Title: Strolling along gauge theory vacua

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Abstract: <p>We consider classical, pure Yang-Mills theory in a box. We show how a set of static electric fields that solve the theory in an adiabatic limit correspond to geodesic motion on the space of vacua, equipped with a particular Riemannian metric that we identify. The vacua are generated by spontaneously broken global gauge symmetries, leading to an infinite number of conserved momenta of the geodesic motion. We show that these correspond to the soft multipole charges of Yang-Mills theory.</p>

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- Gauge theory in the presence of boundaries
  - Global (large) gauge symmetries
  - Moduli space of vacua
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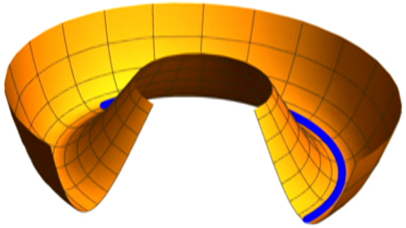

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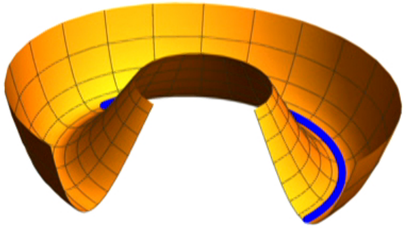

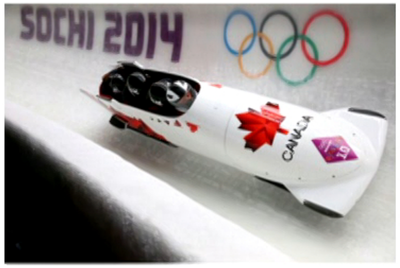

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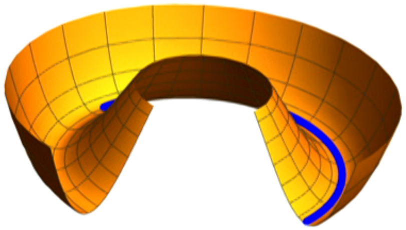

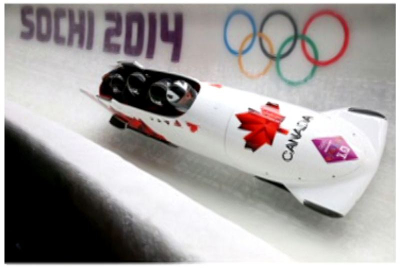

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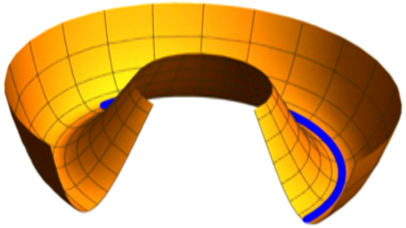

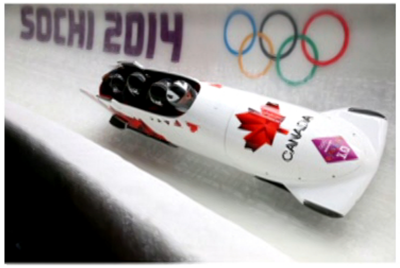

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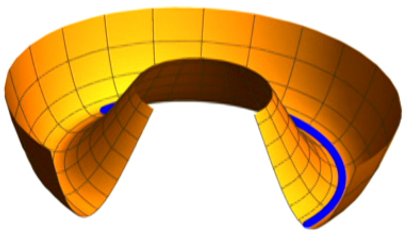

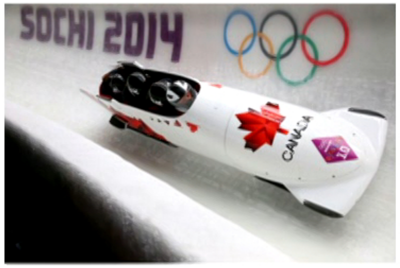

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$\alpha \to \alpha(t)$ Dyon [*Julia, Zee '75*]

Plan of the talk

- Classical pure YM theory in temporal gauge
- Space of vacua and the global gauge symmetries
- Geometry of the space of vacua
- Adiabatic motion on the space of vacua

