Title: Self-dual N=4 theories in four dimensions

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Abstract: <p>Known N=4 theories in four dimensions are characterized by a choice of gauge group, and in some cases some "discrete theta angles", as classified by Aharony, Seiberg and Tachikawa. I will review how this data, for the theories with algebra su(N), is encoded in various familiar realizations of the theory, in particular in the holographic AdS_5 \times S^5 dual and in the compactification of the (2,0) A_N theory on T^2. I will then show how the resulting structure, given by a choice of polarization of an appropriate cohomology group, admits additional choices that, unlike known theories, generically preserve SL(2,Z) invariance in four dimensions.</p>

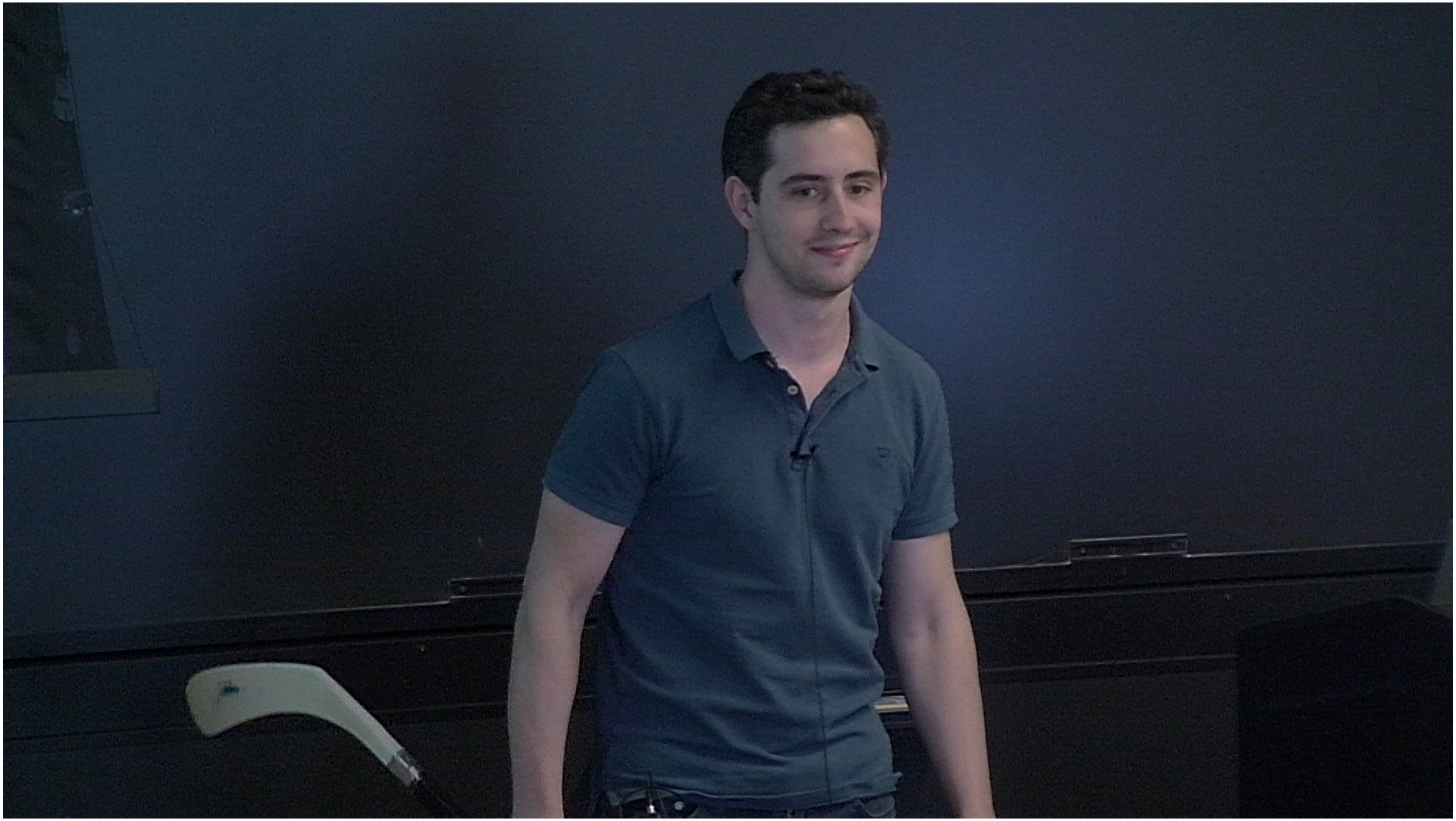
Self-dual $\mathcal{N} = 4$ theories in four dimensions

CERN

Diego Regalado

Perimeter Institute, October 2017

Based on work in progress with I. García-Etxebarria and B. Heidenreich



What defines an $\mathcal{N} = 4$ QFT in 4d?

