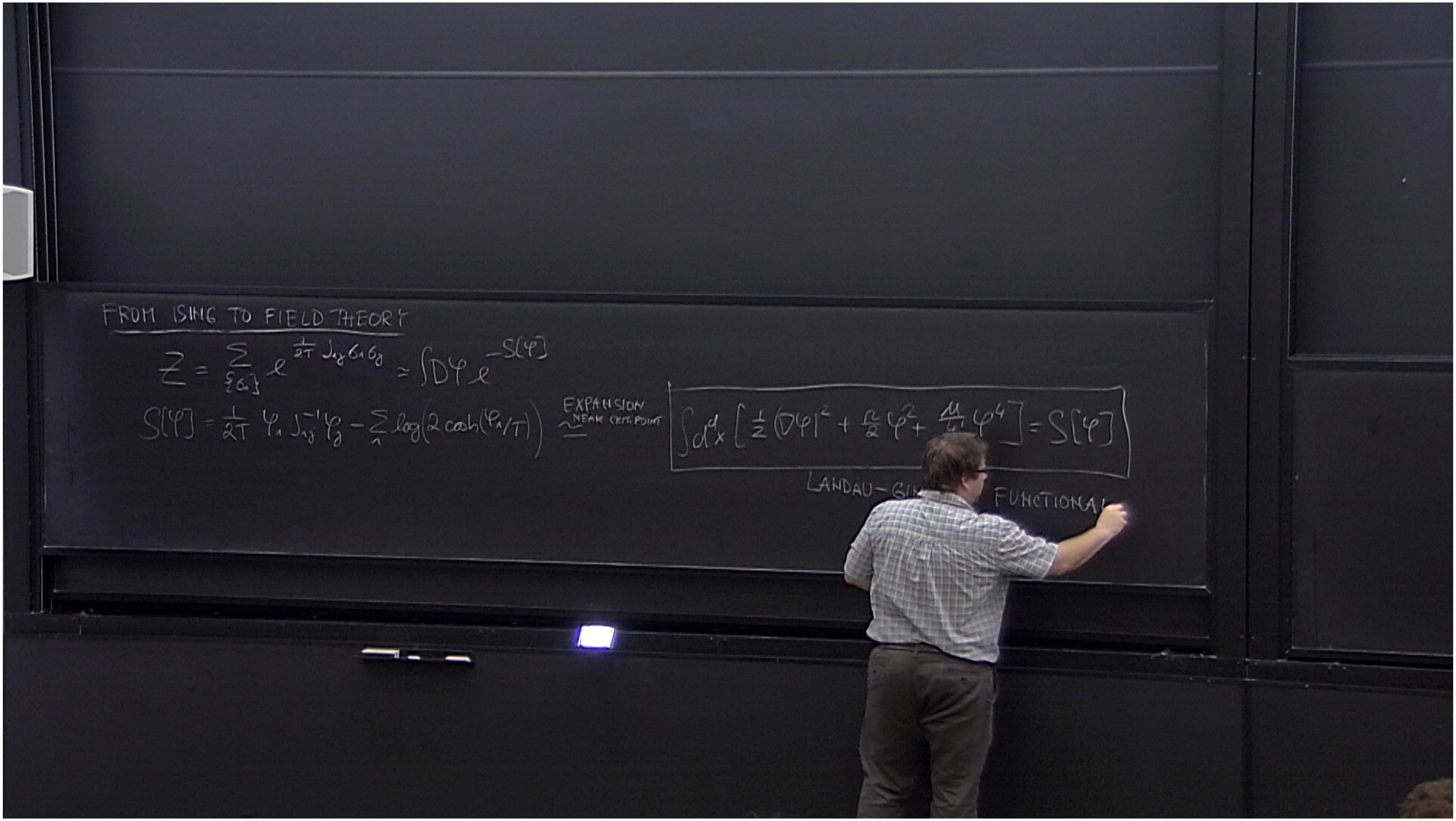
Title: PSI 17/18 - Statistical Mechanics - Lecture 7

Date: Oct 18, 2017 10:45 AM

URL: http://pirsa.org/17100040

Abstract:

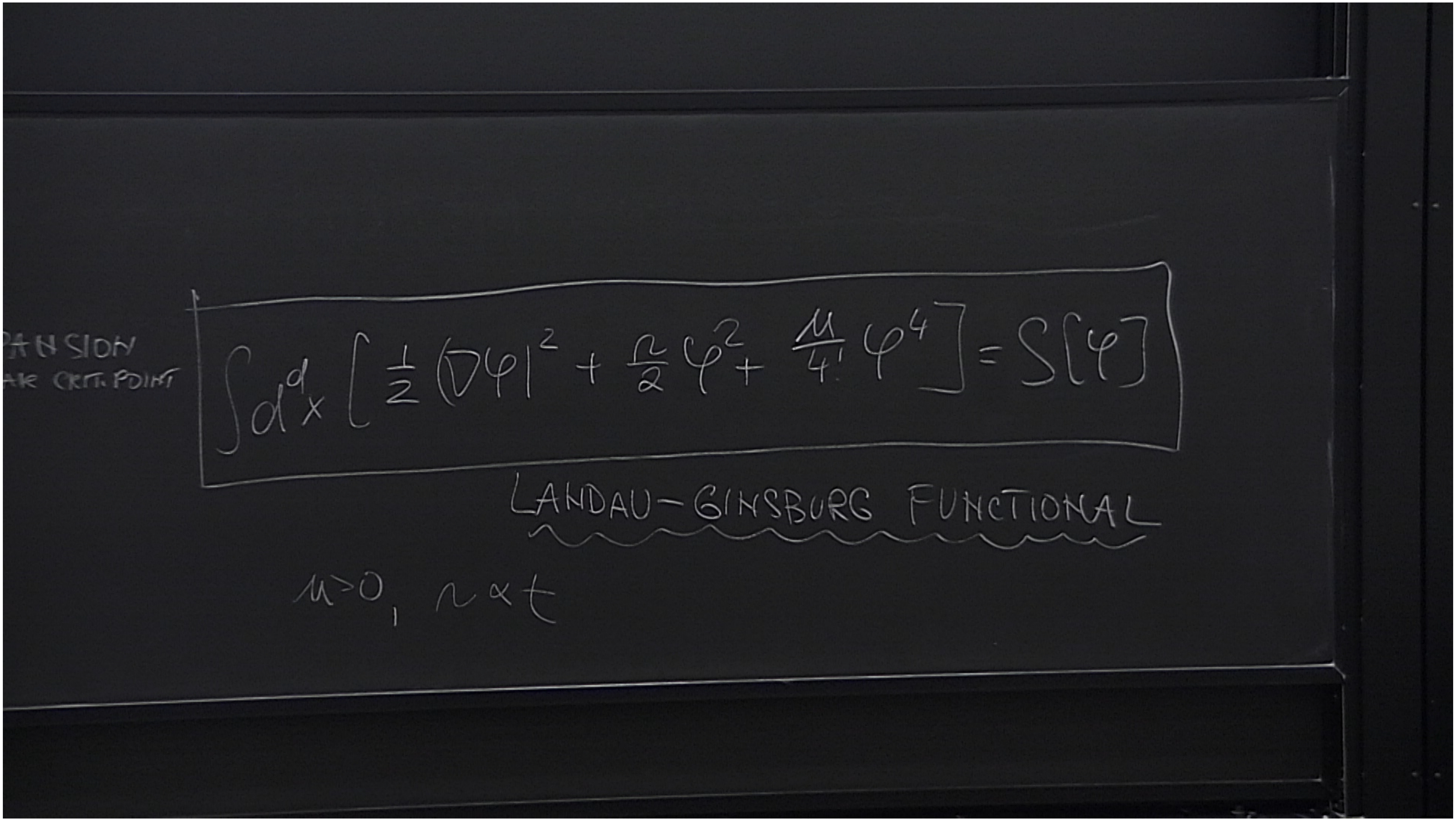

FROM ISING TO FIELD THEORY
$Z = \sum_{\{\sigma_i\}} e^{\frac{1}{2T} J_{ij} \sigma_i \sigma_j} \simeq \int D\varphi \, e^{-S[\varphi]}$
$S[\varphi] = \frac{1}{2T} \varphi_i J^{-1}_{ij} \varphi_j - \sum_i \log\left(2 \cosh(\varphi_i / T)\right)$
EXPANSION NEAR CRIT. POINT
$\boxed{\int d^d x \left[\frac{1}{2} (\nabla\varphi)^2 + \frac{r}{2} \varphi^2 + \frac{u}{4} \varphi^4 \right] = S[\varphi]}$
LANDAU–
FUNCTIONAL

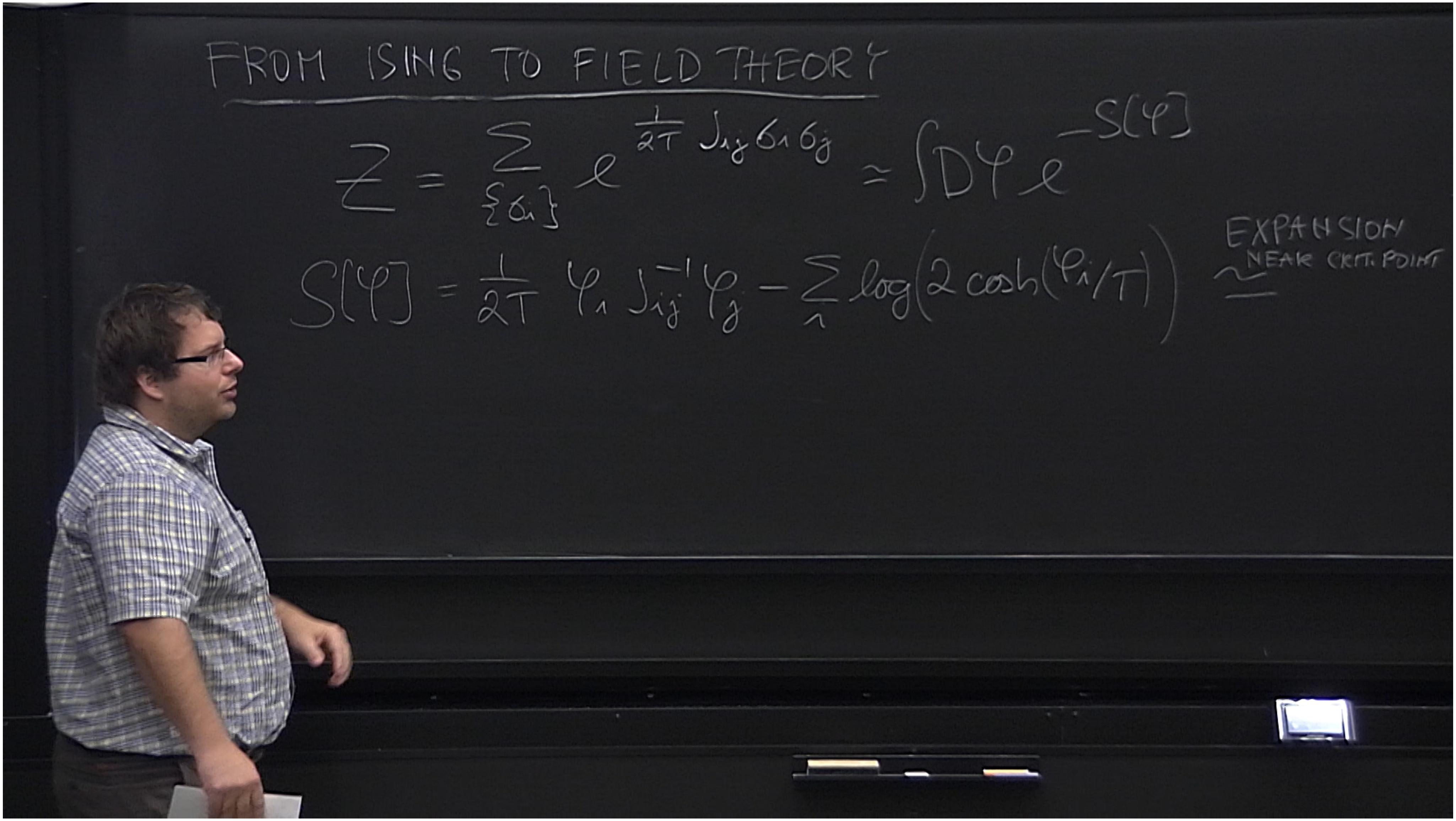
FROM ISING TO FIELD THEORY

$$Z = \sum_{\{\sigma_i\}} e^{\frac{1}{2T} J_{ij}\sigma_i \sigma_j} \simeq \int D\varphi \, e^{-S[\varphi]}$$

$$S[\varphi] = \frac{1}{2T} \varphi_i J^{-1}_{ij} \varphi_j - \sum_i \log\left(2 \cosh(\varphi_i / T)\right) \quad \simeq$$