Yang-Mills theory

- Classical pure Yang-Mills action with group G

$$S = -\frac{1}{2}\,\mathrm{Tr}\,F_{\mu\nu}\,F^{\mu\nu}$$

- Field equations

$$D_\mu F^{\mu\nu} = 0$$

where $D = \partial + [A, \cdot]$

Yang-Mills in temporal gauge

- Temporal gauge $A_0 = 0 \qquad \rightarrow \qquad A = A_i(t,x)dx^i$
- Equations of motion

$$D_i \dot{A}^i = 0, \qquad \ddot{A}_j = D_i F^{ij}$$

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- Lagrangian of the natural form

$$L = T - V$$

where

$$T = \frac{1}{2}\int d^3x \mathrm{Tr} \dot{A}_i \dot{A}^i, \qquad V = \frac{1}{2}\int d^3x \mathrm{Tr} F_{ij}F^{ij}$$

- Residual gauge symmetry group $\mathcal{G}$

$$A \rightarrow g \cdot A \equiv gAg^{-1} + gdg^{-1}, \qquad g = g(x)$$

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- Note. $g(t,x)$ is not a symmetry.

4

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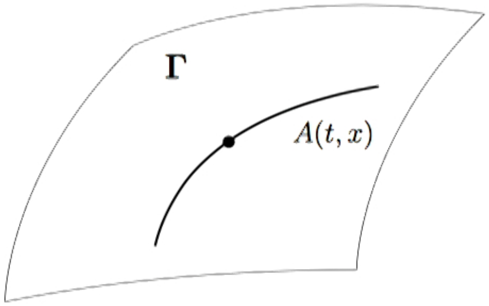
- Time dependent solutions as curves



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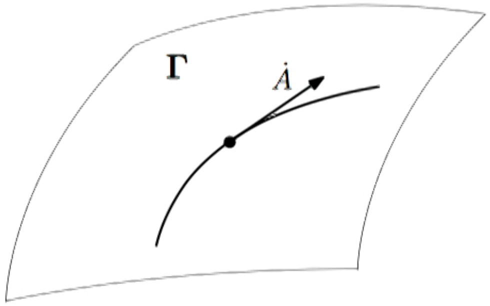

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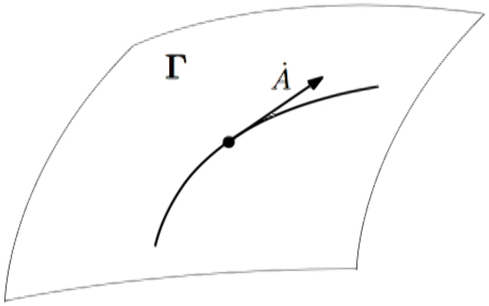

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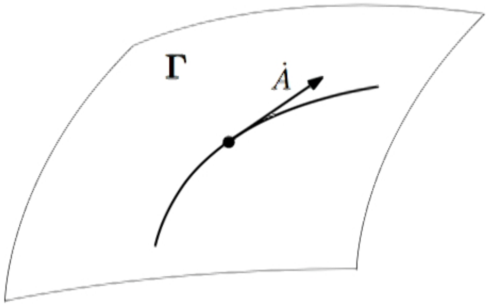
$$T = \int d^3x \, \mathrm{Tr} \, \dot{A}_i \, \dot{A}^i$$

  - Metric on the configuration space

$$g(\delta_1 A, \delta_2 A) = \int d^3x \, \mathrm{Tr} \, \delta_1 A_i \, \delta_2 A^i$$



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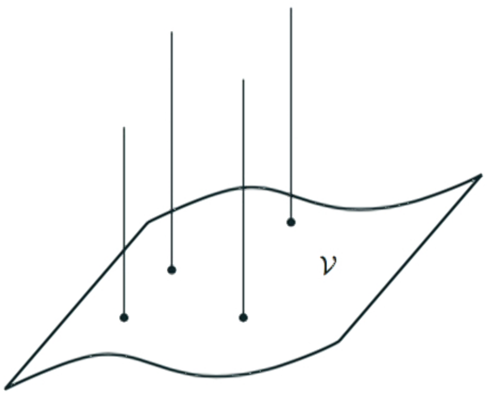

