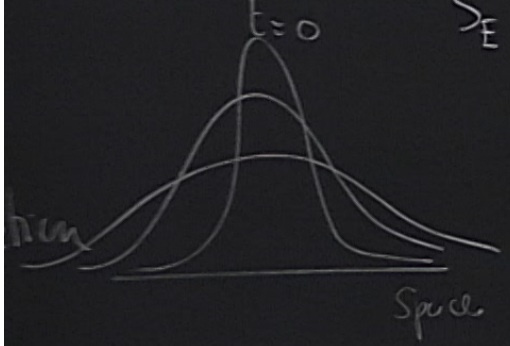


$$e^{-\frac{1}{\hbar}S_E} = \prod_i e^{-\left(\frac{m}{2\hbar\epsilon}(q_{i+1}-q_i)\right)^2}$$

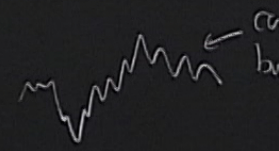
typical trajectory q_i : stochastic process in time

Markov process

$$S_E = \sum_i \frac{m}{2}\frac{(q_{i+1}-q_i)^2}{\epsilon}$$



Brownian Motion



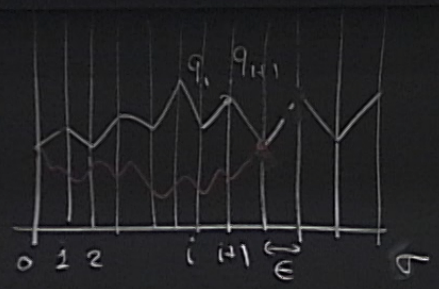
but derivative is ∞

density matrix $\beta = \frac{1}{k_B T}$ mixed state periodic trajectories $\tau = \frac{1}{\hbar}\beta$

Gibbs state

classical

"fluctuations"

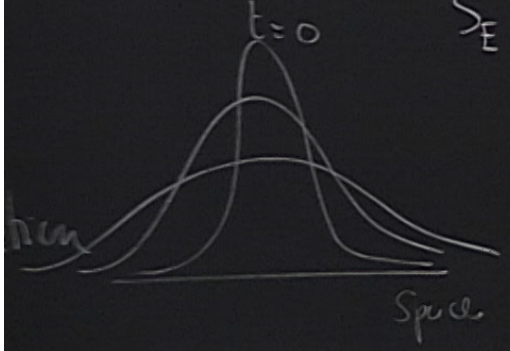


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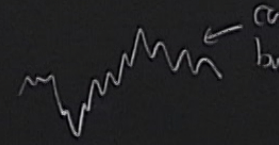
$$S_E = \sum_i \frac{m}{2}\frac{(q_{i+1}-q_i)^2}{\epsilon}$$

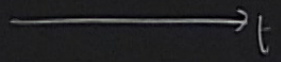
typical trajectory q_i : stochastic process in time

Markov process



Brownian Motion





Quantum Fluctuations at physical time

Stochastic or thermal fluct. at Euclidean time

Schrödinger Eq $i\hbar \frac{\partial}{\partial t}\psi = \cdot \Delta\psi \xrightarrow[\text{Euclidean}]{} -\hbar \frac{\partial}{\partial \tau}\psi = \Delta\psi$ (Fourier)

Quantum Field theory : Functional Integrals

Schrödinger Eq $i\hbar \frac{\partial}{\partial t}\psi = -\Delta\psi$ $\xrightarrow[\text{Euclidean}]{}$ $-\hbar \frac{\partial}{\partial \tau}\psi = \Delta\psi$ (Fourier)

Quantum Field theory : Functional Integrals

Free scalar field (Klein-Gordon Field)

Relativistic mass m (particles)



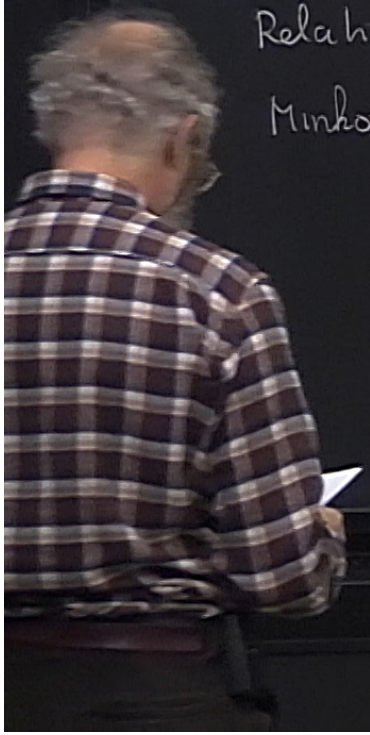
Schrödinger Eq $i\hbar \frac{\partial}{\partial t}\psi = \cdot \Delta\psi$ $\xrightarrow[\text{Euclidean}]{}$ $-\hbar \frac{\partial}{\partial \tau}\psi = \Delta\psi$ (Fourier)

Quantum Field theory : Functional Integrals

Free scalar field (Klein-Gordon Field)

Relativistic mass m (particles)

Minkowski $\mathbb{M}^{1,d-1}$ $ds^2 = -dt^2 + d\vec{x}^2 = h_{\mu\nu} dx^\mu dx^\nu$ $h = \begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix}$



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Quantum Field theory : Functional Integrals

Free scalar field (Klein-Gordon Field)

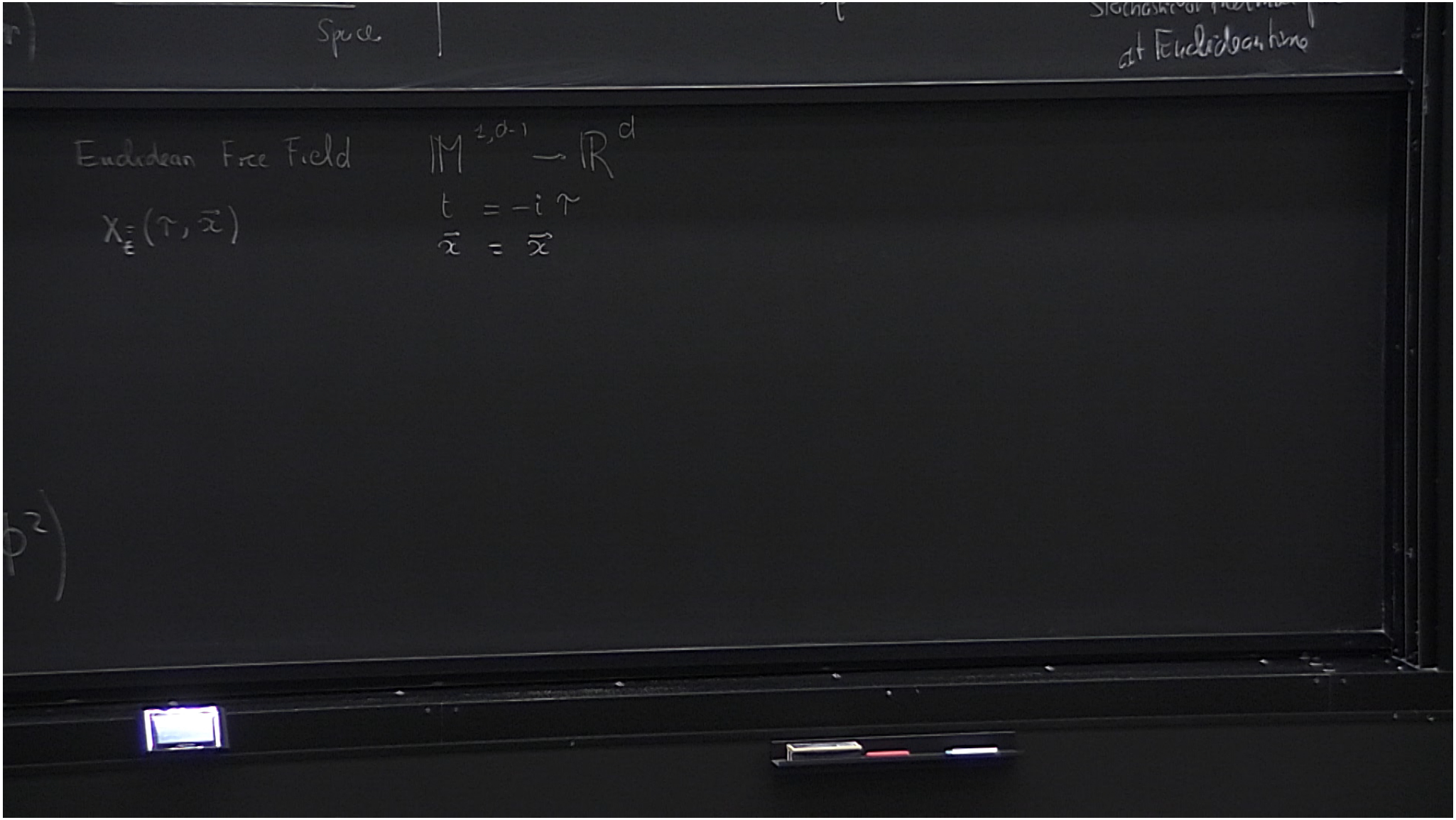
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Classical Field $\quad \phi = \{\phi(x) : x \in \mathbb{M}^{1,d-1}\}$

Action $\int dt \int d^{d-1}\vec{x} \left[\frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 - \left(\frac{1}{2}(\vec{\nabla}_{\vec{x}}\phi)^2 + \frac{m}{2}\phi^2\right)\right] = \int d^d x \left(\frac{-1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2\right)$

$X = (t, \vec{x}) = (x^\mu)$ Poincaré Symm. obvious


Spec
at Euclidean time
Euclidean Free Field
$\mathbb{M}^{1,d-1} \to \mathbb{R}^d$
$t = -i\tau$
$\vec{x} = \vec{x}$
$X_\varepsilon(\tau, \vec{x})$
$p^2)$

Space

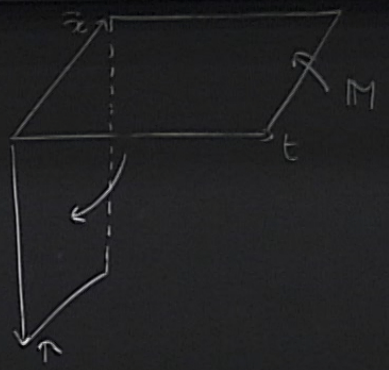
at Euclidean time

Euclidean Free Field $\mathbb{M}^{1,d-1} \to \mathbb{R}^d$

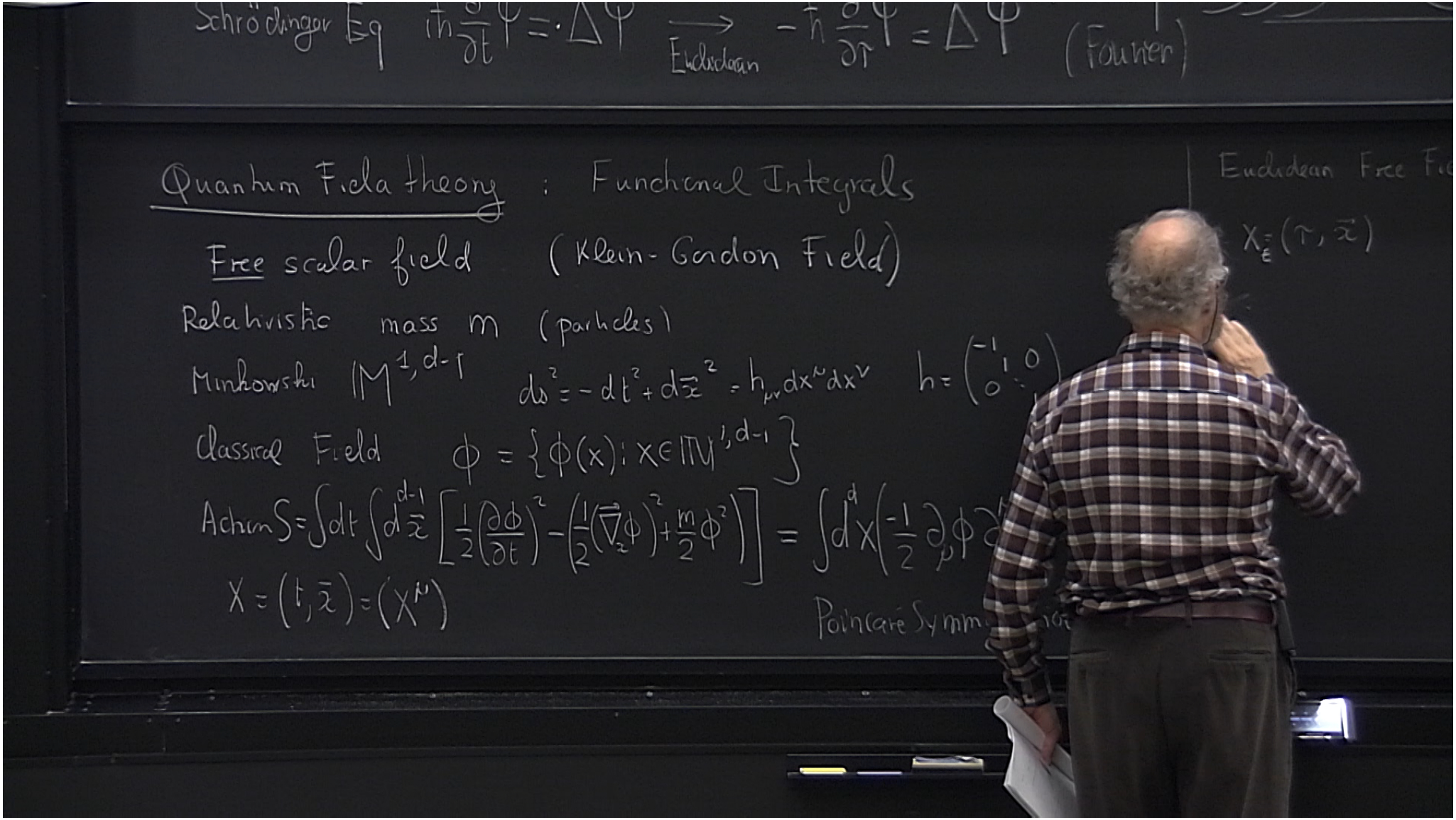
$X_E(\tau, \vec{x})$

$t = -i\tau$

$\vec{x} = \vec{x}$



Black Board = Complex time


Schrödinger Eq $i\hbar\frac{\partial}{\partial t}\psi = -\Delta\psi$
Euclidean
$-\hbar\frac{\partial}{\partial \tau}\psi = \Delta\psi$
(Fourier)
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$X = (t, \vec{x}) = (x^\mu)$
Poincaré Symm
Euclidean Free Fi
$X_E = (\tau, \vec{x})$

