Title: Beyond Geometric Invariant Theory

Date: Oct 04, 2017 11:00 AM

URL: http://pirsa.org/17100070

Abstract: <p>Geometric invariant theory (GIT) is an essential tool for constructing moduli spaces in algebraic geometry. Its advantage, that the construction is very concrete and direct, is also in some sense a draw-back, because semistability in the sense of GIT is often more complicated to describe than related intrinsic notions of semistability in moduli problems. Recently a theory has emerged which treats the results and structures of geometric invariant theory in a broader context. The theory of Theta-stability applies directly to moduli problems without the need to approximate a moduli problem as an orbit space for a reductive group on a quasi-projective scheme. I will discuss some new progress in this program: joint with Jarod Alper and Jochen Heinloth, we give a simple necessary and sufficient criterion for an algebraic stack to have a good moduli space. This leads to the construction of good moduli spaces in many new examples, such as the moduli of Bridgeland semistable objects in derived categories. Time permitting, I will also discuss applications to enumerative geometry and wall crossing formulas.</p>

Q: Why study moduli problems?

(A) classification leads to deeper understanding of geometric objects

Lie groups $\rightsquigarrow$ theory of root data

(B) explosion of mathematical physics in 80's

$\hookrightarrow$ can construct new invariants of a manifold

$$X \longmapsto \mathcal{M}(X) = \text{some moduli problem}$$

$$\text{invariant} = \#\mathcal{M}(X)$$

Q: Why study moduli problems?

(A) classification leads to deeper understanding of geometric objects

Lie groups $\rightsquigarrow$ theory of root data

(B) explosion of mathematical physics in 80's

↳ can construct new invariants of a manifold

$$X \longmapsto \mathcal{M}(X) = \text{some moduli problem}$$

$$\text{invariant} = \# \mathcal{M}(X)$$

Ex: pseudo-holomorphic curves, gauge theory

geometric objects

Lie groups $\rightsquigarrow$ theory of root data

$X \longmapsto \mathcal{M}(X)$ = some moduli problem

invariant = $\underline{\#\mathcal{M}(X)}$

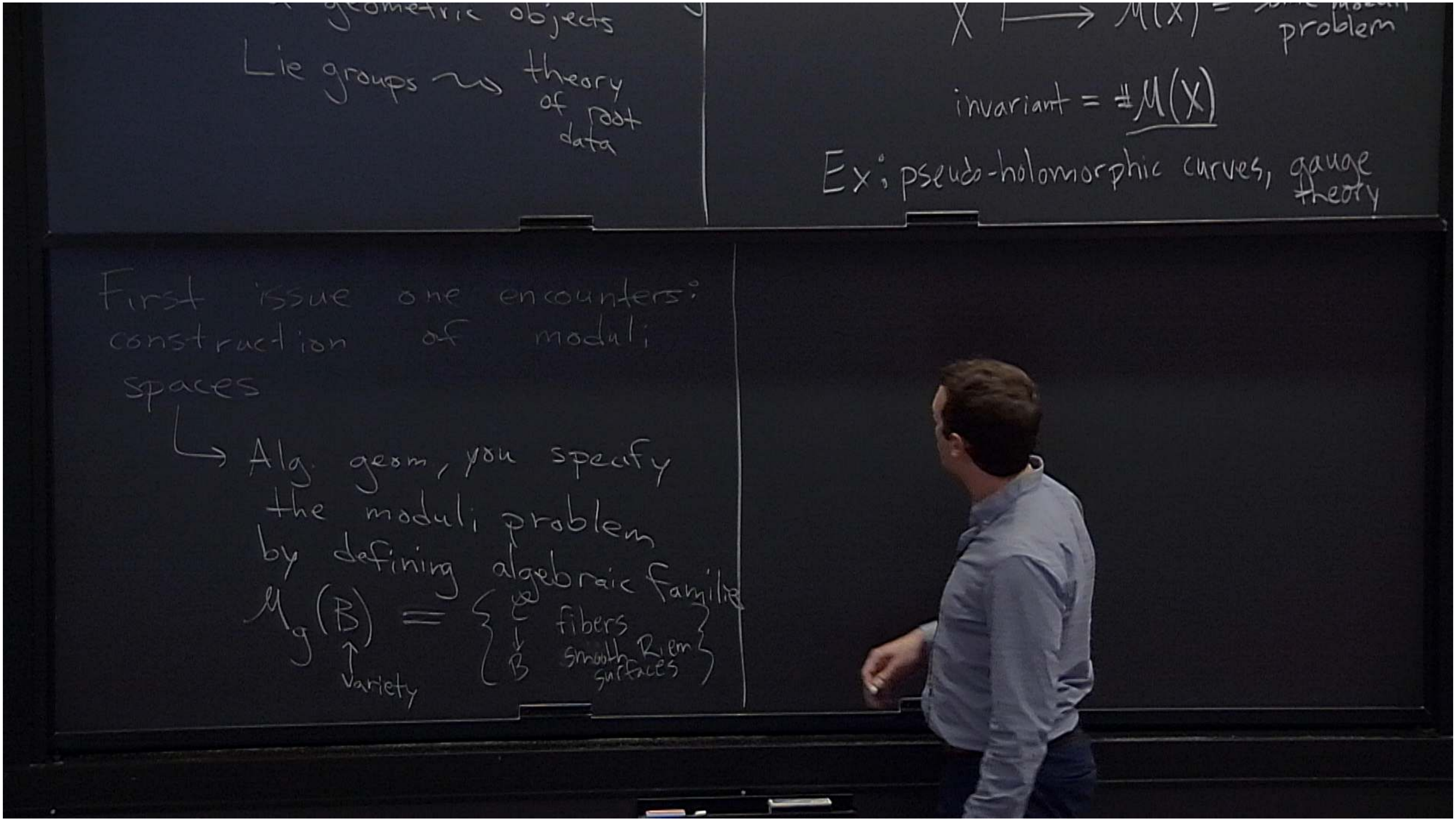
Ex: pseudo-holomorphic curves, gauge theory