- Classically, a Lie algebra (sum of commuting compact simple and u(1) subalgebras)

$$\mathcal{L} = \mathrm{tr}\left(-\frac{1}{2g^2} F \wedge *F + \dots\right)$$

- At the quantum level, we need a Lie group, not just an algebra.

$$Z = \sum_i Z_i$$

i labels the topological sector

- This might not be enough. Discrete theta angles (or line operators)

[Aharony, Seiberg, Tachikawa '13]

$$Z = \sum_i a_i Z_i$$

a_i are coefficients that depend on i

Not every set of coefficients a_i defines a consistent QFT.

An example: $\mathfrak{su}(2)$

We may consider the following gauge groups:

- $SU(2)$: The topological sectors are labeled by the instanton number.

$$Z_{SU(2)} = \sum_{\nu \in \mathbb{Z}} Z_\nu \qquad \nu = \frac{1}{8\pi^2} \int_M \mathrm{tr}\, F \wedge F$$

- $SO(3)$: The different sectors are labeled by the instanton number and the Stiefel-Whitney class (or 'magnetic flux') $w_2 \in H^2(M, \mathbb{Z}_2)$.

[Aharony, Seiberg, Tachikawa '13]

  - $SO(3)^+$: $a_{\nu, w_2} = 1$
  - $SO(3)^-$: $a_{\nu, w_2} = \exp\left(2\pi i \frac{w_2^2}{4}\right)$ (no torsion in cohomology)

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These three theories transform as a triplet under $SL(2, \mathbb{Z})$

$$T \circlearrowright SU(2) \overset{S}{\longleftrightarrow} SO(3)^+ \overset{T}{\longleftrightarrow} SO(3)^- \circlearrowleft S$$

None of these is invariant under the full $SL(2, \mathbb{Z})$

Are there more possibilities?

It is natural to ask whether these are all the possibilities, or if there can be other $\mathcal{N}=4$ theories which do not fall into this classification.

- Argyres and Martone have proposed [Argyres, Martone '16] the existence of a new $\mathcal{N}=4$ theory with algebra $\mathfrak{su}(2)$ which is invariant under $SL(2,\mathbb{Z})$.

- In this talk I will argue that such an $SL(2,\mathbb{Z})$-invariant $\mathfrak{su}(2)$ theory exists (in fact $\mathfrak{su}(N)$ for all N).

A wrong argument

Type IIB on $Q \times \mathbb{C}^2/\mathbb{Z}_N$ is described at low energies by the (2,0) theory of type A_{N-1} on Q. [Witten '95]

(Since $H^2(\mathbb{C}^2/\mathbb{Z}_N, \mathbb{Z}) = \mathbb{Z}^{\oplus(N-1)}$, by (2,0) theory of type A_{N-1}, I mean the theory that has $N-1$ free tensors on its tensor branch. It is $\mathfrak{su}(N)$, not $\mathfrak{u}(N)$).

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The problem with this argument is that Type IIB has a **self-dual** 4-form C_4 and it is not guaranteed that the quantization of this sector is invariant under diffeomorphisms. (More on this at the end).

Strategy and outline

Look at different ways to engineer $\mathcal{N}=4$ SYM with gauge algebra $\mathfrak{su}(N)$, paying special attention to the origin of the global structure of the theory.

- Holography: Type IIB on $X \times S^5$ with $\partial X = M$. [Witten '98]
- (2,0) of type A_{N-1} on $M \times T^2$ for small T^2. [Witten '98] [Tachikawa '14]
- Type IIB on $M \times T^2 \times \mathbb{C}^2/\mathbb{Z}_N$ for small T^2.

Holographic dual

[Witten '98]

- Roughly speaking, Type IIB on $X \times S^5$, with N units of F_5 flux, is dual to $\mathcal{N} = 4$ SYM on $M = \partial X$ with gauge algebra $\mathfrak{su}(N)$.

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- How can we distinguish the different variants ($SU(N)$, $SU(N)/\mathbb{Z}_N$, etc)?

  Boundary conditions for the fields on the gravity side.

