Title: Mirror symmetry for moduli spaces of Higgs bundles via p-adic integration

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Abstract: <p>I will talk about a recent proof, joint with M. GrÃ¶chenig and D. Wyss, of a conjecture of Hausel and Thaddeus which predicts the equality of suitably defined Hodge numbers of moduli spaces of Higgs bundles with SL(n)- and PGL(n)-structure. The proof, inspired by an argument of Batyrev, proceeds by comparing the number of points of these moduli spaces over finite fields via p-adic integration.</p>

joint / M. Groechenig + D. Wyss

SYZ-duality:

mirror pairs $(M, \check{M})$ should look like:

$$\begin{tikzcd} M \arrow[dr] & & \check{M} \arrow[dl] \\ & N & \end{tikzcd}$$

sm surj →

sm mfld →

← generic fibers should be:
- Lagrangian
- pairwise dual complex tori

- $\exists$ generalization for orbifolds involving $\mathbb{G}_m$-gerbes

Expect:
- $h^{p,q}(M) = h^{\dim M - p, q}(\check{M})$
- For M hyperkähler, $h^{p,q}(M) = h^{\dim - p, q}(M)$

$\rightsquigarrow$ Can expect $h^{p,q}(M) = h^{p,q}(\check{M})$

Higgs bundles:

X sm. proj. curve $/\mathbb{C}$

K = can bundle

Def: A Higgs bundle is a pair (E, ϕ).

E v.b on X

$\phi: E \to E \otimes K$

Rmk: $\lambda \in \mathbb{C}$

$\nabla: E \to E \otimes K$

$f \in \mathcal{O}_X$, $e \in E$:

$\nabla(fe) = f\nabla(e) + \lambda\, e \otimes df$

Fix $n, d \geq 1$ $(n,d) = 1$

$\rightsquigarrow \exists$ $M^d_{GL_n}$ mod. space of semistable Higgs bundles of rk n, deg d,

a smooth q-proj. variety

Hitchin fibration:

$(E, \phi) \rightsquigarrow$ char. poly

$x^n + \sum_{i=0}^{n-1} a_i x^i$

$a_i \in H^0(X, K^{n-i})$

Thm: (Hitchin)

$\pi : M^d_{GL_n} \to \bigoplus_{i=0}^{n-1} H^0(X, K^{n-i})$

is proper + surj, gen fibers are torsor under ab. varieties

General str. groups

G red grp scheme / X

A G-Higgs bdle: (E, ϕ),

E G-torsor / X

$\phi \in \mathrm{End}(E) \otimes K$

$\rightsquigarrow$ twist SL_n:

Fix L line bundle on X of deg d

$\rightsquigarrow$ SL_n^L-Higgs b:

(E, ϕ) GL_n-Higgs b

+ $\det(E) \cong L$

st $\mathrm{Tr}(\phi) = 0$

$\rightsquigarrow$ sm. q-proj. moduli space $M_{SL_n^L}$

PGL_n:

$\rightsquigarrow$ twist SL_n:

Fix L line bundle on X of deg d

$\rightsquigarrow$ SL_n^L-Higgs b:

(E, ϕ) GL_n-Higgs b

$+$ $\det(E) \cong L$

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$\rightsquigarrow$ sm. q-proj. moduli space (variety)