First issue one encounters: construction of moduli spaces

$\hookrightarrow$ Alg. geom, you specify the moduli problem by defining algebraic families

$\mathcal{M}_g(B)$

$\uparrow$ variety


geometric objects
Lie groups ⇝ theory of root data
X ⟼ M(X) = some moduli problem
invariant = #M(X)
Ex: pseudo-holomorphic curves, gauge theory
First issue one encounters: construction of moduli spaces
↳ Alg. geom, you specify the moduli problem by defining algebraic families
M_g(B) = { E → B fibers smooth Riem surfaces }
variety

geometric objects

Lie groups $\rightsquigarrow$ theory of root data

$X \longmapsto \mathcal{M}(X) =$ ... moduli problem

invariant $= \# \underline{\mathcal{M}(X)}$

Ex: pseudo-holomorphic curves, gauge theory

First issue one encounters: construction of moduli spaces

↳ Alg. geom, you specify the moduli problem by defining algebraic families

$$\mathcal{M}_g(\underset{\substack{\uparrow \\ \text{variety}}}{B}) = \left\{ \begin{array}{l} E \\ \downarrow \\ B \end{array} \;\; \text{fibers smooth Riem surfaces} \right\}$$

ideal solution: a space $\underline{M}$ such that

$$\mathcal{M}_g(B) = \mathrm{Map}(B, \underline{M})$$

Even in this example, idea is not achieved.

geometric objects

Lie groups $\rightsquigarrow$ theory of root data

$X \longmapsto \mathcal{M}(X)$ = some moduli problem

invariant = $\#\underline{\mathcal{M}(X)}$

Ex: pseudo-holomorphic curves, gauge theory

First issue one encounters: construction of moduli spaces

$\hookrightarrow$ Alg. geom, you specify the moduli problem by defining algebraic families

$$\mathcal{M}_g(\underset{\substack{\uparrow \\ \text{variety}}}{B}) = \left\{ \begin{array}{c} \mathcal{E} \\ \downarrow \\ B \end{array} \text{ fibers smooth Riem surfaces} \right\}$$

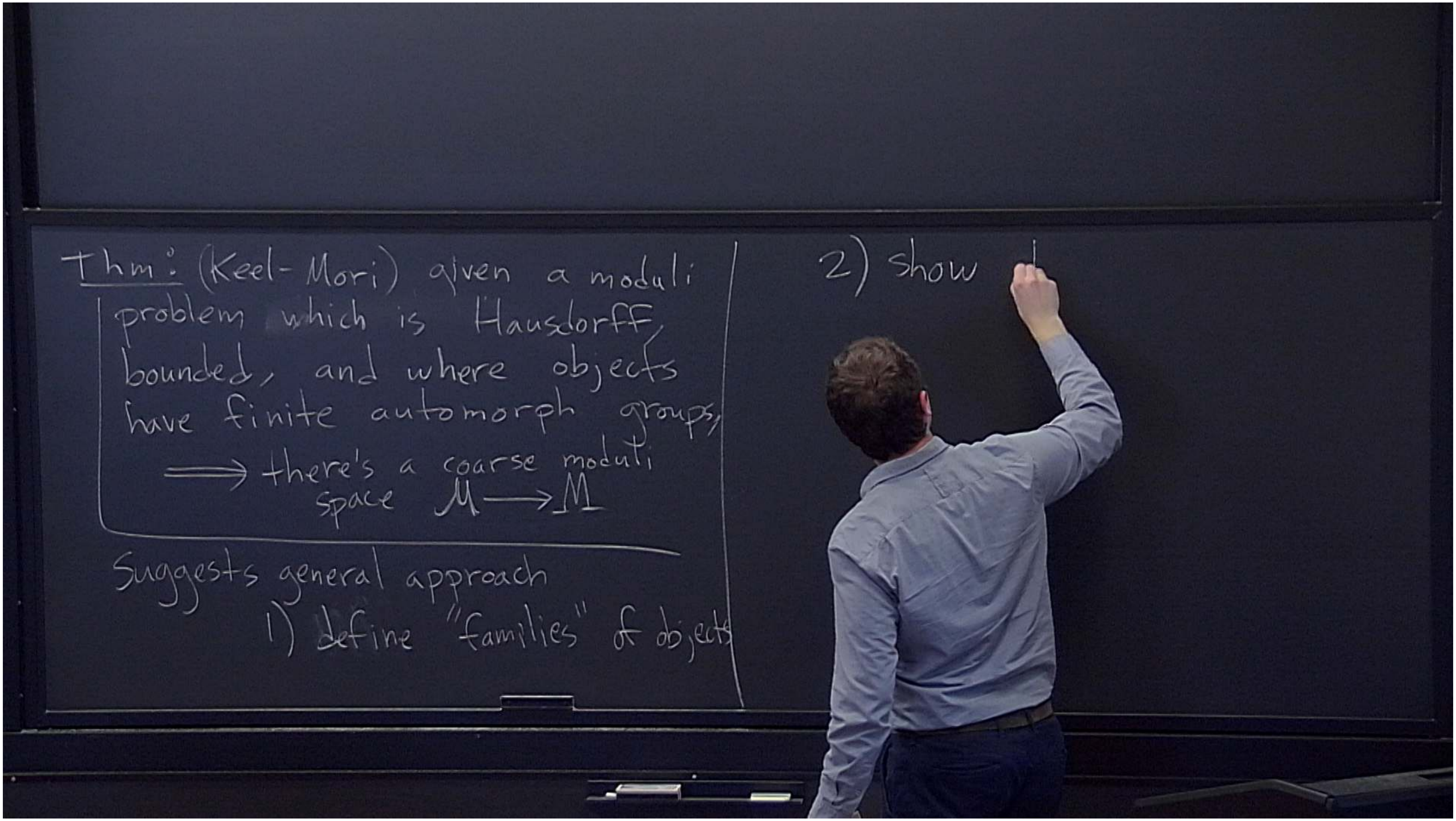
ideal solution: a space $\underline{M}$ such that