  A somewhat trivial illustration of this is the coupling constant. For fixed global structure, the $\mathcal{N}=4$ theories are parametrized by τ. Likewise, Type IIB on $X \times S^5$ is parametrized by the asymptotic value of the axiodilaton.

  The same is true for the global structure, although the details are more subtle.

Boundary data

[Witten '98]

The different variants are really different when M is such that $H^2(M, \mathbb{Z})$ is non-zero. In this case, we have to specify extra data to completely fix the Type IIB theory.

$$\alpha_B(\sigma) = \int_\sigma B$$

$$\alpha_C(\sigma) = \int_\sigma C$$

B and C are the NSNS and RR 2-form potentials

σ is an arbitrary surface in M (non-trivial in $H_2(M, \mathbb{Z})$)

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Naively, we would like to fix all of these quantities. Then, the partition function would be a function of them.

However, this is not possible since they are canonically conjugate variables,

$$[\alpha_B, \alpha_C] \neq 0$$

TQFT for the BC system [Witten '98]

At low energies, the BC system is governed by the action

$$S_{BC} = \frac{iN}{2\pi} \int_X B \wedge dC \qquad \text{where } X \simeq \mathbb{R} \times M \, .$$

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- At the classical level, the fields B and C are conjugate to each other.

$$\Phi_B(\sigma) = e^{i\int_\sigma B} \qquad \Phi_B(\sigma)\Phi_C(\sigma')\Phi_B^{-1}(\sigma)\Phi_C^{-1}(\sigma') = e^{2\pi i(\sigma\cdot\sigma')/N}$$

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The operators $\Phi_B(\sigma)$, $\Phi_C(\sigma')$ generate a symmetry group W,

$$0 \to \mathbb{Z}_N \to W \to H_2(M, \mathbb{Z}_N) \times H_2(M, \mathbb{Z}_N) \to 0$$

- In order to canonically quantize, we need to pick a **polarization**, i.e. a maximal subgroup of commuting Φ's.
- Given a polarization, we find a **Hilbert space** of states. Each state corresponds to a particular boundary condition.

Back to the gauge theory

Polarization	→	Global structure
Example $N=2$ $\langle\Phi_B(\sigma)\rangle$		$SU(2)$
$\langle\Phi_C(\sigma)\rangle$		$SO(3)^+$
$\langle\Phi_B(\sigma)\Phi_C(\sigma)\rangle$		$SO(3)^-$

This choice breaks $SL(2,\mathbb{Z})$ on both sides.

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Boundary conditions (state in Hilbert space) → Background field for 1-form global symmetry

The $SU(2)$, $SO(3)^+$and $SO(3)^-$ have global 1-form $\mathbb{Z}_2$ symmetries, which can be coupled to a background 2-form in $H^2(M,\mathbb{Z}_2)$.

[Gaiotto, Kapustin, Seiberg, Willet'15]

Example

A polarization determines a specific $\mathcal{N} = 4$ theory with algebra $\mathfrak{su}(N)$. So in order to find new theories, we need to look for new polarizations.

Let's consider a simple example first: $N = 2$ and $M = S^2 \times S^2$.

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The relevant homology group to look at is $H_2(M,\mathbb{Z}_2)=\mathbb{Z}_2\oplus\mathbb{Z}_2$, with intersection product $\sigma\cdot\sigma=1$, $\sigma^2=\sigma^2=0$.

The symmetry group W is generated by Φ_B, Φ_B, Φ_C, Φ_C with relations

$$\Phi_B\Phi_C\Phi_B^{-1}\Phi_C^{-1}=-1,\ {\Phi_B}^2=1,\ldots$$

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Usual polarizations:	$\langle\Phi_B,\Phi_B\rangle$	$\langle\Phi_C,\Phi_C\rangle$	$\langle\Phi_B\Phi_C,\Phi_B\Phi_C\rangle$
	$SU(2)$	$SO(3)^+$	$SO(3)^-$

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But there are more!	$\langle\Phi_B,\Phi_C\rangle$	$\langle\Phi_B,\Phi_C\rangle$	$\langle\Phi_B\Phi_B,\Phi_C\Phi_C\rangle$

New polarizations

The reason we found other polarizations in the previous example is that $H_2(S^2 \times S^2, \mathbb{Z}_2)$ contains a **maximal isotropic** subspace A, i.e. a maximal subgroup such that its elements do not intersect each other.