EXPANSION
NEAR CRIT. POINT


PANSION
AR CRIT. POINT
$\int d^d x \left[\frac{1}{2} (\nabla \varphi)^2 + \frac{r}{2} \varphi^2 + \frac{u}{4!} \varphi^4 \right] = S[\varphi]$
LANDAU-GINSBURG FUNCTIONAL
$u > 0, \; r \propto t$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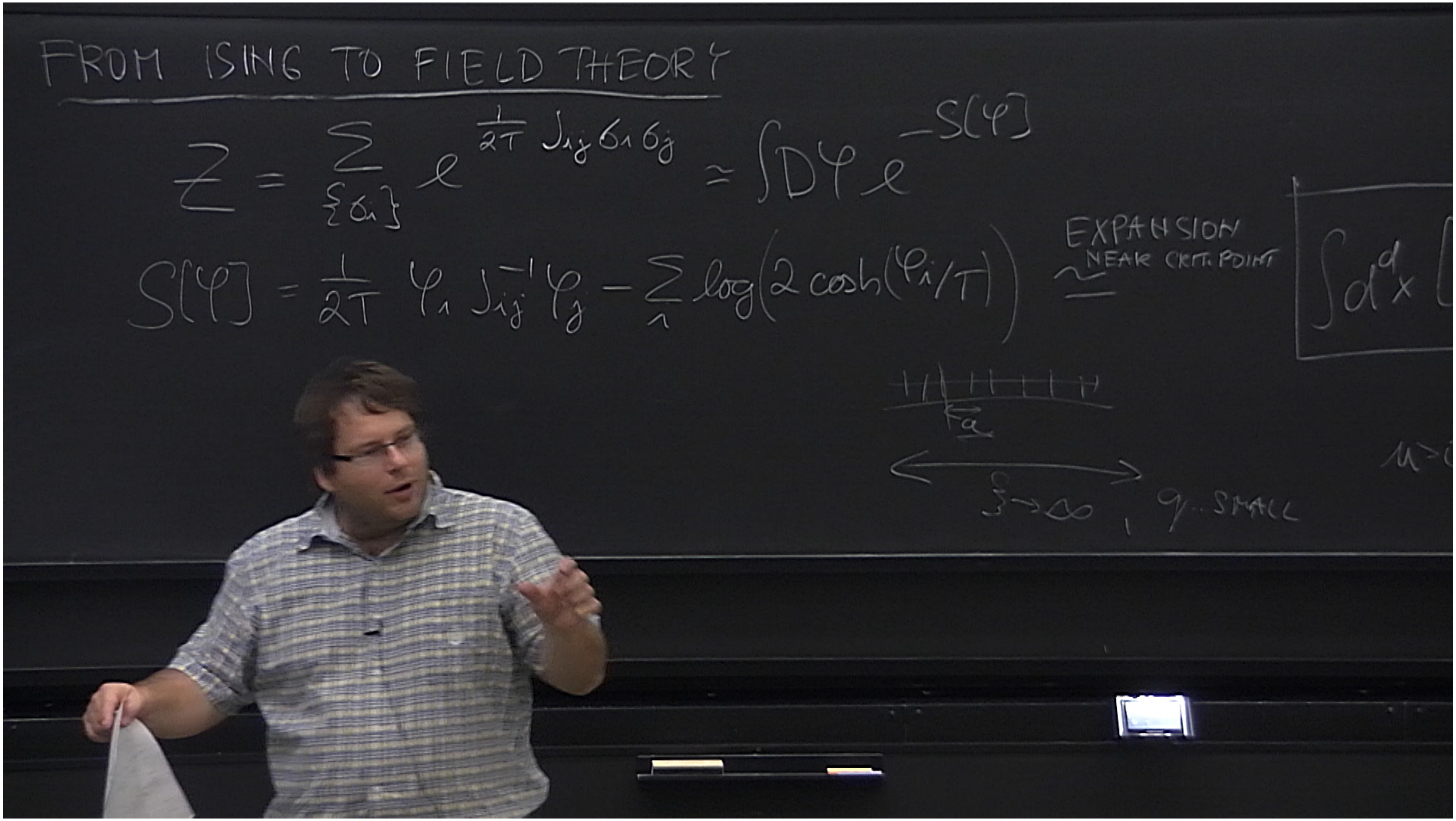
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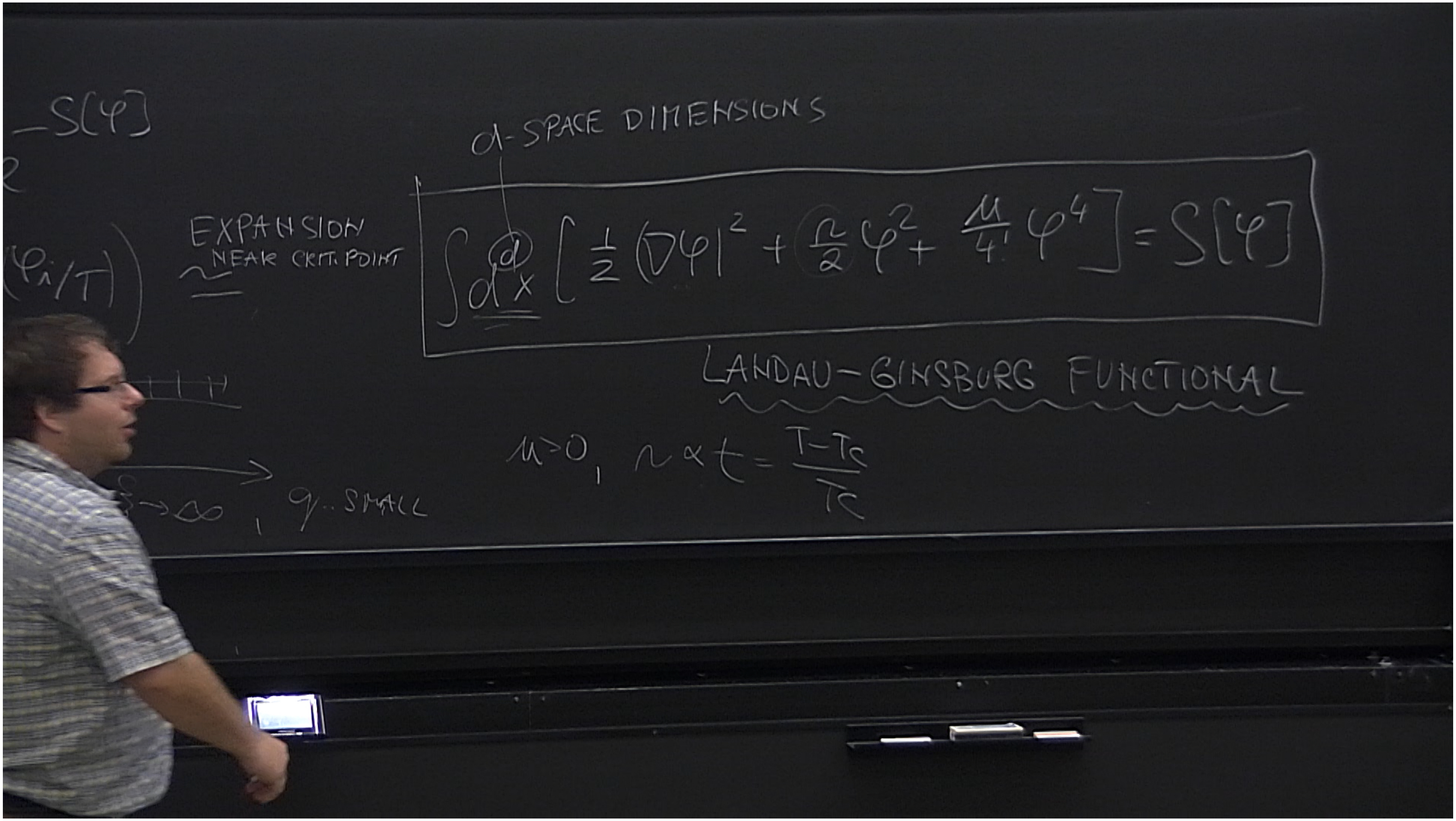
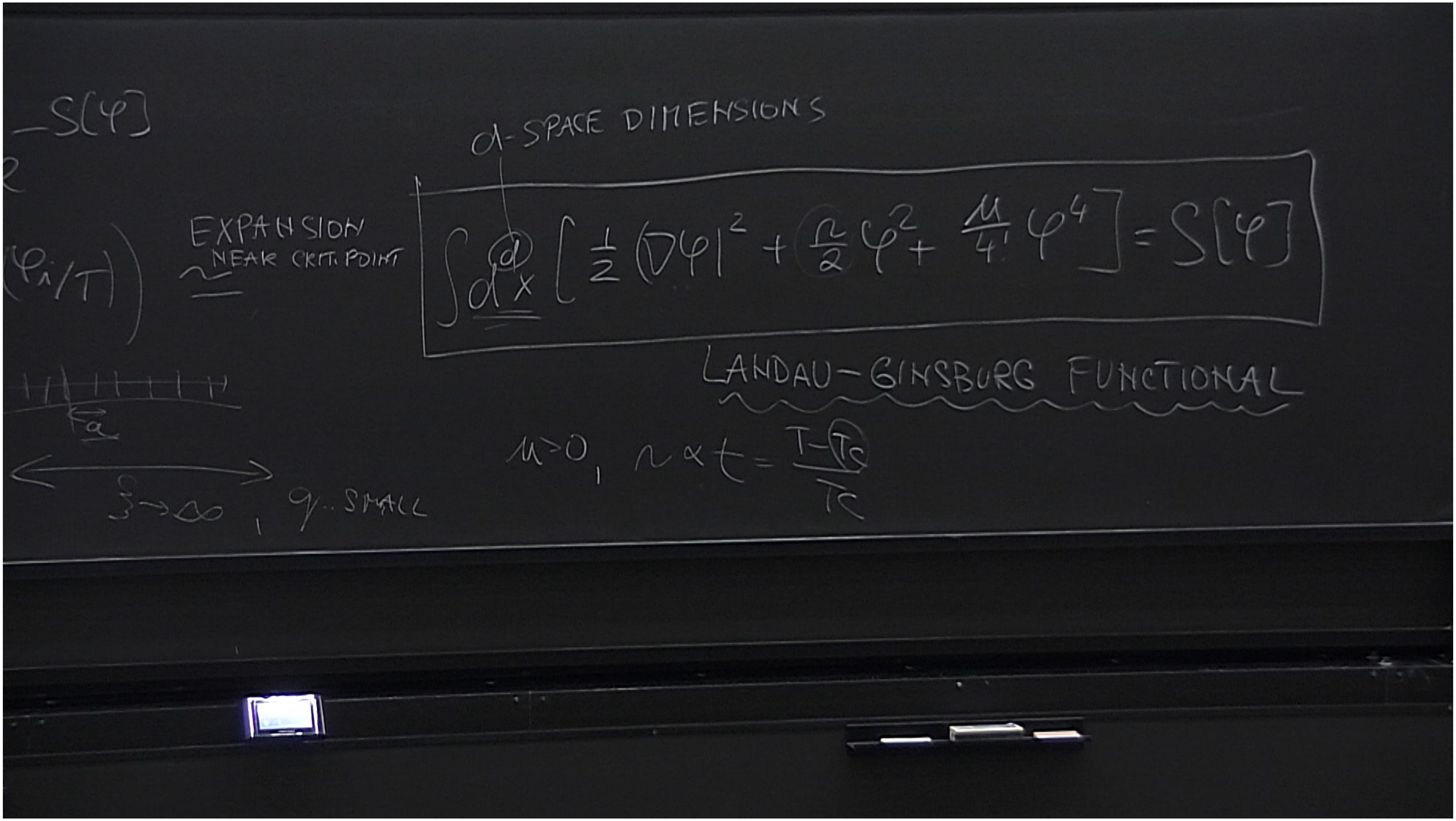
EXPANSION NEAR CRIT. POINT
$\int d^dx$
a
$\xi \to \infty$, q ... SMALL


