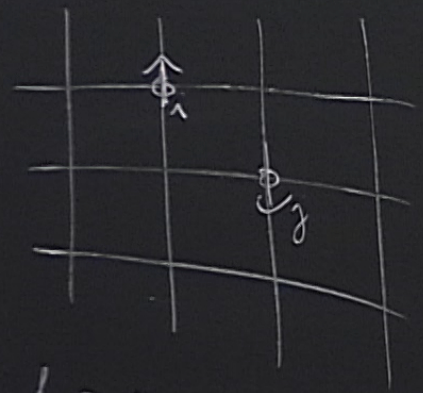
Title: PSI 17/18 - Statistical Mechanics - Lecture 10

Date: Oct 23, 2017 10:45 AM

URL: http://pirsa.org/17100043

Abstract:

LAST WEEK: ISING MODEL OF FERROMAGNETISM



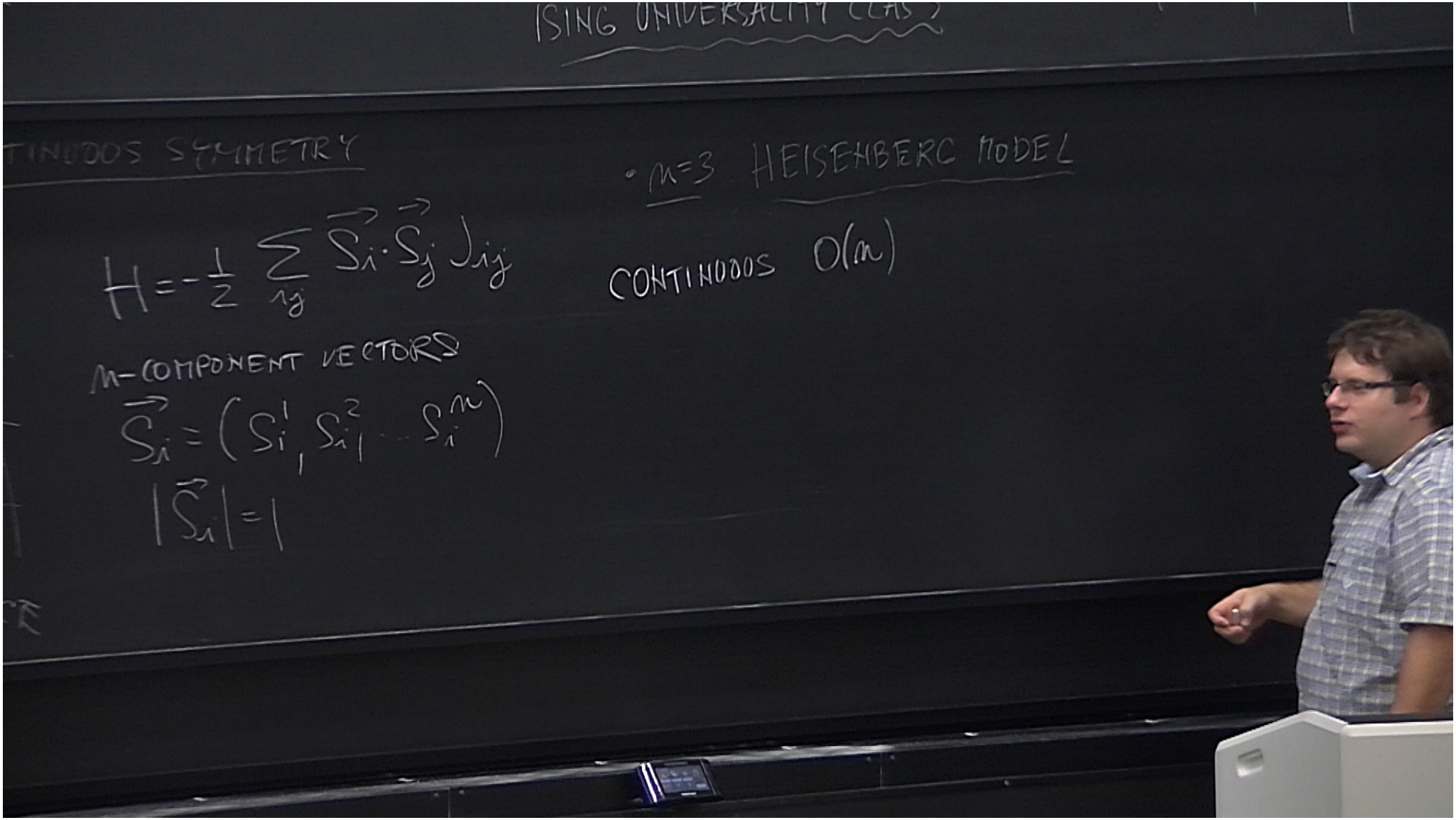
d-DIM CUBIC LATTICE

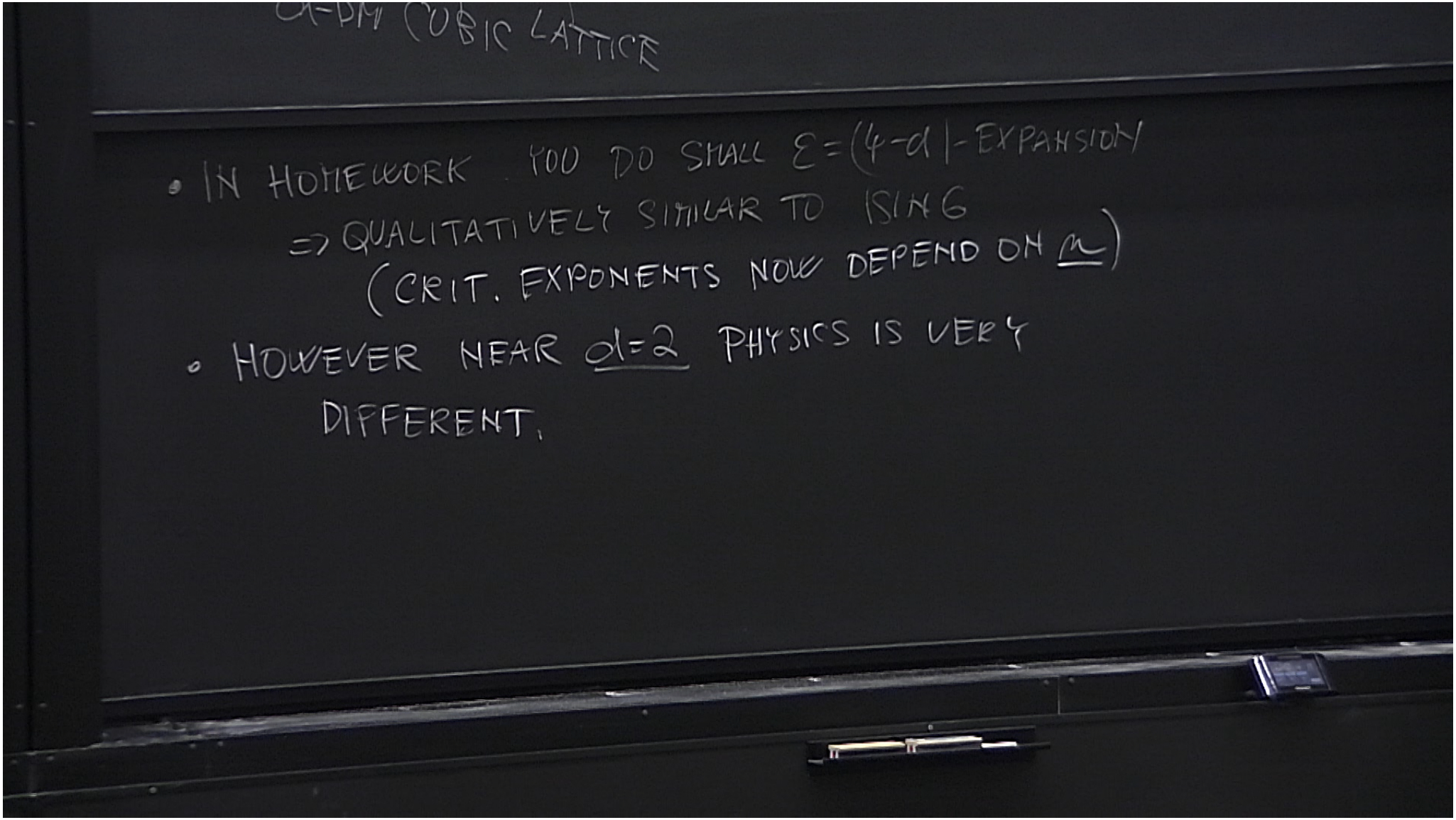
$$H = -\frac{1}{2}\sum_{ij} J_{ij}\sigma_i\sigma_j \qquad \sigma_i = \pm 1$$

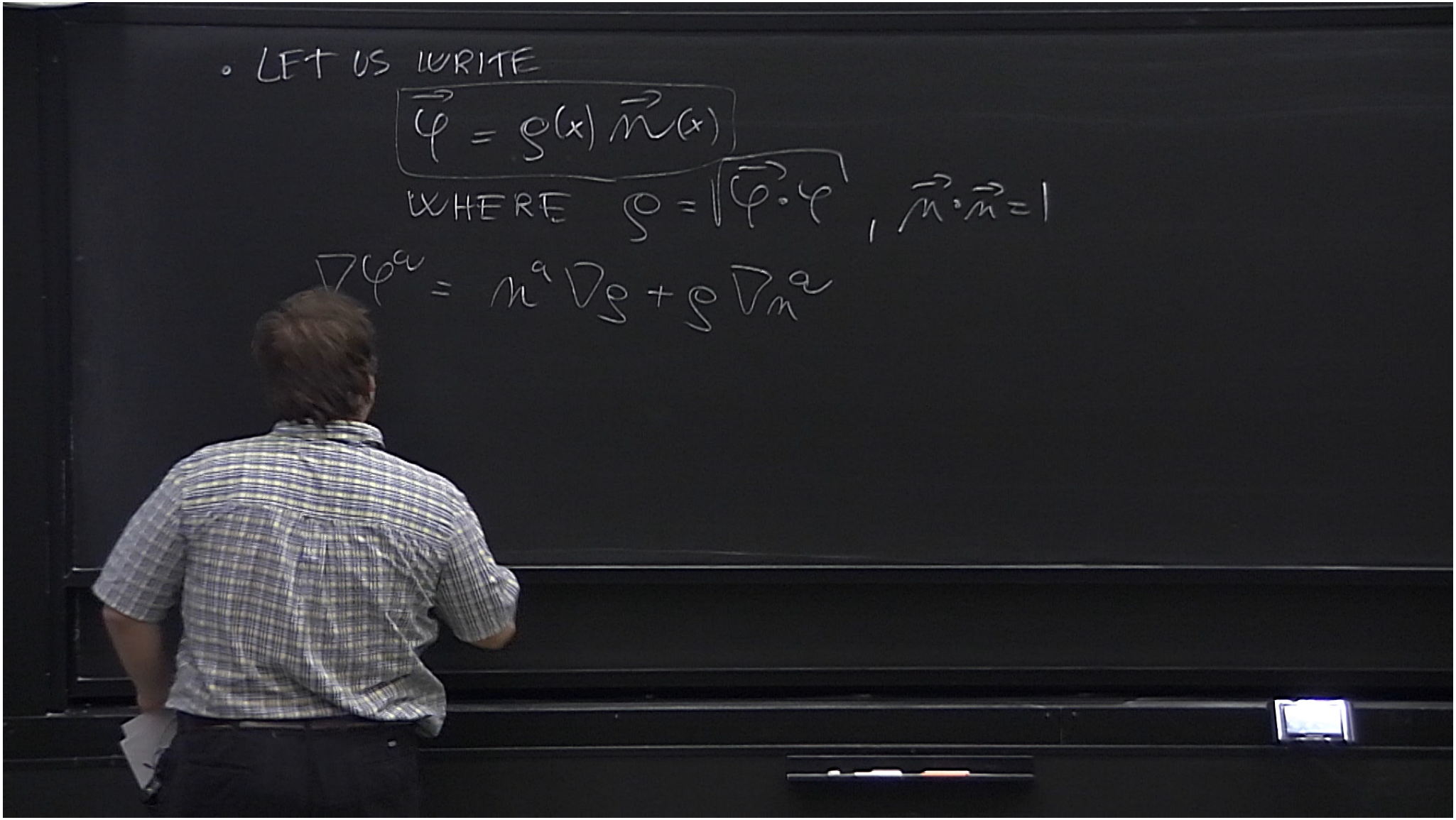
$$\rightarrow \quad S = \int d^dx \left(\frac{1}{2}(\nabla\varphi)^2 + \frac{r}{2}\varphi^2 + \right.$$

LANDAU-GINSBURG

- ORDER PARAMETER IS SCALAR
- DISCRETE Z_2 SYMMETRY


ISING UNIVERSALITY CLASS
TINUOUS SYMMETRY
$H = -\frac{1}{2}\sum_{ij} \vec{S}_i \cdot \vec{S}_j J_{ij}$
n-COMPONENT VECTORS
$\vec{S}_i = (S_i^1, S_i^2 \cdots S_i^n)$
$|\vec{S}_i| = 1$
• n=3 HEISENBERG MODEL
CONTINUOUS O(n)


d-DIM CUBIC LATTICE
• IN HOMEWORK YOU DO SMALL $\varepsilon=(4-d)$-EXPANSION
⇒ QUALITATIVELY SIMILAR TO ISING
(CRIT. EXPONENTS NOW DEPEND ON n)
• HOWEVER NEAR $d=2$ PHYSICS IS VERY
DIFFERENT.