For $A \subset H_2(M, \mathbb{Z}_N)$ maximal isotropic, then $\langle \Phi_B(\sigma), \Phi_C(\sigma) \rangle_{\sigma \in A}$ is a polarization of W.

$$0 \to \mathbb{Z}_N \to W \to H_2(M, \mathbb{Z}_N) \times H_2(M, \mathbb{Z}_N) \to 0$$

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For every compact, smooth, orientable, spin manifold M (without torsion), $H_2(M, \mathbb{Z}_N)$ has a maximal isotropic subspace.*

*We have a proof for every prime N and for low values of N.

Intersection form

For every compact, smooth, orientable, spin manifold M (without torsion) the intersection form on $H_2(M,\mathbb{Z})$ is [Donaldson '83, '87]

$$Q = (-\mathcal{C}(E_8))^{\oplus m} \oplus H^{\oplus n}$$

$\mathcal{C}(E_8)$ is the E_8 Cartan form

$$H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

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- In general, $H_2(M,\mathbb{Z})$ has no maximal isotropic subspace, since $\mathcal{C}(E_8)$ is negative definite.

- However, $H_2(M,\mathbb{Z}_N)$ does have one for every N.* In fact, over $\mathbb{Z}_N$, the intersection form is

$$Q_{\mathbb{Z}_N} = H^{\oplus(4m+n)}$$

So, if we work over $\mathbb{Z}_N$, every M behaves essentially like $S^2 \times S^2$.

Partition function

We would like to compute the partition function of the theory associated to the polarization $\langle \Phi_B(\sigma), \Phi_C(\sigma) \rangle_{\sigma \in A}$ and show that it is invariant under $SL(2,\mathbb{Z})$.

For a theory with gauge group G, the partition function is roughly

$$Z(\tau) = \sum_{\text{bundles of } G} a_i \, Z_i(\tau)$$

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We want to find the analogous formula for the polarization associated to the maximal isotropic subspace of $H_2(M, \mathbb{Z}_N)$.

The result shows that there is not really a notion of gauge group.

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For the special case $w = 0$, we recover the usual $SU(N)$ partition function. For other values, we are coupling the 1-form global $\mathbb{Z}_N$ symmetry to a background 2-form field $B_{bg} = w$.

Partition function of $SU(N)/\mathbb{Z}_N$

For any other polarization, we find other partition functions, depending on the boundary conditions in AdS (or background fields in the CFT).

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For the polarization $\langle \Phi_B(\sigma), \Phi_C(\sigma) \rangle_{\sigma \in A}$, with A a maximal isotropic subspace of $H_2(M, \mathbb{Z}_N)$, the invariant vector is

$$|\Omega_\bullet\rangle = \sum_{w \in A^*} |w\rangle$$

so the corresponding partition function is

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Notice that the polarization $\langle \Phi_B(\sigma), \Phi_C(\sigma) \rangle_{\sigma \in A}$ is invariant under $SL(2, \mathbb{Z})$. We expect that $Z_\bullet(\tau)$ is also $SL(2, \mathbb{Z})$ invariant.

$SL(2,\mathbb{Z})$ invariance of the partition function

The behavior of the functions $Z_w(\tau)$ under $SL(2,\mathbb{Z})$ is [Vafa,Witten '94]

$$Z_w(\tau+1) = e^{2\pi i \frac{w^2}{2N}} Z_w(\tau)$$

$$Z_w(-1/\tau) = N^{-b_2/2} \sum_{u \in H^2(M,\mathbb{Z}_N)} e^{\frac{2\pi i}{N} u \cdot w} Z_u(\tau)$$

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where we used that $\displaystyle \sum_{w \in A^*} e^{\frac{2\pi i}{N} u\cdot w} = \begin{cases} N^{b_2/2} & u \in A^* \\ 0 & u \notin A^* \end{cases}$

Partition function on K3

Typically, it is hard to compute the functions $Z_w(\tau)$. For $K3$, these are known [Vafa Witten '94]. In that case, one can check explicitly that the partition function $Z_\bullet(\tau)$ is invariant under $SL(2,\mathbb{Z})$.

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For $N=2$, there are three different blocks:

$$Z_w(\tau) = \begin{cases} Z_0 = \frac{1}{4}G(q^2) + \frac{1}{2}[G(q^{1/2}) + G(-q^{1/2})] & \text{if } w = 0 \\ Z_e = \frac{1}{2}[G(q^{1/2}) + G(-q^{1/2})] & \text{if } w \neq 0,\ w^2 = 0 \mod 4 \\ Z_o = \frac{1}{2}[G(q^{1/2}) - G(-q^{1/2})] & \text{if } w^2 = 2 \mod 4 \end{cases}$$

with $G(q) \equiv \eta(q)^{-24}$, $q = \exp 2\pi i\tau$.