d-SPACE DIMENSIONS
EXPANSION
NEAR CRIT. POINT
$\int d^d x \left[\frac{1}{2} (\nabla \varphi)^2 + \frac{r}{2} \varphi^2 + \frac{u}{4!} \varphi^4 \right] = S[\varphi]$
LANDAU-GINSBURG FUNCTIONAL
$u > 0, \quad r \propto t = \frac{T - T_c}{T_c}$
$\xi \to \infty, \quad q$.. SMALL


S[φ]
d-SPACE DIMENSIONS
EXPANSION
NEAR CRIT. POINT
∫ d^d x [1/2 (∇φ)^2 + r/2 φ^2 + u/4! φ^4] = S[φ]
LANDAU-GINSBURG FUNCTIONAL
a
u>0, r ∝ t = (T−Tc)/Tc
ξ→∞, q.. SMALL

- HIGHER POWERS OF φ OR ∇ AR IRRELEVANT AND SO THIS IS EVERYTHING YOU CAN WRITE DOWN.

- DIMINSIONAL ANALYSIS "UNITS OF ENERGY"

$$[k] = \dim(k) = +1$$

$$[x] = -1$$

$\to\infty$, g.. SMALL

$$[S] = 0 = \underbrace{[dx^d]}_{-d} + \underbrace{[\nabla^2]}_{+2} + \underbrace{[\varphi^2]}_{2\times[\varphi]}$$

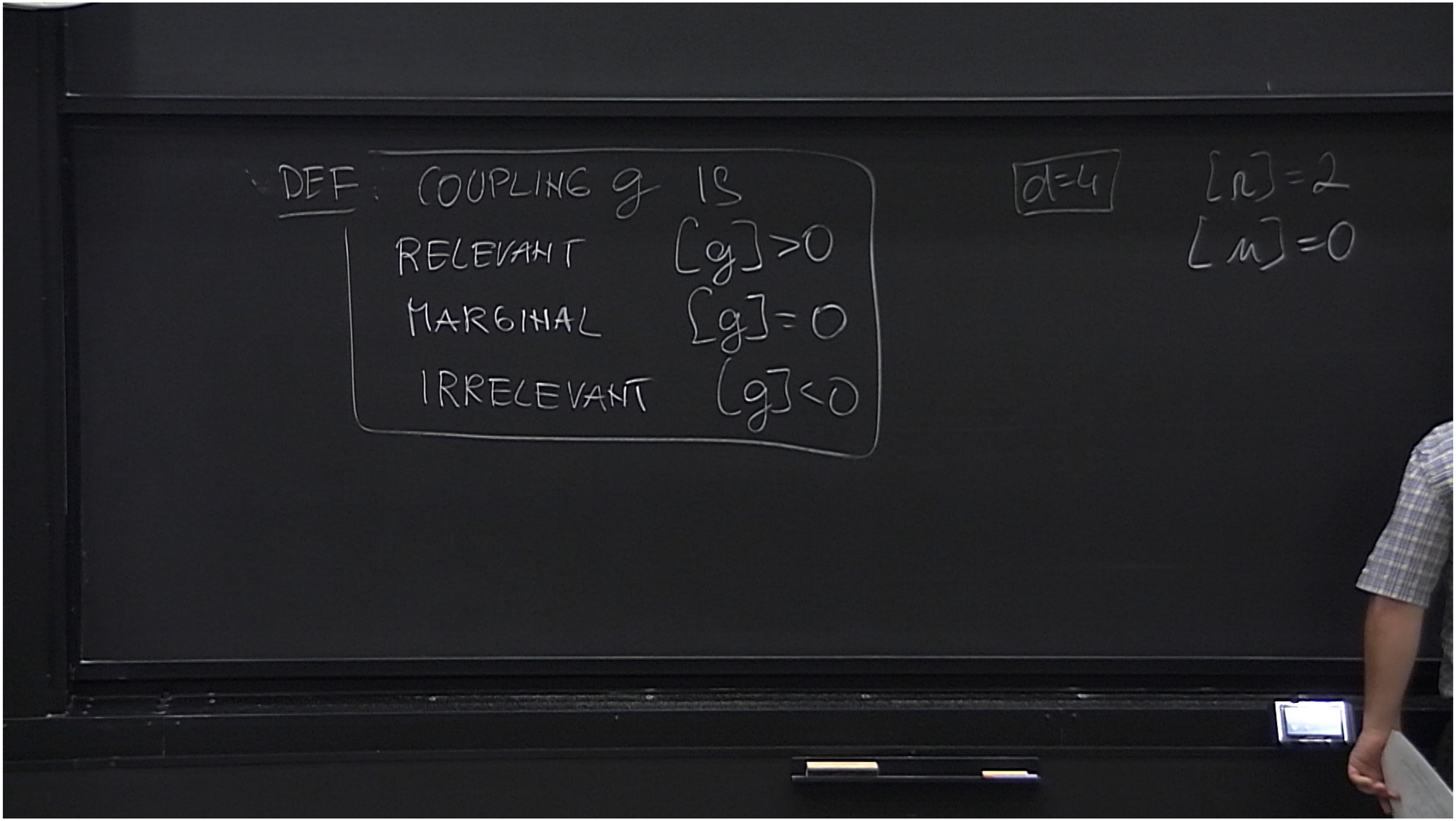
$$\Rightarrow \boxed{[\varphi] = \frac{d-2}{2}}$$

$$\boxed{[r] = +2}$$

$$[d^dx] + [u] + 4\times[\varphi] = 0$$

$$\underbrace{}_{-d} + [u] + 4\times\frac{d-2}{2} = 0$$

$$\boxed{[u] = 4-d = \varepsilon}$$


DEF: COUPLING g IS
RELEVANT $[g]>0$
MARGINAL $[g]=0$
IRRELEVANT $[g]<0$
$\boxed{d=4}$
$[\Omega]=2$
$[m]=0$

$$\boxed{[m] = 4 - d = \varepsilon}$$

$\boxed{d=4}$

$[\lambda] = 2$

$[m] = 0$

$+\sigma \varphi^6$

$[\sigma] = 6 - 2d = \underline{-2}$

L-G AS TRUNCATED HERE IS ONLY GOOD IN $\underline{d=4}$.

$[m] = 4 - d = \varepsilon$

$\boxed{d=4}$ $[\lambda] = 2$

$[m] = 0$

$+\sigma\varphi^6$

$[\sigma] = 6 - 2d = \underline{-2}$

L-G AS TRUNCATED HERE IS ONLY GOOD IN $\underline{d=4}$

- HOW TO CALCULATE $\underline{Z}$?

$\boxed{[u] = 4 - d = \varepsilon}$

$\boxed{d=4}$

$[r] = 2$

$[u] = 0$

$+ v \varphi^6$

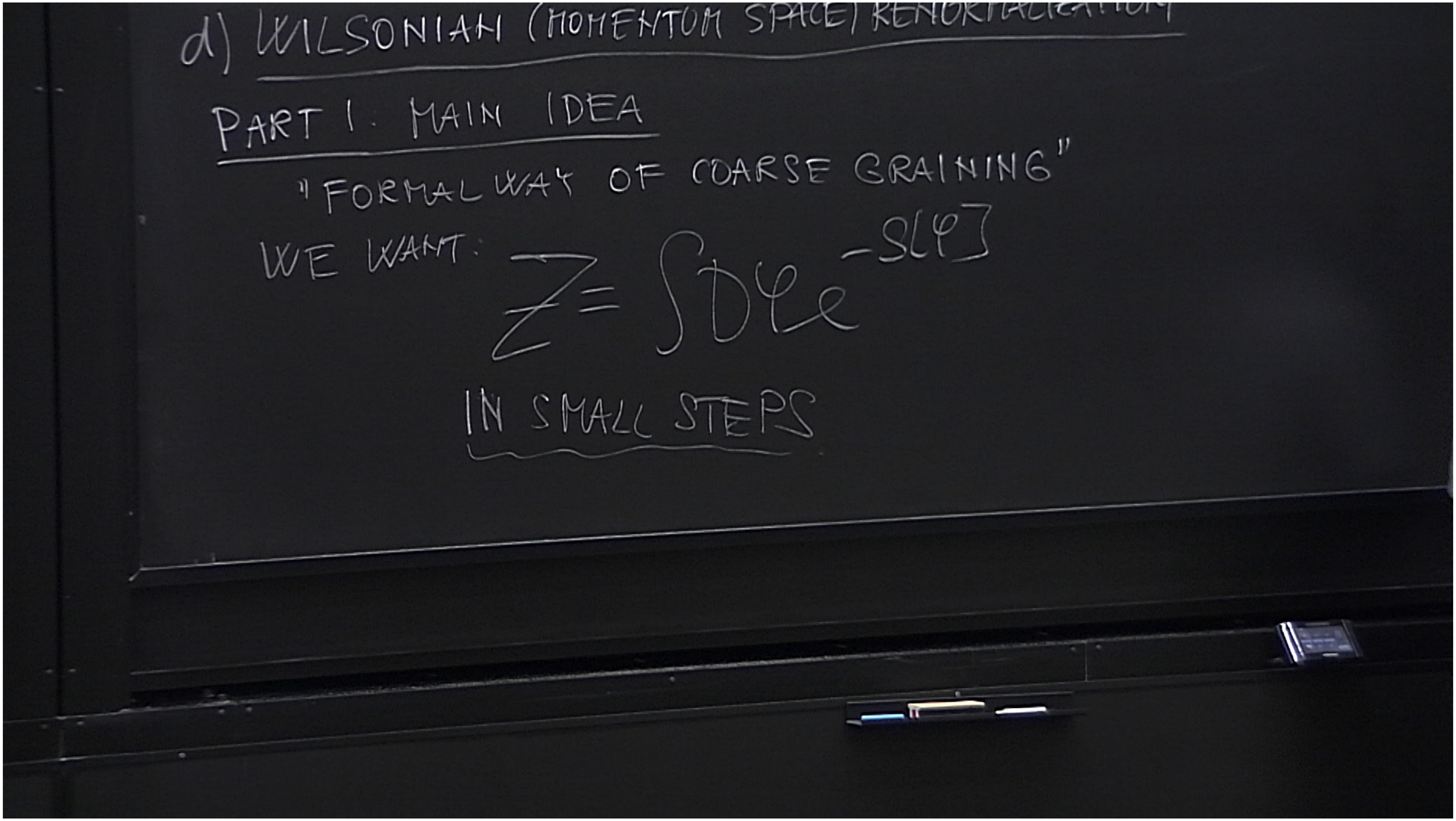
$[v] = 6 - 2d = -2$

L-G AS TRUNCATED HERE IS ONLY GOOD IN $d=4$

- HOW TO CALCULATE ε ? — PERT. THEORY IN u ↯

  P.T. FLUCTUATIONS $\delta\varphi$ ↯

RIGHT WAY: PERT. TH. IN SMALL ε.