$$\mathcal{M}_g(B) = \mathrm{Map}(B, \underline{M})$$

Even in this example, idea is not achieved.

$\hookrightarrow$ still has coarse moduli space

$$\mathcal{M}_g \longrightarrow \underline{M}$$


Thm: (Keel-Mori) given a moduli problem which is Hausdorff, bounded, and where objects have finite automorph groups, ⟹ there's a coarse moduli space $\mathcal{M} \longrightarrow M$
Suggests general approach
1) define "families" of objects
2) show

Thm: (Keel-Mori) given a moduli problem which is Hausdorff, bounded, and where objects have finite automorph groups,

$\Longrightarrow$ there's a coarse moduli space $\mathcal{M} \longrightarrow M$

Suggests general approach

1) define "families" of objects

2) show it defines an algebraic stack

$\hookrightarrow$ Artin's criteria

3) check Hausdorff, bounded, finite automorph

$\overset{}{\underset{KM}{\Longrightarrow}}$ existence of a moduli space

Problem: (3) fails in many situations (relevant to both Ⓐ and Ⓑ)

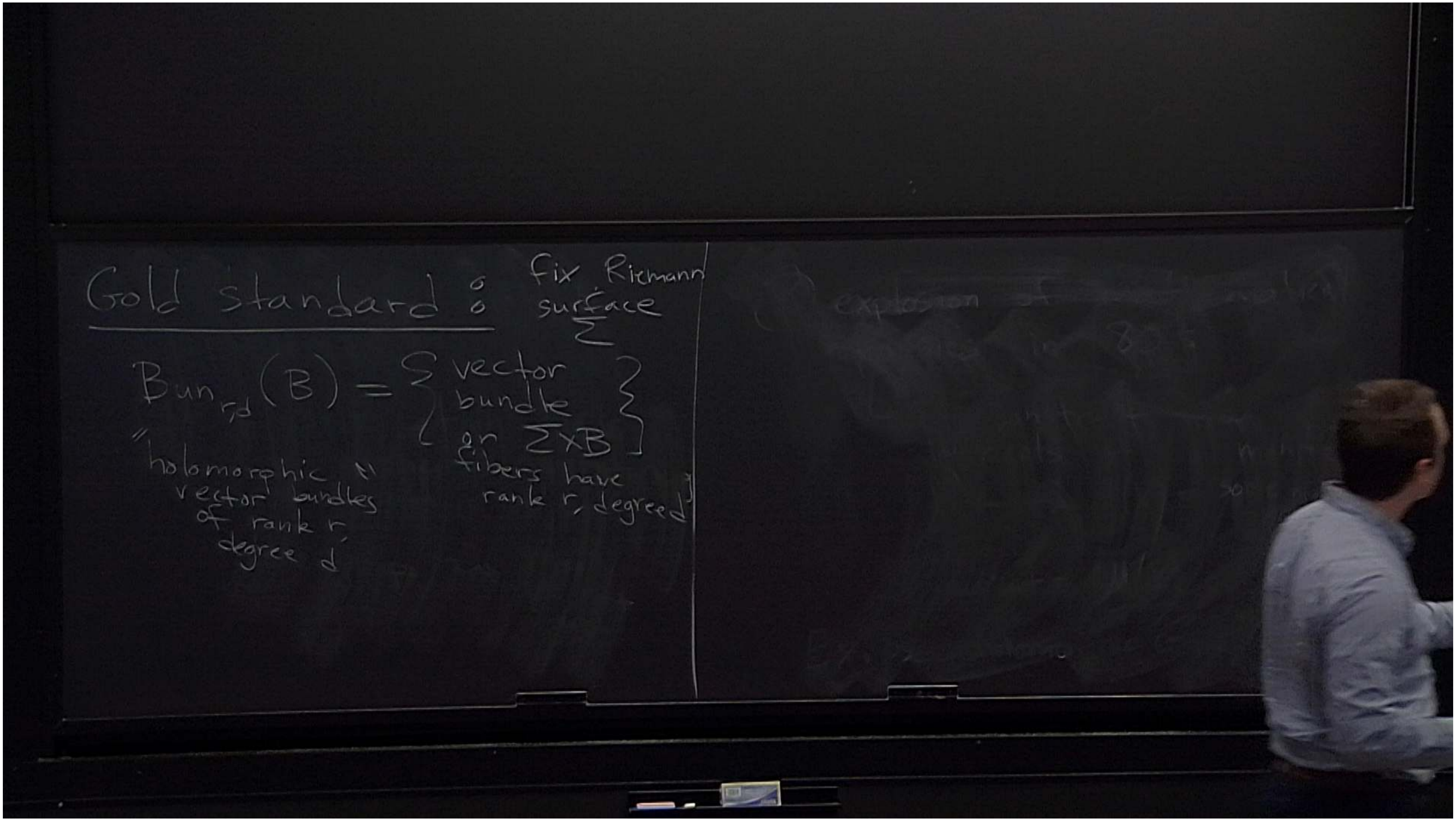
Today: a modification of this approach which works for "arbitrary" moduli problems