Space

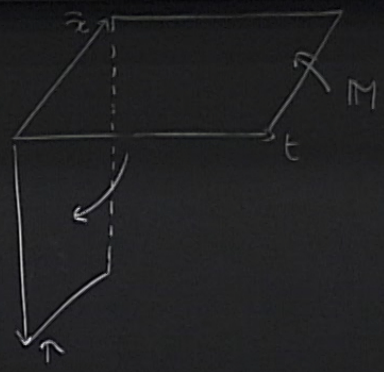
at Euclidean time

Euclidean Free Field

$\mathbb{M}^{1,d-1} \to \mathbb{R}^d$

$t = -i\tau$

$\vec{x} = \vec{x}$



Black Board = Complex time

$X_E(\tau, \vec{x})$

$\phi(X_E)$

$S_E[\phi_E]$

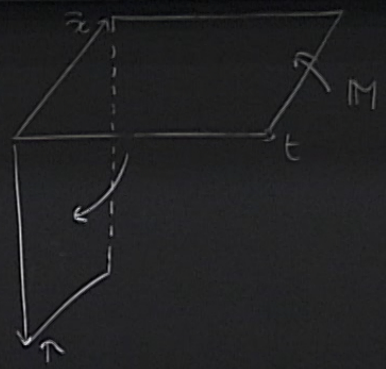
Space

at Euclidean time

Euclidean Free Field $\mathbb{M}^{1,d-1} \to \mathbb{R}^d$

$t = -i\tau$

$\vec{x} = \vec{x}$



Black Board = Complex time

$X_E(\tau, \vec{x})$

$\Phi(X_E)$

$S_E[\phi_E] = \int d\tau \int d^{d-1}\vec{x}$

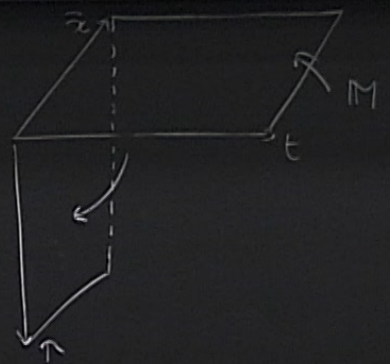
Space

at Euclidean time

Euclidean Free Field $\mathbb{M}^{1,d-1} \to \mathbb{R}^d$

$t = -i\tau$

$\vec{x} = \vec{x}$



Black Board = Complex time

$X_E(\tau, \vec{x})$

$\phi(X_E)$

$$S_E[\phi_E] = \int d\tau \int d^{d-1}\vec{x}\ \frac{1}{2}\left(\frac{\partial \phi_E}{\partial \tau}\right)^2 + \frac{1}{2}\left(\nabla_{\vec{x}}\phi_E\right)^2 + \frac{1}{2}m^2\phi^2$$

Space

at Euclidean time

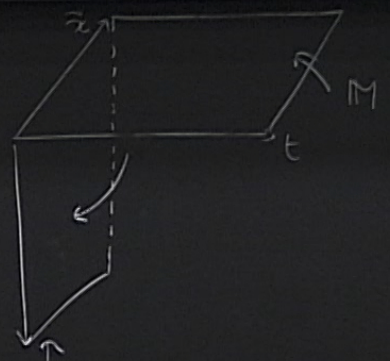
Euclidean Free Field $\mathbb{M}^{1,d-1} \to \mathbb{R}^d$

$t = -i\tau$

$\vec{x} = \vec{x}$

$X_E(\tau, \vec{x})$

$\Phi(X_E)$



Black Board = Complex time

$$S_E[\phi_E] = \int d\tau \int d^{d-1}\vec{x}\ \frac{1}{2}\left(\frac{\partial \phi_E}{\partial \tau}\right)^2 + \frac{1}{2}\left(\nabla_{\vec{x}}\phi_E\right)^2 + \frac{1}{2}m^2\phi^2$$

$$= \int_{\mathbb{R}^d} d^d X_E$$

Schrödinger Eq $i\hbar \frac{\partial}{\partial t}\psi = -\Delta\psi$ $\xrightarrow[\text{Euclidean}]{}$ $-\hbar \frac{\partial}{\partial \tau}\psi = \Delta\psi$ (Fourier)

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$X = (t, \vec{x}) = (x^\mu)$

Poincaré Symm. obvious

Euclidean Free Fi

$X_E = (\tau, \vec{x})$

$\phi(X_E)$

$S_E[\phi_E] =$

Space

at Euclidean time

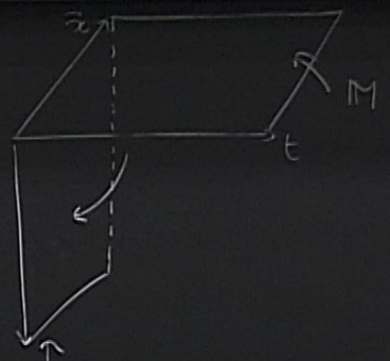
Euclidean Free Field $\mathbb{M}^{1,d-1} \to \mathbb{R}^d$

$t = -i\tau$

$\vec{x} = \vec{x}$

$X_E(\tau, \vec{x})$

$\phi(X_E)$



Black Board = Complex time

$$S_E[\phi_E] = \int d\tau \int d^{d-1}\vec{x}\ \frac{1}{2}\left(\frac{\partial \phi_E}{\partial \tau}\right)^2 + \frac{1}{2}\left(\nabla_{\vec{x}}\phi_E\right)^2 + \frac{1}{2}m^2\phi^2$$

$$= \int_{\mathbb{R}^d} d^d X_E \left(\frac{1}{2}\partial_\nu \phi_E \partial^\nu \phi_E + \frac{m^2}{2}\phi^2\right)$$

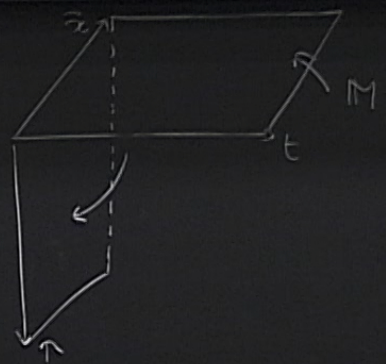
Space

at Euclidean time

Euclidean Free Field $\quad \mathbb{M}^{1,d-1} \to \mathbb{R}^d$

$t = -i\tau$

$\vec{x} = \vec{x}$



Black Board = Complex time

$X_E(\tau, \vec{x})$

$\Phi(X_E)$

$$S_E[\phi_E] = \int d\tau \int d^{d-1}\vec{x}\ \frac{1}{2}\left(\frac{\partial \phi_E}{\partial \tau}\right)^2 + \frac{1}{2}\left(\nabla_{\vec{x}}\phi_E\right)^2 + \frac{1}{2}m^2\phi^2$$

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Euclidean Symm.

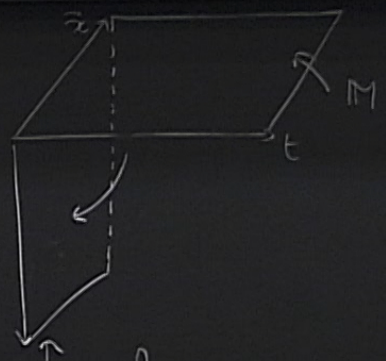
Space

at Euclidean time

Euclidean Free Field $\quad \mathbb{M}^{1,d-1} \rightsquigarrow \mathbb{R}^d$

$t = -i\tau$

$\vec{x} = \vec{x}$



Black Board = Complex time

$X_E(\tau, \vec{x})$

$\Phi(X_E)$

$$S_E[\phi_E] = \int d\tau \int d^{d-1}\vec{x}\ \frac{1}{2}\left(\frac{\partial \phi_E}{\partial \tau}\right)^2 + \frac{1}{2}\left(\nabla_{\vec{x}}\phi_E\right)^2 + \frac{1}{2}m^2\phi^2$$

$$= \int_{\mathbb{R}^d} d^d X_E \left(\frac{1}{2}\partial_\nu \phi_E \partial^\nu \phi_E + \frac{m^2}{2}\phi^2\right)$$

Euclidean Symm

Classical e.o.m

$$-\Delta \phi_E + m^2 \phi_E = 0$$

$$\Delta = \sum_{\mu=1}^{d} \left(\frac{\partial}{\partial X_E^\mu}\right)^2$$

Space

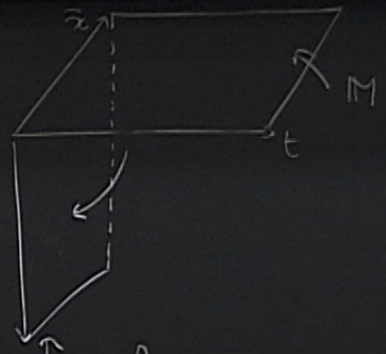
at Euclidean time

Euclidean Free Field

$\mathbb{M}^{1,d-1} \to \mathbb{R}^d$

$t = -i\tau$

$\vec{x} = \vec{x}$



Black Board = Complex time

$X_E(\tau, \vec{x})$

Classical

$\phi(X_E)$

$$S_E[\phi_E] = \int d\tau \int d^{d-1}\vec{x}\ \frac{1}{2}\left(\frac{\partial \phi_E}{\partial \tau}\right)^2 + \frac{1}{2}\left(\vec{\nabla}_{\vec{x}}\phi_E\right)^2 + \frac{1}{2}m^2\phi^2$$