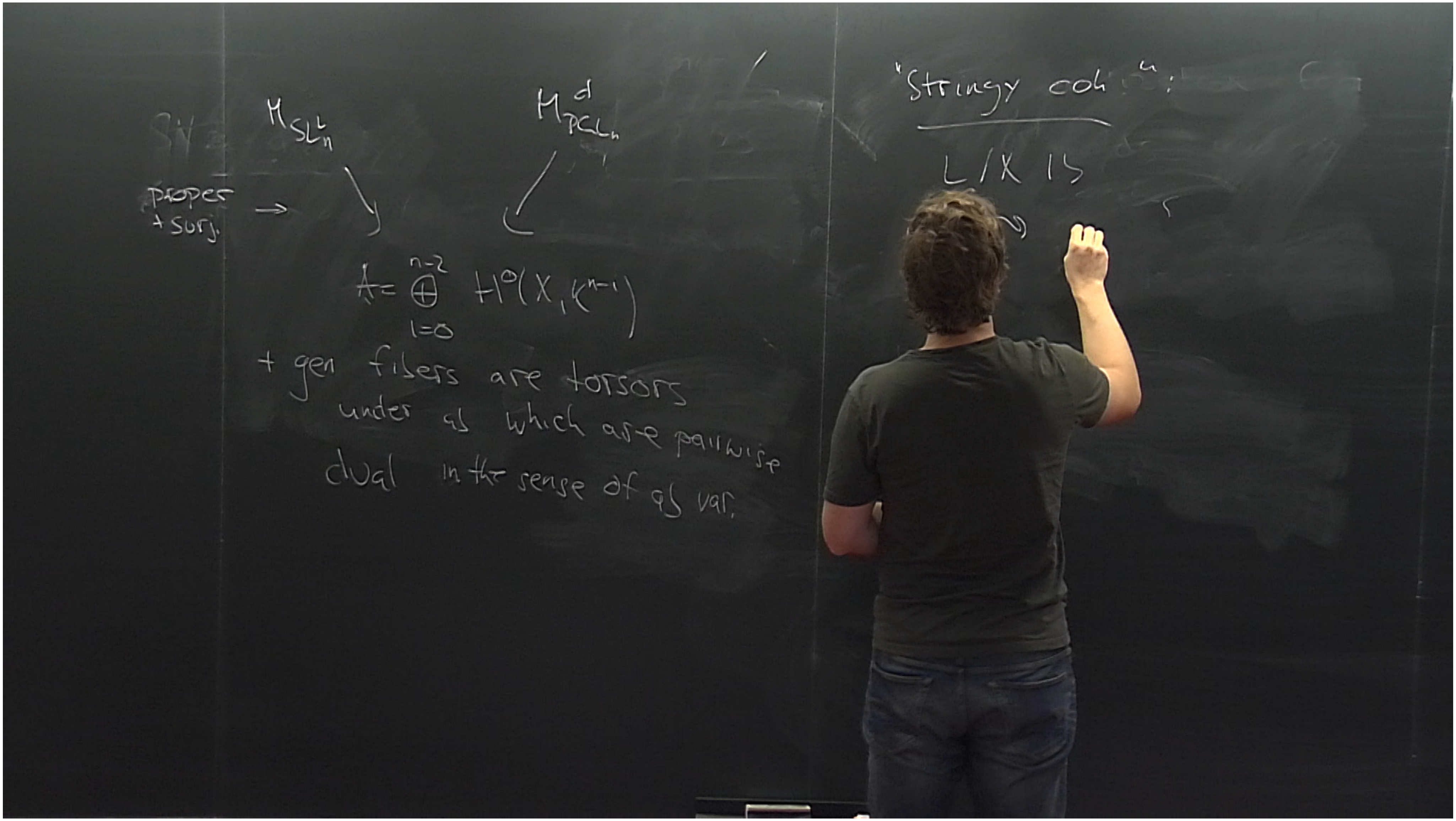
$M_{SL_n^L}$

PGL_n:

$\rightsquigarrow$ $M^d_{PGL_n}$ sm. q-proj stack

$\Gamma = \mathrm{Pic}^0(X)[n] \curvearrowright M_{SL_n^L}$ by $\otimes$

$\rightsquigarrow$ $M^d_{PGL_n} = \left[M_{SL_n^L} / \Gamma \right]$



M_{SL_n^L}
M^d_{PGL_n}
proper + surj. →
A = ⊕ H^0(X, K^{n-i})
i=0 n-2
+ gen fibers are torsors under AS which are pairwise dual in the sense of AS var.
"Stringy coh":
L/X is

...sors
...se pairwise
...f as var.

"Stringy coh.":

L/X is

$\rightsquigarrow \alpha^L$ $\mathbb{G}_m$-gerbe on $M^d_{PGL_n}$

$\in H^2(M^d_{PGL_n}, \mathbb{G}_m)$

$\rightsquigarrow H^*_c(M^d_{PGL_n}, \alpha^L)$ "stringy" coh.

grp equipped w/ H str.