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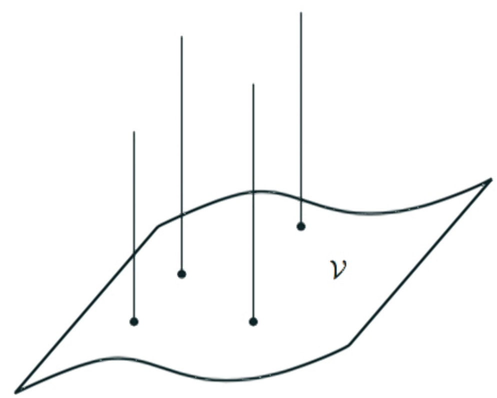
- Bundle structure
- Choose coordinates z^a on the space of physical vacua $\mathcal{V}$



6

Adiabatic motion on vacua

- Manton approximation [Manton'82, Stuart'07]

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- First characterization of global gauge symmetries

$$\delta_\sigma \bar{A}_o \qquad \text{is physical} \qquad \text{iff} \qquad \sigma \in \mathfrak{s}$$

Second characterization of global gauge symmetries

- Scalar product on gauge algebra

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- **Remark.** $\mathfrak{s}$ does not form an algebra w.r.t the usual bracket!

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Proof [First isomorphism thm.] If $\phi : G \to H$ is a homomorphism,

$$G/\ker \phi \cong \phi(G)$$

Take ϕ to be the restriction to the boundary $\phi(\mathcal{G}) = \mathcal{G}|_{\partial M}$.

Inner product on global gauge symmetries

- Given the identification $\mathfrak{s} \cong \boldsymbol{g}\big|_{\partial M}$, one can naively define the inner product

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- $\mathbb{D}$ can be extended into the bulk by further requiring that

$$\mathbb{D}\sigma \in \mathfrak{s}, \qquad \forall \sigma \in \mathfrak{s}$$

12

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- That is $\ker \mathbb{D} = \mathbb{k}$, the algebra of $\mathcal{K}$.

Geometry of the space of vacua

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$$\mathcal{V} \cong \mathcal{S}/\mathcal{K} \cong \mathcal{G}_0 \backslash \mathcal{G}/\mathcal{K}$$

- Around the reference vacuum

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- Choose a basis $\{\lambda_{\underline{a}}\}$ for $\boldsymbol{\mathfrak{m}}$

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$$\mathcal{V} \equiv \mathcal{S} \cdot \bar{A}_o$$

- It is a homogeneous space

$$\mathcal{V} \cong \mathcal{S}/\mathcal{K} \cong \mathcal{G}_0 \backslash \mathcal{G}/\mathcal{K}$$

- Around the reference vacuum

$$\boldsymbol{s} = \boldsymbol{m} \oplus \boldsymbol{k}, \qquad T_o\mathcal{V} \cong \boldsymbol{m}$$

- Choose a basis $\{\lambda_{\underline{a}}\}$ for $\boldsymbol{m}$

- Coordinate on $\mathcal{V}$

$$\bar{A}_z = g_z \cdot \bar{A}_o, \qquad g_z = \exp(\lambda_{\underline{a}} z^a)$$

14

Geometry of the space of vacua

- Remind that for adiabatic motion

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Geometry of the space of vacua

Page 86 of 121

Geometry of the space of vacua

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- e plays the role of a vielbein

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- Corresponding spacetime field

$$E = -v\, g\, D_o \lambda_{\underline{a}}\, g^{-1}, \qquad B_i = \frac{1}{2}\epsilon_{ijk} F^{jk} = 0$$

Conserved charges

18

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$$P[\xi] = \frac{\partial L}{\partial \dot{z}^a} \xi^a = g_{ab} \dot{z}^a \xi^b$$