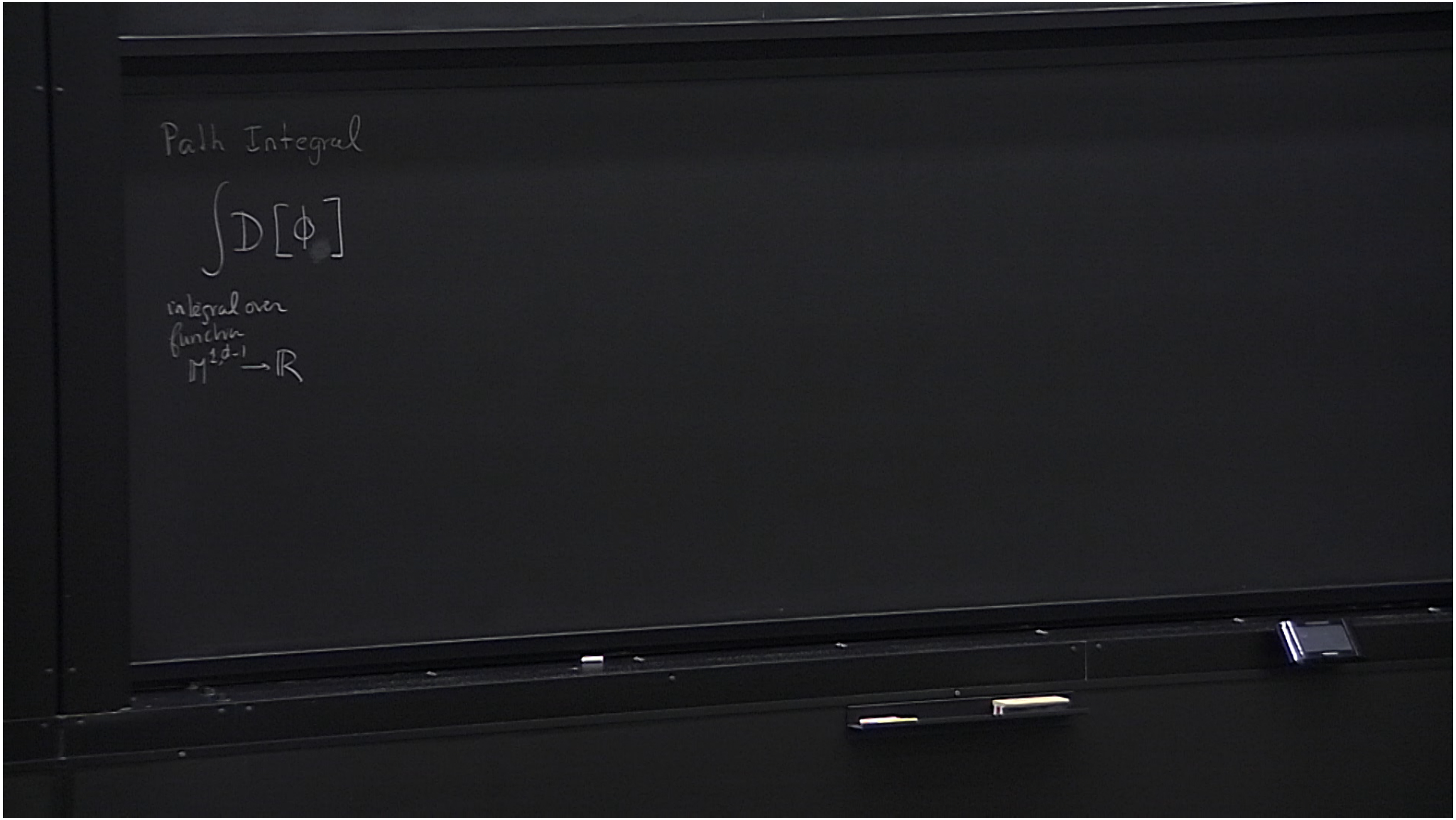
$$= \int_{\mathbb{R}^d} d^d X_E \left(\frac{1}{2}\partial_\nu \phi_E \partial^\nu \phi_E + \frac{m^2}{2}\phi^2\right)$$

Euclidean Symm

Classical e.o.m

$$-\Delta \phi_E + m^2 \phi_E = 0$$

$$\Delta = \sum_{\mu=1}^{d} \left(\frac{\partial}{\partial X_E^\mu}\right)^2$$


Path Integral
$\int D[\phi]$
integral over
functions
$M^{1,d-1} \to \mathbb{R}$

Schrödinger Eq $i\hbar \frac{\partial}{\partial t}\psi = -\Delta\psi$ $\xrightarrow[\text{Euclidean}]{}$ $-\hbar \frac{\partial}{\partial \tau}\psi = \Delta\psi$ (Fourier)

Quantum Field theory : Functional Integrals

Free scalar field (Klein-Gordon Field)

Relativistic mass m (particles)

Minkowski $\mathbb{M}^{1,d-1}$ $\quad ds^2 = -dt^2 + d\vec{x}^2 = h_{\mu\nu}dx^\mu dx^\nu \quad h = \begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix}$

Classical Field $\phi = \{\phi(x) : x \in \mathbb{M}^{1,d-1}\}$ $\quad \phi$ is real, no charges

Action $S = \int dt \int d^{d-1}\vec{x} \left[\frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 - \left(\frac{1}{2}(\vec{\nabla}_{\vec{x}}\phi)^2 + \frac{m}{2}\phi^2\right)\right] = \int_{\mathbb{M}} d^dx \left(\frac{-1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2\right)$

$X = (t, \vec{x}) = (x^\mu)$

Poincaré Symm. obvious

Euclidean Free Fi

$X_E = (\tau, \vec{x})$

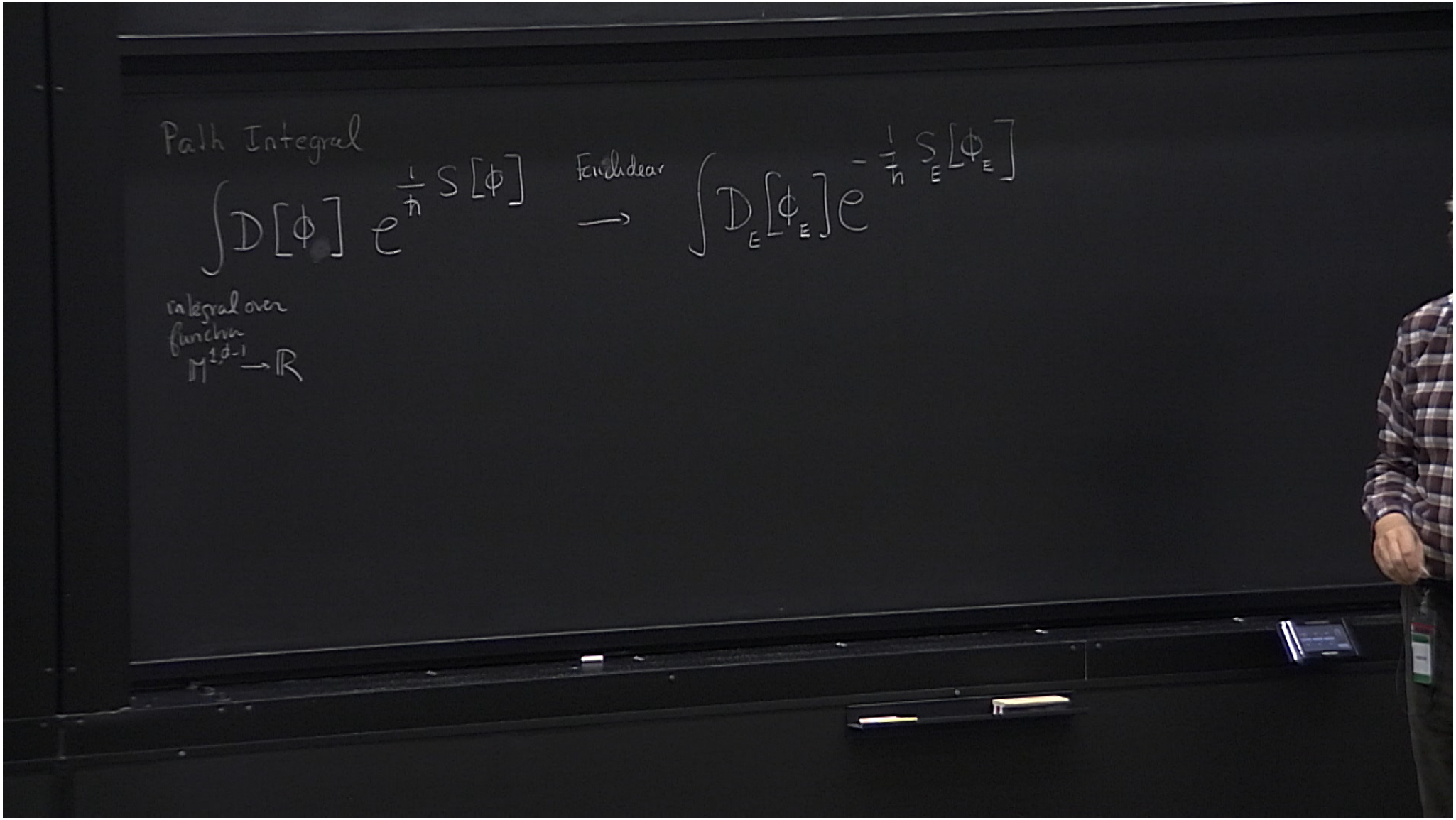
$\phi(X_E)$

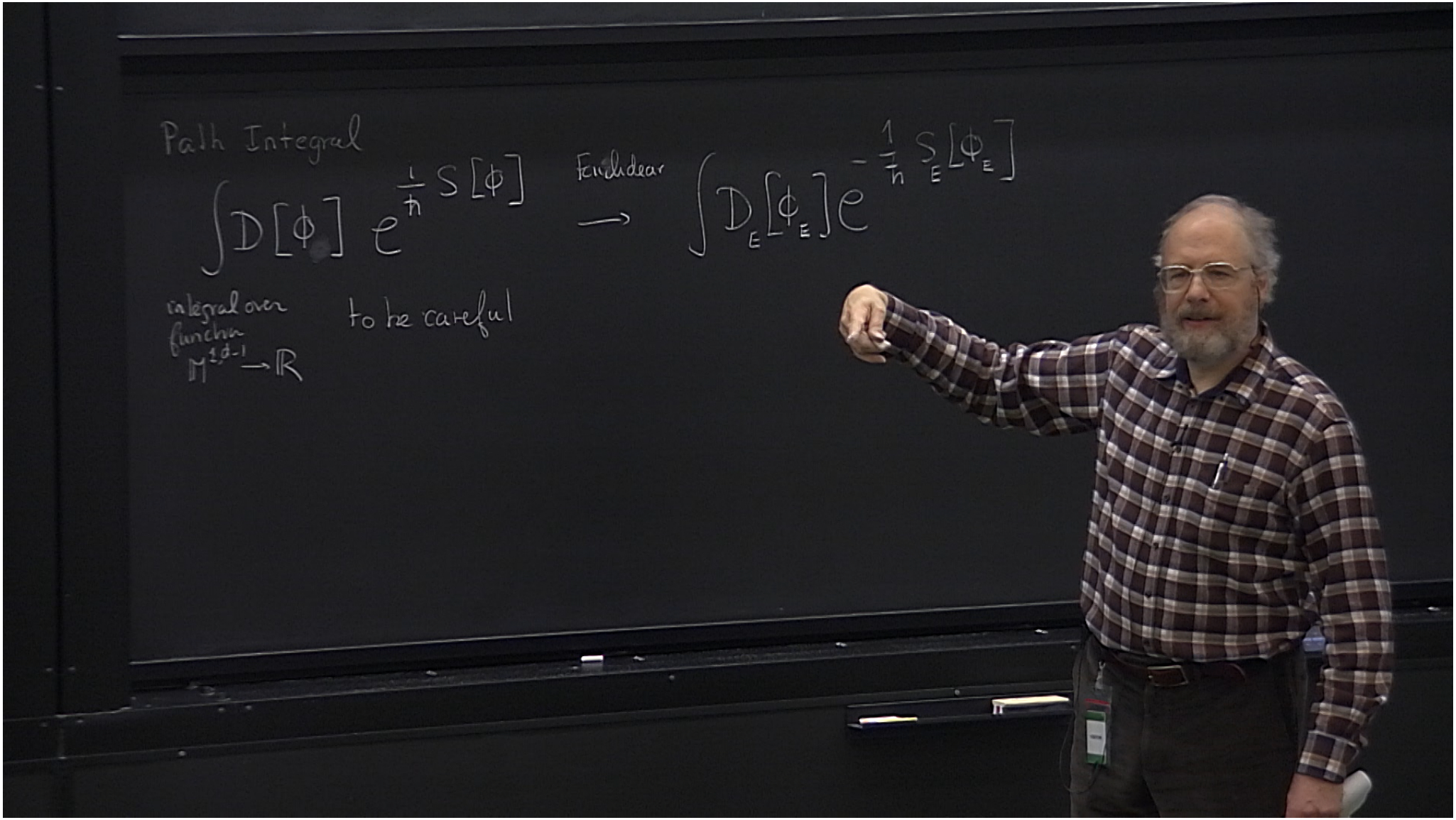
$S_E[\phi_E] = \int d\tau$

$= \int d^d$

$\mathbb{R}^d$

Eucl

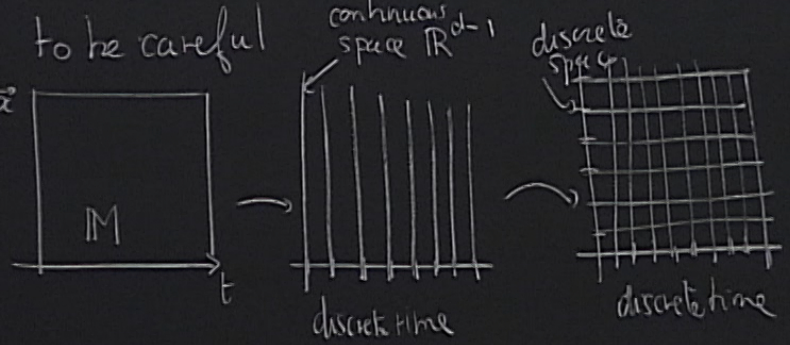

Path Integral
$\int D[\phi]\, e^{\frac{1}{\hbar} S[\phi]}$
Euclidean
$\longrightarrow$
$\int D_E[\phi_E]\, e^{-\frac{1}{\hbar} S_E[\phi_E]}$
integral over
functions
$M^{1,d-1} \to \mathbb{R}$


Path Integral
$\int D[\phi]\, e^{\frac{1}{\hbar} S[\phi]}$
Euclidean
$\longrightarrow$
$\int D_E[\phi_E]\, e^{-\frac{1}{\hbar} S_E[\phi_E]}$
integral over functions $M^{1,d-1} \to \mathbb{R}$
to be careful