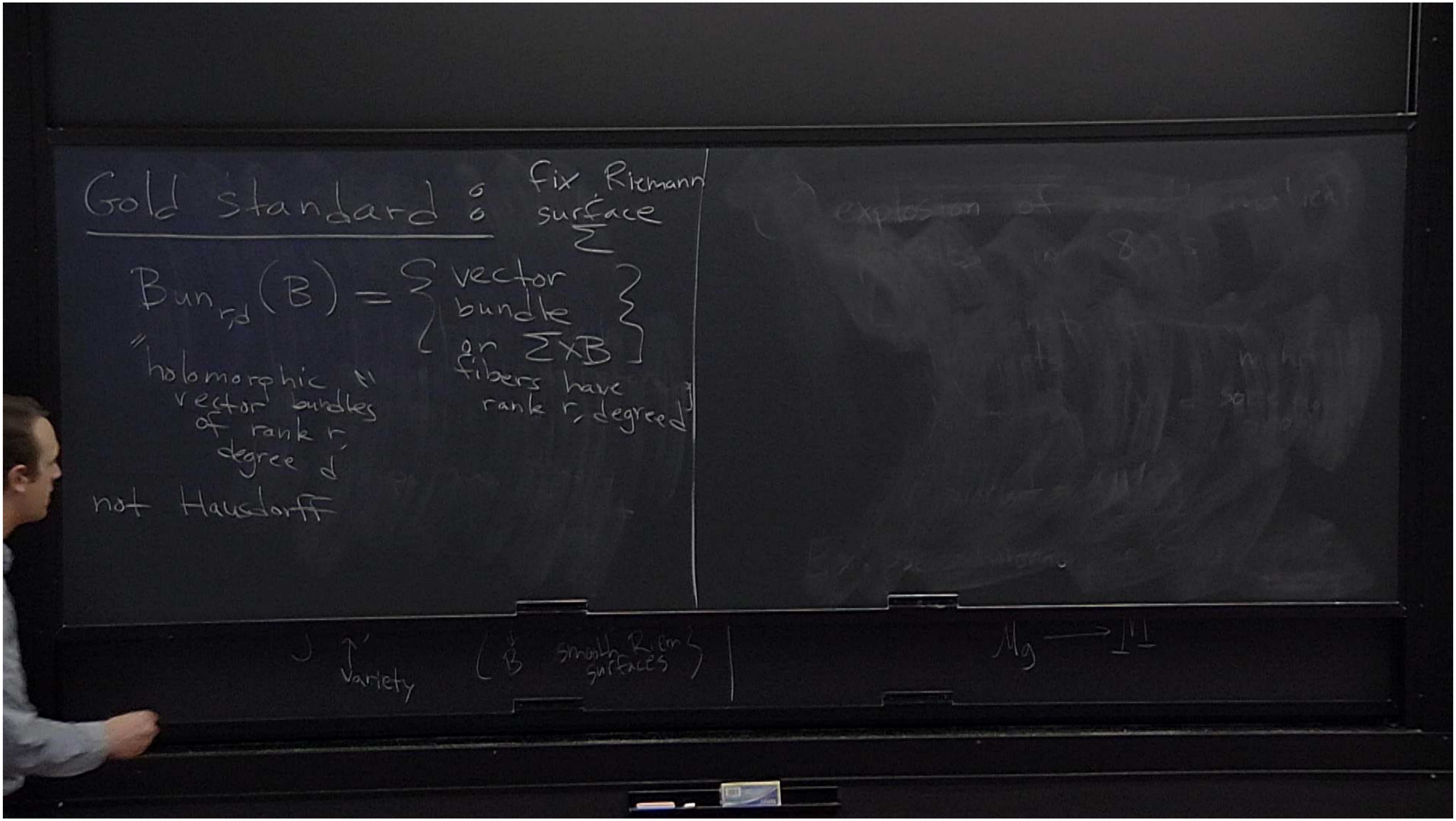
Thm (Alper-HL-Heinloth)
If $\mathcal{X}$ is a bounded algebraic stack (with affine diagonal), and
1) $\mathcal{X}$ is "Θ-reductive"
2) $\mathcal{X}$ has "unpunctured inertia"
3) closed points have linearly reductive stabilizers