d) WILSONIAN (MOMENTUM SPACE) RENORMALIZATION
PART I: MAIN IDEA
"FORMAL WAY OF COARSE GRAINING"
WE WANT: $Z = \int D\varphi \, e^{-S[\varphi]}$
IN SMALL STEPS

d) WILSONIAN (MOMENTUM SPACE) RENORMALIZATION

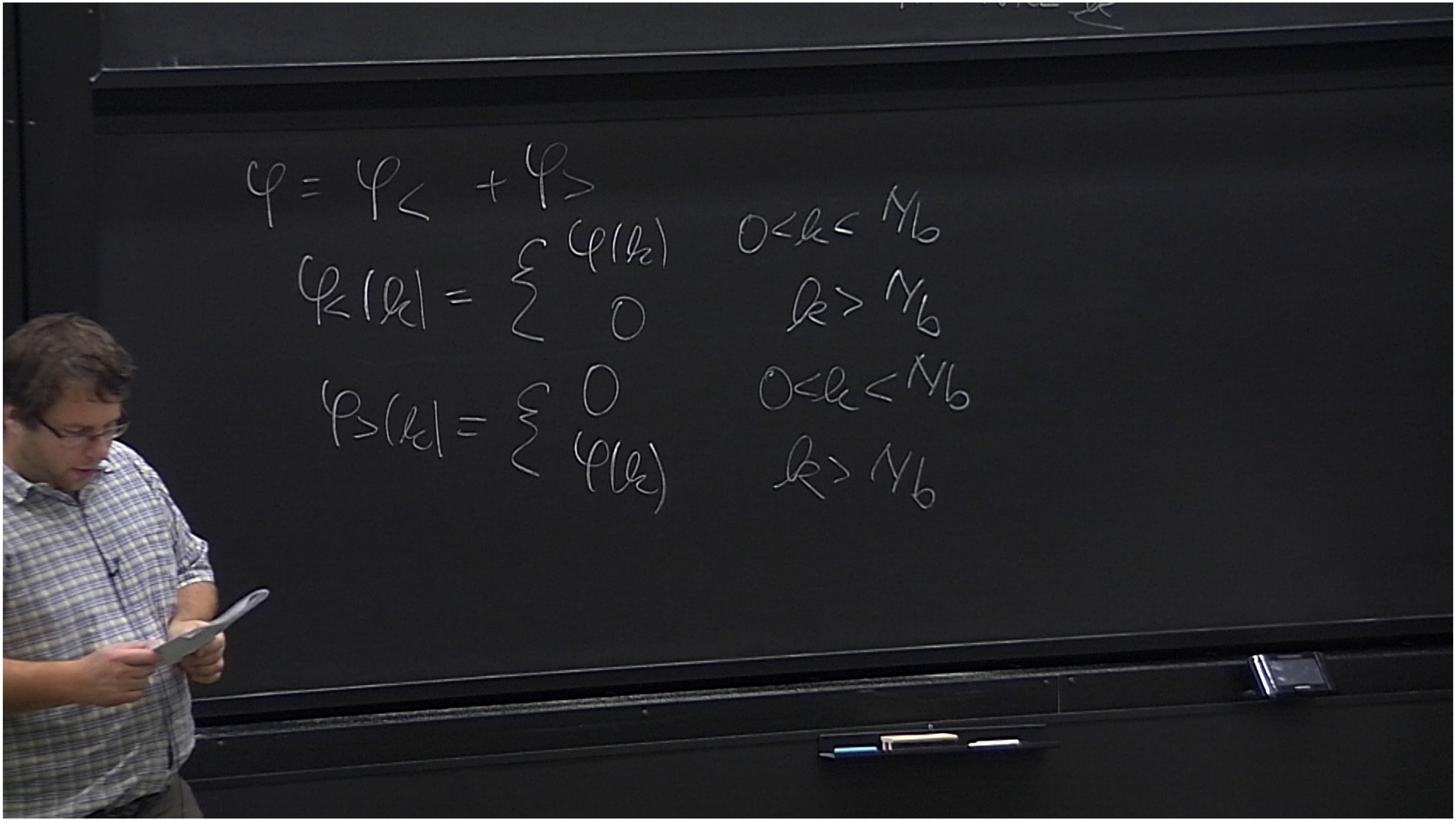
PART I: MAIN IDEA

"FORMAL WAY OF COARSE GRAINING"

WE WANT: $Z = \int D\varphi \, e^{-S[\varphi]}$

IN SMALL STEPS AVOIDING DIVERGENCIES IN SMALL k

• $\varphi($


$\varphi = \varphi_< + \varphi_>$
$\varphi_<(k) = \begin{cases} \varphi(k) & 0<k<\Lambda/b \\ 0 & k>\Lambda/b \end{cases}$
$\varphi_>(k) = \begin{cases} 0 & 0<k<\Lambda/b \\ \varphi(k) & k>\Lambda/b \end{cases}$

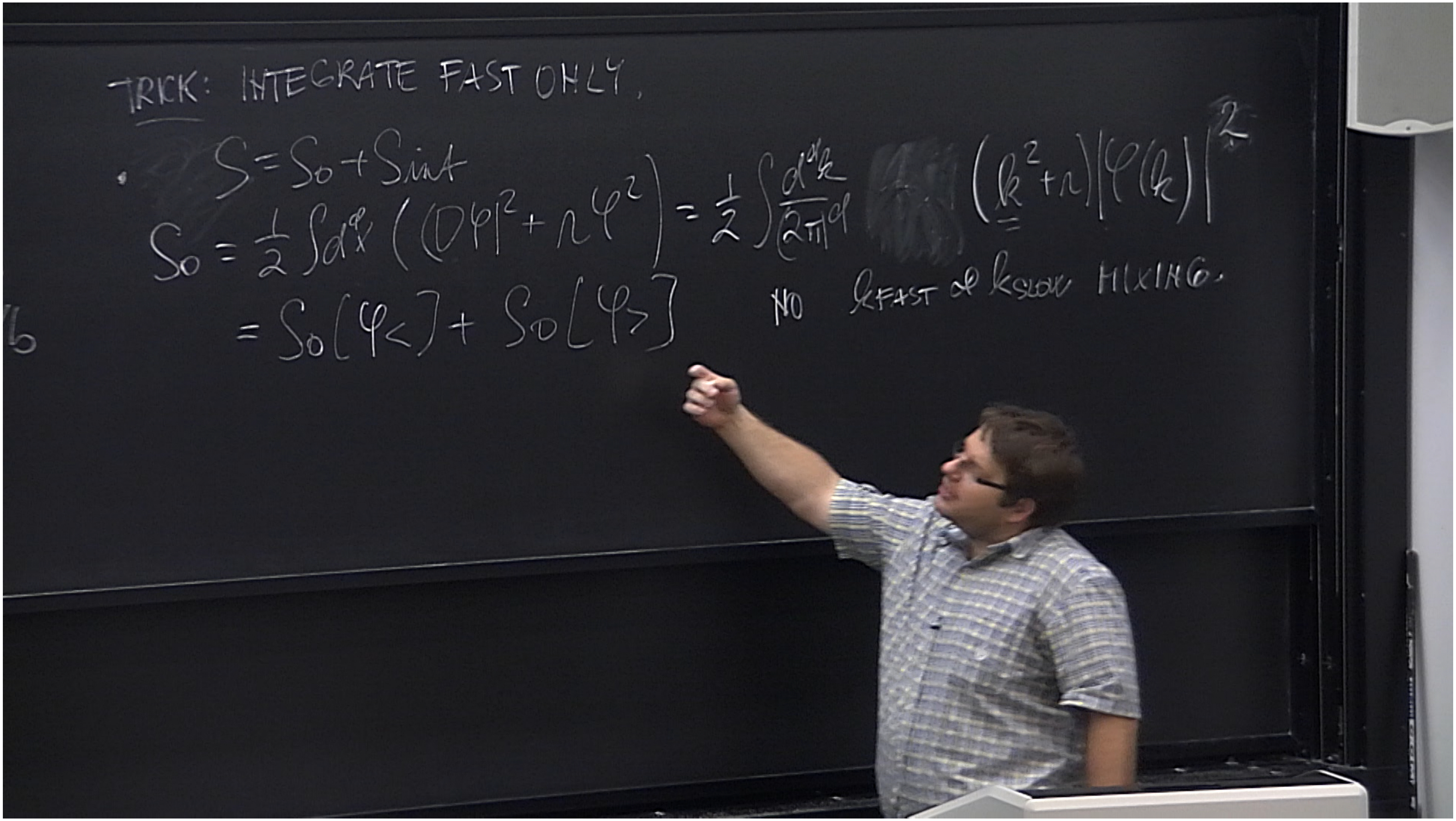
TRICK: INTEGRATE FAST ONLY.

- $S = S_0 + S_{int}$

$$S_0 = \frac{1}{2}\int d^dx \left((\partial\varphi)^2 + r\varphi^2 \right) = \frac{1}{2}\int \frac{d^dk}{(2\pi)^d} (k^2+r)|\varphi(k)|^2$$

$$= S_0[\varphi_<] + S_0[\varphi_>]$$

NO k_{FAST} & k_{SLOW} MIXING.


TRICK: INTEGRATE FAST ONLY.
$S = S_0 + S_{int}$
$S_0 = \frac{1}{2}\int d^d x \left((\nabla\varphi)^2 + r\varphi^2 \right) = \frac{1}{2}\int \frac{d^d k}{(2\pi)^d} (k^2 + r)|\varphi(k)|^2$
$= S_0[\varphi_<] + S_0[\varphi_>]$
NO k_{FAST} & k_{SLOW} MIXING.