Path Integral

$$\int \mathcal{D}[\phi]\, e^{\frac{i}{\hbar} S[\phi]} \quad \text{Euclidean} \longrightarrow \int \mathcal{D}_E[\phi_E]\, e^{-\frac{1}{\hbar} S_E[\phi_E]}$$

to be careful

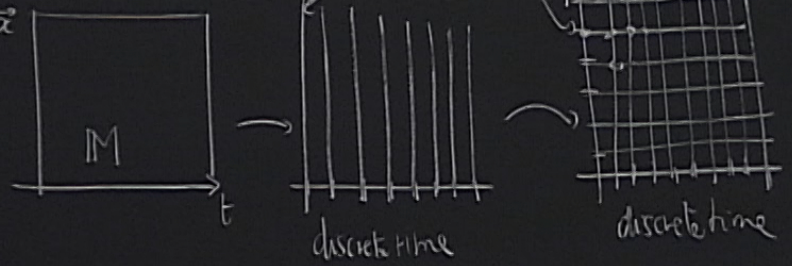


Path Integral

$$\int \mathcal{D}[\phi]\, e^{\frac{i}{\hbar} S[\phi]} \xrightarrow{\text{Euclidean}} \int \mathcal{D}_E[\phi_E]\, e^{-\frac{1}{\hbar} S_E[\phi_E]}$$

integral over functions $M^{1,d-1} \to \mathbb{R}$

to be careful

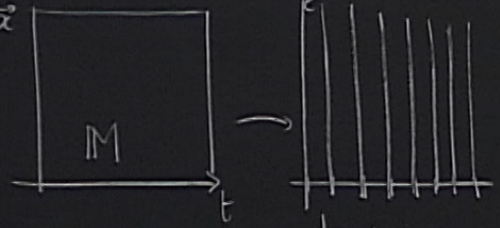


Path Integral

$$\int \mathcal{D}[\phi]\, e^{\frac{i}{\hbar} S[\phi]} \xrightarrow{\text{Euclidean}} \int \mathcal{D}_E[\phi_E]\, e^{-\frac{1}{\hbar} S_E[\phi_E]}$$

integral over functions $M^{1,d-1} \to \mathbb{R}$

to be careful



continuous space $\mathbb{R}^{d-1}$

discrete time

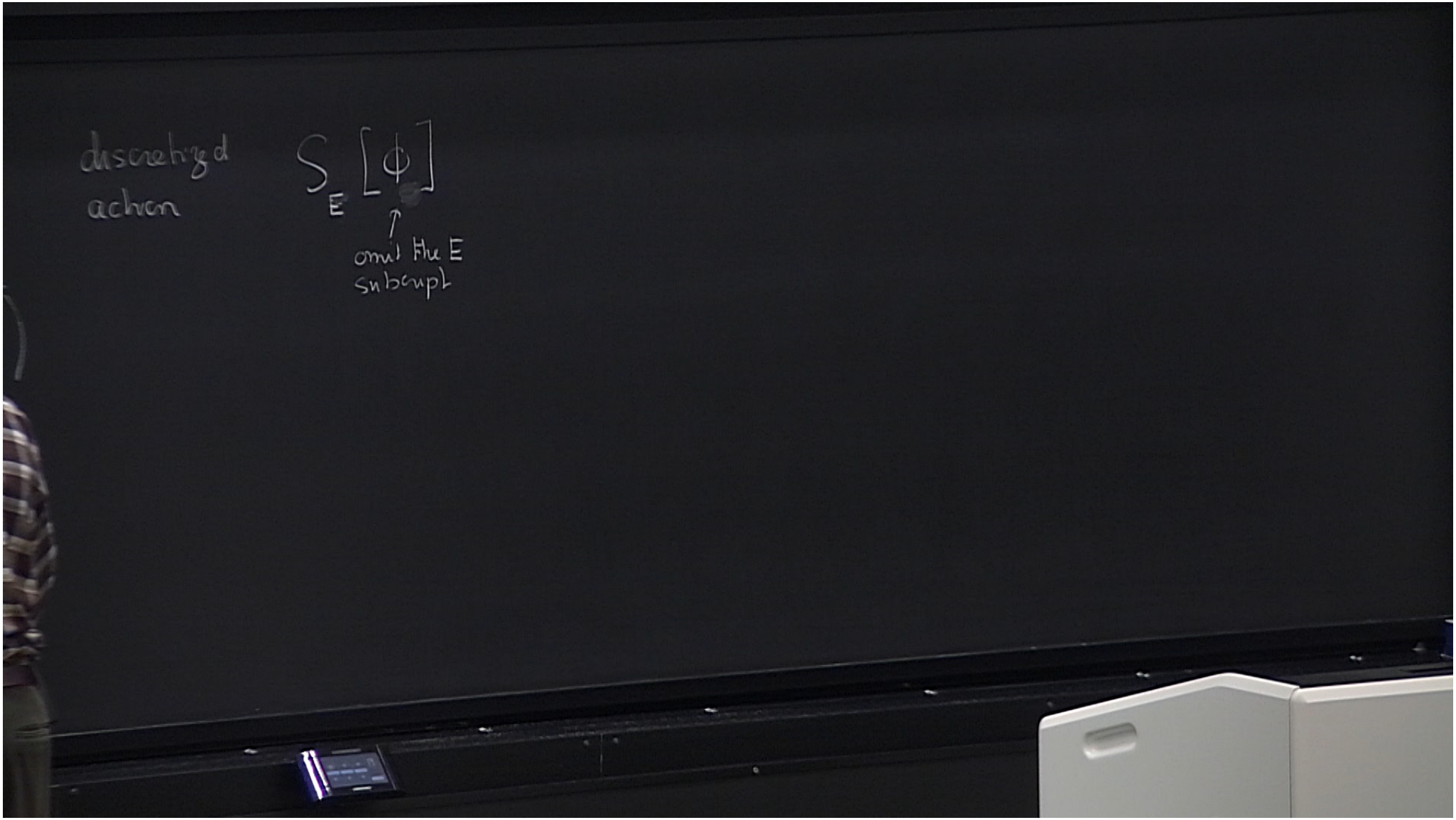
discrete space

discrete time

Lattice (square hypercubic for simplicity)

$\vec{x}$ $\in$ discrete space $\vec{N} = \mathbb{Z}^d$

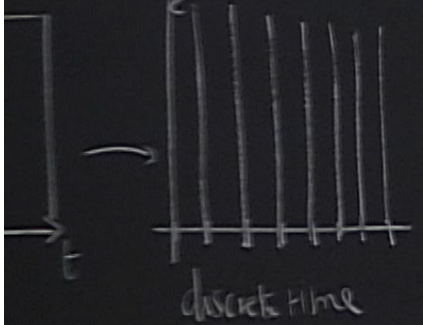
$\vec{t} \in \vec{N}$ discrete time step/interval

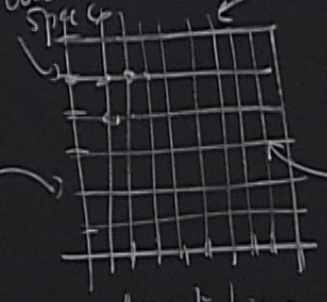

discretized
action
$S_E[\phi]$
omit the E
subscript

$S[\phi]$ Euclidean $\longrightarrow$ $\int D_E[\phi_E] e^{-\frac{1}{\hbar} S_E[\phi_E]}$ discretized action $S_E[\phi]$

omit the E subscript

useful



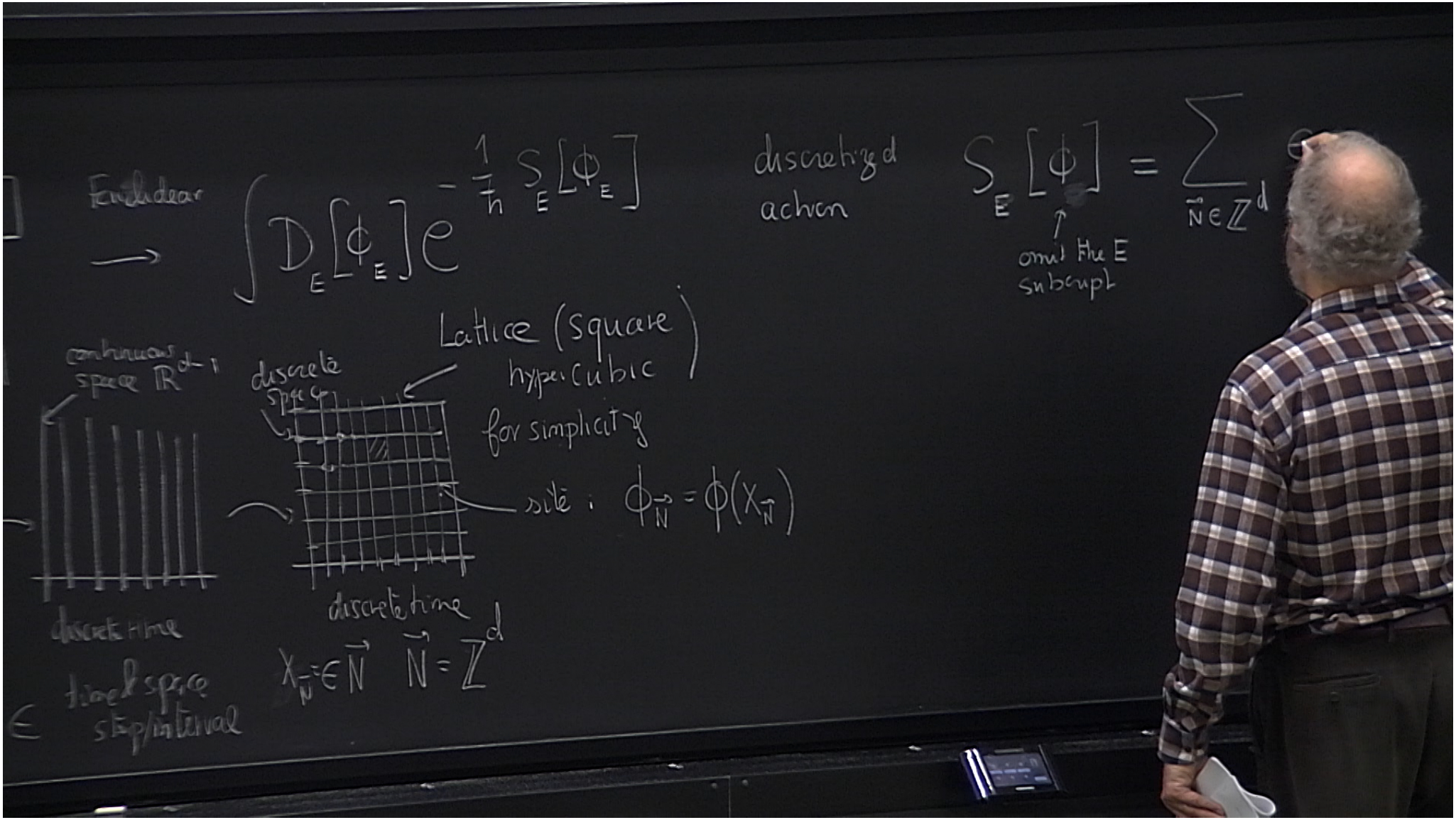


Lattice (square hypercubic) for simplicity

site : $\phi_{\vec{N}} = \phi(X_{\vec{N}})$

ϵ fixed space step/interval

$X_{\vec{N}} = \epsilon \vec{N}$ $\vec{N} = \mathbb{Z}^d$


Euclidean
$\int D_E[\phi_E] e^{-\frac{1}{\hbar} S_E[\phi_E]}$
discretized action
$S_E[\phi] = \sum_{\vec{N} \in \mathbb{Z}^d}$
ϵ
omit the E subscript
continuous space $\mathbb{R}^{d-1}$
discrete time
time/space step/interval
ϵ
discrete space
Lattice (square hypercubic) for simplicity
site : $\phi_{\vec{N}} = \phi(x_{\vec{N}})$
discrete time
$x_{\vec{N}} = \epsilon \vec{N}$ $\vec{N} = \mathbb{Z}^d$

Euclidean

$$\int \mathcal{D}_E[\phi_E]\, e^{-\frac{1}{\hbar} S_E[\phi_E]}$$

discretized action

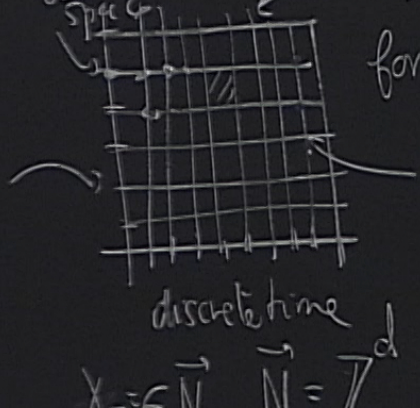
$$S_E[\phi] = \sum_{\vec{N} \in \mathbb{Z}^d} \epsilon^d$$