• LET US WRITE
$\vec{\varphi} = \varrho(x)\,\vec{n}(x)$
WHERE $\varrho = \sqrt{\vec{\varphi}\cdot\vec{\varphi}}$, $\vec{n}\cdot\vec{n} = 1$
$\nabla\varphi^a = n^a \nabla\varrho + \varrho\,\nabla n^a$

- LET US WRITE

$$\vec{\varphi} = \rho(x)\,\vec{n}(x)$$

WHERE $\rho = \sqrt{\vec{\varphi}\cdot\vec{\varphi}}$, $\vec{n}\cdot\vec{n} = 1$

$$\nabla\varphi^a = n^a \nabla\rho + \rho\,\nabla n^a$$

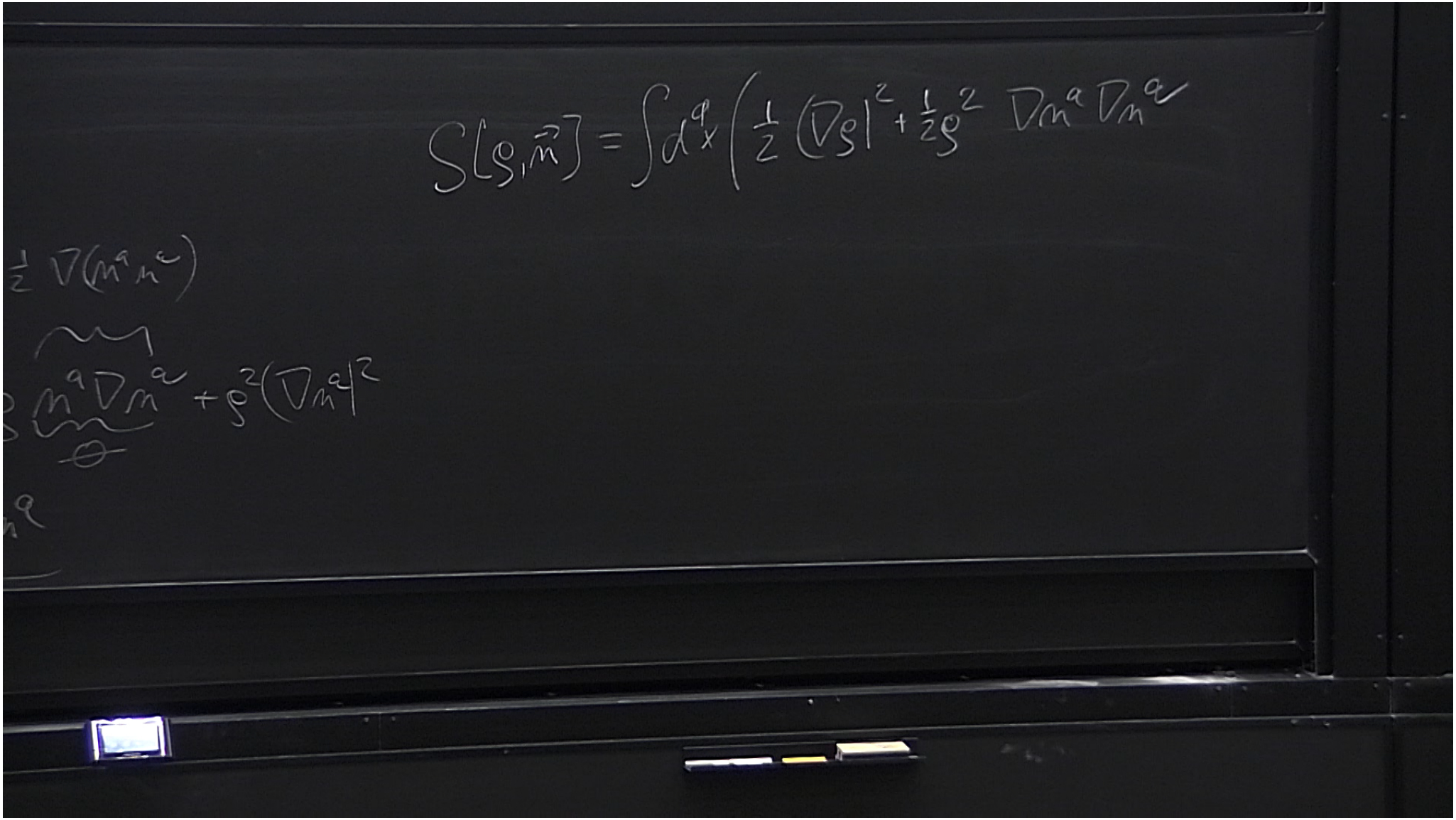
$$(\nabla\vec{\varphi})\cdot(\nabla\vec{\varphi}) = \nabla\varphi^a\,\nabla\varphi^a = n^a n^a (\nabla\rho)^2 + 2\rho\nabla\rho\, n^a\nabla n^a + \rho^2(\nabla n^a)^2$$

$\vec{\varphi}\cdot\vec{\varphi} = \rho^2$

$n^a n^a = 1$

$n^a\nabla n^a = \frac{1}{2}\nabla(n^a n^a) = 0$

$$= (\nabla\rho)^2 + \rho^2\,\nabla n^a \nabla n^a$$


$S[\rho, \vec{n}] = \int d^q x \left(\frac{1}{2} (\nabla \rho)^2 + \frac{1}{2} \rho^2 \nabla n^a \nabla n^a \right.$
$\frac{1}{2} \nabla (n^a n^a)$
$n^a \nabla n^a + \rho^2 (\nabla n^a)^2$
n^a

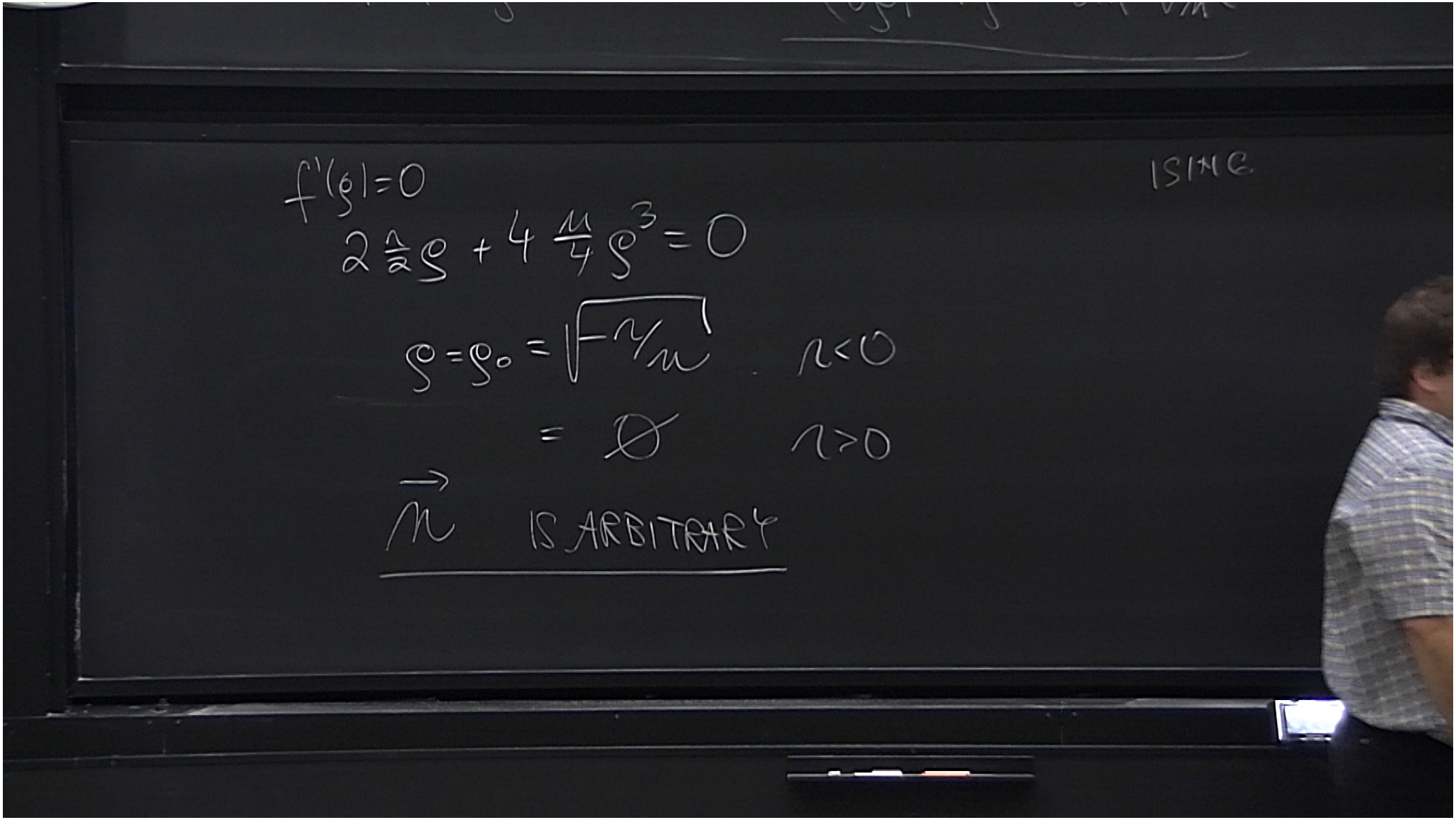
$$S[\rho,\vec{n}] = \int d^q x \left(\frac{1}{2} (\nabla\rho)^2 + \frac{1}{2}\rho^2 \nabla n^a \nabla n^a + \frac{r}{2}\rho^2 + \frac{u}{4}\rho^4 \right)$$