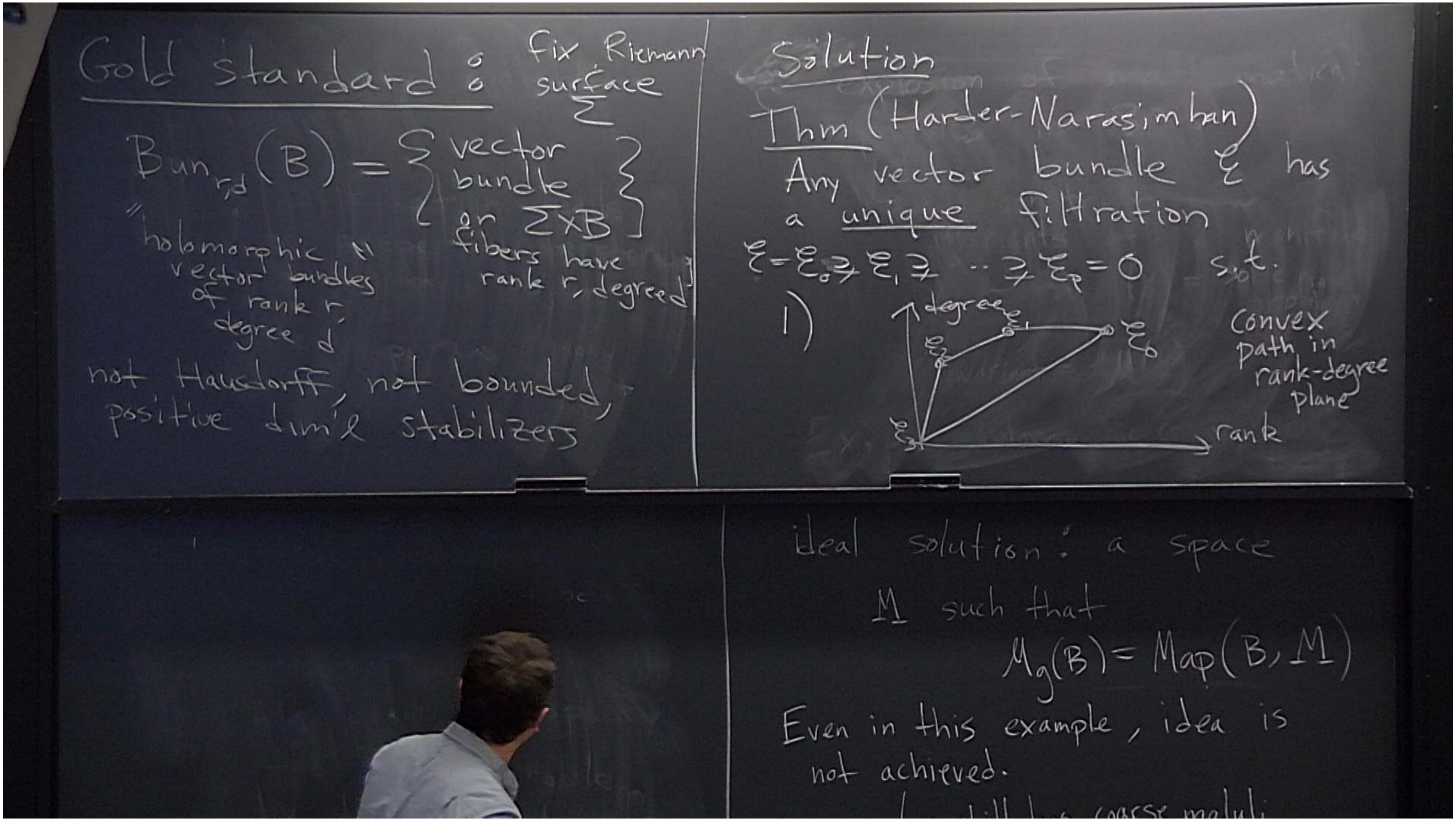
Then $\exists$ a "good moduli space" $\mathcal{X} \xrightarrow{q} X$