ϕ: omit the E subscript

continuous space $\mathbb{R}^{d-1}$

discrete time

time/space step/interval ϵ



$X_{\vec{N}} \in \vec{N}$ $\vec{N} = \mathbb{Z}^d$

Euclidean

$$\int \mathcal{D}_E[\phi_E]\, e^{-\frac{1}{\hbar} S_E[\phi_E]}$$

discretized action

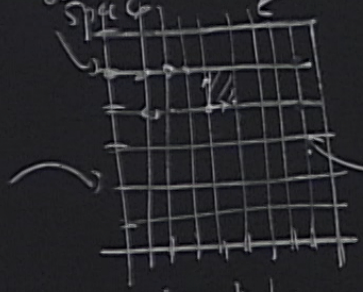
$$S_E[\phi] = \sum_{\vec{N} \in \mathbb{Z}^d} \epsilon^d \frac{1}{2}$$

ϕ: omit the E subscript

continuous space $\mathbb{R}^{d-1}$

discrete time

time / space step/interval ϵ

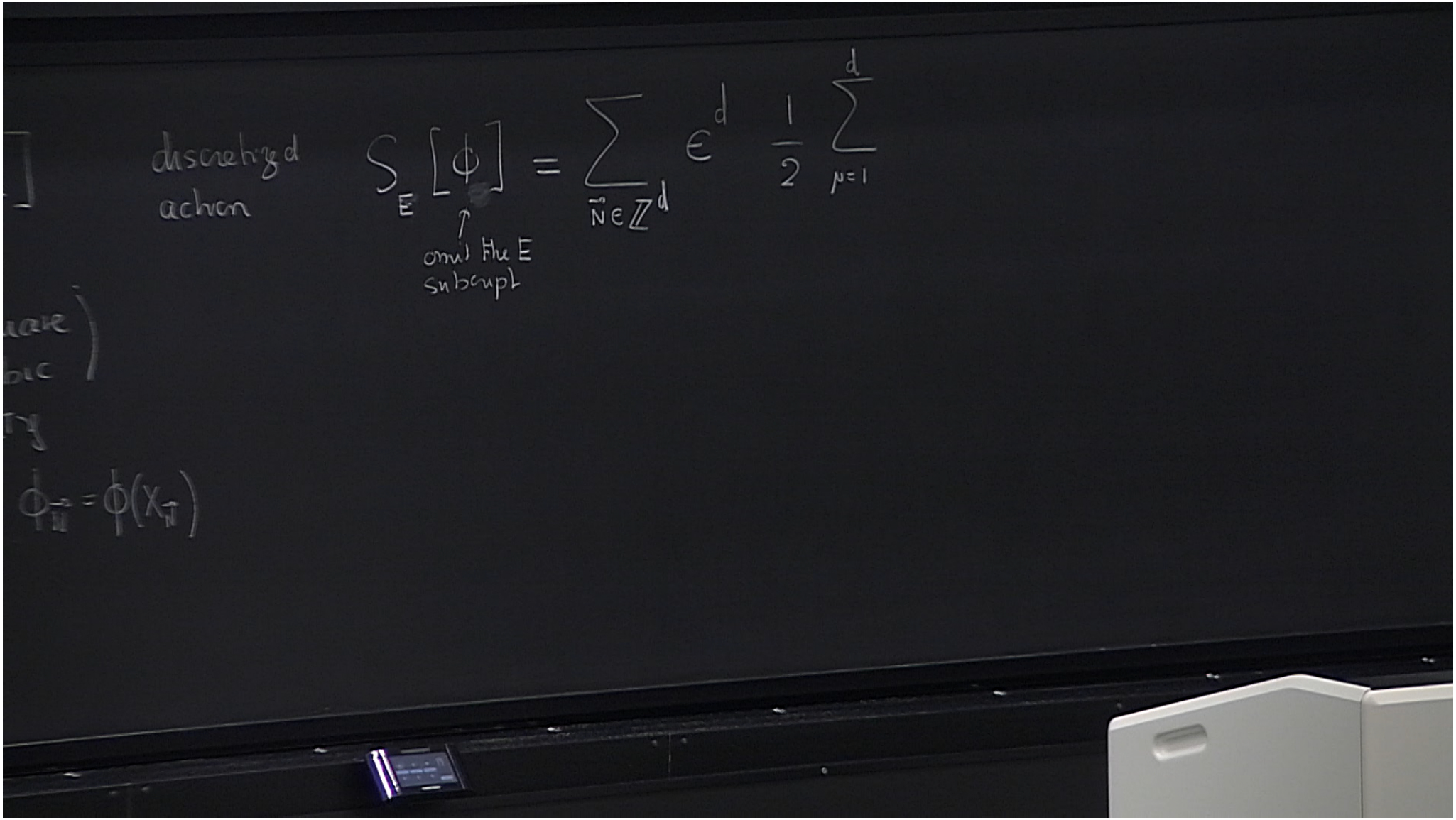


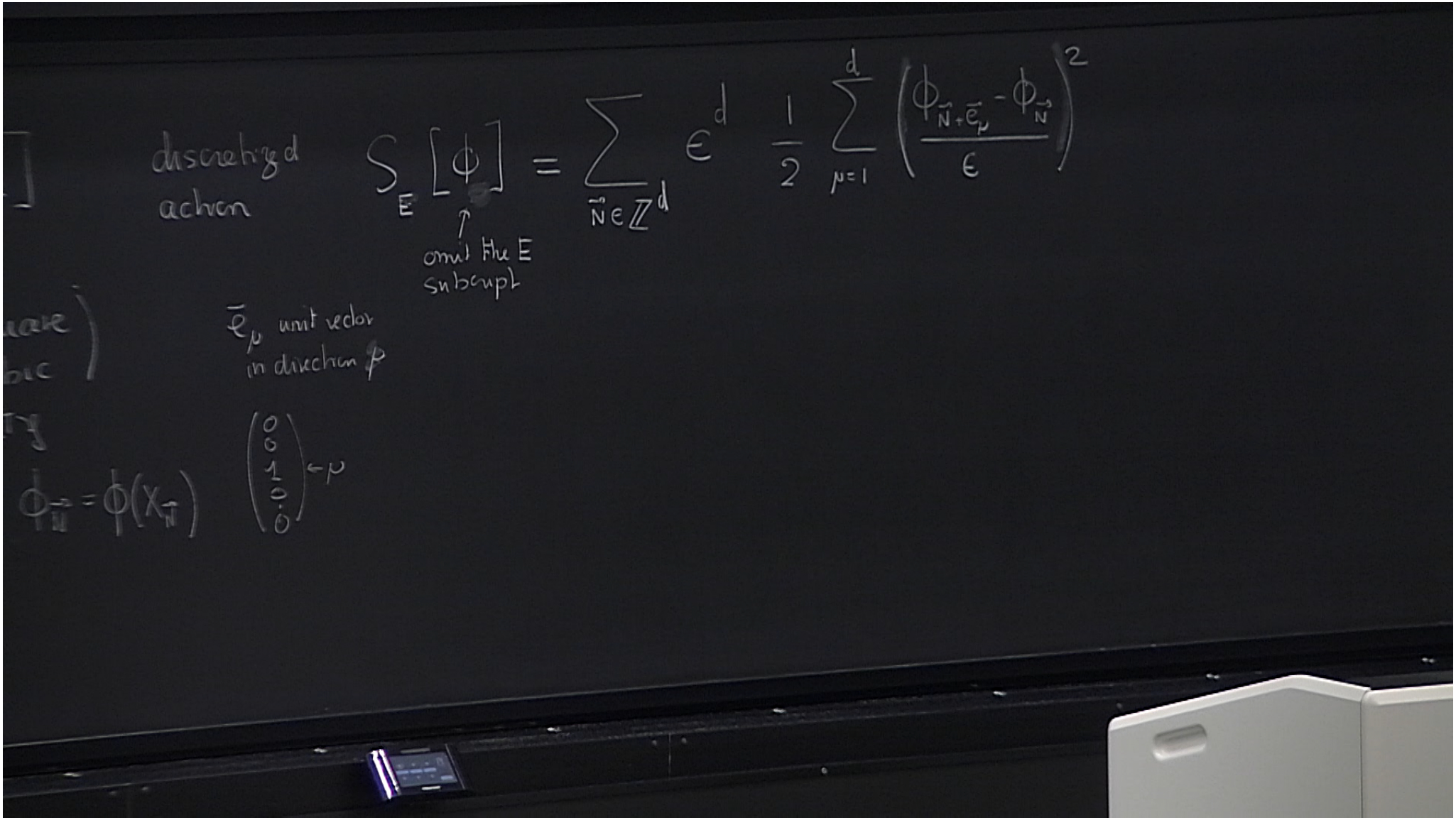
Lattice (square hypercubic) for simplicity

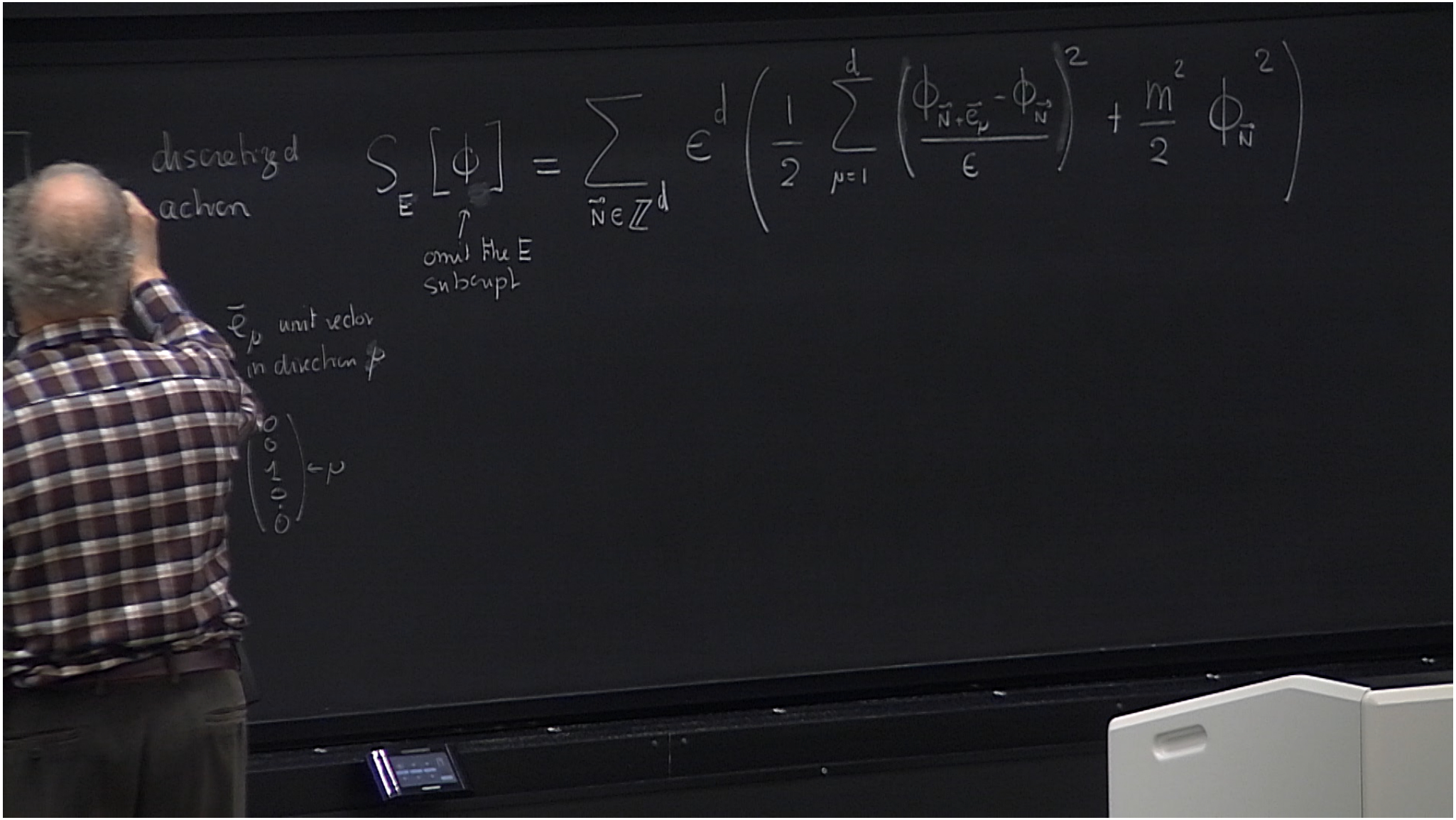
site: $\phi_{\vec{N}} = \phi(X_{\vec{N}})$