TRICK: INTEGRATE FAST ONLY.

$$S = S_0 + S_{int}$$

$$S_0 = \frac{1}{2}\int d^d x \left((\nabla\varphi)^2 + r\varphi^2 \right) = \frac{1}{2}\int \frac{d^d k}{(2\pi)^d} \left(\underline{k}^2 + r\right)|\varphi(k)|^2$$

$$= \underline{S_0[\varphi_<] + S_0[\varphi_>]}$$ NO k_{FAST} & k_{SLOW} MIXING.

$$S_{INT} = \frac{u}{4!}\int d^d x\, \varphi^4 = \frac{u}{4!}\int \frac{d^d k_1}{(2\pi)^d} \cdots \frac{d^d k_4}{(2\pi)^d} \varphi(k_1)\ldots\varphi(k_4)(2\pi)^d \delta(k_1 + \ldots + k_4)$$

$$= S_{INT}[\varphi_<, \varphi_>]$$

WANT TO INTEGRATE OVER FAST MODES

$$Z = \int D\varphi_> D\varphi_< \, e^{-S[\varphi_<, \varphi_>]} = \int D\varphi_< \, e^{-\tilde{S}[\varphi_<]}$$

$$Z = \int D\varphi_> D\varphi_< \, e^{-S[\varphi_<, \varphi_>]} = \int D\varphi_< \, e^{-\tilde{S}[\varphi_<]}$$

$\tilde{S}[\varphi_<]$ ... HAS TO HAVE THE SAME FORM WE STARTED WITH.

$$\tilde{S}[\varphi_<] = \frac{1}{2} \int_0^{\Lambda/b} \frac{d^d k}{(2\pi)^d} \left| \varphi_<(k) \right|^2 \Big($$

WANT TO INTEGRATE OVER FAST MODES

$$Z = \int D\varphi_> D\varphi_< \, e^{-S[\varphi_<,\varphi_>]} = \int D\varphi_< \, e^{-\tilde{S}[\varphi_<]}$$

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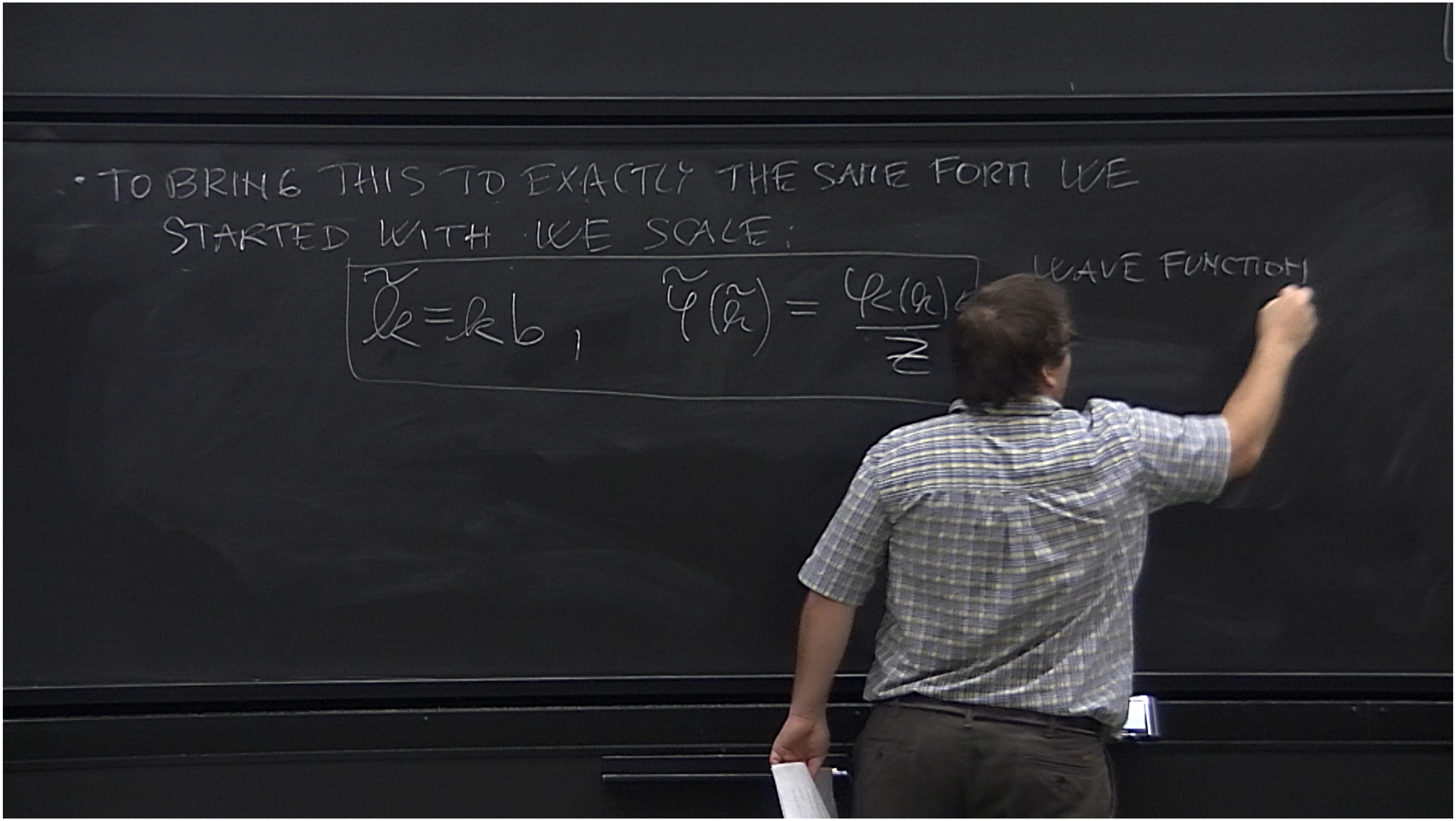
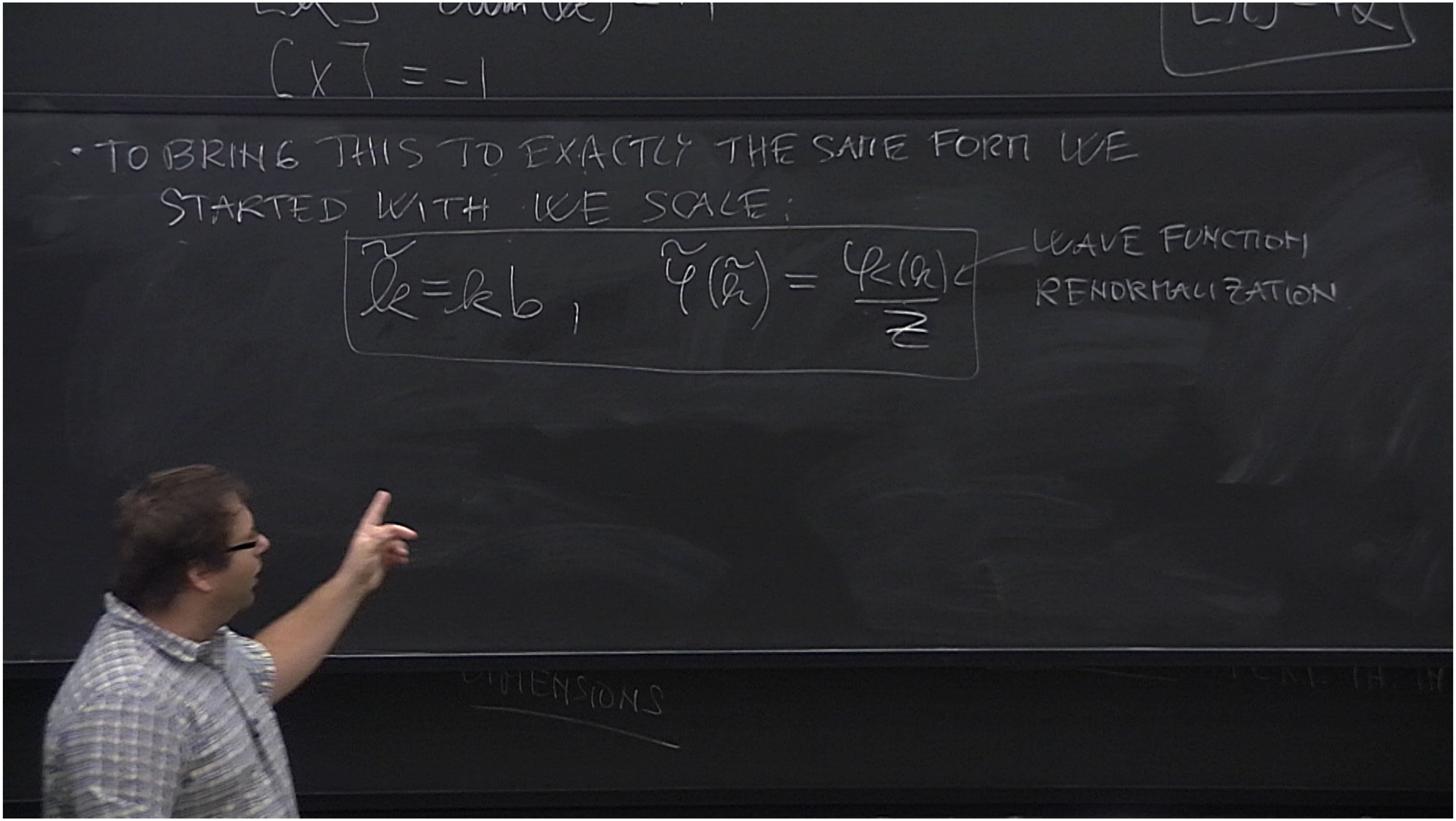
$$\tilde{S}[\varphi_<] = \frac{1}{2} \int_0^{\Lambda/b} \frac{d^dk}{(2\pi)^d} \left|\varphi_<(k)\right|^2 \left(\tilde{r}_2 k^2 + \tilde{r}_0\right)$$

$$e^{-S[\varphi_<]}$$

...E STARTED WITH. (CAN GENERATE MORE TERM BUT THESE ARE IRRELEVANT)

$$\tilde{\lambda}_0\Big) + \frac{\tilde{\lambda}_4}{4!}\int_0^{\Lambda/b} \frac{d^dk_1 \ldots d^dk_4}{(2\pi)^{4d}}\, \varphi_<(k_1)\ldots\varphi_<(k_4)\,(2\pi)^d\,\delta(k_1+k_2+k_3+k_4)$$


• TO BRING THIS TO EXACTLY THE SAME FORM WE
STARTED WITH WE SCALE:
$\tilde{k} = kb\,,\quad \tilde{\varphi}(\tilde{k}) = \frac{\varphi_k(k)}{Z}$
WAVE FUNCTION


[x] = -1
• TO BRING THIS TO EXACTLY THE SAME FORM WE
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$\tilde{k} = kb, \quad \tilde{\varphi}(\tilde{k}) = \frac{\varphi_k(k)}{Z}$
WAVE FUNCTION
RENORMALIZATION
DIMENSIONS