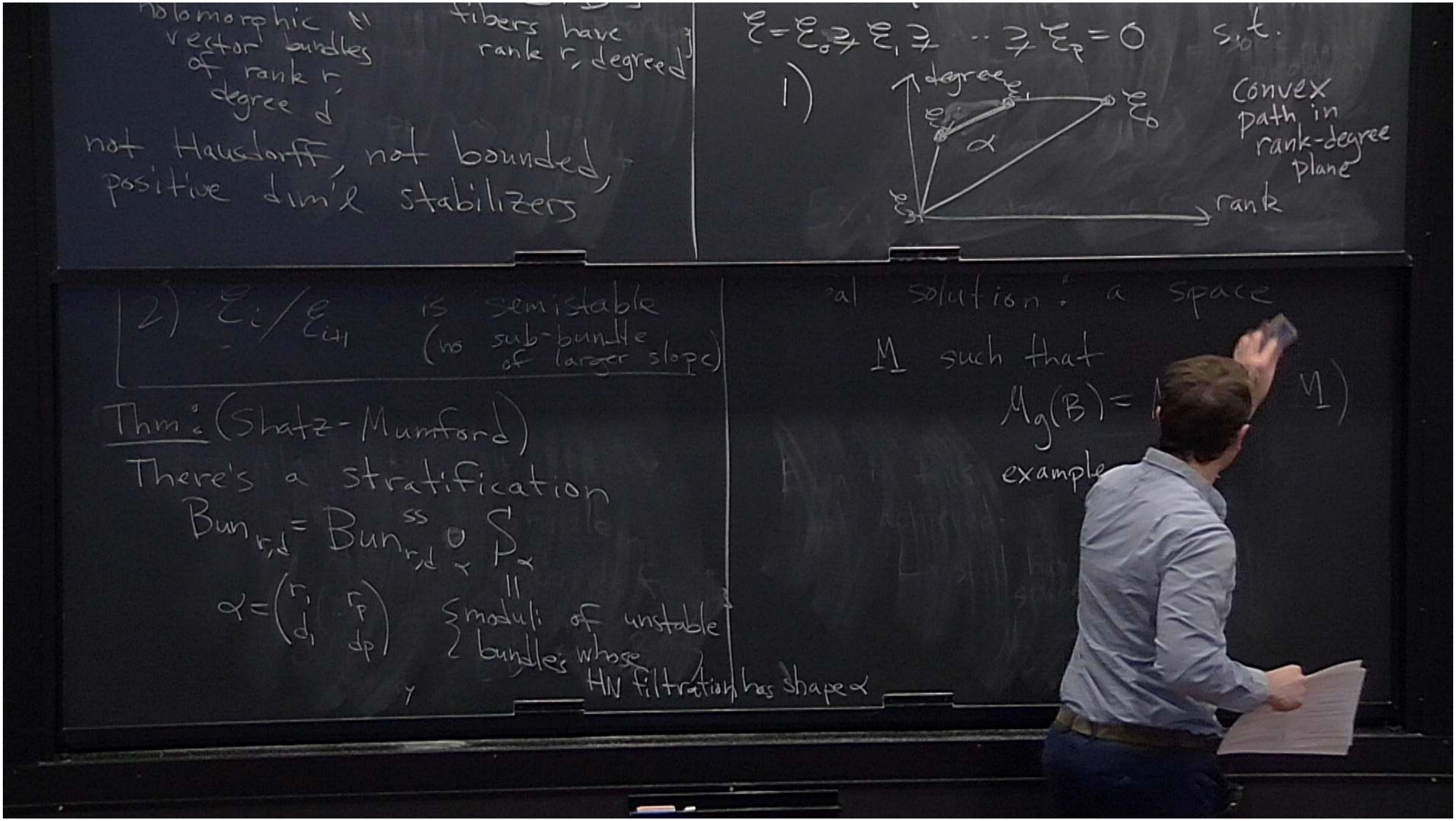
Thm: (Keel-Mori) given a moduli problem which ...

2) show it defines an algebraic stack


Gold standard :
fix Riemann surface Σ
Bun_{r,d}(B) = { vector bundle on Σ×B | fibers have rank r, degree d }
"holomorphic vector bundles of rank r, degree d"


Gold standard :
fix Riemann surface Σ
Bun_{r,d}(B) = { vector bundle on Σ×B fibers have rank r, degree d }
" holomorphic " vector bundles of rank r, degree d
not Hausdorff
variety
B smooth Riem surfaces
explosion of
80's


Gold standard: fix Riemann surface Σ
Bun_{r,d}(B) = { vector bundle on Σ×B, fibers have rank r, degree d }
" holomorphic " vector bundles of rank r, degree d
not Hausdorff, not bounded, positive dim'l stabilizers
Solution
Thm (Harder-Narasimhan)
Any vector bundle ℰ has a unique filtration
ℰ = ℰ₀ ⊋ ℰ₁ ⊋ ... ⊋ ℰ_p = 0 s.t.
1)
degree
rank
convex path in rank-degree plane
ideal solution: a space M such that
M_g(B) = Map(B, M)
Even in this example, idea is not achieved.


holomorphic vector bundles of rank r, degree d
fibers have rank r, degree d
not Hausdorff, not bounded, positive dim'l stabilizers
E = E0 ⊋ E1 ⊋ ... ⊋ Ep = 0 s.t.
1)
degree
rank
α
convex path in rank-degree plane
2) Ei/Ei+1 is semistable (no sub-bundle of larger slope)
Thm: (Shatz-Mumford)
There's a stratification
Bun r,d = Bun ss r,d ∪ ∪α Pα
α = (r1 ... rp / d1 ... dp)
{moduli of unstable bundles whose HN filtration has shape α}
solution: a space
M such that
Mg(B) =
example

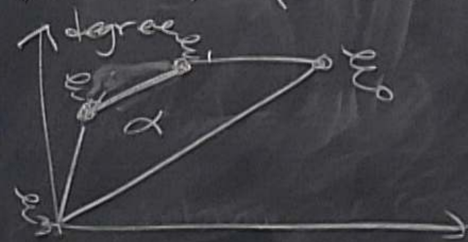
holomorphic vector bundles of rank r, degree d

fibers have rank r, degree d

not Hausdorff, not bounded, positive dim'l stabilizers

$\mathcal{E} = \mathcal{E}_0 \supsetneq \mathcal{E}_1 \supsetneq \cdots \supsetneq \mathcal{E}_p = 0$ s.t.