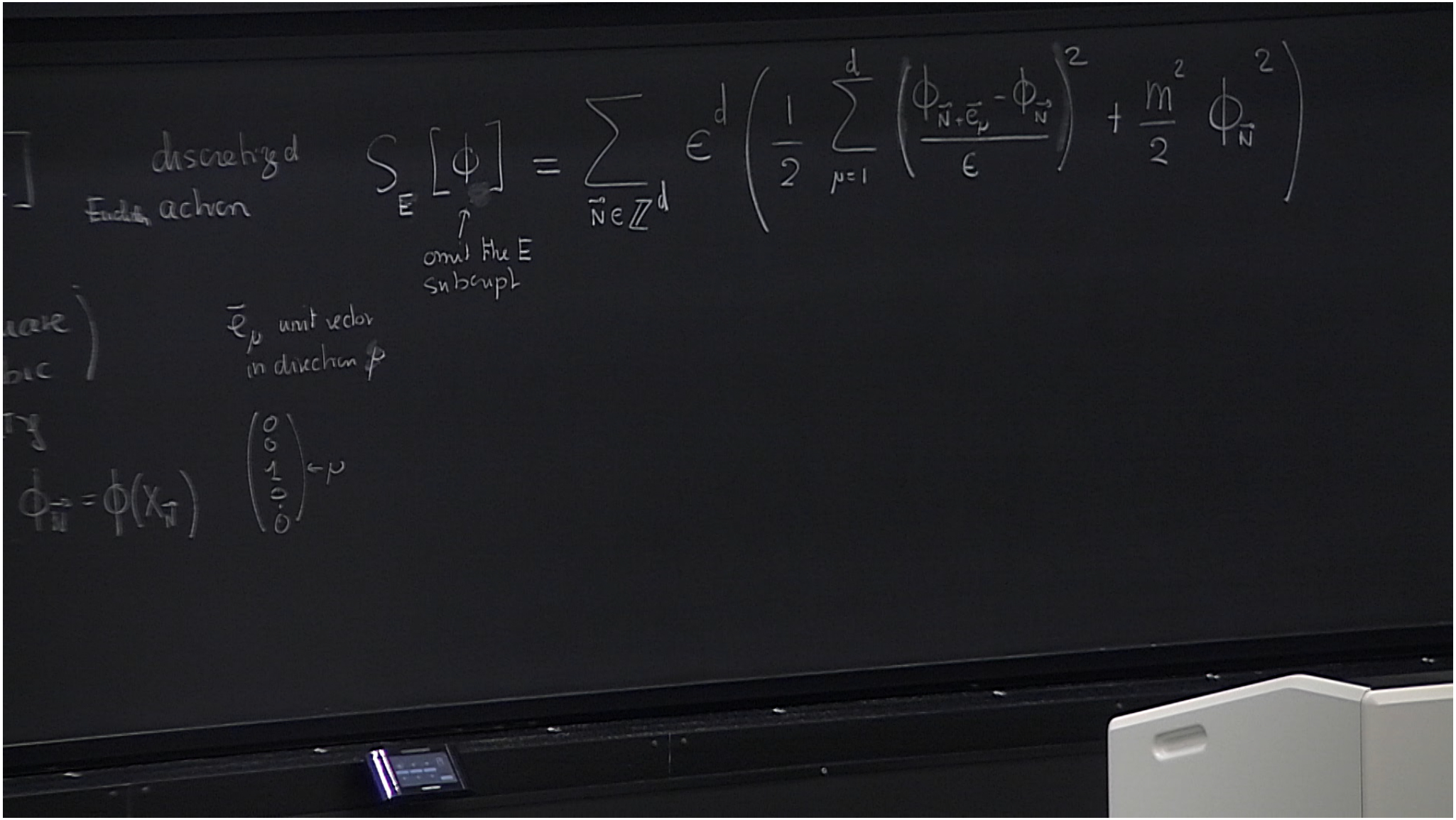
discrete time

$X_{\vec{N}} = \epsilon \vec{N} \quad \vec{N} = \mathbb{Z}^d$


discretized
action
$S_E[\phi] = \sum_{\vec{N} \in \mathbb{Z}^d} \epsilon^d \frac{1}{2} \sum_{\mu=1}^{d}$
omit the E
subscript
$\phi_{\vec{N}} = \phi(x_{\vec{N}})$


discretized action
$S_E[\phi] = \sum_{\vec{N} \in \mathbb{Z}^d} \epsilon^d \frac{1}{2} \sum_{\mu=1}^{d} \left(\frac{\phi_{\vec{N}+\vec{e}_\mu} - \phi_{\vec{N}}}{\epsilon} \right)^2$
omit the E subscript
$\vec{e}_\mu$ unit vector in direction μ
$\phi_{\vec{N}} = \phi(x_{\vec{N}})$
$\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow \mu$


discretized action
$S_E[\phi] = \sum_{\vec{N} \in \mathbb{Z}^d} \epsilon^d \left(\frac{1}{2} \sum_{\mu=1}^{d} \left(\frac{\phi_{\vec{N}+\vec{e}_\mu} - \phi_{\vec{N}}}{\epsilon} \right)^2 + \frac{m^2}{2} \phi_{\vec{N}}^2 \right)$
omit the E subscript
$\vec{e}_\mu$ unit vector in direction μ
$\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \leftarrow \mu$


discretized
Euclidean action
$S_E[\phi] = \sum_{\vec{N} \in \mathbb{Z}^d} \epsilon^d \left(\frac{1}{2} \sum_{\mu=1}^{d} \left(\frac{\phi_{\vec{N}+\vec{e}_\mu} - \phi_{\vec{N}}}{\epsilon} \right)^2 + \frac{m^2}{2} \phi_{\vec{N}}^2 \right)$
omit the E subscript
$\vec{e}_\mu$ unit vector in direction μ
$\phi_{\vec{N}} = \phi(x_{\vec{N}})$
$\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow \mu$

discretized Euclidean action

$$S_E[\phi] = \sum_{\vec{N} \in \mathbb{Z}^d} \epsilon^d \left(\frac{1}{2} \sum_{\mu=1}^{d} \left(\frac{\phi_{\vec{N}+\vec{e}_\mu} - \phi_{\vec{N}}}{\epsilon} \right)^2 + \frac{m^2}{2} \phi_{\vec{N}}^2 \right)$$

↑ omit the E subscript

$\vec{e}_\mu$ unit vector in direction μ

$$\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow \mu$$

$\phi_{\vec{N}} = \phi(x_{\vec{N}})$

$$S_E[\phi] = \sum_{\vec{N}\in\mathbb{Z}^d} \epsilon^d \left(\frac{1}{2} \sum_{\mu=1}^{d} \left(\frac{\phi_{\vec{N}+\vec{e}_\mu} - \phi_{\vec{N}}}{\epsilon} \right)^2 + \frac{m^2}{2} \phi_{\vec{N}}^2 \right)$$

↑ omit the E subscript

Measure

$$D_E[\phi] = \prod_{\vec{N}\in\mathbb{Z}^d} \left[d\phi_{\vec{N}} \left(\frac{2\pi\hbar}{\epsilon^{d-2}} \right)^{-1/2} \right]$$

product of local flat measure at each site

$$S_E[\phi] = \sum_{\vec{N}\in\mathbb{Z}^d} \epsilon^d \left(\frac{1}{2} \sum_{\mu=1}^{d} \left(\frac{\phi_{\vec{N}+\vec{e}_\mu} - \phi_{\vec{N}}}{\epsilon} \right)^2 + \frac{m^2}{2} \phi_{\vec{N}}^2 \right)$$

omit the E subscript

unit vector direction μ

Measure

$$D_E[\phi] = \prod_{\vec{N}\in\mathbb{Z}^d} \left[d\phi_{\vec{N}} \left(\frac{2\pi\hbar}{\epsilon^{d-2}} \right)^{-1/2} \right]$$

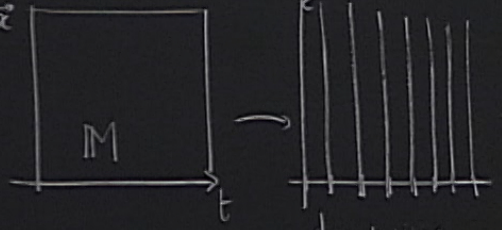
product of local flat measure at each site

Path Integral

$$\int \mathcal{D}[\phi]\, e^{\frac{1}{\hbar} S[\phi]}$$

integral over function $M^{1,d-1} \to \mathbb{R}$

to be careful



$\vec{X}$

Euclidean

$$\int \mathcal{D}_E[\phi_E]\, e^{-\frac{1}{\hbar} S_E[\phi_E]}$$

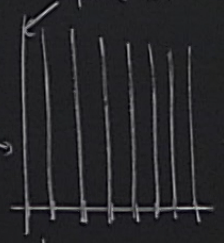
discretized Euclidean action

$$S_E[\phi] = \Delta$$

$\vec{N}$

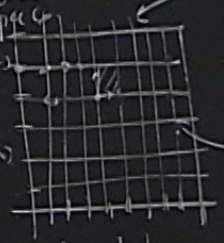
omit the E subscript

continuous space $\mathbb{R}^{d-1}$



$\in$ time/space step/interval

Lattice (square hypercubic for simplicity)



site : $\phi_{\vec{N}} = \phi(X_{\vec{N}})$

$X_{\vec{N}} \in \vec{N}$ $\vec{N} = \mathbb{Z}^d$

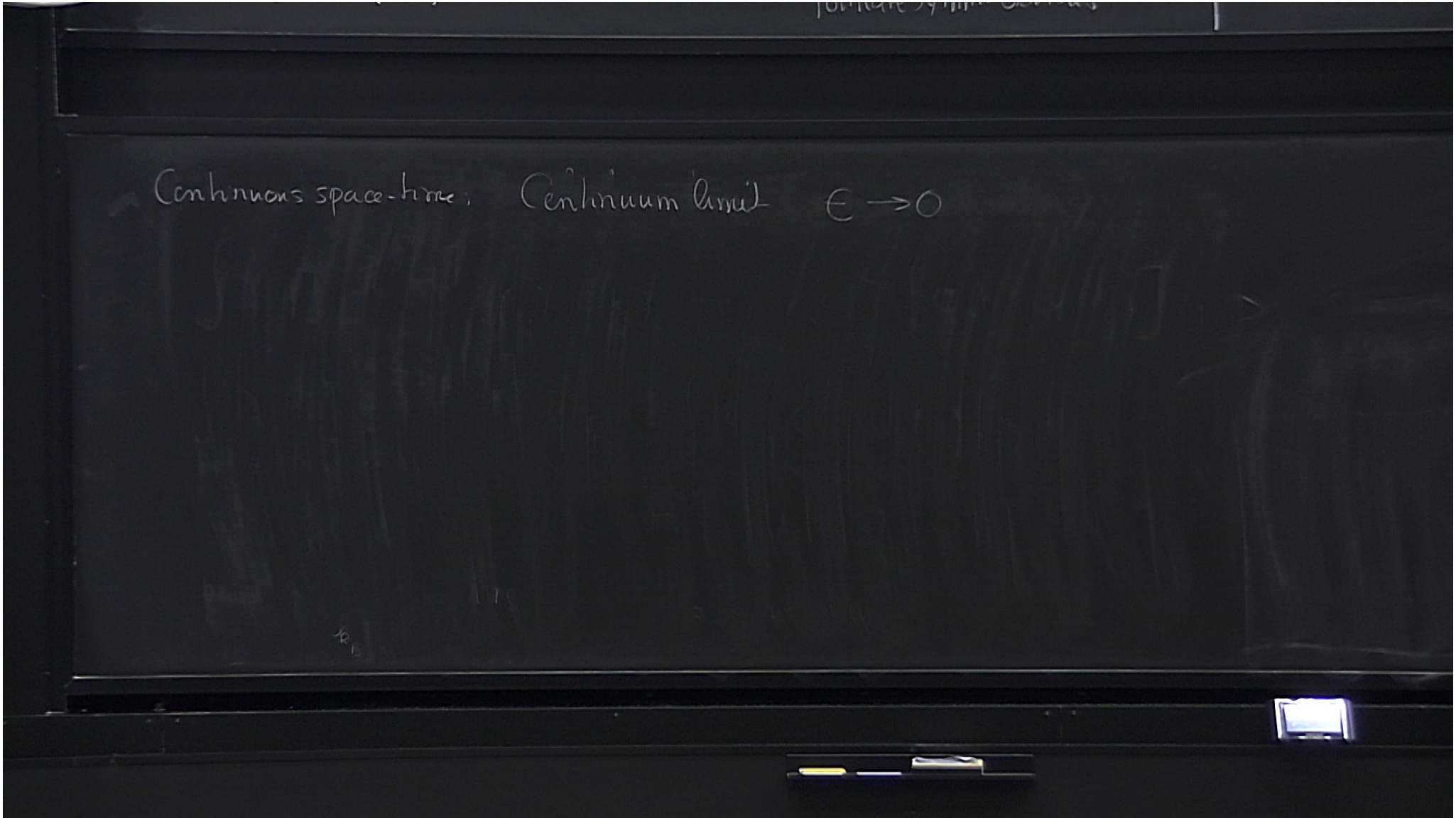
$\hat{e}_\mu$ unit vector in direction μ

$$\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \leftarrow \mu$$

Measure

$$D_E[\phi] =$$

product of loc


Continuous space-time: Continuum limit $\epsilon \to 0$

Continuous space-time: Continuum limit $\epsilon \to 0$

discrete symmetries of the lattice $\xrightarrow{?}$ Poincaré / Euclidean Symm.

Continuous space-time: Continuum limit $\epsilon \to 0$ Free Field. no anomalies

discrete symmetries of the lattice $\xrightarrow{?}$ Poincaré / Euclidean Symm.