Fact: The H. str. on $H^*_c(M_{SL_n})$ and $H^*_c(M^d_{PGL_n}, \alpha^L)$ are pure

Thm.

$\forall p,q \quad h^{p,q}(M_{SL_n}^{L}) = h^{p,q}_{st}(M^{d}_{PGL_n}, \alpha^{L})$

This was conjectured by Hausel–Thaddeus who proved it for $n=2,3$

Ingredients of pf:

Weil conj:

X sm proj. var. $/\mathbb{F}_q$

$\rightsquigarrow \forall n \quad |X(\mathbb{F}_{q^n})|$ finite

Thm (Grothendieck, Deligne, ...):

$Z_X(T) = \exp\left(\sum_{n\geq 1} \frac{|X(\mathbb{F}_{q^n})|}{n} T^n\right) \in \mathbb{Q}[[T]]$

(1) $Z_X(T) \in \mathbb{Q}(T)$

(2) $Z_X(T) = \dfrac{P_1(T) \cdots P_{2g-1}(T)}{P_0(T) \cdots P_{2g}(T)}$

s.t. (1) $P_i(T) \in \mathbb{Z}[T]$

1st. term 1

(2) Every cplx root of $P_i(T)$ has $|\ | = q^{i/2}$.

$\hookrightarrow$ this determines the P_i.

Furthermore:

Let $\mathbb{Q}_p \subset F$ finite over field

$\rightsquigarrow \mathcal{O}_F \subset F$

$k_F = \mathcal{O}_F / \mathfrak{m}_F$ res. field

Say $k_F = \mathbb{F}_q$

$\mathcal{X}$ / $\mathrm{Spec}(\mathcal{O}_F)$ sm. proj. scheme

$\rightsquigarrow X := \mathcal{X}_{k_F}$

Choose $F \subset \mathbb{C}$

$\rightsquigarrow \mathcal{X}(\mathbb{C})$ cplx mfld

Then: $\deg P_i(T) = \dim H^i(\mathcal{X}(\mathbb{C}), \mathbb{Q})$

PG

$\rightsquigarrow$

Γ

Sat: (1) $P_i(T) \in \mathbb{Z}[T]$

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$\mathfrak{X}$ / Spec($\mathcal{O}_F$) sm. proj. scheme

$\rightsquigarrow X_i = \mathfrak{X}_{k_F}$

Choose $F \subset \mathbb{C}$

$\rightsquigarrow \mathfrak{X}(\mathbb{C})$ cpl

Then: $\deg P_i(T) = \dim$ H

Upshot: For such $\mathfrak{X}, \mathcal{Y}$ / Spec

to compare Betti numbers

its enough to compar

pt. counts / $\mathbb{F}_q$

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c.t. term 1

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Upshot: For such $\mathcal{X}, \mathcal{Y} / \mathrm{Spec}(\mathcal{O}_F)$

to compare Betti numbers / $\mathbb{C}$

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pt. counts / $\mathbb{F}_q$

Ho

$\mathcal{X}$ / Spec($\mathcal{O}_F$) sm. proj. scheme

$\rightsquigarrow X_i = \mathcal{X}_{k_F}$

Choose $F \subset \mathbb{C}$

$\rightsquigarrow \mathcal{X}(\mathbb{C})$ cplx mfld

Then. $\deg P_i(T) = \dim H^i(\mathcal{X}(\mathbb{C}), \mathbb{Q})$

Upshot: For such $\mathcal{X}, \mathcal{Y}$ / Spec($\mathcal{O}_F$)

to compare Betti numbers / $\mathbb{C}$

its enough to compar

pt. counts / $\mathbb{F}_q$

How to get these.