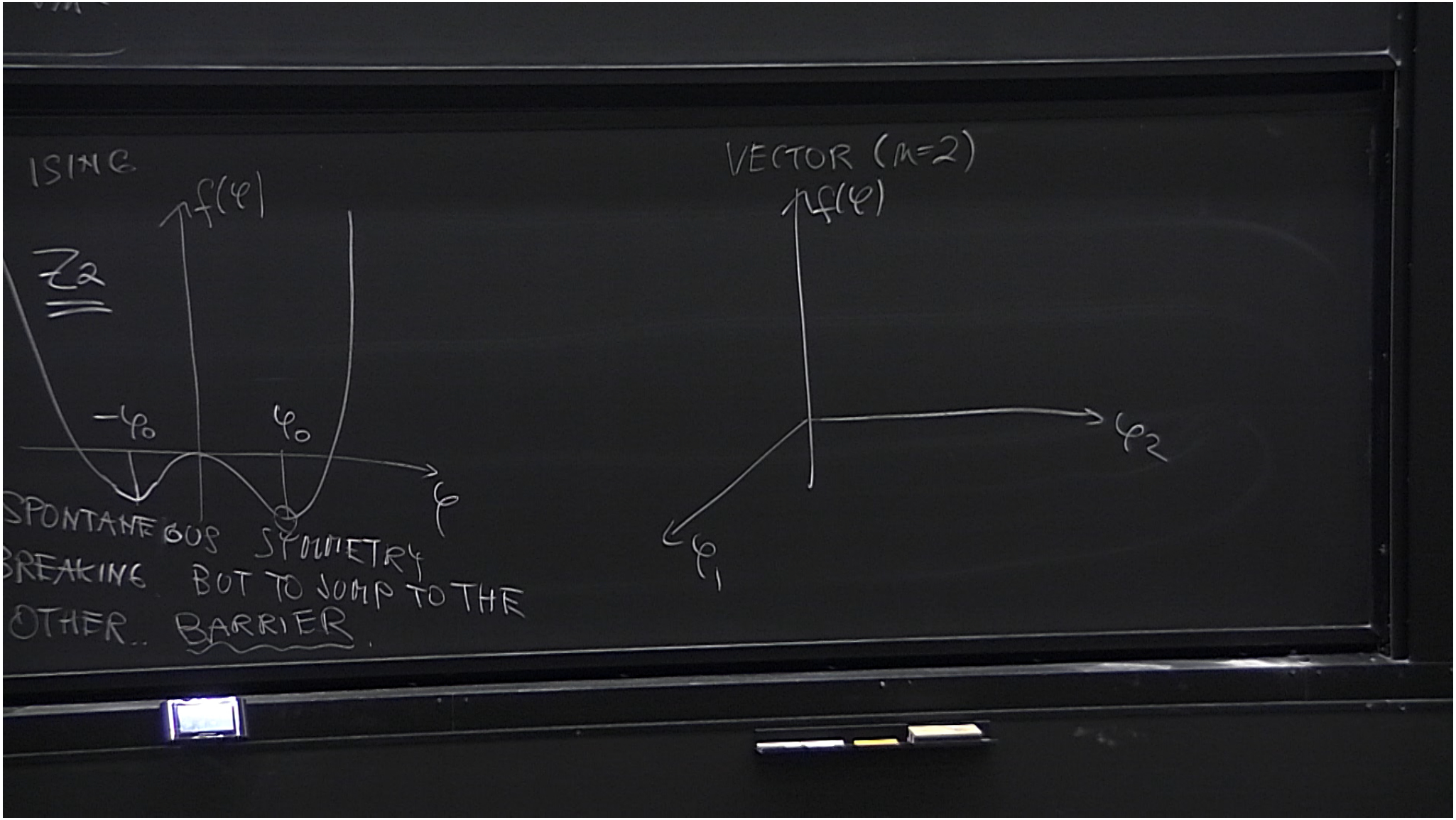
LANDAU'S THEORY

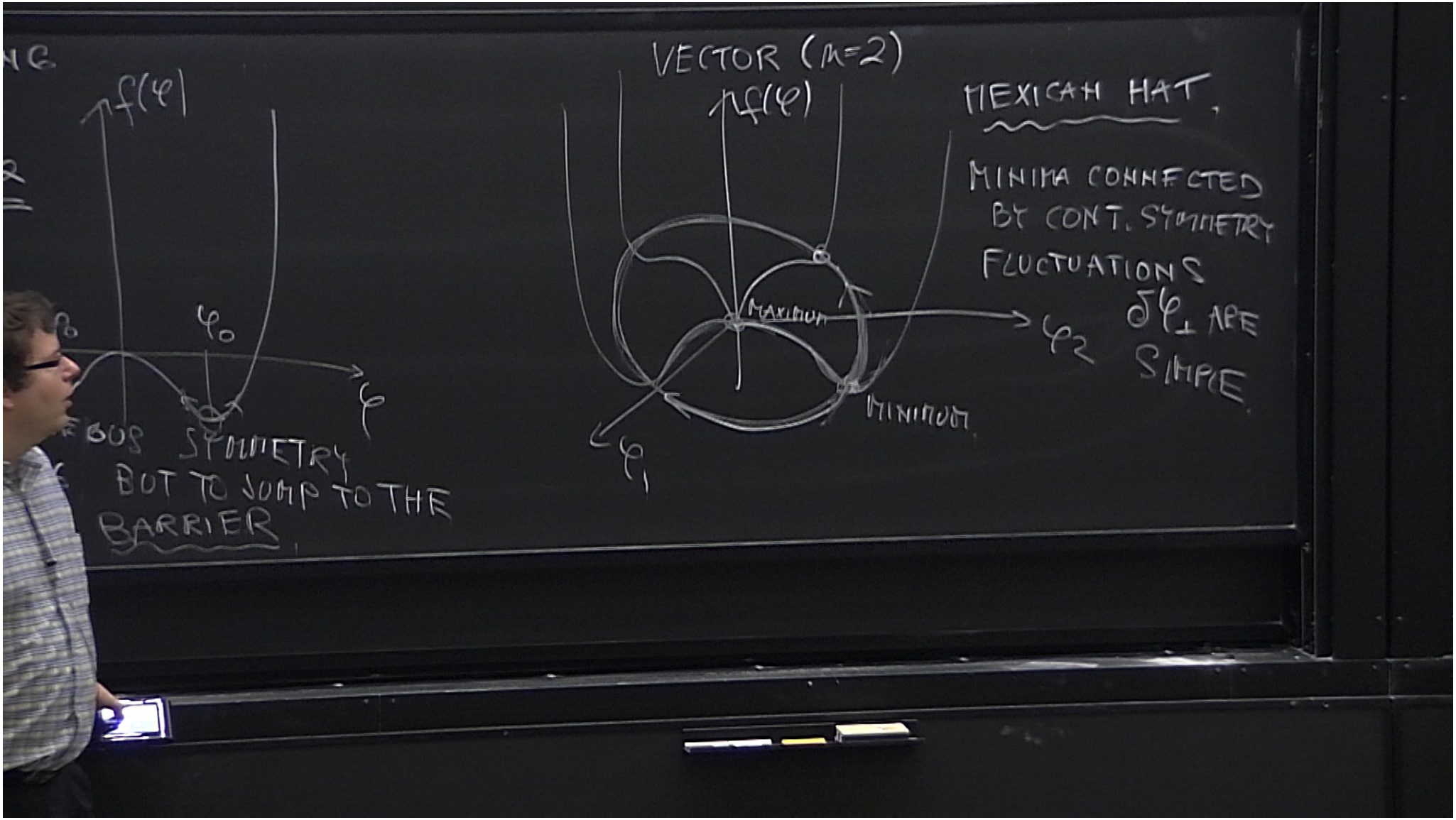
$$f(\rho) = \frac{r}{2}\rho^2 + \frac{u}{2}\rho^4$$

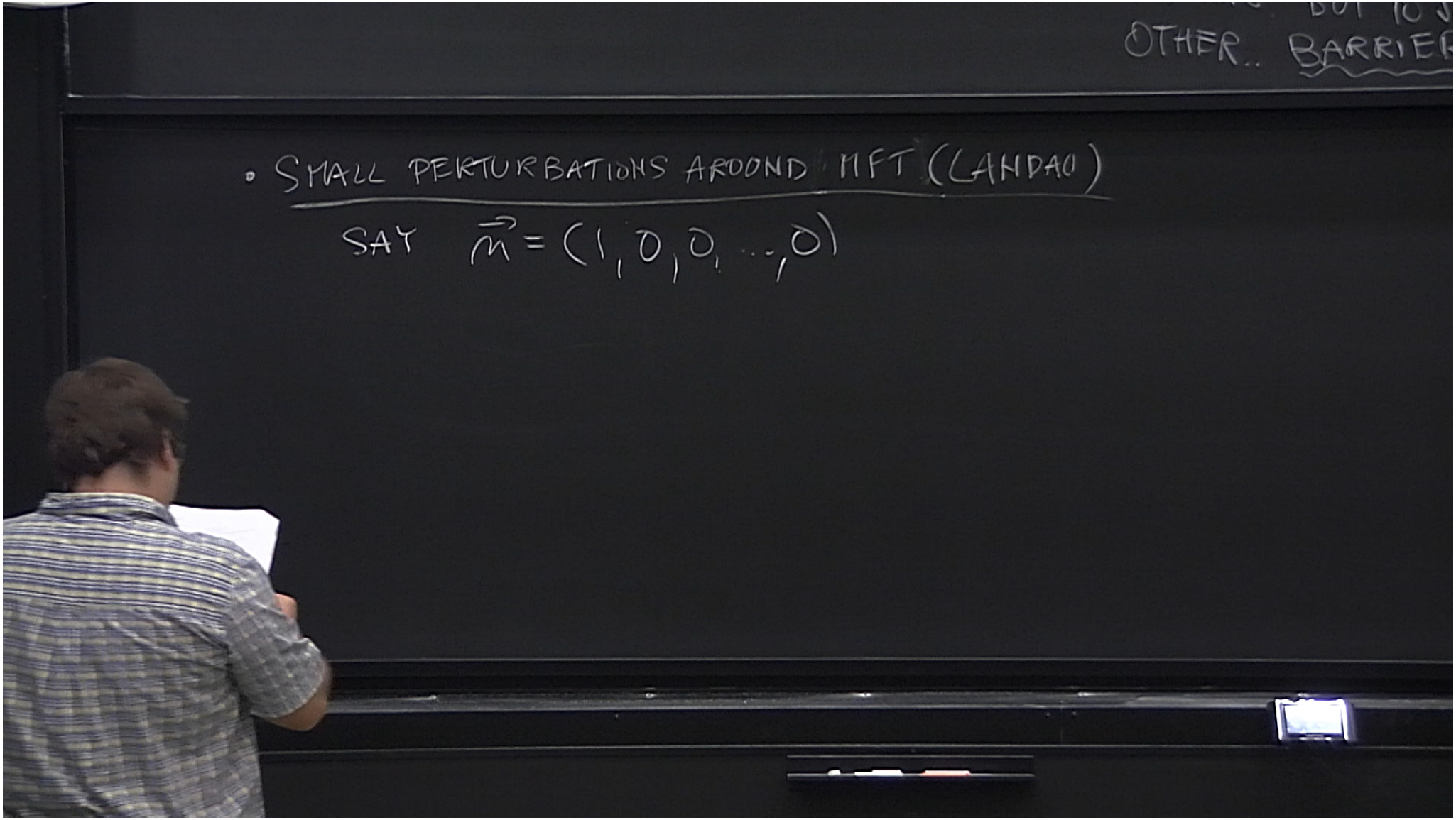
$\nabla(n^a n^a)$

$n^a \nabla n^a + \rho^2 (\nabla n^a)^2$


ISING
$f'(\varphi)=0$
$2\frac{r}{2}\varphi + 4\frac{u}{4}\varphi^3 = 0$
$\varphi=\varphi_0=\sqrt{-r/u}$ $r<0$
$=0$ $r>0$
$\vec{n}$ IS ARBITRARY


ISING
Z2
f(φ)
−φ0
φ0
φ
SPONTANEOUS SYMMETRY
BREAKING BOT TO JUMP TO THE
OTHER.. BARRIER.
VECTOR (N=2)
f(φ)
φ2
φ1


VECTOR (n=2)
f(φ)
MEXICAN HAT.
MINIMA CONNECTED BY CONT. SYMMETRY
FLUCTUATIONS $\delta\varphi_\perp$ ARE SIMPLE.
MAXIMUM
MINIMUM
φ_1
φ_2
f(φ)
φ_0
φ
BUT TO JUMP TO THE BARRIER.


OTHER.. BARRIER
• SMALL PERTURBATIONS AROUND MFT (LANDAU)
SAY $\vec{m} = (1, 0, 0, \ldots, 0)$

OTHER.. BARRIER.

- SMALL PERTURBATIONS AROUND MFT (LANDAU)

SAY $\vec{n} = (1, 0, 0, \ldots, 0)$ & WRITE

$$\vec{\varphi}(x) = \underbrace{\rho_0 \vec{n}}_{\text{MFT}} + \underbrace{\rho_0 \vec{\delta\varphi}}_{\text{FLUCTUATION}} \qquad \vec{\delta\varphi} = \vec{n}\,\delta\varphi_1 + \vec{\delta\varphi}_\perp$$

OTHER.. BARRIER.