In this case, since $w^2 = 0 \mod 4$ for all $w \in A^* \subset H^2(K3, \mathbb{Z}_2)$ we find

$$Z_\bullet(\tau) = Z_0(\tau) + (2^{11} - 1)Z_e(\tau)$$

This is indeed (the unique combination) invariant under $SL(2,\mathbb{Z})$.

$SL(2,\mathbb{Z})$ and diffeomorphisms

The maximal isotropic subspace of $H_2(M,\mathbb{Z}_N)$ leads to an $\mathcal{N}=4$ theory with algebra $\mathfrak{su}(N)$ which is invariant under $SL(2,\mathbb{Z})$. However, we do not expect an arbitrary diffeomorphism of M to leave the isotropic subspace invariant.

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The link between $SL(2,\mathbb{Z})$ and large diffs of the 4d space M becomes even more clear in the Type IIB and (2,0) constructions that we sketch in the following.

(2,0) of Type A_N

It is believed that the (2,0) theory in six dimensions of type A_N on $M \times T^2$ flows to $\mathcal{N} = 4$ SYM of type $\mathfrak{su}(N)$ on M, when T^2 is small.

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- When $Q = \partial Y$ has $H_3(Q, \mathbb{Z}) \neq 0$, we have to specify the boundary value of

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- Not all of them can be specified at the same time,

$$S \sim N \int_Y C \wedge dC \longrightarrow [C, C] \neq 0 \longrightarrow$$

Polarization of W

$$0 \to \mathbb{Z}_N \to W \to H_3(Q, \mathbb{Z}_N) \to 0$$

(2,0) of Type A_N

The choice of polarization, or maximal isotropic subspace of $H_3(Q,\mathbb{Z}_N)$ breaks invariance under diffeomorphisms (that act at infinity) generically.

(Small diffs are also broken anyway [Kraus, Larsen '05][Solodukhin '05])

When $Q = M \times T^2$, we have that (assuming $H_1(M) = 0$ and no torsion)

$$\begin{aligned} H_3(Q,\mathbb{Z}_N) &= H_2(M,\mathbb{Z}_N) \otimes H_1(T^2,\mathbb{Z}_N) \\ &= H_2(M,\mathbb{Z}_N) \oplus H_2(M,\mathbb{Z}_N) \end{aligned}$$

We recover the same picture as before:

- [Tachikawa '14] There exist polarizations that preserve invariance under the diffs of M. This reproduces [Aharony, Seiberg, Tachikawa '13].

- The self-dual polarization is also possible, which generically breaks invariance under large diffs of M.

Type IIB on $\mathbb{C}^2/\mathbb{Z}_N$ (work in progress)

Consider Type IIB on $Q \times \mathbb{C}^2/\mathbb{Z}_N$, which is described at low energies by the (2,0) theory of type A_{N-1}.

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- In order to specify boundary conditions, we first need to pick a **polarization** of W. This breaks invariance under large diffeomorphisms acting at infinity.

Summary

We typically think of an $\mathcal{N}=4$ theory to be defined by a choice of gauge group (plus some additional discrete theta angles).

We have argued, by looking at the holographic dual as well as the (2,0) theory on T^2 and Type IIB on $M \times T^2 \times \mathbb{C}^2/\mathbb{Z}_N$, that it might be better to think of the global data required to define an $\mathcal{N}=4$ theory as a **Lie algebra + polarization**.

For $\mathfrak{su}(N)$, this reproduces the known $\mathcal{N}=4$ theories (AST). By taking different polarizations we may construct **new theories**.

In particular, we have shown that there is a global version of the $\mathfrak{su}(N)$ theory that is invariant under $SL(2,\mathbb{Z})$ (but not invariant under large diffeomorphisms). This follows from the fact that $H_2(M,\mathbb{Z}_N)$ always has a **maximal isotropic subspace**.

Open questions

- Prove that the maximal isotropic subspace of $H_2(M, \mathbb{Z}_N)$ actually exists for every N.
- Classification of all the possible variants (polarizations).
- Include torsion in $H_2(M, \mathbb{Z})$.
- Generalization to other algebras.
- Generalization to class $\mathcal{S}$.
- Understand the gluing axiom better for the (2,0) theory. The anomaly theory has appeared recently [Monnier '16],[Monnier '17]. Despite being non-invertible, the anomaly theory obeys the gluing axioms. We would like to understand this better.