* Continuous space-time: Continuum limit $\epsilon \to 0$ Free Field. no anomalies

discrete symmetries of the lattice $\xrightarrow{?}$ Poincaré / Euclidean Symm.

* Discrete Euclidean Functional Integral for the Free Field $=$ Partition Function of a lattice of harmonic oscillators with linear couplings between n.n.

* Continuous space-time. Continuum limit $\epsilon \to 0$ Free Field. no anomalies

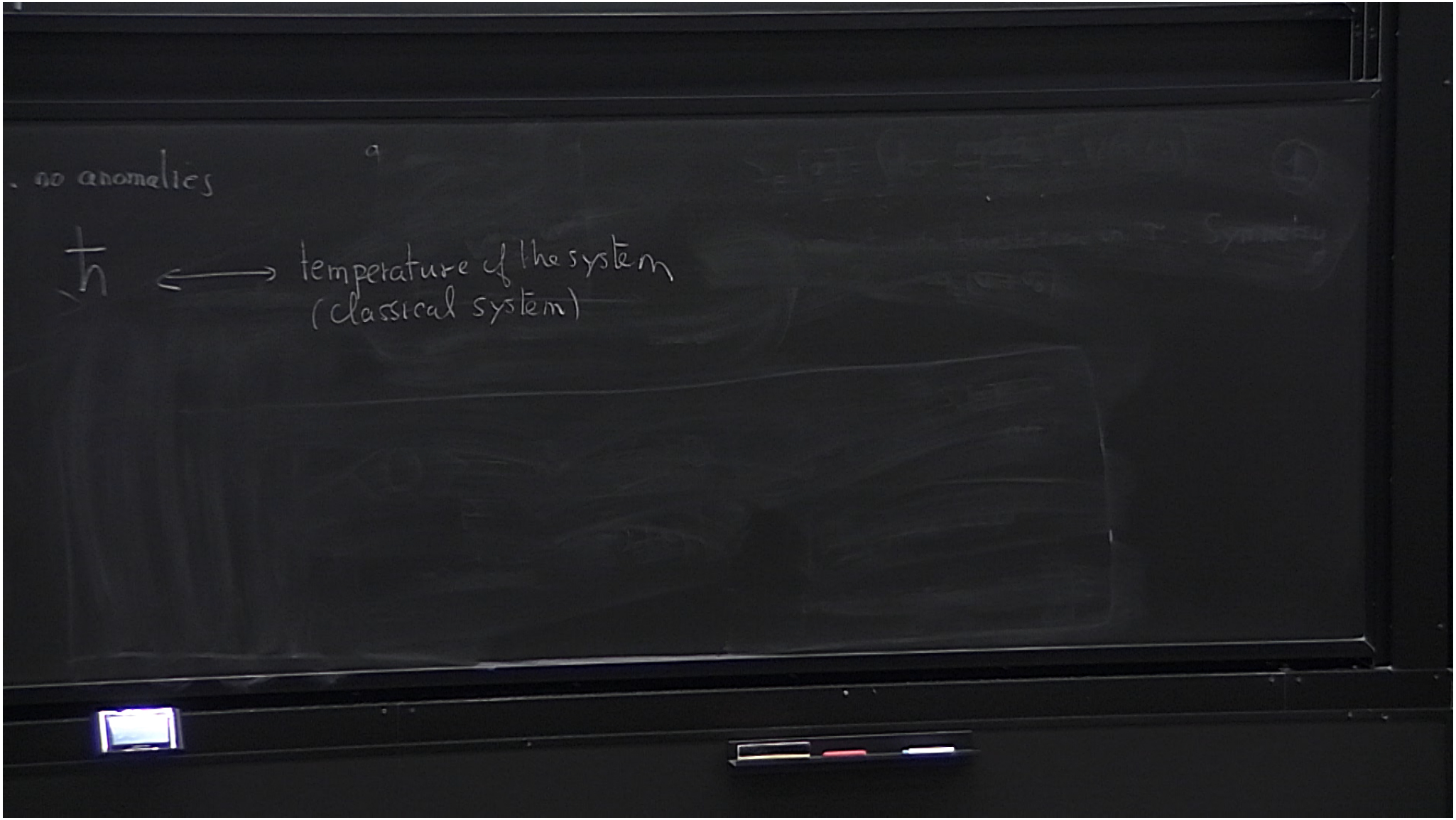
discrete symmetries of the lattice $\xrightarrow{?}$ Poincaré / Euclidean Symm.

* Discrete Euclidean Functional Integral for the Free Field $=$ Partition Function of a lattice of harmonic oscillators with linear couplings between n.n.

$X_{\vec{N}} \longleftrightarrow \vec{N}$ site on the lattice

$\phi_{\vec{N}} \longleftrightarrow$ d.o.f of the classical oscillator

$S[\phi] \longleftrightarrow$ classical energy of a micro-state


no anomalies
$\hbar$ ⟷ temperature of the system
(classical system)

* Continuous space-time: Continuum limit $\epsilon \to 0$ Free Field. no anomalies

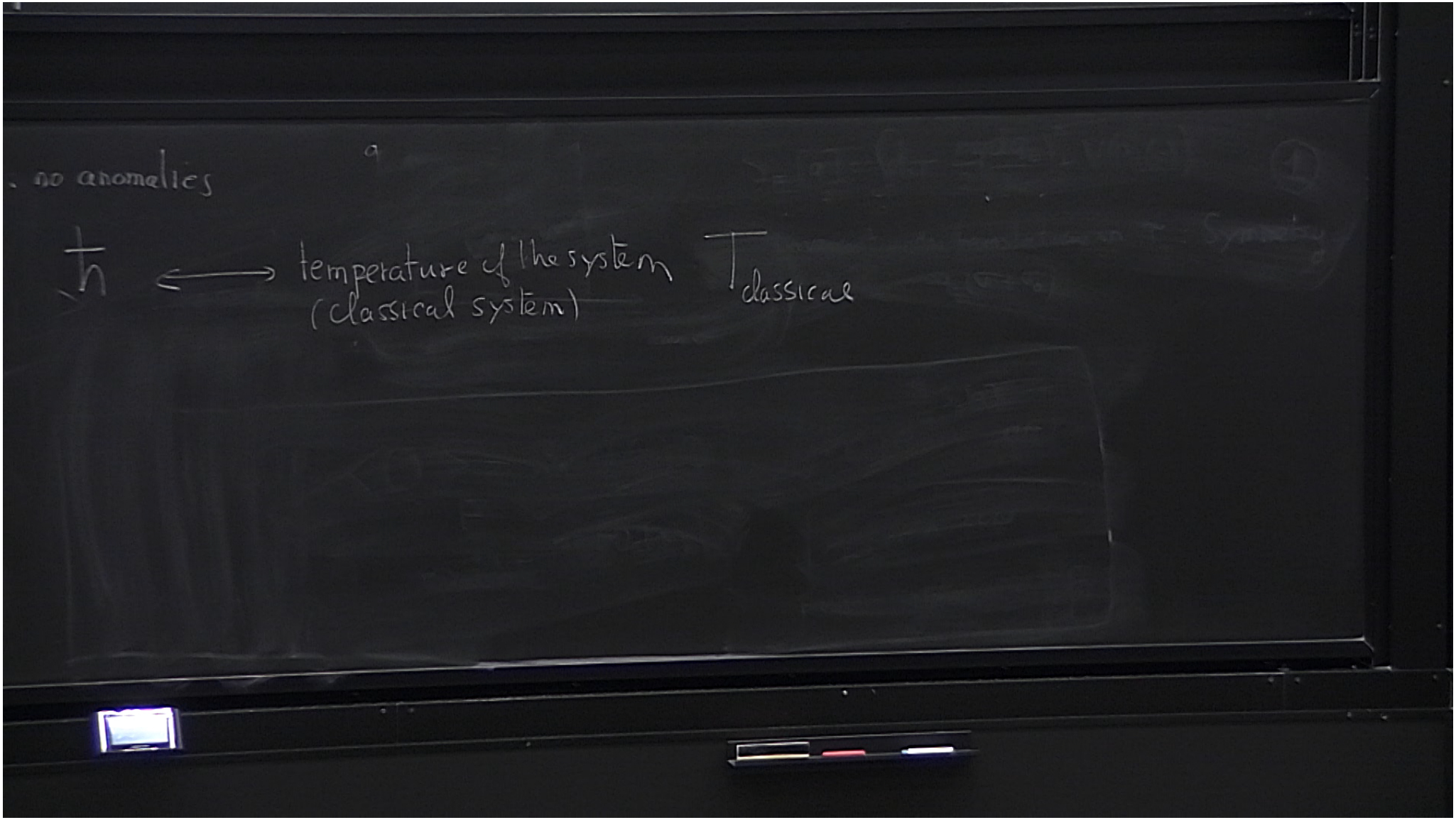
discrete symmetries of the lattice $\xrightarrow{?}$ Poincaré / Euclidean Symm. $\xleftarrow{\hbar}$

* Discrete Euclidean Functional Integral for the Free Field Quantum System at 0-temperature $=$ Partition Function of a lattice of harmonic oscillators with linear couplings between n.n.

$X_{\bar{N}} \longleftarrow \bar{N}$ site on the lattice

$\phi_{\bar{N}} \longleftarrow$ d.o.f of the classical oscillator

$S[\phi] \longleftarrow$ classical energy of a micro-state


no anomalies
$\hbar$ ⟷ temperature of the system (classical system) $T_{classical}$

* Continuous space-time: Continuum limit $\epsilon \to 0$ Free Field. no anomalies

discrete symmetries of the lattice $\xrightarrow{?}$ Poincaré / Euclidean Symm. $\hbar \longleftrightarrow$

* Discrete Euclidean Functional Integral for the Free Field $=$ Partition Function of a lattice of harmonic oscillators with linear couplings between n.n.