- The momentum is equal to the corresponding Noether charge in the full theory

$$P_\sigma = \oint d\Sigma n_i \mathrm{Tr} \sigma E^i = Q_\sigma$$

18

Examples

Maxwell theory

- Global gauge symmetries

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19

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19

Maxwell continued

- Metric

$$ds^2 = R \sum_{\ell \geq 1, m} \ell \, (dz^{\ell m})^2$$

20

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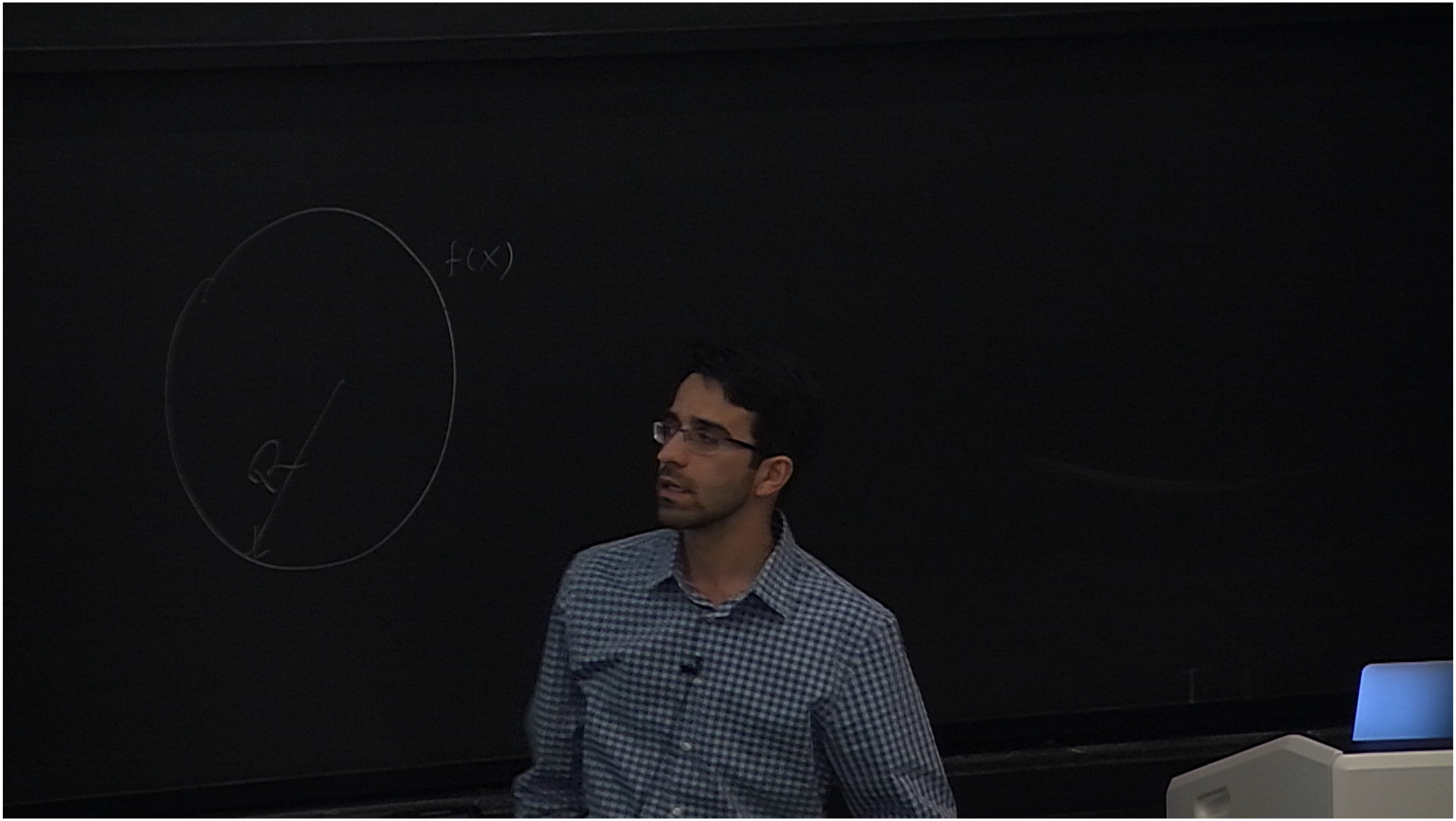
Adiabatic motion on the space of vacua corresponds to spacetime field which is a source-free electrostatic solution.