NDAO)

RITE

$$\delta\vec{\varphi} = \vec{n}\,\delta\varphi_1 + \delta\vec{\varphi}_\perp$$

TRANSVERSE DIR.

$$\vec{n}\cdot\delta\vec{\varphi}_\perp = 0$$

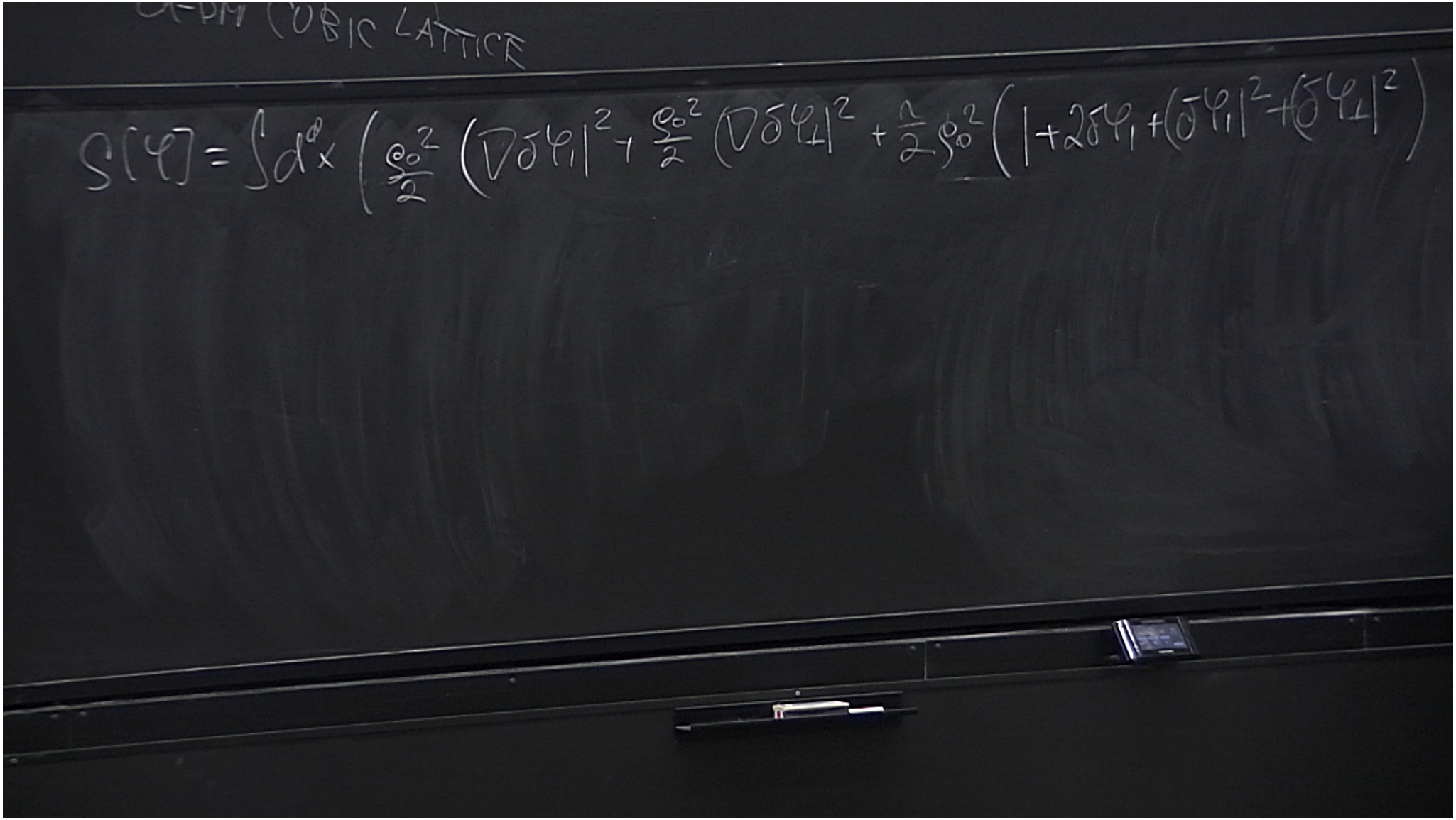
$$\vec{\varphi}\cdot\vec{\varphi} = (\rho_0\vec{n} + \rho_0\delta\vec{\varphi})\cdot(\rho_0\vec{n} + \rho_0\delta\vec{\varphi})$$

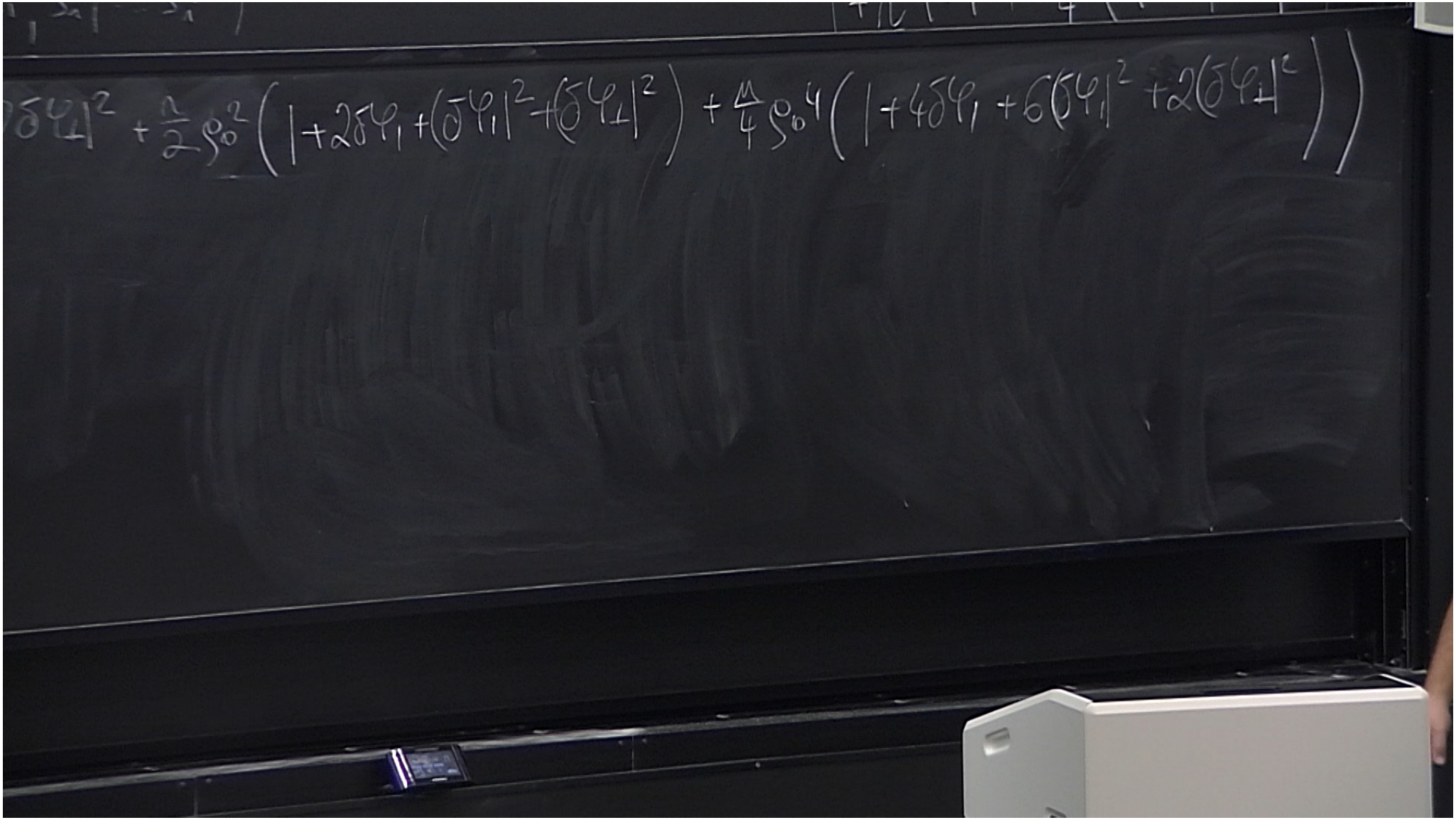
$$= \rho_0^2 + 2\rho_0^2\,\vec{n}\cdot\delta\vec{\varphi} + \rho_0^2\,\delta\vec{\varphi}\cdot\delta\vec{\varphi}$$

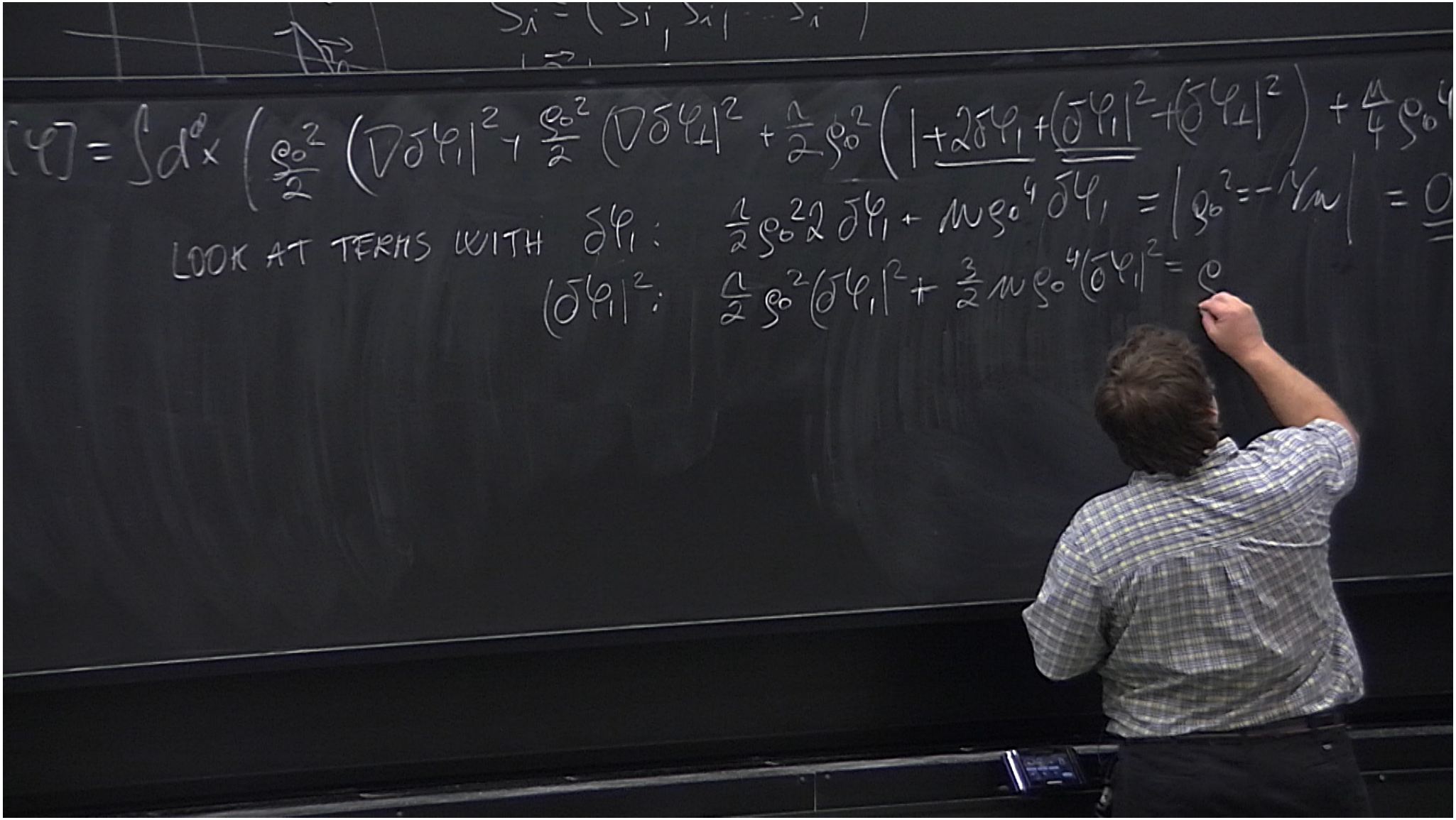
$$= \rho_0^2 + 2\rho_0^2\,\delta\varphi_1 + \rho_0^2(\delta\varphi_1)^2$$