$[x] = -1$

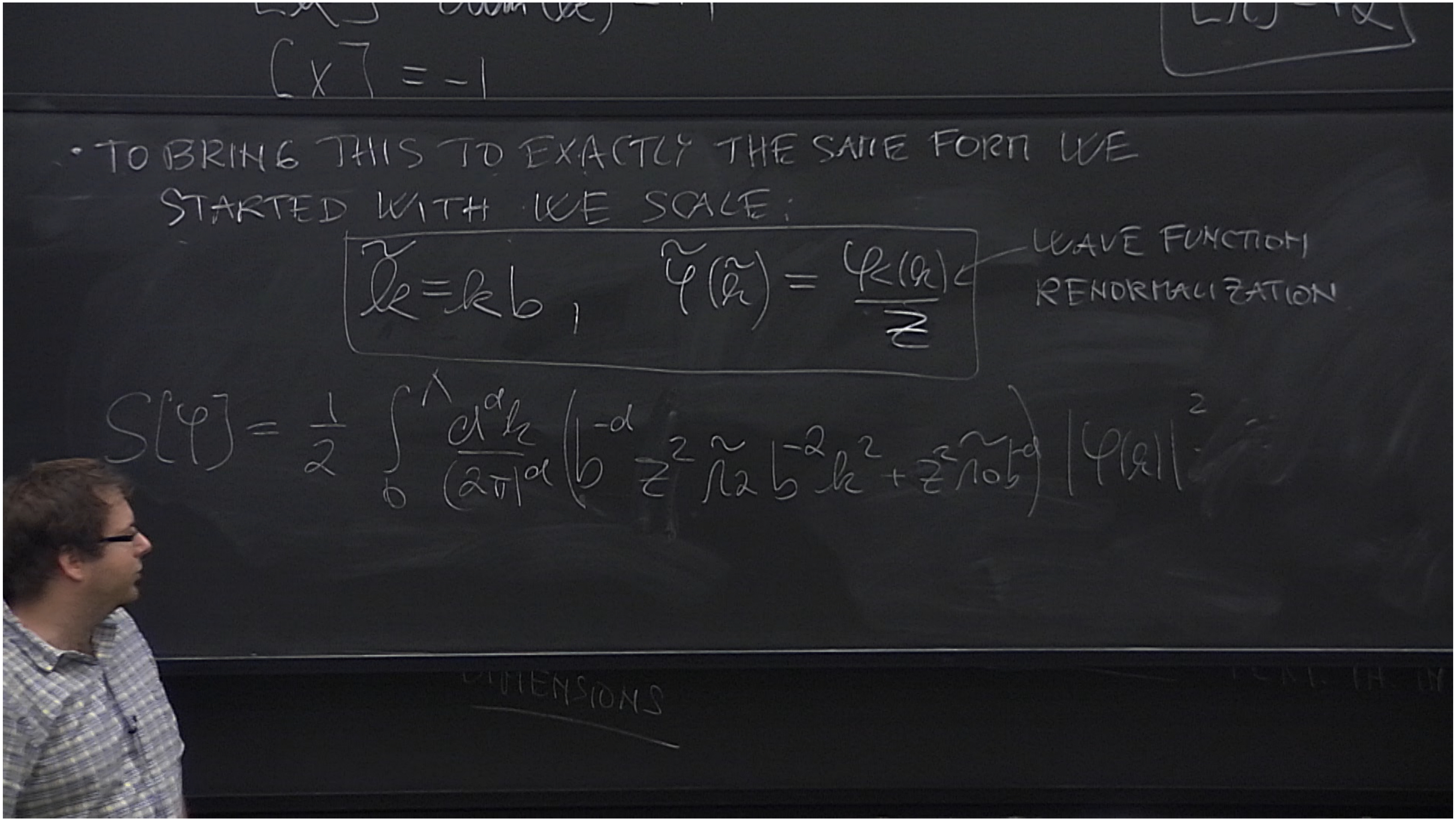
• TO BRING THIS TO EXACTLY THE SAME FORM WE STARTED WITH WE SCALE:

$$\boxed{\tilde{k} = kb\,, \quad \tilde{\varphi}(\tilde{k}) = \frac{\varphi(k)}{Z}}$$

WAVE FUNCTION RENORMALIZATION

$$S[\varphi] = \frac{1}{2}\int_0^{\Lambda} \frac{d^d k}{(2\pi)^d}\, b^{-d}\left(\tilde{r}_2 b^{-2} k^2 + \tilde{r}_0\right)|\varphi(k)|^2 Z^2$$

DIMENSIONS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$\tilde{k} = kb, \quad \tilde{\varphi}(\tilde{k}) = \frac{\varphi(k)}{Z}$
WAVE FUNCTION RENORMALIZATION
$S[\varphi] = \frac{1}{2}\int_0^\Lambda \frac{d^dk}{(2\pi)^d}\left(b^{-d} z^2 \tilde{\Omega}_2 b^{-2} k^2 + z^2 \tilde{\Omega}_0 b^{-d}\right)|\varphi(k)|^2$
DIMENSIONS

$[x] = -1$

• TO BRING THIS TO EXACTLY THE SAME FORM WE STARTED WITH WE SCALE:

$$\boxed{\tilde{k} = kb, \quad \tilde{\varphi}(\tilde{k}) = \frac{\varphi_<(k)}{Z}}$$

WAVE FUNCTION RENORMALIZATION

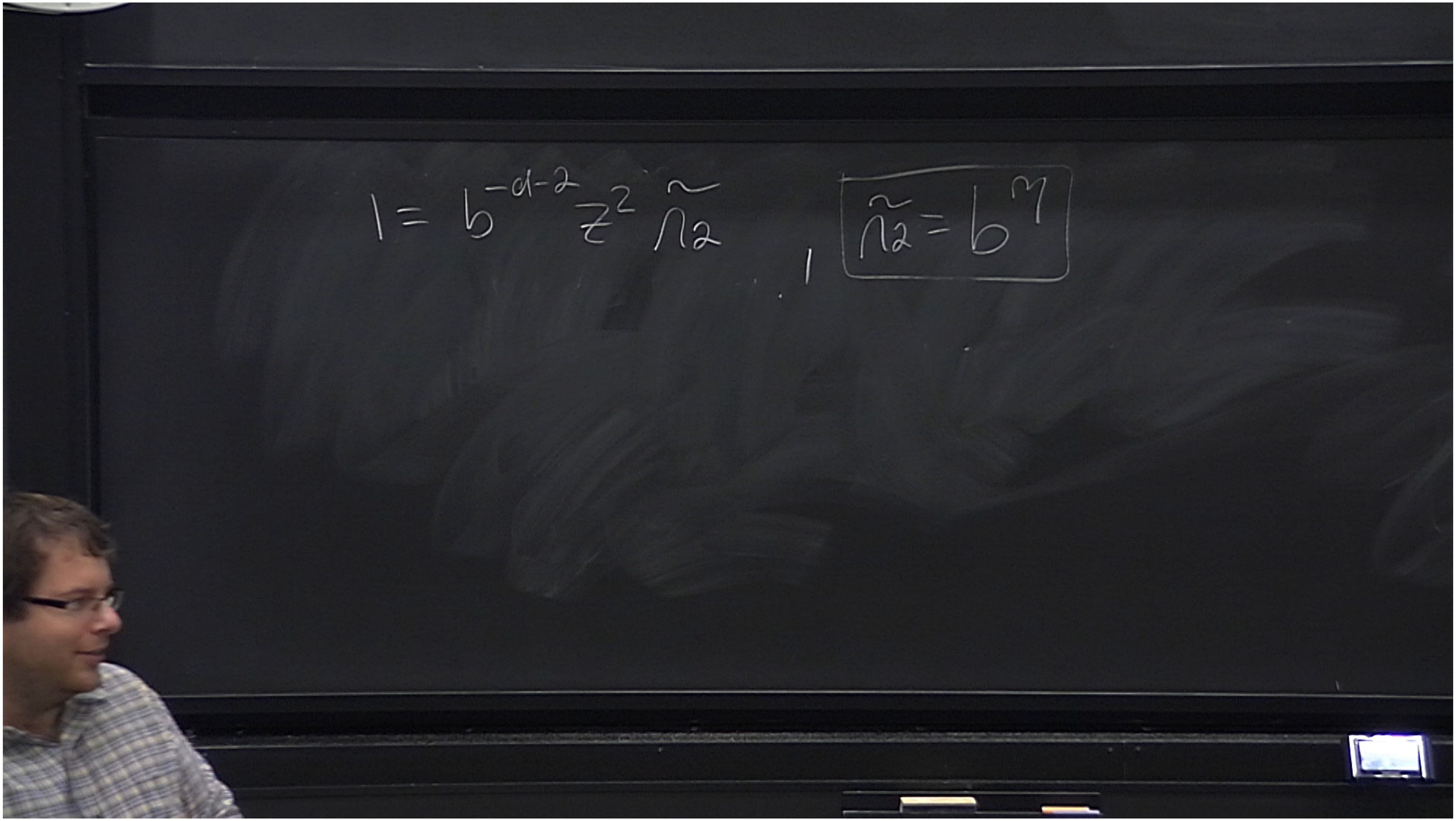
$$S[\varphi] = \frac{1}{2}\int_0^{\Lambda} \frac{d^dk}{(2\pi)^d}\left(b^{-d} Z^2 \tilde{r}_2 b^{-2} k^2 + Z^2 \tilde{r}_0 b^{-d}\right)|\varphi(\ldots|^2 \ldots \tilde{r}_4 b^{-4d} \ldots$$