1)



convex path in rank-degree plane

2) $\mathcal{E}_i/\mathcal{E}_{i+1}$ is semistable (no sub-bundle of larger slope)

Thm: (Shatz-Mumford)

There's a stratification

$$Bun_{r,d} = \underline{Bun^{ss}_{r,d}} \sqcup \bigsqcup_{\alpha} S_\alpha$$

$$\alpha = \begin{pmatrix} r_1 & \cdots & r_p \\ d_1 & \cdots & d_p \end{pmatrix}$$

S_α = {moduli of unstable bundles whose HN filtration has shape α}

Furthermore $Bun^{ss}_{r,d}$ has a good moduli space

$$S_\alpha \xrightarrow{gr} \underbrace{Bun^{ss}_{r_1,d_1} \times \cdots \times Bun^{ss}_{r_p,d_p}}$$

Gold standard: fix Riemann surface Σ

$$\mathrm{Bun}_{r,d}(B) = \left\{ \begin{array}{l} \text{vector bundle} \\ \text{on } \Sigma \times B \end{array} \right\}$$

fibers have rank r, degree d

"holomorphic vector bundles of rank r, degree d"

not Hausdorff, not bounded, positive dim'l stabilizers

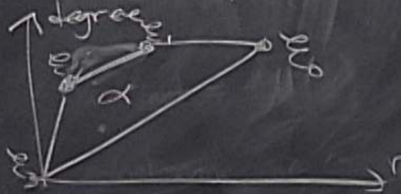
Solution

Thm (Harder-Narasimhan)

Any vector bundle $\mathcal{E}$ has a unique filtration

$$\mathcal{E} = \mathcal{E}_0 \supsetneq \mathcal{E}_1 \supsetneq \cdots \supsetneq \mathcal{E}_p = 0 \quad \text{s.t.}$$

1)



convex path in rank-degree plane

2) $\mathcal{E}_i/\mathcal{E}_{i+1}$ is semistable (no sub-bundle of larger slope)

Thm: (Shatz-Mumford)

There's a stratification

$$\mathrm{Bun}_{r,d} = \mathrm{Bun}^{ss}_{r,d} \cup \bigcup_\alpha S_\alpha$$

$$\alpha = \begin{pmatrix} r_1 & \cdots & r_p \\ d_1 & \cdots & d_p \end{pmatrix}$$

$S_\alpha = \{$moduli of unstable bundles whose HN filtration has shape $\alpha\}$

Furthermore $\mathrm{Bun}^{ss}_{r,d}$ has a good moduli space

$$S_\alpha \xrightarrow[gr]{} \underbrace{\mathrm{Bun}^{ss}_{r_1,d_1} \times \cdots \times \mathrm{Bun}^{ss}_{r_p,d_p}}$$

$$\mathcal{E}_i \supset \mathcal{E}_{i+1} \supset \cdots \supset$$

Θ-stability: immitates this approach for more general moduli problems

↳ maps from nodal curves to V/G

what is a filtration?

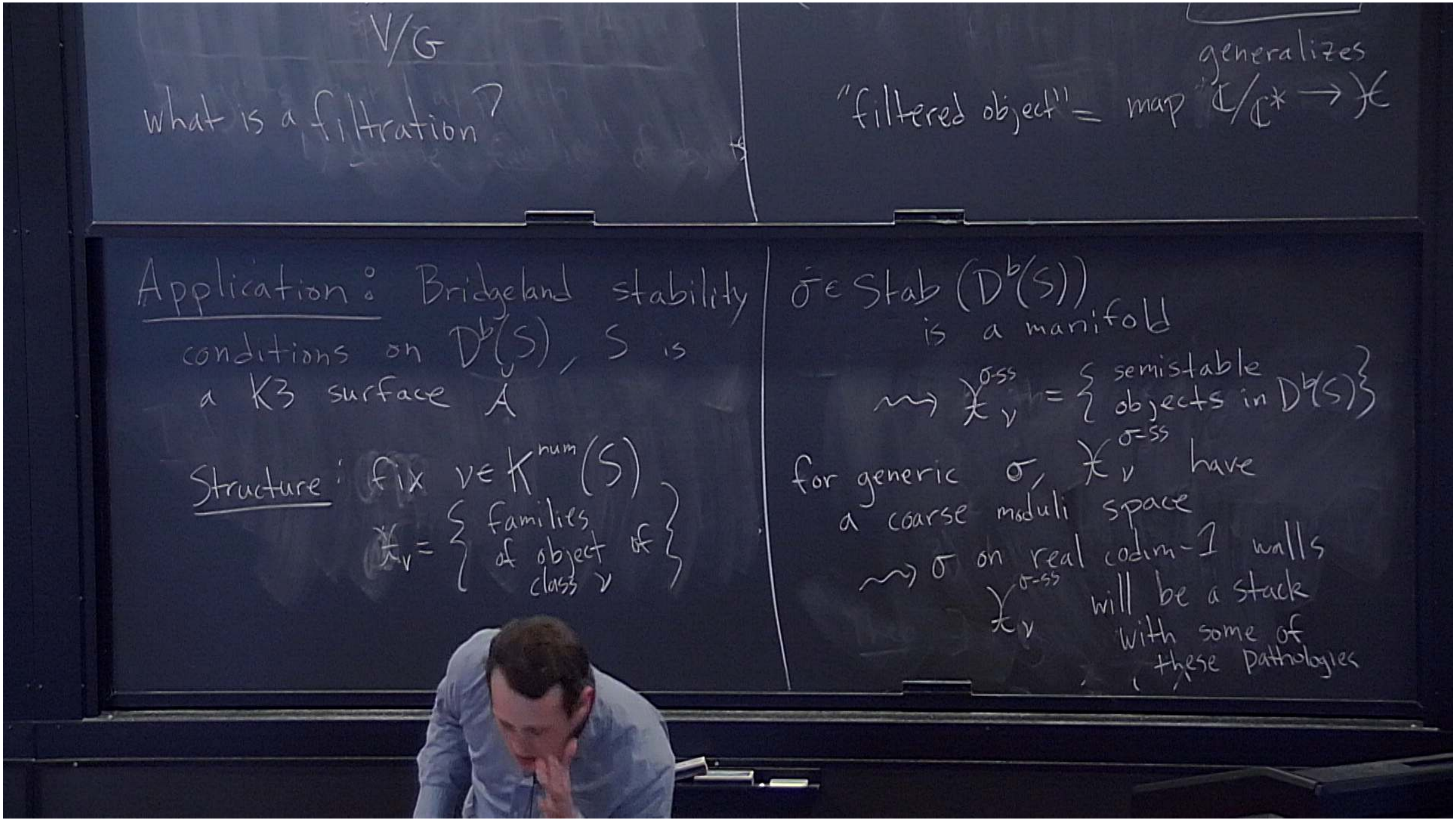
Rees construction:

$$\left(\begin{matrix}(\mathbb{Z}\text{-weighted})\\ \text{filtered}\\ \text{vector}\\ \text{bundles}\end{matrix}\right) \longleftrightarrow \underbrace{\left(\begin{matrix}\mathbb{C}^*\text{-equivariant}\\ \text{bundles on}\\ \mathbb{C}\times\Sigma\end{matrix}\right)}_{\text{generalizes}}$$

"filtered object" = map $\mathbb{C}/\mathbb{C}^* \to \mathcal{X}$

Problem: (3) fails in many situations (relevant to both)

Thm (Alper-HL-Heinloth)


V/G
what is a filtration?
generalizes
"filtered object" = map $\mathbb{C}/\mathbb{C}^* \to \mathfrak{X}$
Application: Bridgeland stability conditions on $D^b(S)$, S is a K3 surface
Structure: fix $v \in K^{num}(S)$
$\mathfrak{X}_v = \{$ families of object of class $v \}$
$\sigma \in \mathrm{Stab}(D^b(S))$ is a manifold
$\rightsquigarrow \mathfrak{X}_v^{\sigma\text{-ss}} = \{$ semistable objects in $D^b(S) \}$
for generic σ, $\mathfrak{X}_v^{\sigma\text{-ss}}$ have a coarse moduli space
$\rightsquigarrow$ σ on real codim-1 walls
$\mathfrak{X}_v^{\sigma\text{-ss}}$ will be a stack with some of these pathologies

... a stratification

$$\mathrm{Bun}_{r,d} = \underline{\mathrm{Bun}^{ss}_{r,d}} \sqcup \bigcup_{\alpha} P_\alpha$$

$\alpha = \begin{pmatrix} r_1 & \cdots & r_p \\ d_1 & \cdots & d_p \end{pmatrix}$ $\quad P_\alpha = \left\{ \begin{array}{l} \text{moduli of unstable} \\ \text{bundles whose} \\ \text{HN filtration has shape } \alpha \end{array} \right.$

gr ...

$\mathcal{E}_i \supset \mathcal{E}_{i+1} \supset \cdots \supset \cdots$

Cor: $\mathcal{X}_v^{\sigma\text{-ss}}$ has good moduli space $\forall \underbrace{\sigma}_{\text{non-generic}}$

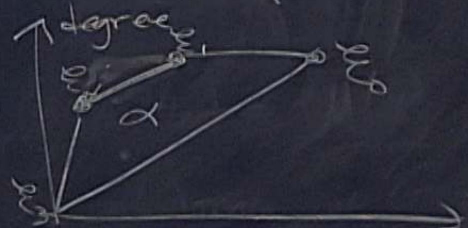
Picture:

$$\begin{tikzcd}
\mathcal{X}_v^{\sigma_-\text{-ss}} \arrow[r, phantom, "\subset"] & \mathcal{X}_v^{\sigma_0\text{-ss}} \arrow[d] \arrow[r, phantom, "\supset"] & \mathcal{X}_v^{\sigma_+\text{-ss}} \\
M^{\sigma_-\text{-ss}} \arrow[rr, dashed] \arrow[dr] & & M^{\sigma_+\text{-ss}} \arrow[dl] \\
& M^{\sigma_0\text{-ss}} &
\end{tikzcd}$$

Solution

Thm (Harder-Narasimhan)
Any vector bundle $\mathcal{E}$ has a **unique** filtration

$\mathcal{E} = \mathcal{E}_0 \supsetneq \mathcal{E}_1 \supsetneq \cdots \supsetneq \mathcal{E}_p = 0$ s.t.

1) 

convex path in rank-degree plane