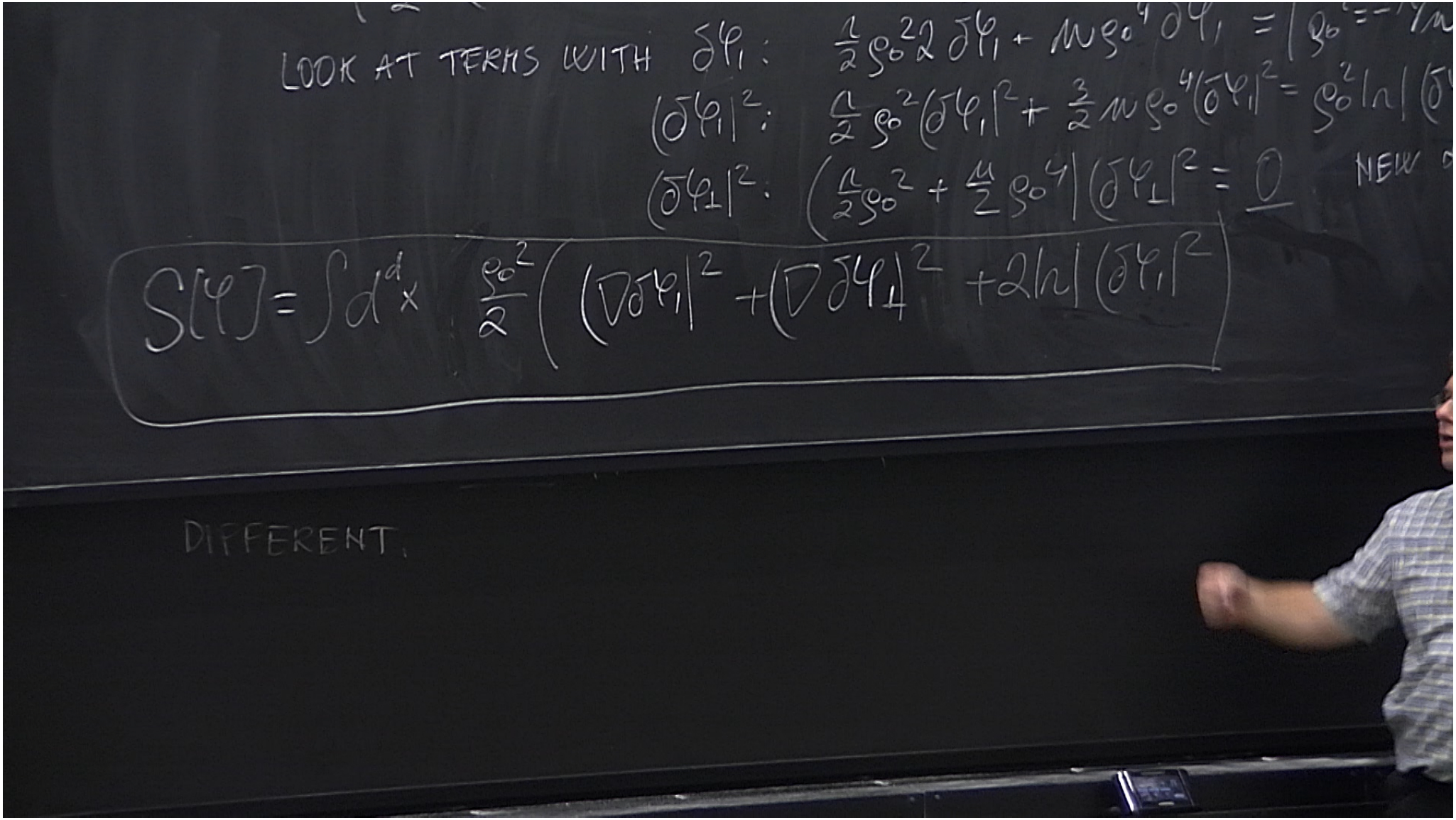
$$= \rho_0^2\left(1 + 2\delta\varphi_1 + (\delta\varphi_1)^2 + (\delta\varphi_\perp)^2\right)$$

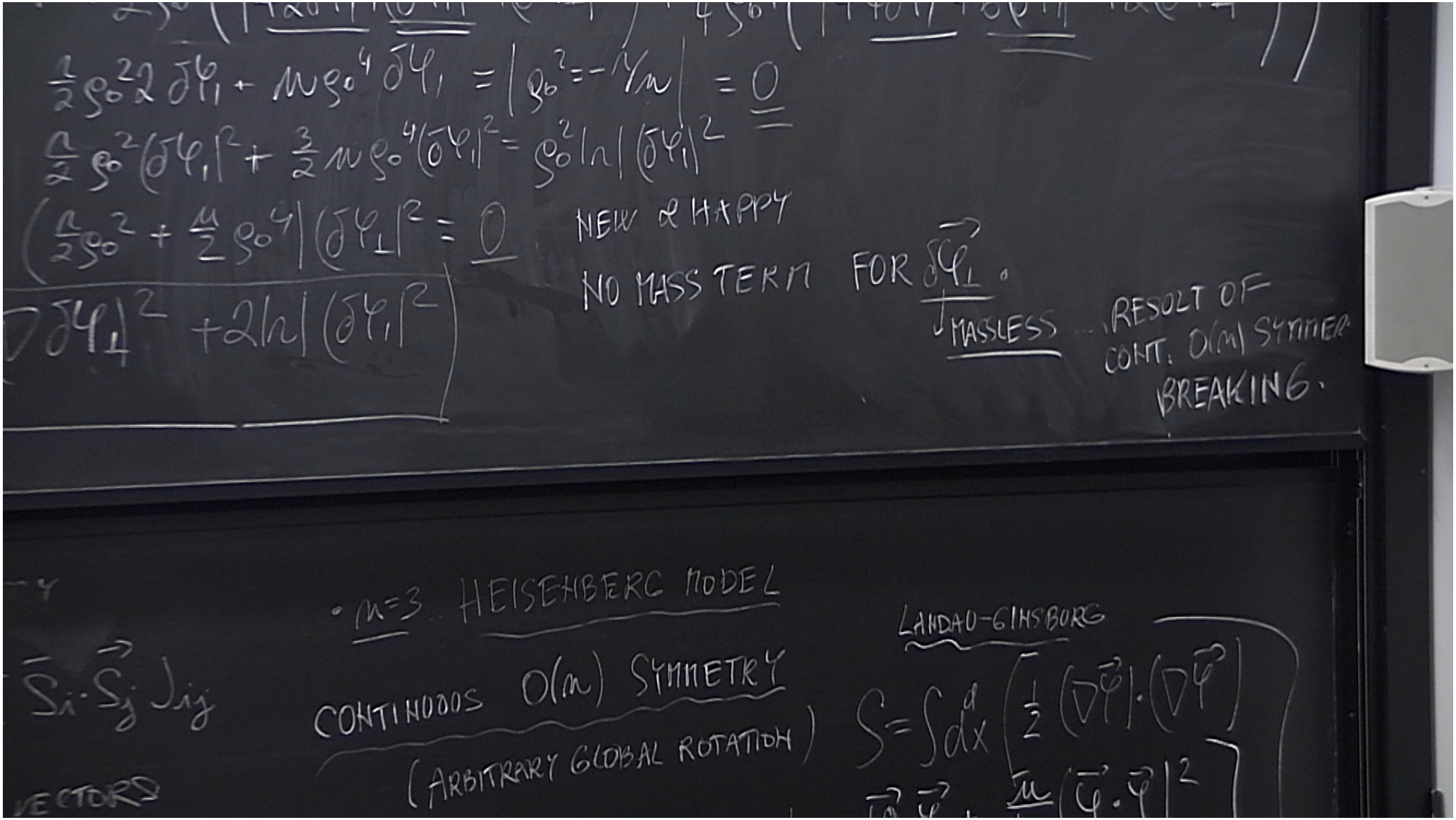
$$(\vec{\varphi}\cdot\vec{\varphi})^2 = \rho_0^4\left(1 + 4\delta\varphi_1 + 6(\delta\varphi_1)^2 + 2(\delta\varphi_\perp)^2\right)$$


CUBIC LATTICE
$S[\varphi] = \int d^4x \left(\frac{\rho_0^2}{2} |D\delta\varphi_\parallel|^2 + \frac{\rho_0^2}{2} |D\delta\varphi_\perp|^2 + \frac{\Lambda}{2}\rho_0^2 \left(1 + 2\delta\varphi_\parallel + |\delta\varphi_\parallel|^2 + |\delta\varphi_\perp|^2 \right) \right.$


$|\delta\psi_\perp|^2 + \frac{1}{2}\rho_0^2\left(1 + 2\delta\psi_1 + |\delta\psi_1|^2 + |\delta\psi_\perp|^2\right) + \frac{1}{4}\rho_0^4\left(1 + 4\delta\psi_1 + 6|\delta\psi_1|^2 + 2|\delta\psi_\perp|^2\right)\Big)$


LOOK AT TERMS WITH $\delta\varphi_1$:
$\frac{1}{2}\rho_0^2 2\delta\varphi_1 + \mu\rho_0^4\delta\varphi_1 = |\rho_0^2 = -\mu/u| = 0$
$|\delta\varphi_1|^2$:
$\frac{1}{2}\rho_0^2|\delta\varphi_1|^2 + \frac{3}{2}u\rho_0^4|\delta\varphi_1|^2 = 0$


LOOK AT TERMS WITH $\delta\varphi_1$:
$|\delta\varphi_1|^2$:
$|\delta\varphi_\perp|^2$:
NEW
$S[\varphi] = \int d^d x \; \frac{\rho_0^2}{2}\left(|\nabla\delta\varphi_1|^2 + |\nabla\delta\varphi_\perp|^2 + 2h|\delta\varphi_1|^2 \right)$
DIFFERENT.

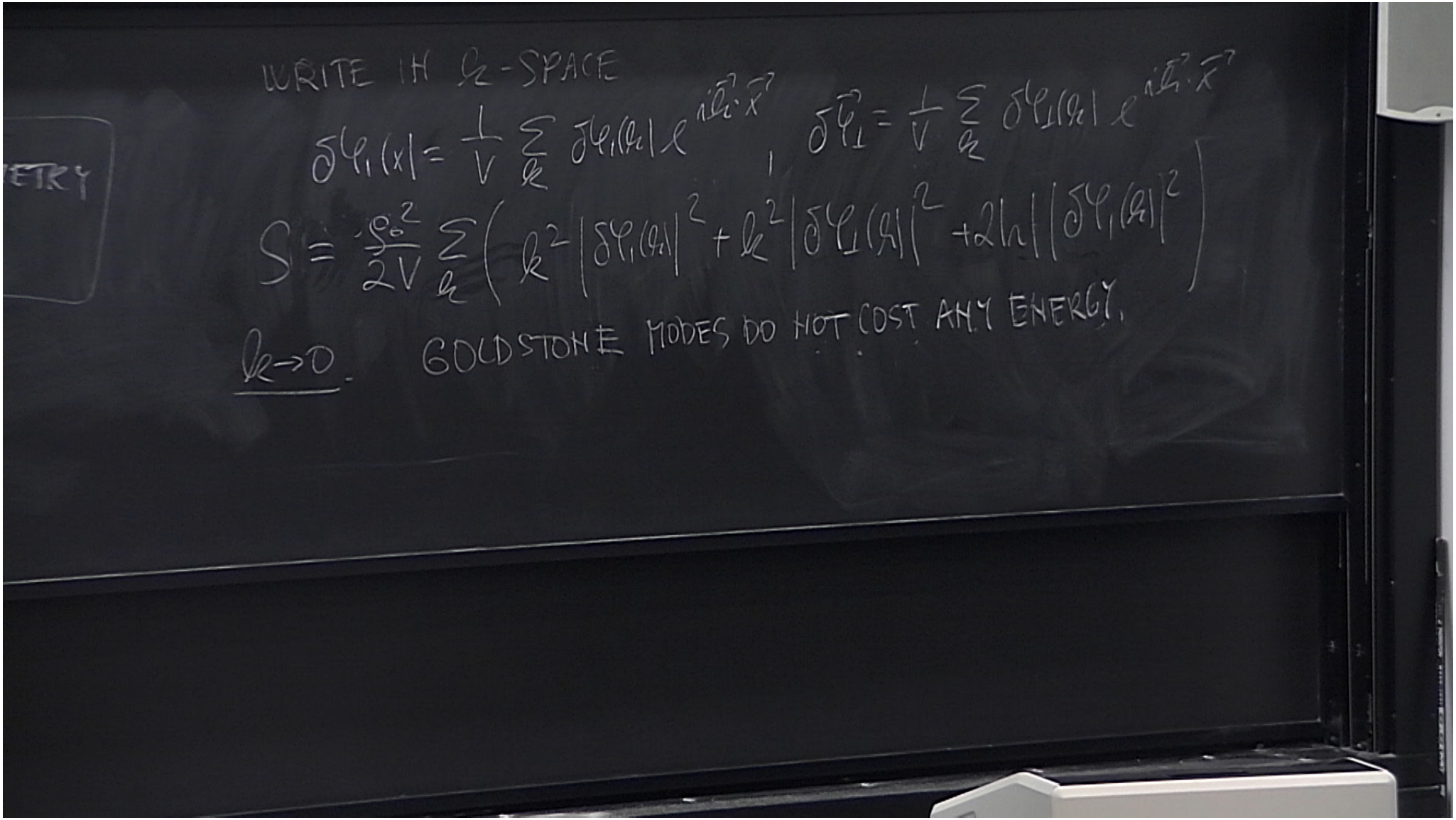

$\frac{1}{2}\rho_0^2 2\,\delta\varphi_1 + n\rho_0^4\,\delta\varphi_1 = |\rho_0^2 = -r/n| = 0$
$\frac{r}{2}\rho_0^2(\delta\varphi_1)^2 + \frac{3}{2}n\rho_0^4(\delta\varphi_1)^2 = \rho_0^2 |n| (\delta\varphi_1)^2$
$\left(\frac{r}{2}\rho_0^2 + \frac{u}{2}\rho_0^4\right)|\delta\varphi_\perp|^2 = 0$
NEW & HAPPY
NO MASS TERM FOR $\delta\vec{\varphi}_\perp$.
MASSLESS
RESULT OF CONT. O(n) SYMMETRY BREAKING.
$\delta\varphi_\perp|^2 + 2|n|(\delta\varphi_1|^2$
$\cdot n=3$ HEISENBERG MODEL
CONTINUOUS O(n) SYMMETRY
(ARBITRARY GLOBAL ROTATION)
LANDAU-GINSBURG
$S = \int d^d x \left[\frac{1}{2}(\nabla\vec{\varphi}_1)\cdot(\nabla\vec{\varphi}\right.$
$\vec{\varphi}\cdot\frac{u}{}(\vec{\varphi}\cdot\vec{\varphi}|^2\right]$
$\vec{S}_i\cdot\vec{S}_j J_{ij}$
VECTORS

GOLDSTONE MODES (BOSONS)

ARISE WHEN CONTINUOUS SYMMETRY IS SPONTANEOUSLY BROKEN.