Quantum System at 0-temperature

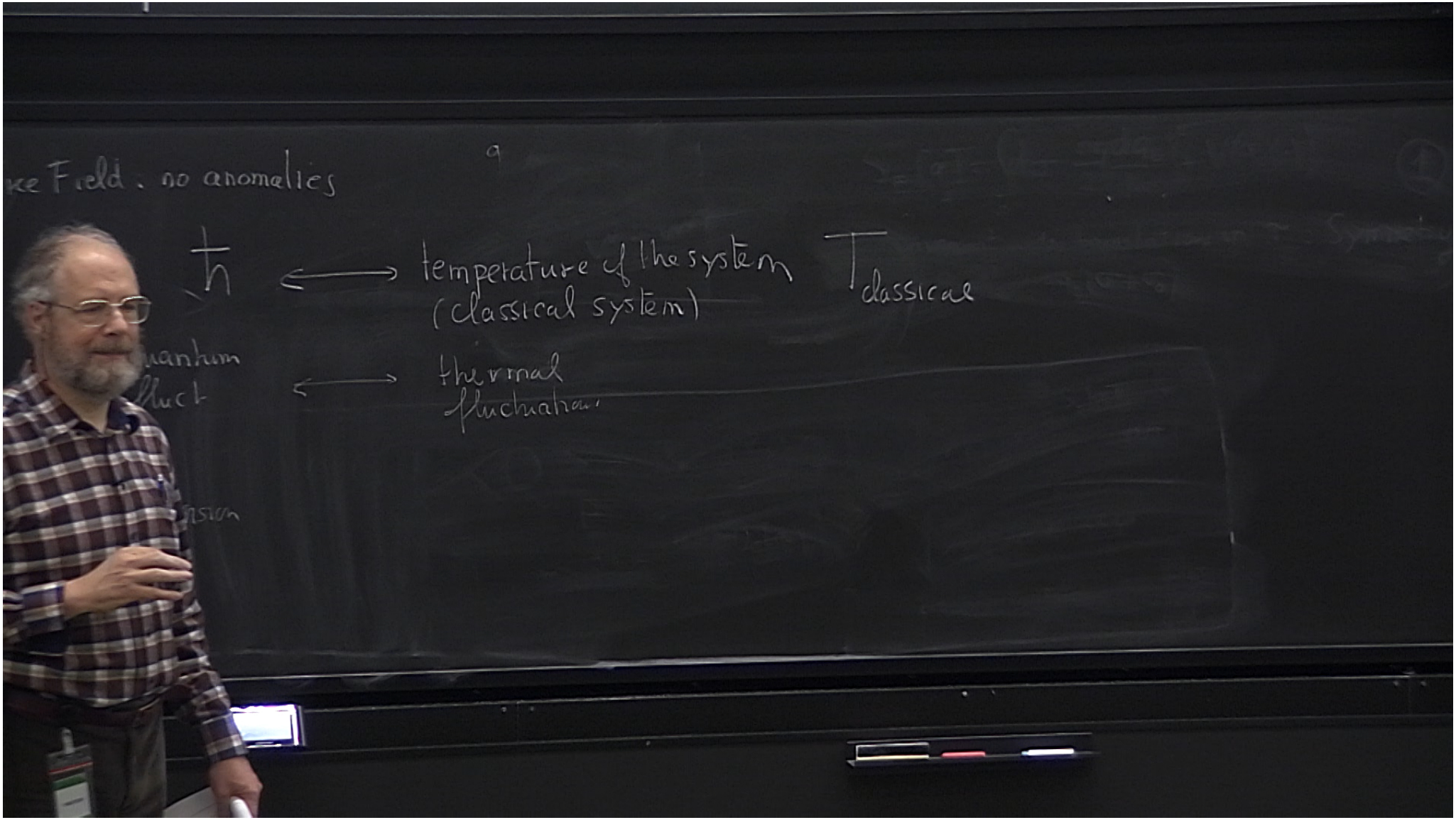
1 time

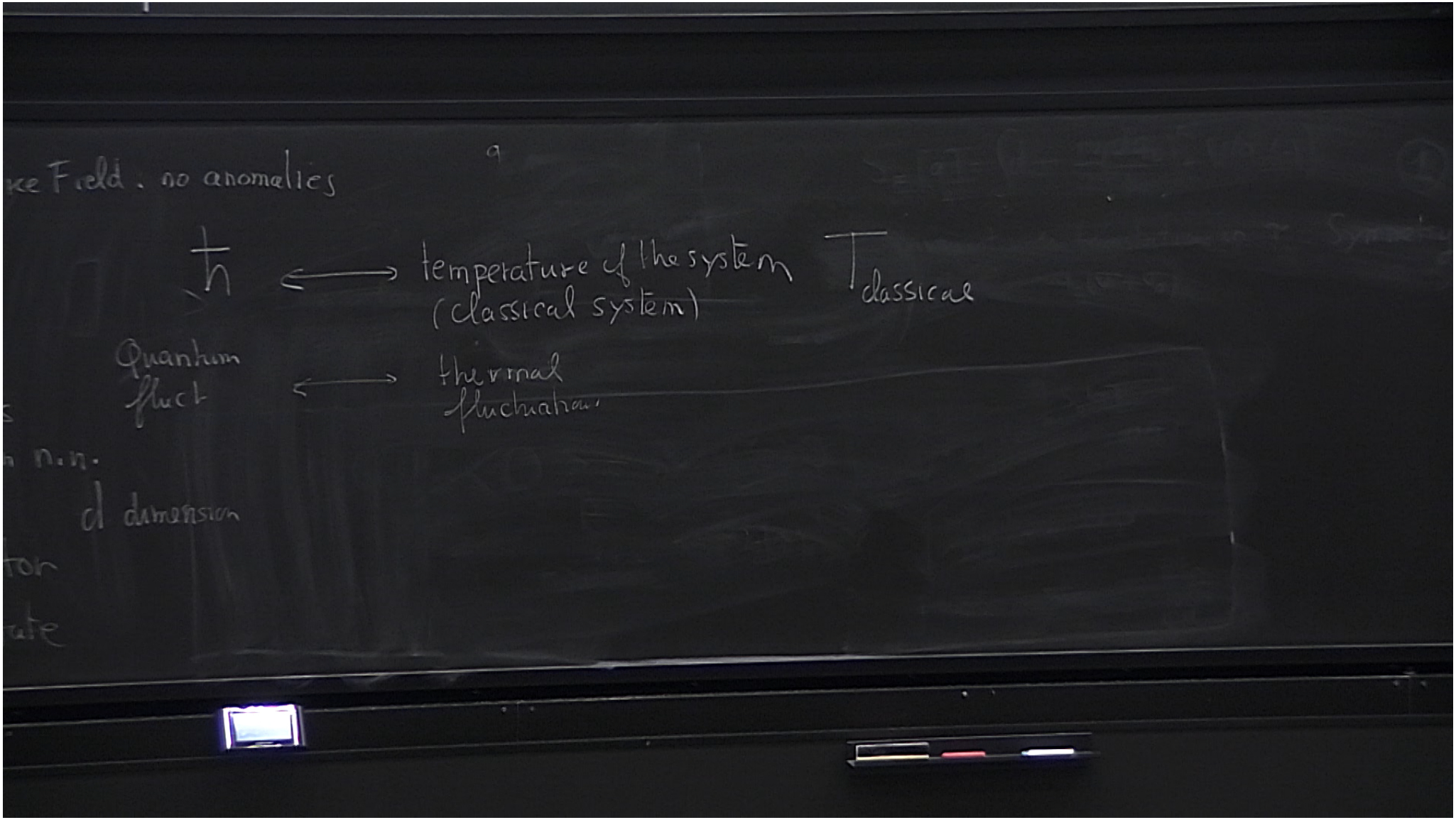
d-1 space

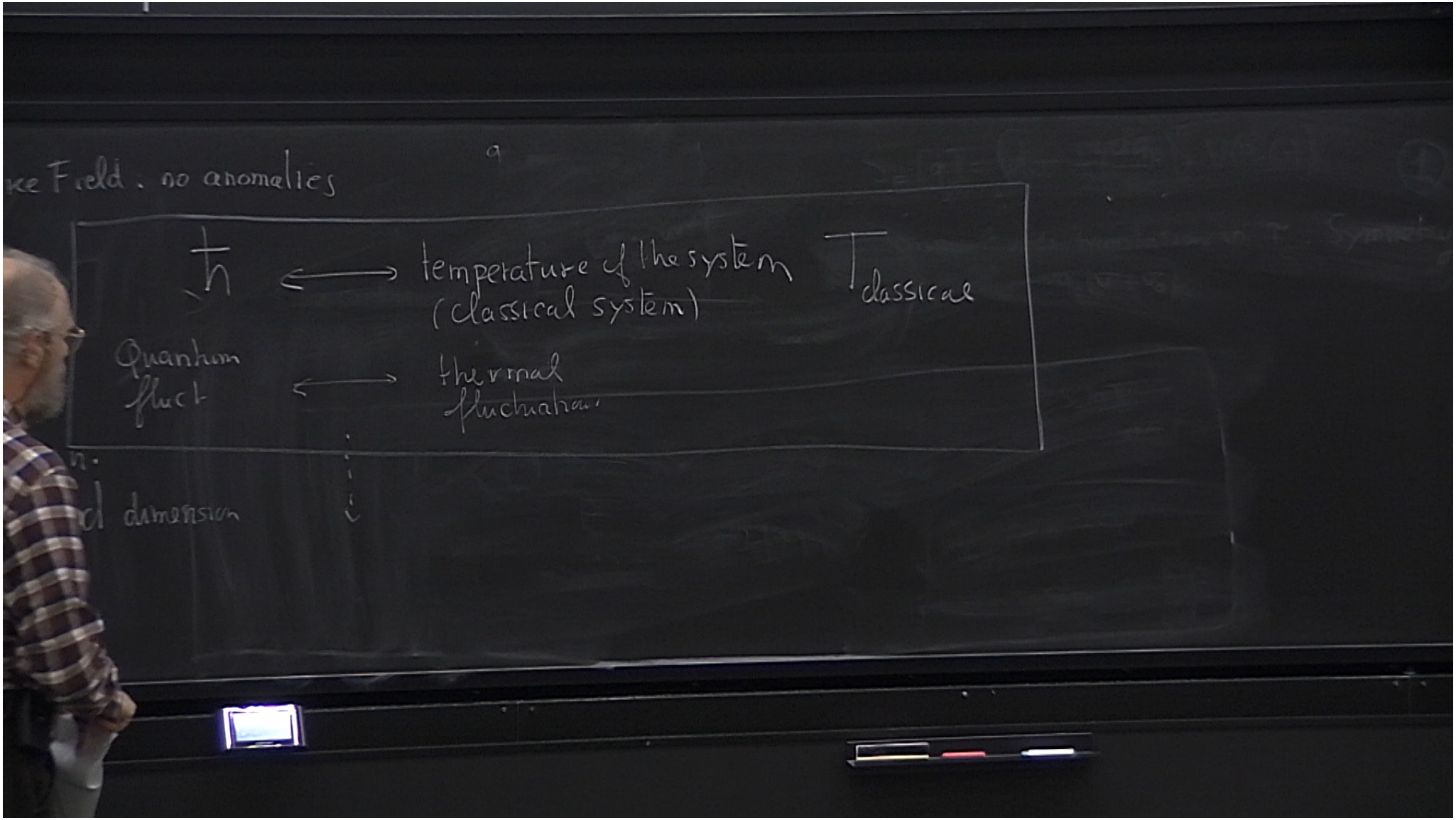
$X_{\vec{N}} \longleftrightarrow \vec{N}$ site on the lattice d. dimension

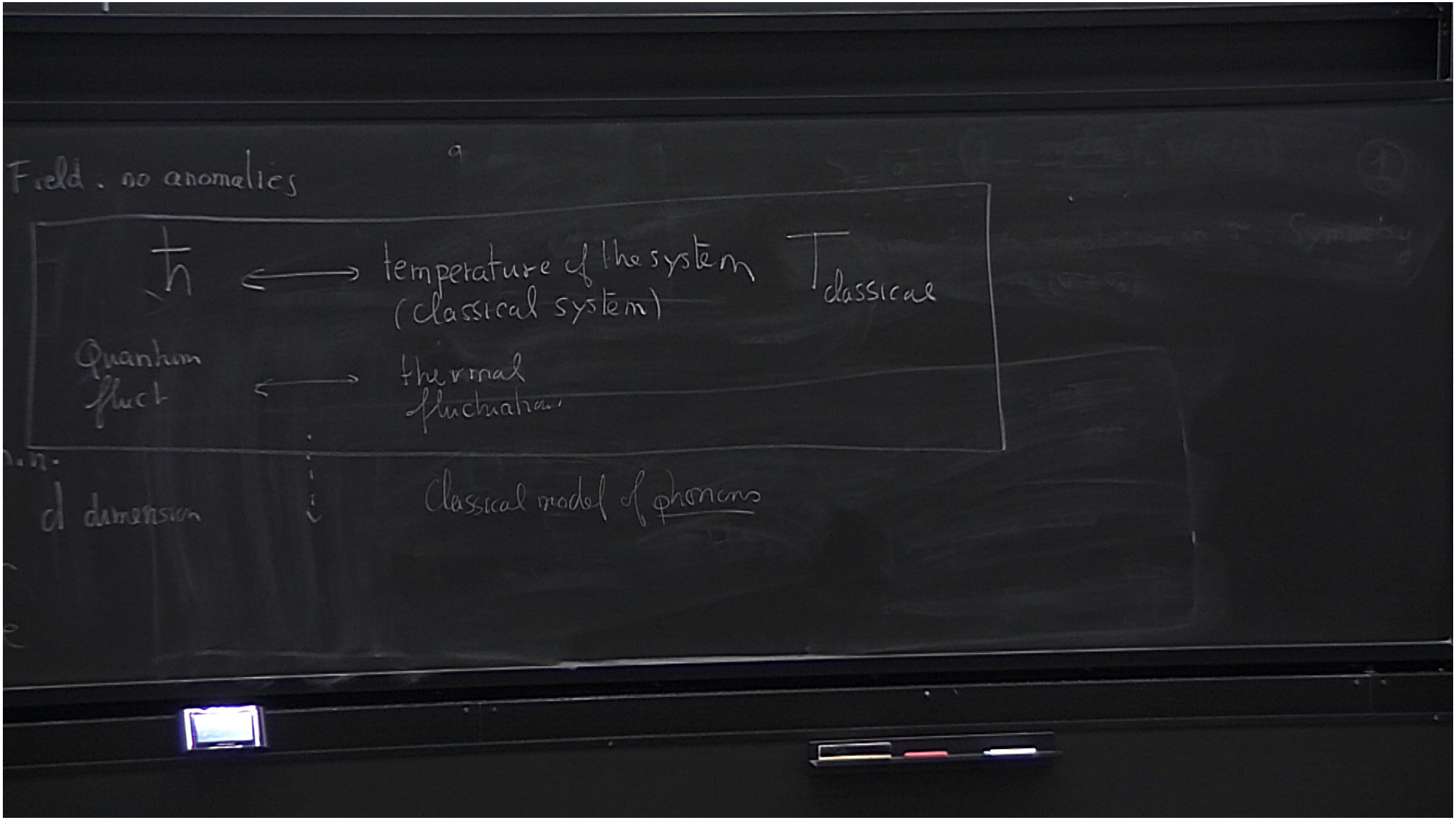
$\phi_{\vec{N}} \longleftrightarrow$ d.o.f of the classical oscillator

$S[\phi] \longleftrightarrow$ classical energy of a micro-state


ke Field. no anomalies
ħ ⟷ temperature of the system (classical system) T classical
thermal fluctuations


ke Field. no anomalies
9
$\hbar$ ⟷ temperature of the system (classical system) $T_{classical}$
Quantum fluct ⟷ thermal fluctuation
n.n.
d dimension
tor
ute


ce Field . no anomalies
9
$\hbar$ ⟷ temperature of the system (classical system) $T_{classical}$
Quantum fluct ⟷ thermal fluctuation
dimension


Field . no anomalies
$\hbar$ ⟷ temperature of the system (classical system) $T_{classical}$
Quantum fluct ⟷ thermal fluctuations
d dimension
Classical model of phonons