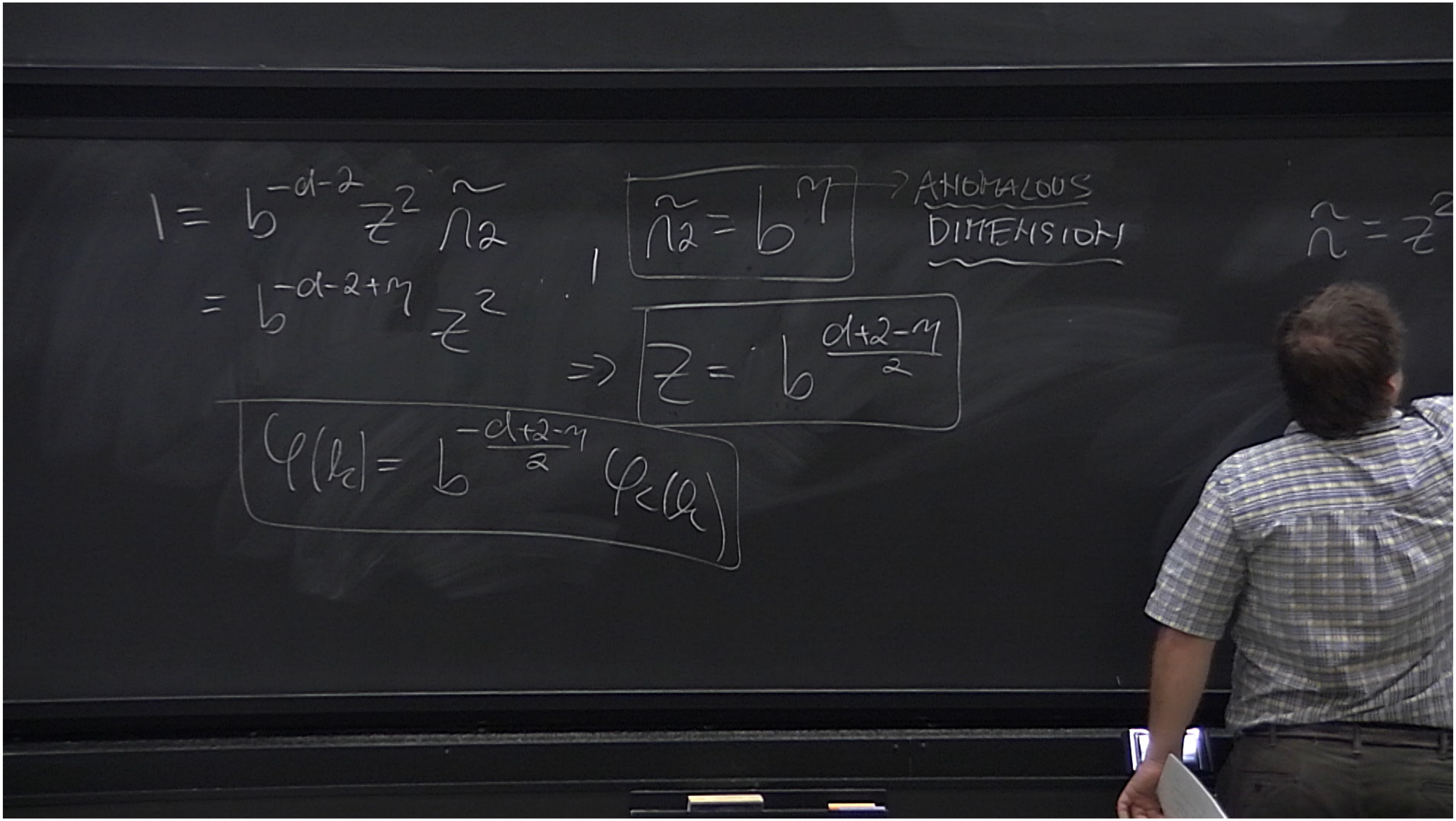
DIMENSIONS

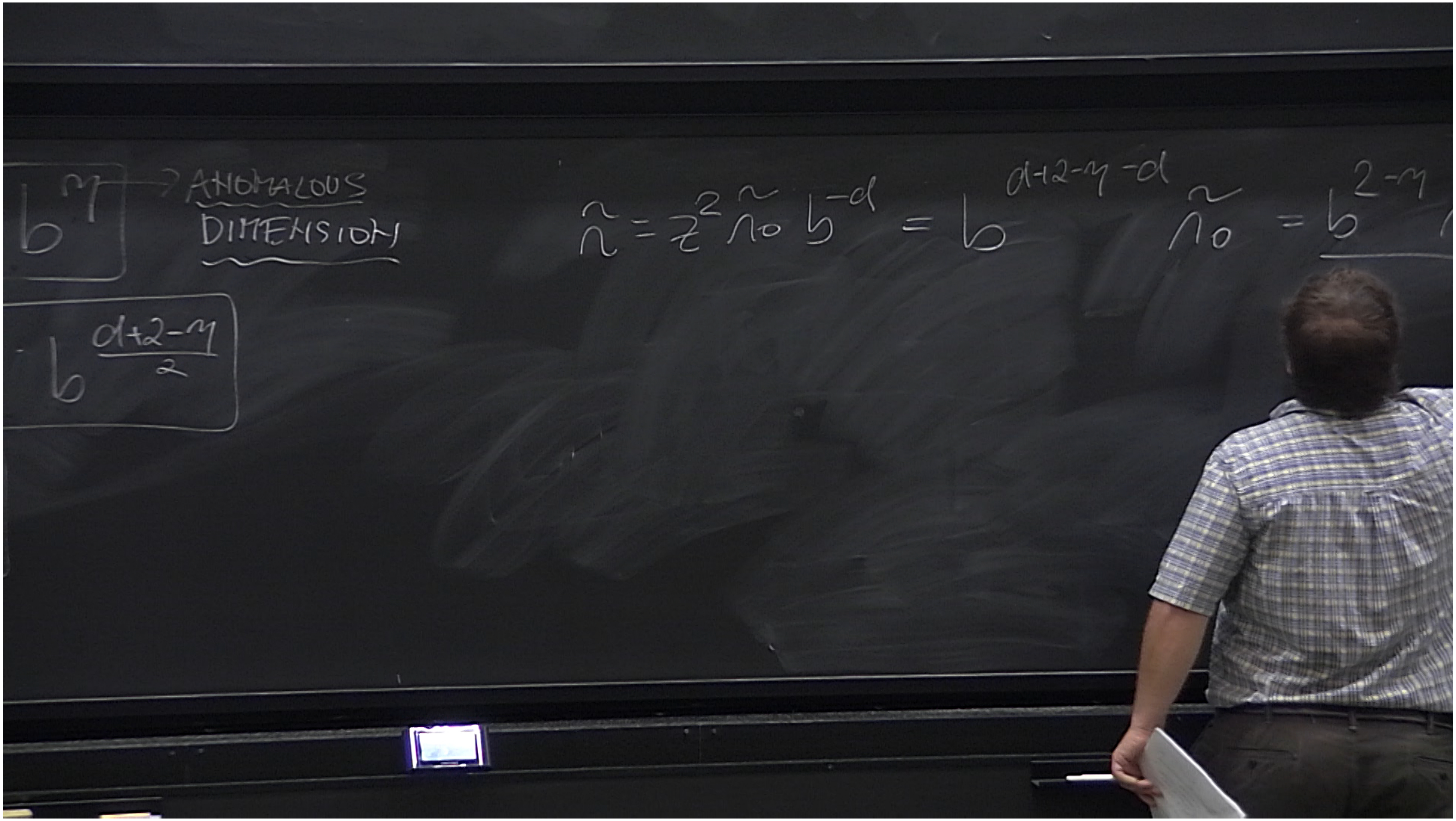
STARTED WITH. (CAN GENERATE MORE TERM BUT THESE ARE IRRELEVANT)

$$+\ \frac{\tilde{\lambda}_4}{4!}\int_0^{\Lambda/b}\frac{d^dk_1\ldots d^dk_4}{(2\pi)^{4d}}\,\varphi_<(k_1)\ldots\varphi_<(k_4)\,(2\pi)^d\,\delta(k_1+k_2+k_3+k_4)$$

$$\delta(f(x)) = \sum \frac{\delta(x-x_i)}{|f'(x_i)|}$$


$1 = b^{-d-2} z^2 \tilde{n}_2$, $\boxed{\tilde{n}_2 = b^{\eta}}$


$1 = b^{-d-2} z^2 \tilde{n}_2$
$= b^{-d-2+\eta} z^2$
$\tilde{n}_2 = b^{\eta}$
ANOMALOUS DIMENSION
$\Rightarrow z = b^{\frac{d+2-\eta}{2}}$
$\varphi(k) = b^{-\frac{d+2-\eta}{2}} \varphi_<(k)$
$\tilde{n} = z^2$


b^{η}
ANOMALOUS DIMENSION
$b^{\frac{d+2-\eta}{2}}$
$\tilde{n} = z^2 \tilde{n}_0 b^{-d} = b^{d+2-\eta-d} \tilde{n}_0 = b^{2-\eta}$

$$\tilde{r} = z^2 \tilde{\lambda}_0 b^{-d} = b^{d+2-\eta-d}\,\tilde{\lambda}_0 = \boxed{b^{2-\eta}\,\tilde{\lambda}_0 = \tilde{r}}$$

$$\boxed{\tilde{u} = b^{4-2\eta-d}\,\tilde{\lambda}_4}$$

...US

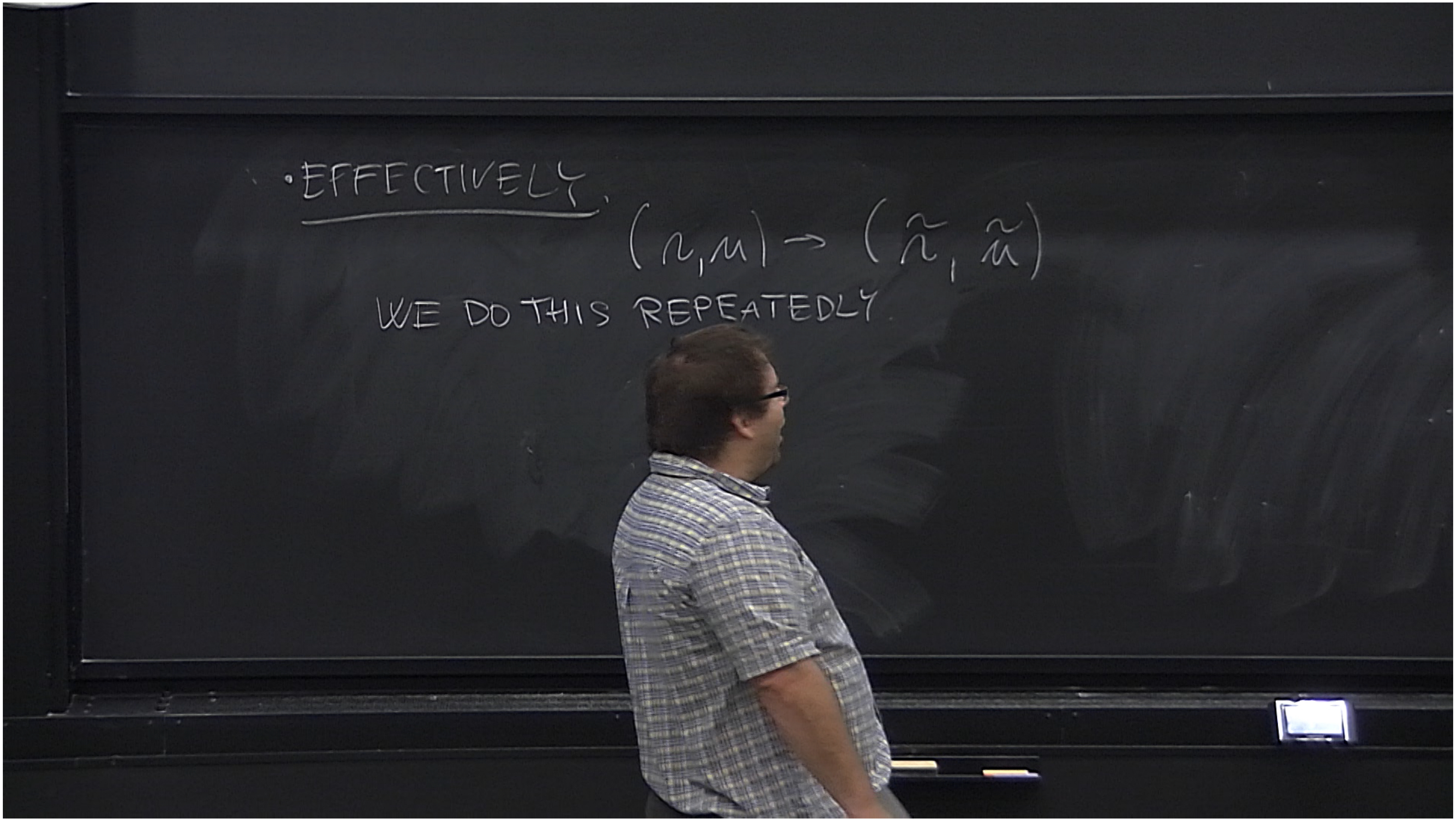
...ION

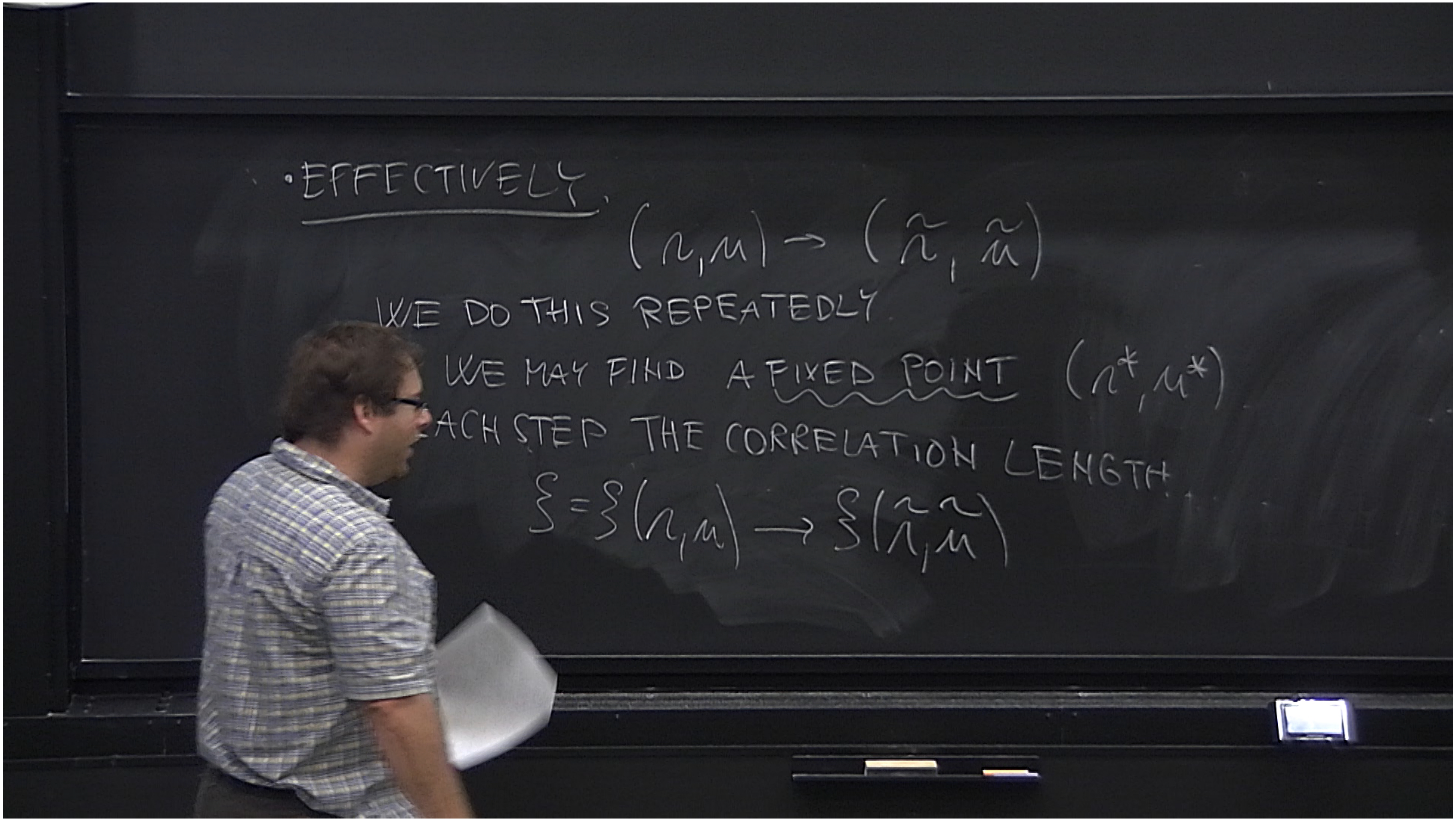
$$\tilde{r} = z^2 \tilde{r}_0 b^{-d} = b^{d+2-\eta-d} \tilde{r}_0 = \boxed{b^{2-\eta} \tilde{r}_0 = \tilde{r}}$$