WRITE IN k-SPACE

$$\delta\varphi_i(x) = \frac{1}{V}\sum_k \delta\varphi_i(k)\, e^{ikx}$$


WRITE IN k-SPACE
$\delta\varphi_\parallel(x) = \frac{1}{V}\sum_k \delta\varphi_\parallel(k)\, e^{i\vec{k}\cdot\vec{x}}$, $\delta\vec{\varphi}_\perp = \frac{1}{V}\sum_k \delta\vec{\varphi}_\perp(k)\, e^{i\vec{k}\cdot\vec{x}}$
$S = \frac{\rho_0^2}{2V}\sum_k \left(k^2 |\delta\varphi_\parallel(k)|^2 + k^2 |\delta\vec{\varphi}_\perp(k)|^2 + 2h |\delta\varphi_\parallel(k)|^2 \right)$
$k \to 0$. GOLDSTONE MODES DO NOT COST ANY ENERGY.

IS SPONTANEOUSLY BROKEN.

$S = \frac{\rho_0^2}{2V} \sum_k \left(k \right.$

- EX 2: ACOUSTIC PHONONS
  (BREAKING THE CONT. TRANSLAT.S. AS THE CRYSTAL FORMS A LATTICE)

$k \to 0$. GOLDS

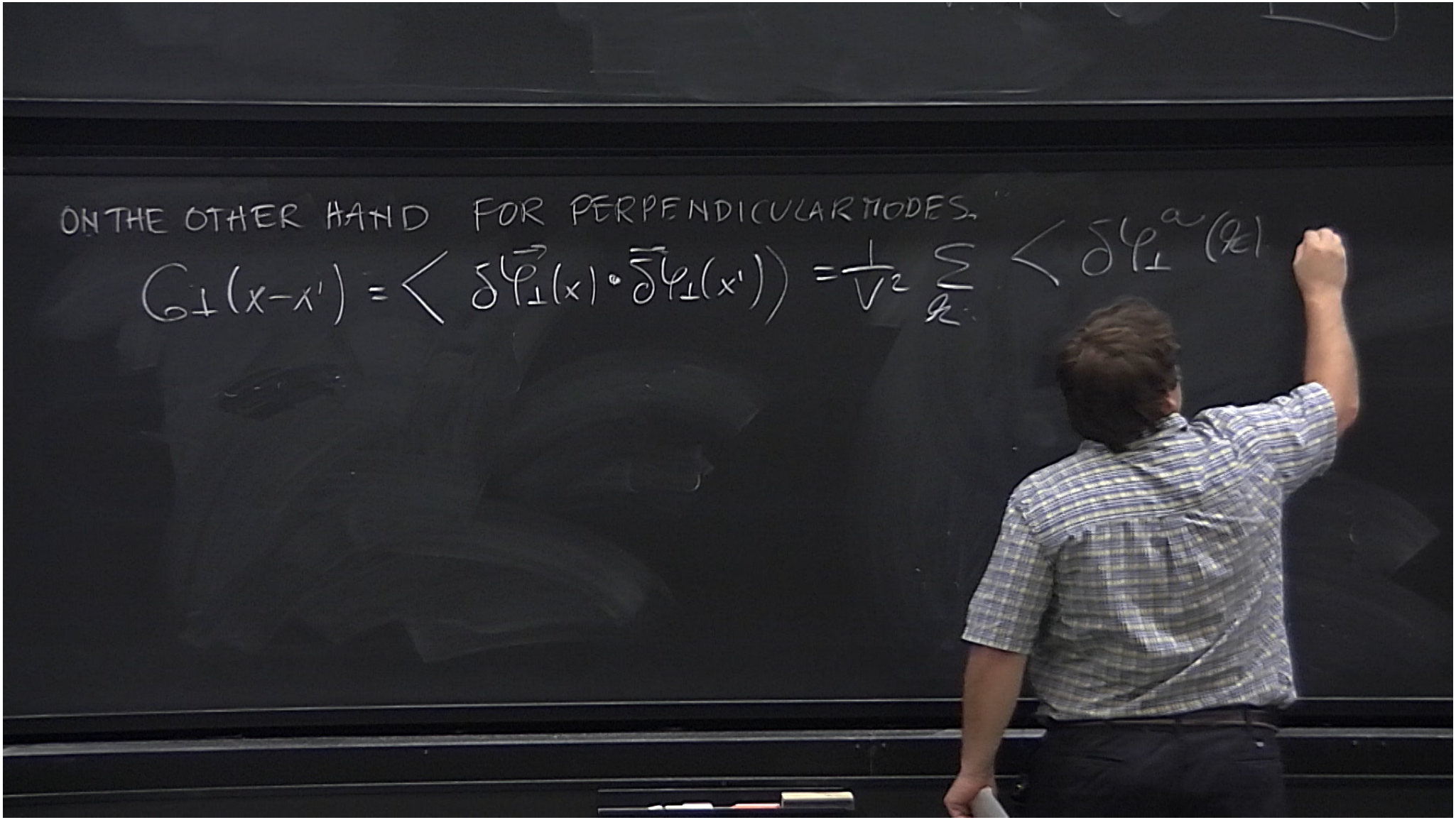
$$E_k = \hbar c_s k$$

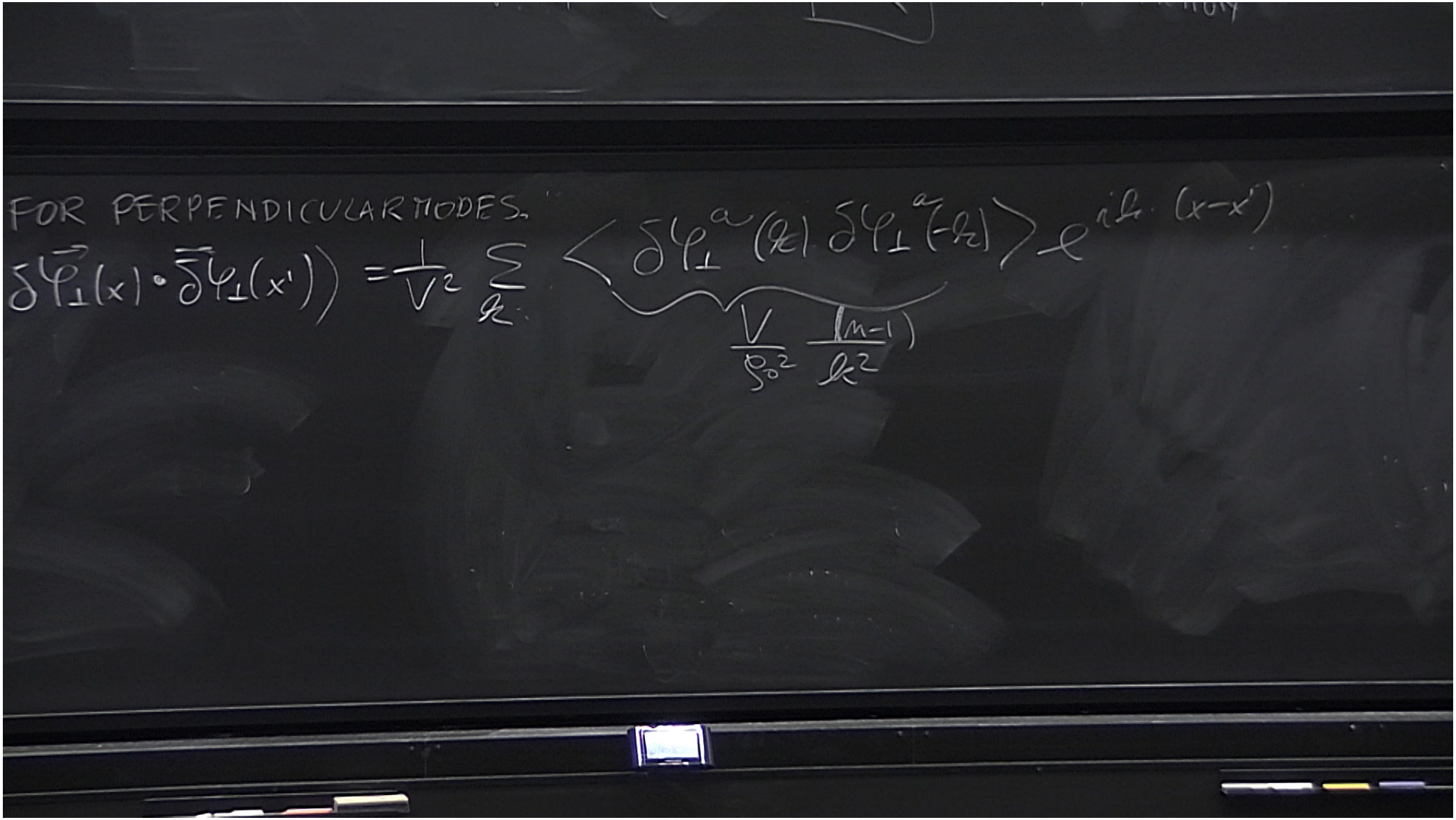
OTHER.. BARRIER.

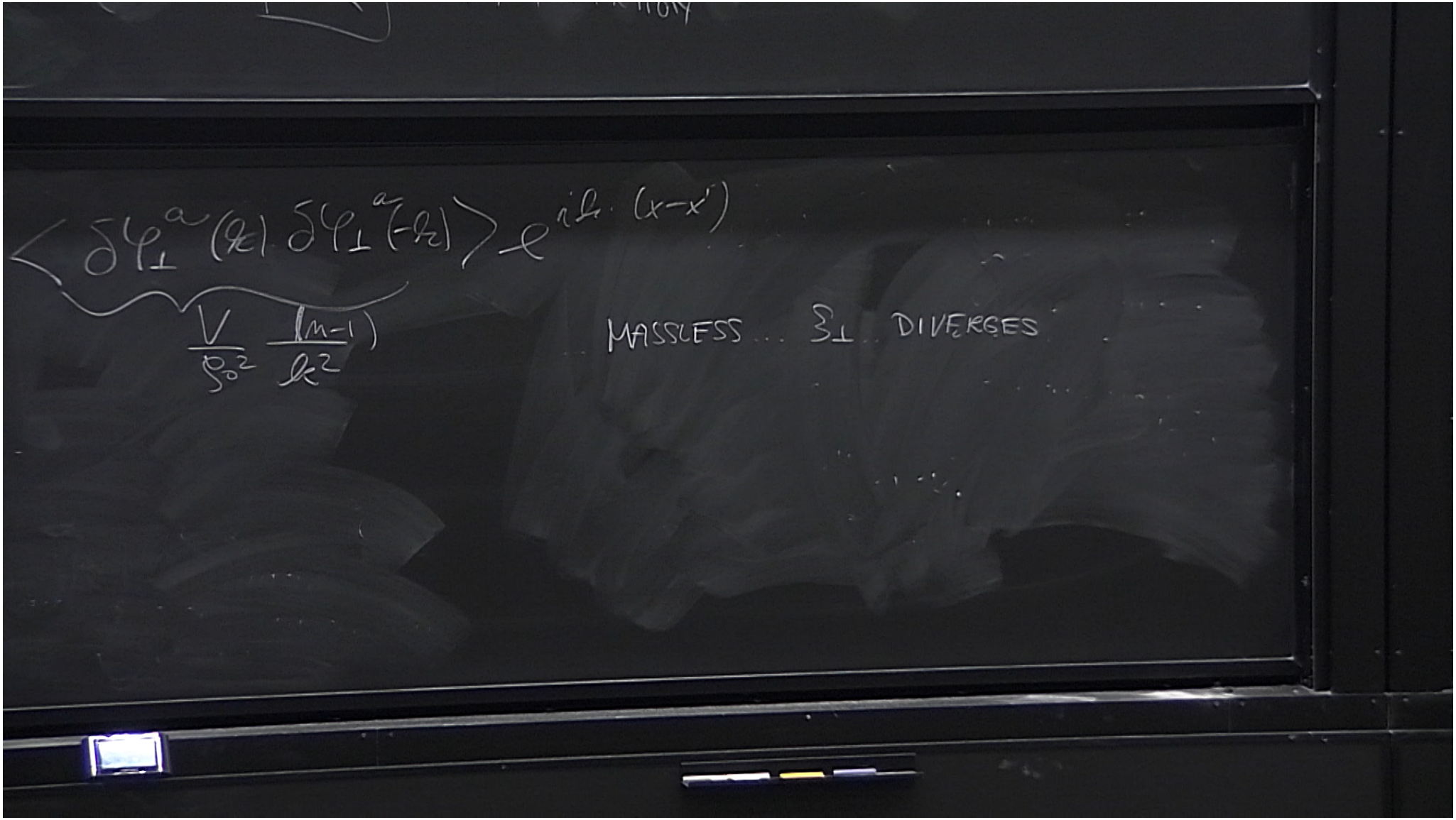
- CORRELATION FUNCTION

$$\langle \varphi \rangle = \varphi_0 \vec{n}$$

$$G_i(x-x') = \langle \delta\varphi_i(x)\, \delta\varphi_i(x') \rangle = \frac{1}{V^2} \sum_k \langle \delta\varphi_i(k)\, \delta\varphi_i(-k) \rangle\, e^{ik(x-x')}$$