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Yang-Mills theory

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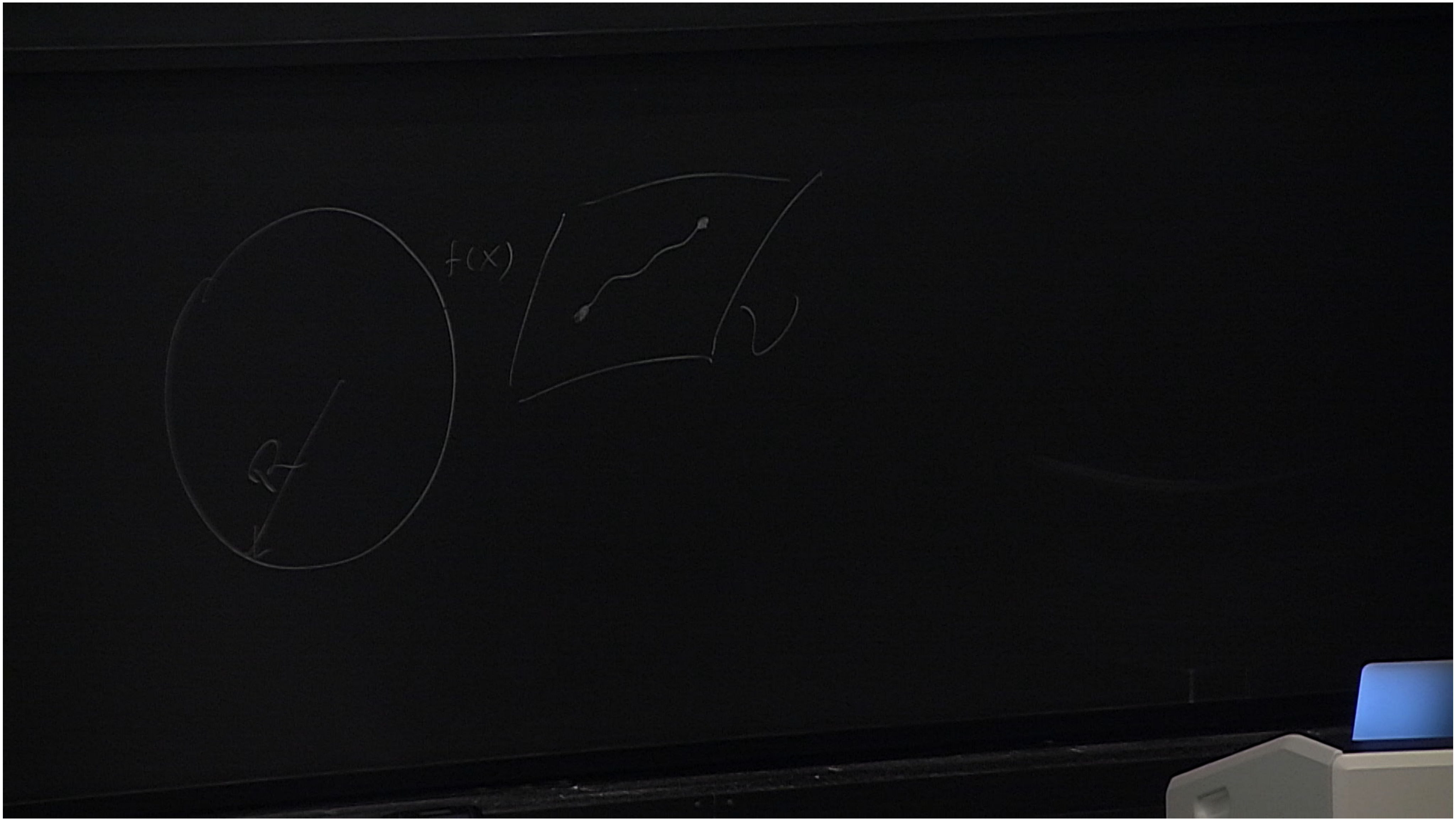
- Spacetime fields are again source free electrostatic solutions dressed by global gauge symmetries

Discussion

22

Discussion

- Memory effect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Discussion

- Memory effect
- The large R limit

$$Q \propto Rv, \qquad T \propto Rv^2$$

Thank you for your attention

In memory of Maryam Mirzakhani (1977-2017)