Given fin many varieties / $\mathbb{C}$

+ some morphisms between them

+ some properties

$\rightsquigarrow$ Can describe everything w/ fin many polynomials w/ fin. many coeff

$\rightsquigarrow \mathbb{Z} \subset R \subset \mathbb{C}$

gen. by all coeff

→ the vas. + morph
descend to $\mathrm{Spec}(R)$

→ By localizing R get
properties / $\mathrm{Spec}(R)$

Can look at $R \to \mathcal{O}_F$
for local fields F

→ by Chebotarev density
thm + comparison th.
from p-adic Hodge theory
(Faltings),
if $\mathcal{X}, \mathcal{Y}$ / $\mathrm{Spec}(R)$)
have same pt. count / k_F
for suff. many $R \hookrightarrow \mathcal{O}_F$,
$\mathcal{X}(\mathbb{C})$ $\mathcal{Y}(\mathbb{C})$ have same H. #,

density
...ison th.
...dge theorem

...Spec($\mathbb{R}$))

...point / k_F

...$\hookrightarrow \mathcal{O}_F$,

...same H. #,

$\rightsquigarrow$ Reduce to $\mathcal{X}$, $\mathcal{Y}$ / Spec($\mathcal{O}_F$),

Want to compare pt. count

<u>Weil:</u> $\mathcal{X}$ / Spec($\mathcal{O}_F$)
smooth sch.

$\mathcal{X}(\mathcal{O}_F)$ locally $\cong$ $\mathcal{O}_F^n$

$\uparrow$ Haar measure

by locally integrating a volume form on $\mathcal{X}/\mathcal{O}_F$ against this Haar measure

get a can measure on $\mathcal{X}(\mathcal{O}_F)$

$\rightsquigarrow$ $\mathrm{vol}(\mathcal{X}(\mathcal{O}_F)) = \dfrac{|\mathcal{X}(k_F)|}{|k_F|^{\dim X}}$

$\rightsquigarrow$ can compare measures!

Pf of thm.

Look at situation $/\mathcal{O}_F$

$M_1 := M_{SL_n^L}$ $\quad M_2 = M^d_{PGL_n}$

Get. $\cdot$ p-adic measure on $M_i^c(\mathcal{O}_F)$ (+ coarse mod.)

$\cdot$ $f_i : M_i^c(\mathcal{O}_F) \to \mathbb{C}$

s.t. enough to show:

$$\int_{M_1^c(\mathcal{O}_F)} f_1 \, d\mu_1 = \int_{M_2^c(\mathcal{O}_F)} f_2 \, d\mu_2$$

$$M_1 \xrightarrow{\pi_1} A \xleftarrow{\pi_2} M_2$$

Compute fibrewise:

$$\int_{M_1^c(\mathcal{O}_F)} f_1 \, d\mu_1 = \int_{a \in A(\mathcal{O}_F)} \left(\int_{\pi_1^{-1}(a)(\mathcal{O}_F)} f_1 \, d\mu_1 \right) d\mu_A$$

$\rightsquigarrow$ Enough to show:

$\forall a \in A(\mathcal{O}_F)$ s.t. over $a|_F \in A(F)$ have torsors under ab. var

$$\int_{\pi_1^{-1}(a)(F)} f_1 = \int_{\pi_2^{-1}(a)(F)} f_2$$

$P_1 \subset M_1 \qquad M_2 \supset P_2$

$\pi_1 \searrow_A \swarrow \pi_2$

Compute fibrewise:

$$\int_{\pi_1^c(\mathcal{O}_F)} f_1 \, d\mu_1 = \int_{a \in A(\mathcal{O}_F)} \left(\int_{\pi_1^{-1}(a)(\mathcal{O}_F)} f_1 \, d\mu_1 \right) d\mu_A$$

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$$\int_{\pi_1^{-1}(a)(F)} f_1 = \int_{\pi_2^{-1}(a)(F)} f_2$$

Fix such $a \in A(\mathcal{O}_F)$

Say $\pi_i^{-1}(a)(F) \neq \emptyset$:

Fix $\pi_i^{-1}(a)(F) \cong P_i(F)$

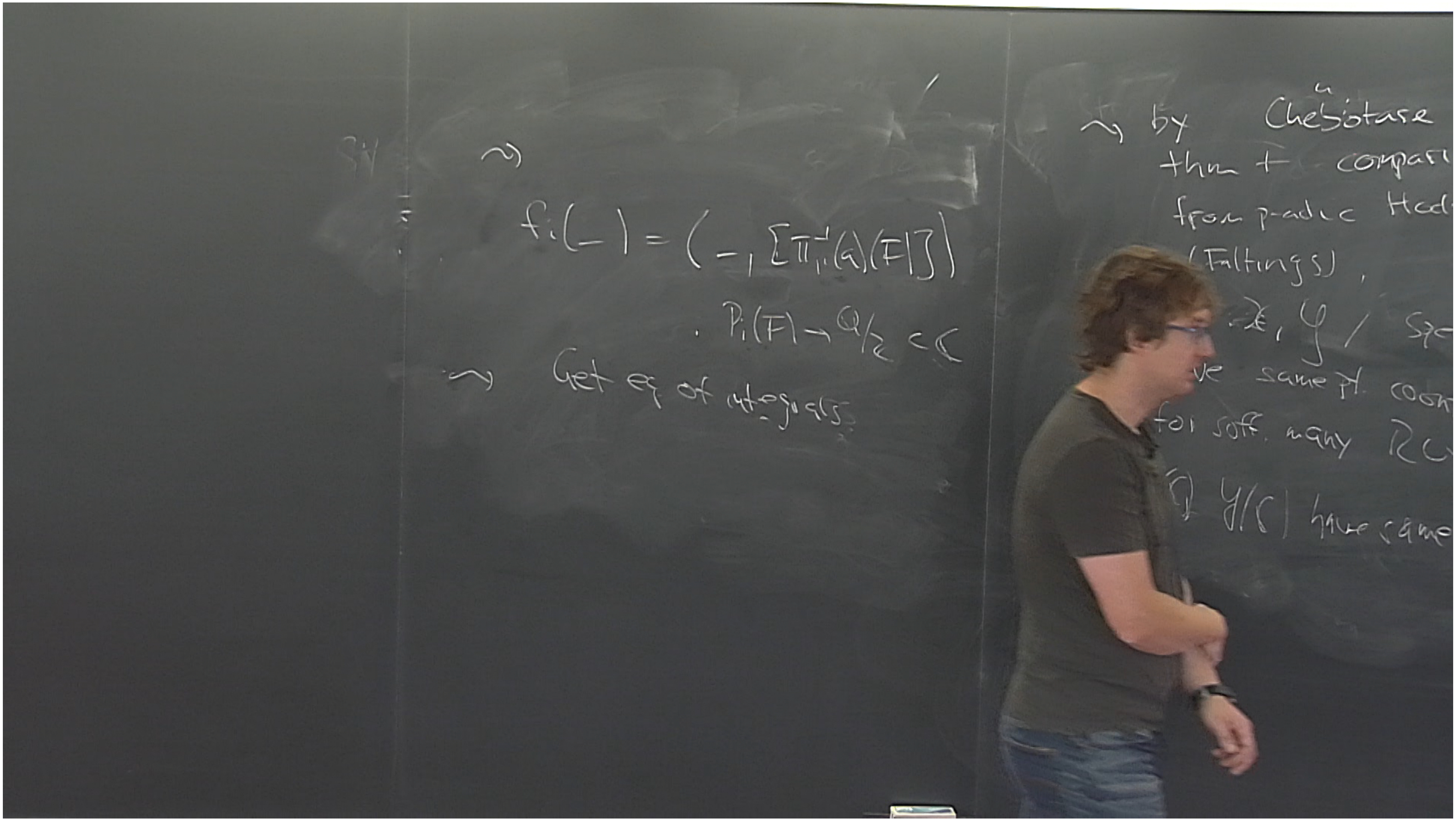
$f_{i'} : P_i(F) \to \mathbb{C} \qquad \{i', i\} = \{1, 2\}$

can be described as follows.

Tate duality. $\quad [\pi_{i'}^{-1}(a)] \in$

$$(\ ,\ ) : P_i(F) \times H^1(F, P_{i'}) \to \mathbb{Q}/\mathbb{Z}$$

perfect pairing


$f_i(-) = (-, [\pi_1^{-1}(a)(F)])$
$P_1(F) \to \mathbb{Q}/\mathbb{Z} \subset \mathbb{C}$
Get eq of integrals
by Chebotare
thm + compari
from p-adic Hod
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