ON THE OTHER HAND FOR PERPENDICULAR MODES.
$G_\perp(x-x') = \langle \delta\vec{\varphi}_\perp(x) \cdot \delta\vec{\varphi}_\perp(x') \rangle = \frac{1}{V^2} \sum_k \langle \delta\varphi_\perp^a(k)$


FOR PERPENDICULAR MODES.
$\delta\vec{\varphi}_\perp(x) \cdot \delta\vec{\varphi}_\perp(x')) = \frac{1}{V^2} \sum_k$
$\langle \delta\varphi_\perp^a(k) \delta\varphi_\perp^a(-k) \rangle e^{ik\cdot(x-x')}$
$\frac{V}{\rho_0^2} \frac{(N-1)}{k^2}$

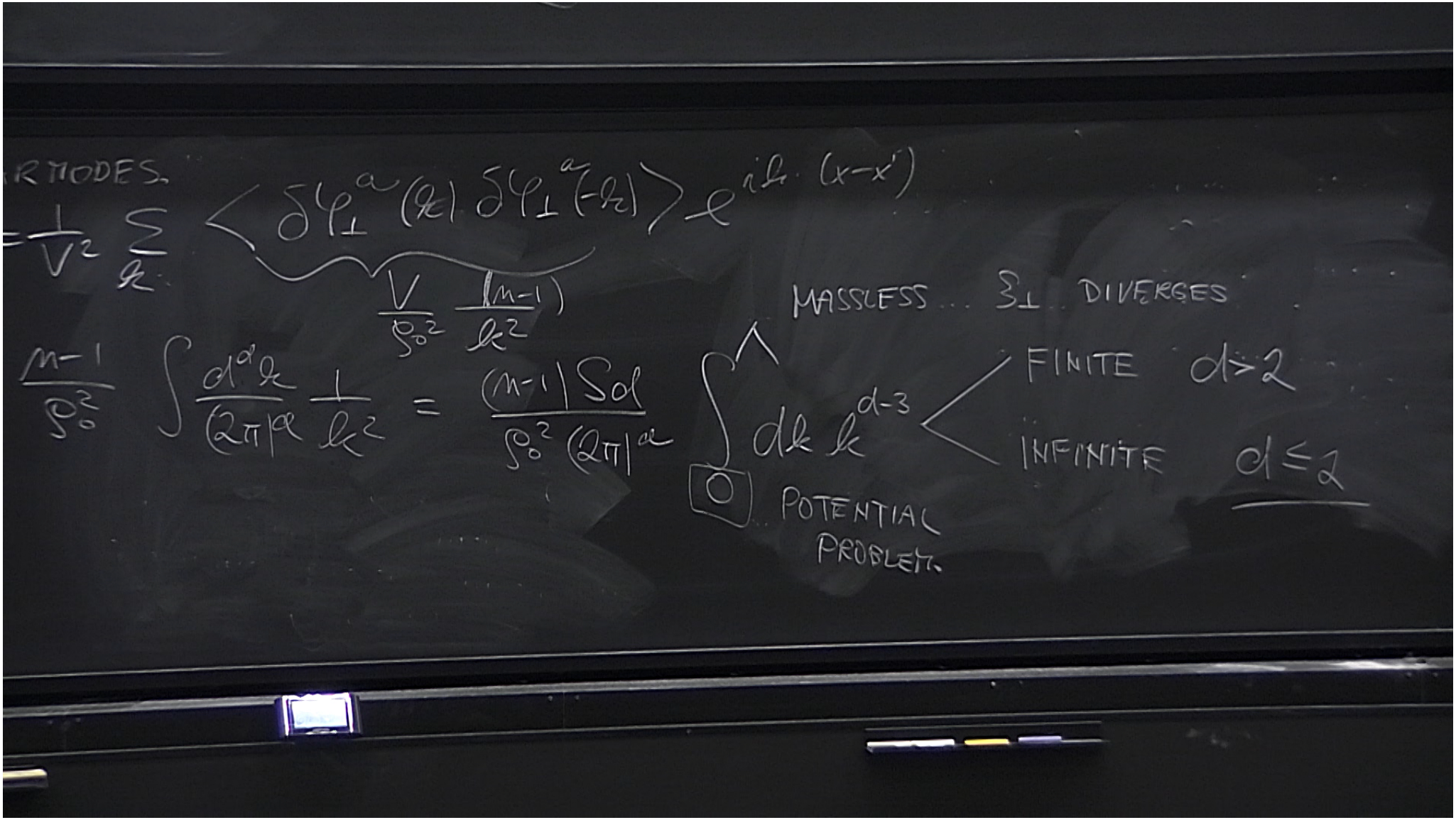

$\langle \delta\varphi_\perp^a(k) \delta\varphi_\perp^a(-k) \rangle e^{ik\cdot(x-x')}$
$\frac{V}{\rho_0^2} \frac{(n-1)}{k^2}$
MASSLESS ... $\Sigma_\perp$ DIVERGES

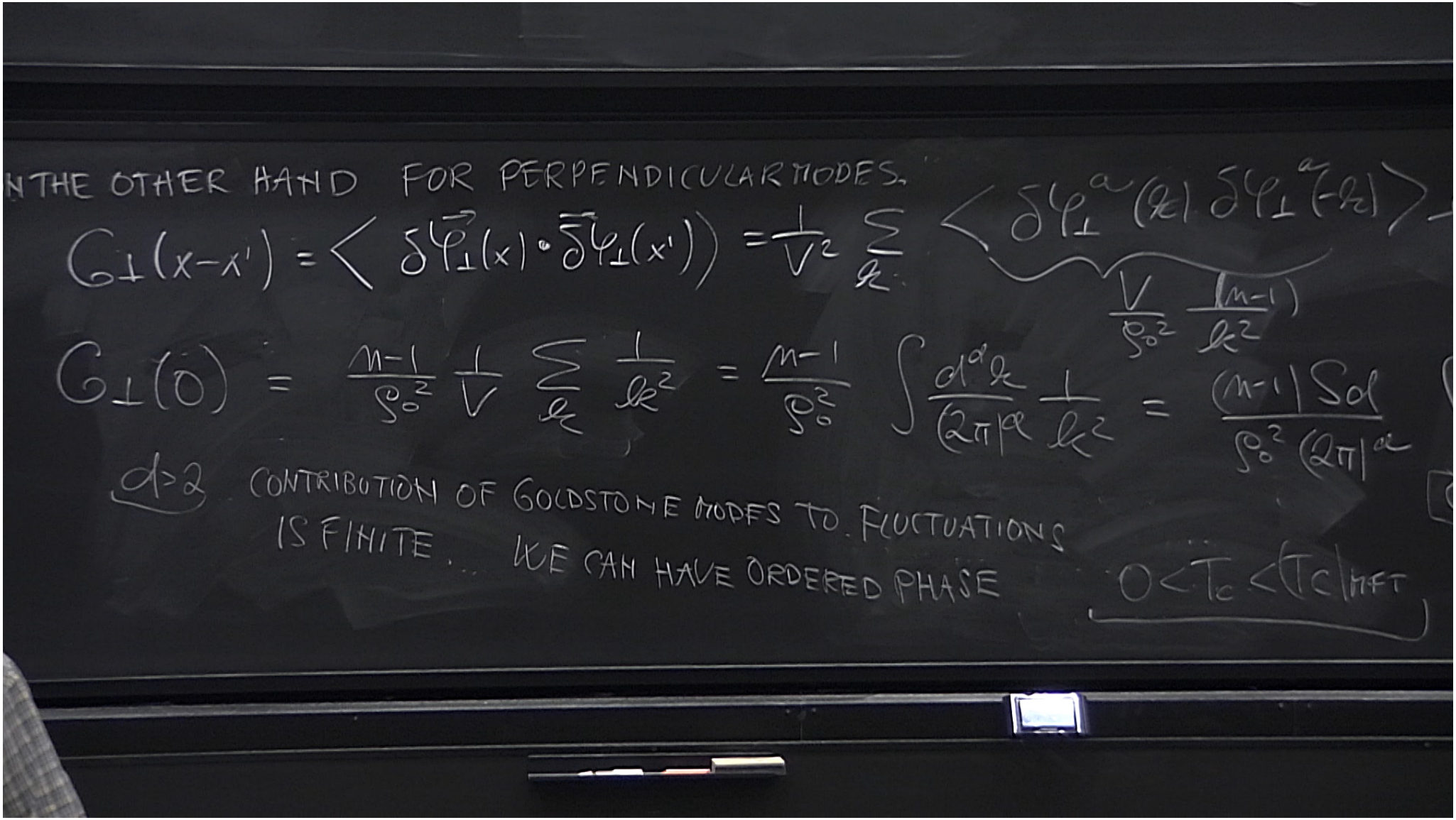
FOR PERPENDICULAR MODES.

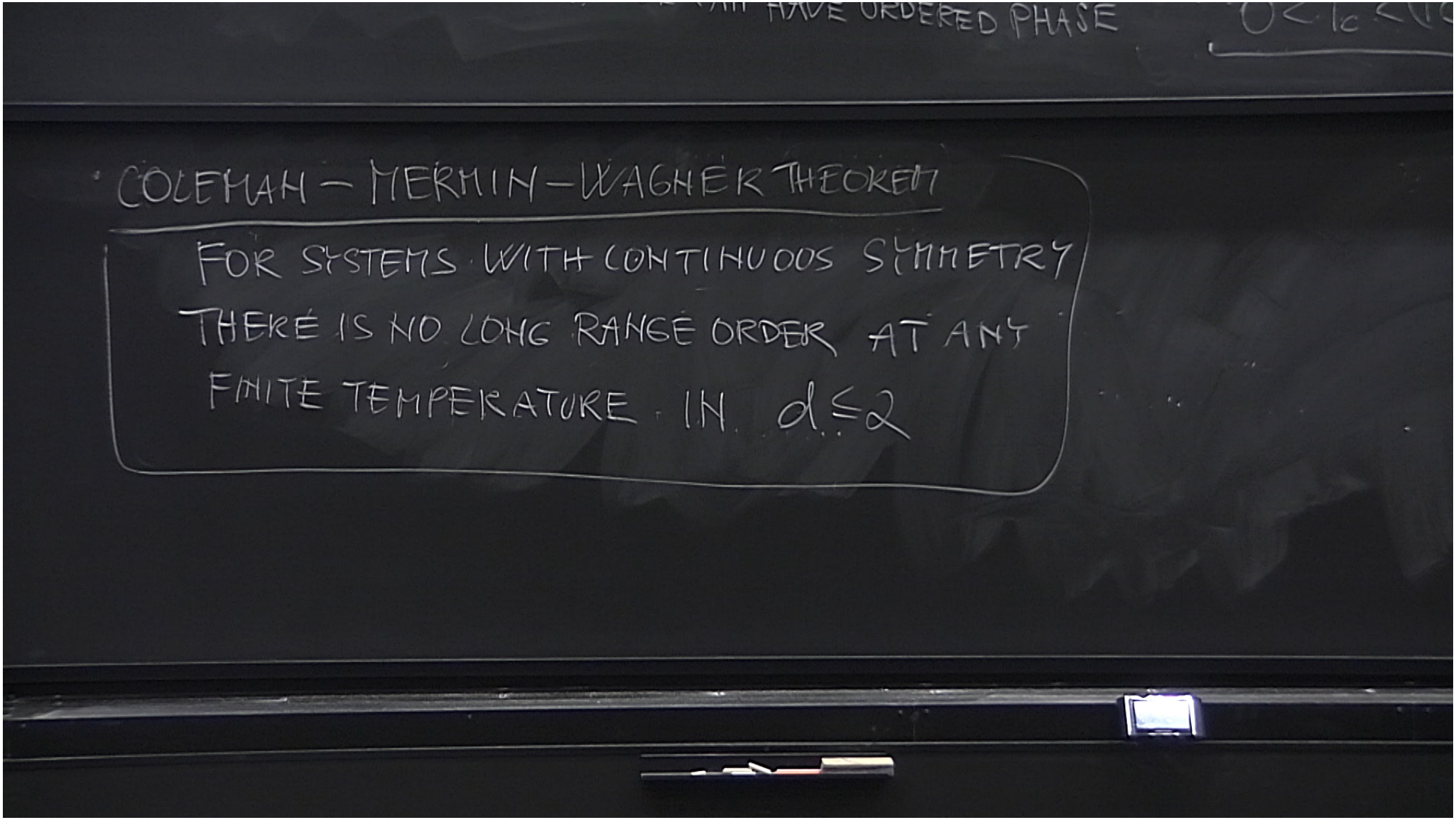
$$\langle \delta\vec{\varphi}_\perp(x) \cdot \delta\vec{\varphi}_\perp(x') \rangle = \frac{1}{V^2} \sum_k \underbrace{\langle \delta\varphi_\perp^a(k)\, \delta\varphi_\perp^a(-k) \rangle}_{\frac{V}{\rho_0^2}\frac{(n-1)}{k^2}} e^{ik\cdot(x-x')}$$

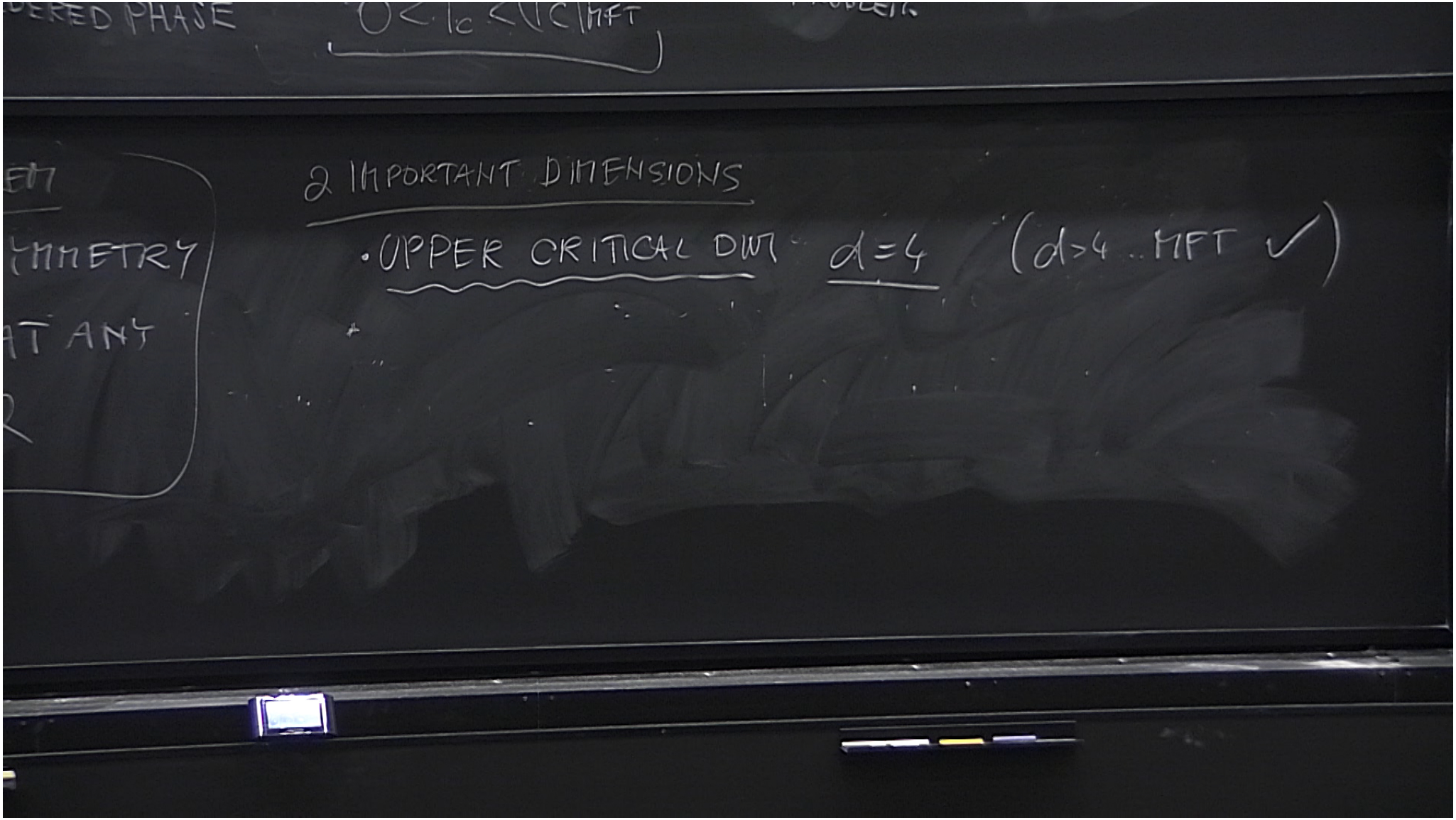
MASSLESS ... $\xi_\perp$

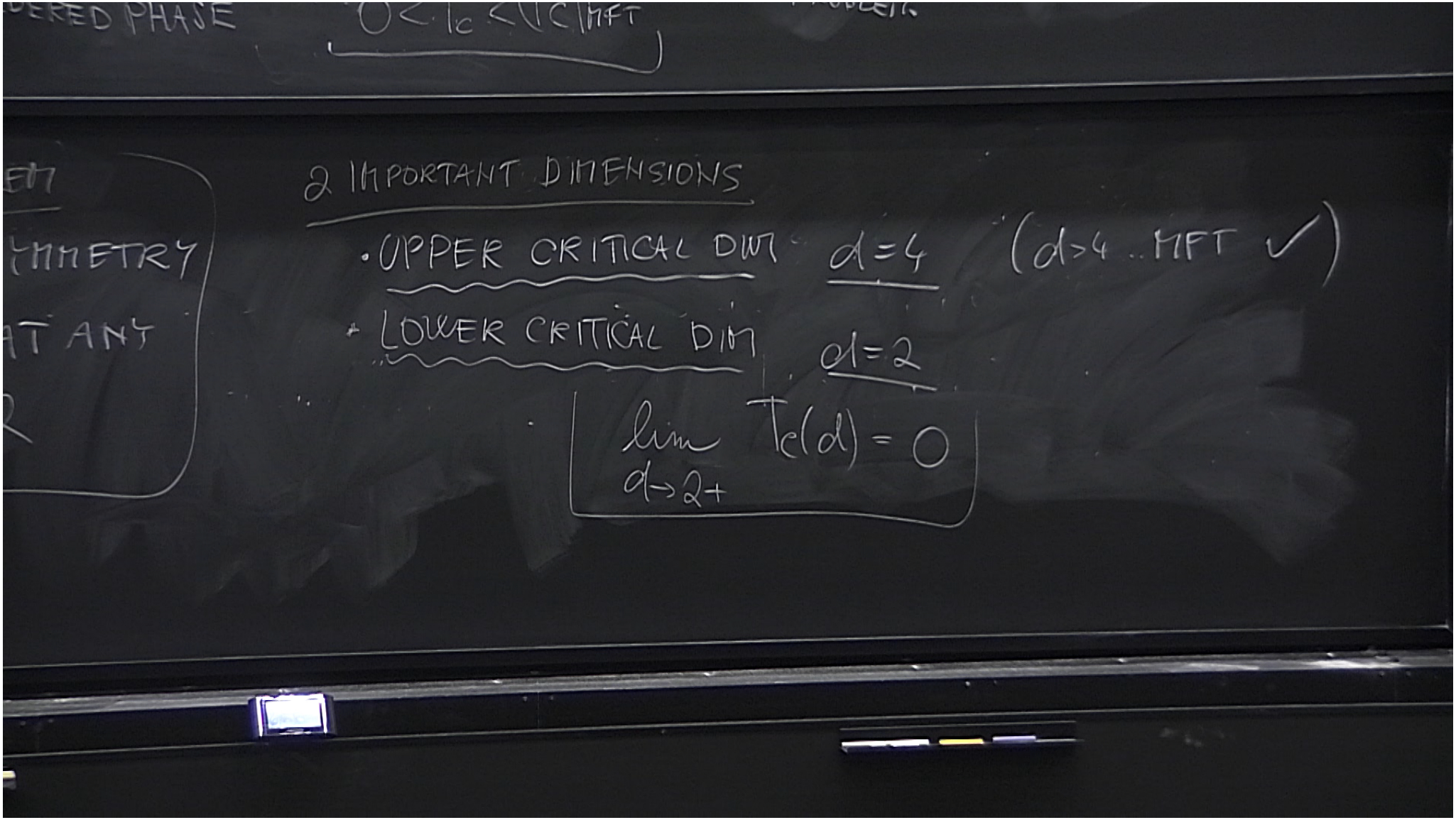
$$\frac{n-1}{\rho_0^2}\frac{1}{V}\sum_k \frac{1}{k^2} = \frac{n-1}{\rho_0^2}\int \frac{d^d k}{(2\pi)^d}\frac{1}{k^2} = \frac{(n-1)\,|S_{d-1}|}{\rho_0^2\,(2\pi)^d}\int_0^\Lambda dk\, k^{d-3}$$


R MODES.
$= \frac{1}{V^2} \sum_k$
$\langle \delta\varphi_\perp^a(k) \delta\varphi_\perp^a(-k) \rangle e^{ik\cdot(x-x')}$
$\frac{V}{\rho_0^2} \frac{(n-1)}{k^2}$
$\frac{n-1}{\rho_0^2} \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2} = \frac{(n-1)\,S_d}{\rho_0^2 (2\pi)^d}$
MASSLESS ... $\Sigma_\perp$ DIVERGES
$\int dk\, k^{d-3}$
FINITE $d > 2$
INFINITE $d \leq 2$
POTENTIAL PROBLEM.


IN THE OTHER HAND FOR PERPENDICULAR MODES.
$G_\perp(x-x') = \langle \delta\vec{\varphi}_\perp(x) \cdot \delta\vec{\varphi}_\perp(x') \rangle = \frac{1}{V^2} \sum_k \langle \delta\varphi_\perp^a(k)\, \delta\varphi_\perp^a(-k) \rangle$
$\frac{V}{\rho_0^2} \frac{(n-1)}{k^2}$
$G_\perp(0) = \frac{n-1}{\rho_0^2} \frac{1}{V} \sum_k \frac{1}{k^2} = \frac{n-1}{\rho_0^2} \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2} = \frac{(n-1)\,|S_d|}{\rho_0^2\,(2\pi)^d}$
$d>2$ CONTRIBUTION OF GOLDSTONE MODES TO FLUCTUATIONS IS FINITE.. WE CAN HAVE ORDERED PHASE
$0 < T_c < (T_c)_{MFT}$


HAVE ORDERED PHASE
COLEMAN – MERMIN – WAGNER THEOREM
FOR SYSTEMS WITH CONTINUOUS SYMMETRY
THERE IS NO LONG RANGE ORDER AT ANY
FINITE TEMPERATURE IN $d \leq 2$


ERED PHASE
$0 < T_c < T_{c\,MFT}$
2 IMPORTANT DIMENSIONS
• UPPER CRITICAL DIM $d=4$ ($d>4$ .. MFT ✓)


ERED PHASE
$0 < T_c < T_c^{MFT}$
2 IMPORTANT DIMENSIONS
• UPPER CRITICAL DIM $d=4$ $(d>4$ .. MFT ✓)
• LOWER CRITICAL DIM $d=2$
$\lim_{d \to 2+} T_c(d) = 0$