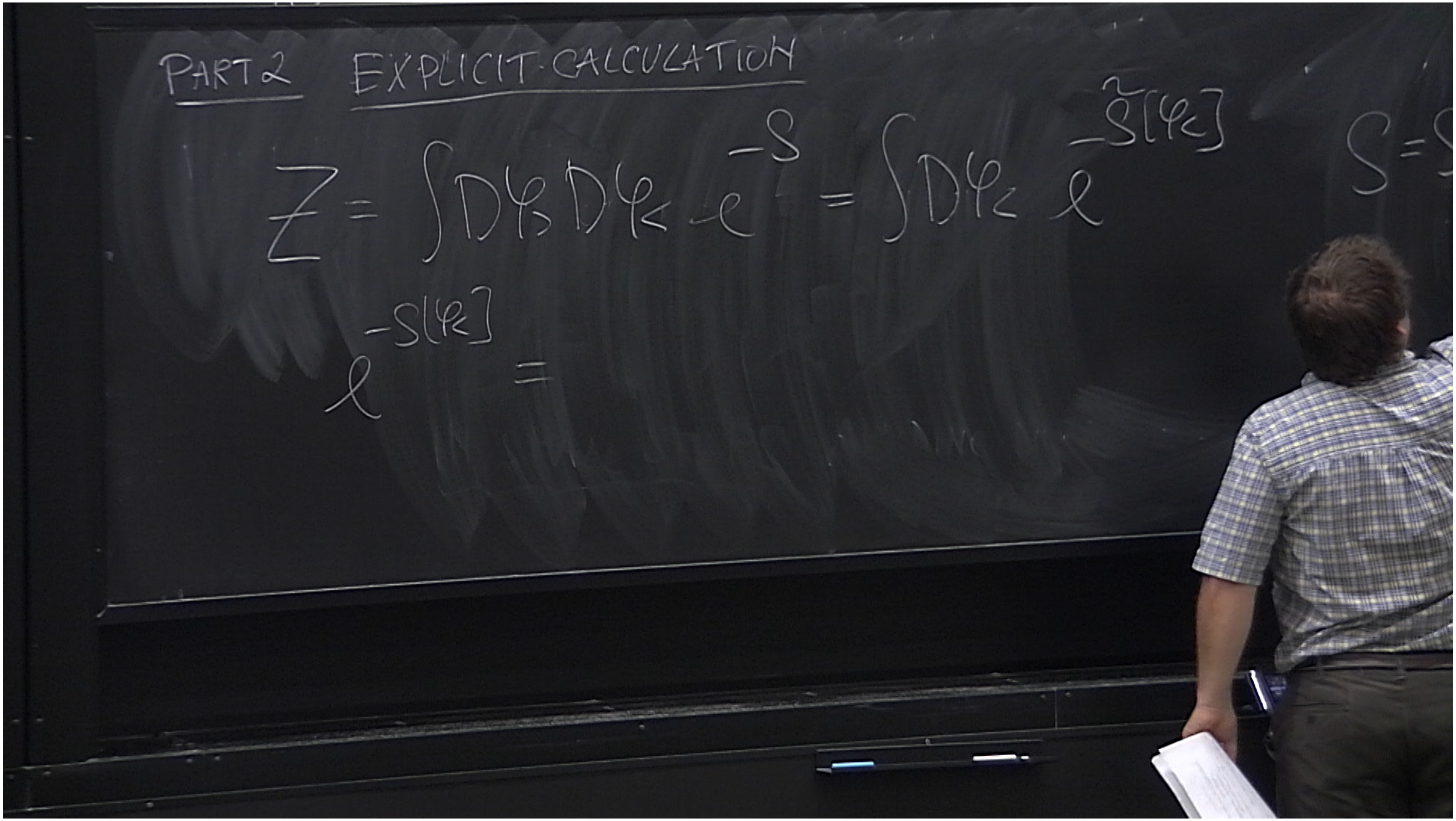
$$\boxed{\tilde{u} = b^{4-2\eta-d} \tilde{\lambda}_4}$$

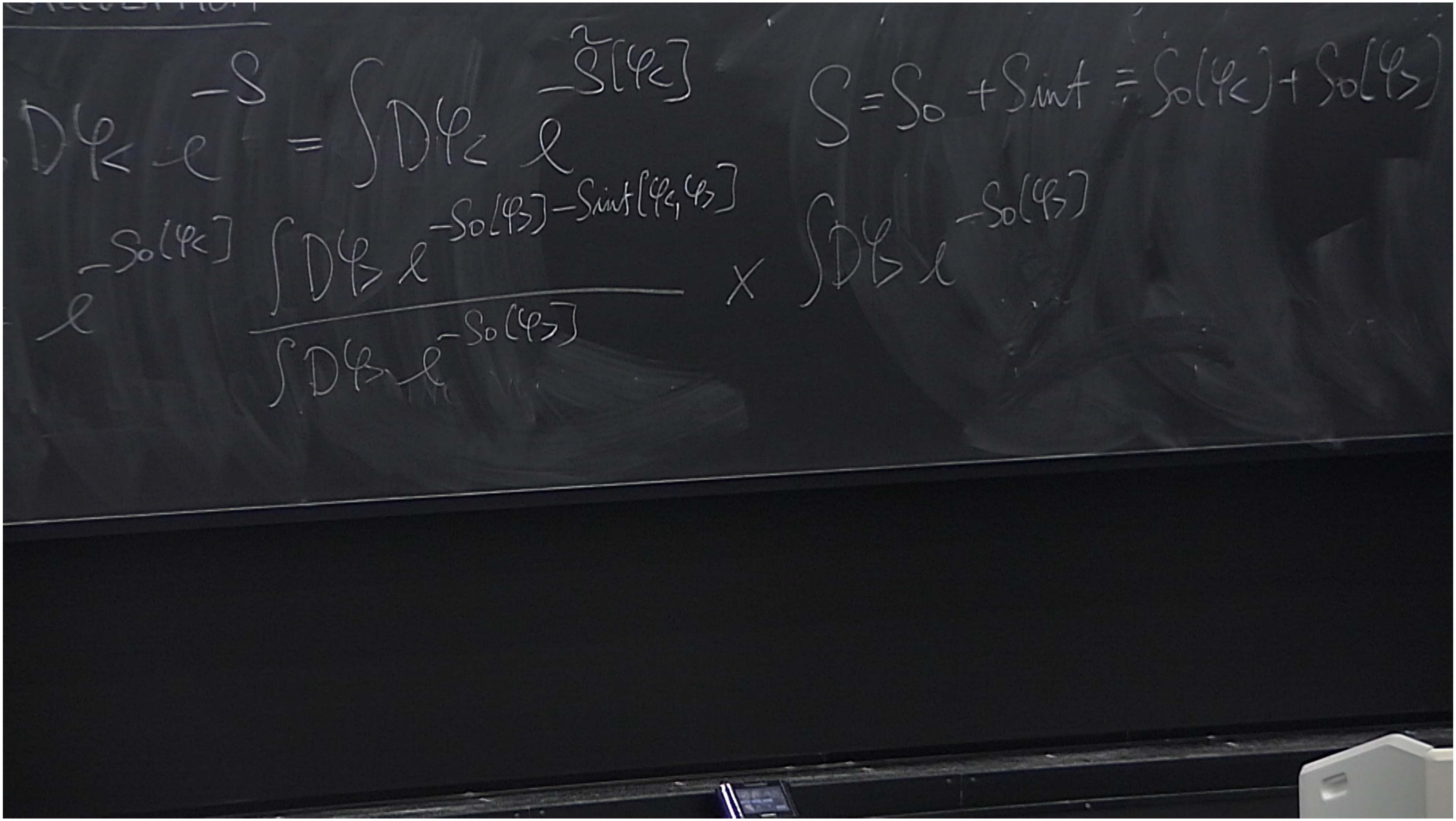
HAVE NOW

$$S[\varphi] = \int d^d x \left[\frac{1}{2} (\nabla\varphi)^2 + \frac{\tilde{r}}{2} \varphi^2 + \frac{\tilde{u}}{4!} \varphi^4 \right]$$


• EFFECTIVELY:
$(r, u) \to (\tilde{r}, \tilde{u})$
WE DO THIS REPEATEDLY


• EFFECTIVELY
$(r,u) \to (\tilde{r},\tilde{u})$
WE DO THIS REPEATEDLY
WE MAY FIND A FIXED POINT (r^*,u^*)
ACH STEP THE CORRELATION LENGTH
$\xi = \xi(r,u) \to \xi(\tilde{r},\tilde{u})$


PART 2 EXPLICIT CALCULATION
$Z = \int D\varphi_> D\varphi_< \, e^{-S} = \int D\varphi_< \, e^{-\tilde{S}[\varphi_<]}$
$S =$
$e^{-\tilde{S}[\varphi_<]} =$


$\int D\varphi_< \, e^{-S} = \int D\varphi_< \, e^{-\tilde{S}[\varphi_<]}$
$S = S_0 + S_{int} = S_0[\varphi_<] + S_0[\varphi_>]$
$e^{-S_0[\varphi_<]} \dfrac{\int D\varphi_> \, e^{-S_0[\varphi_>] - S_{int}[\varphi_<, \varphi_>]}}{\int D\varphi_> \, e^{-S_0[\varphi_>]}} \times \int D\varphi_> \, e^{-S_0[\varphi_>]}$

$$\tilde{S}[\varphi_<] = S_0[\varphi_<] - \log \langle e^{-S_{int}} \rangle_{FAST} \cancel{- \log Z_0}$$

CONSTANT

CUMULANT EXPANSION

Ω ... RANDOM VARIABLE

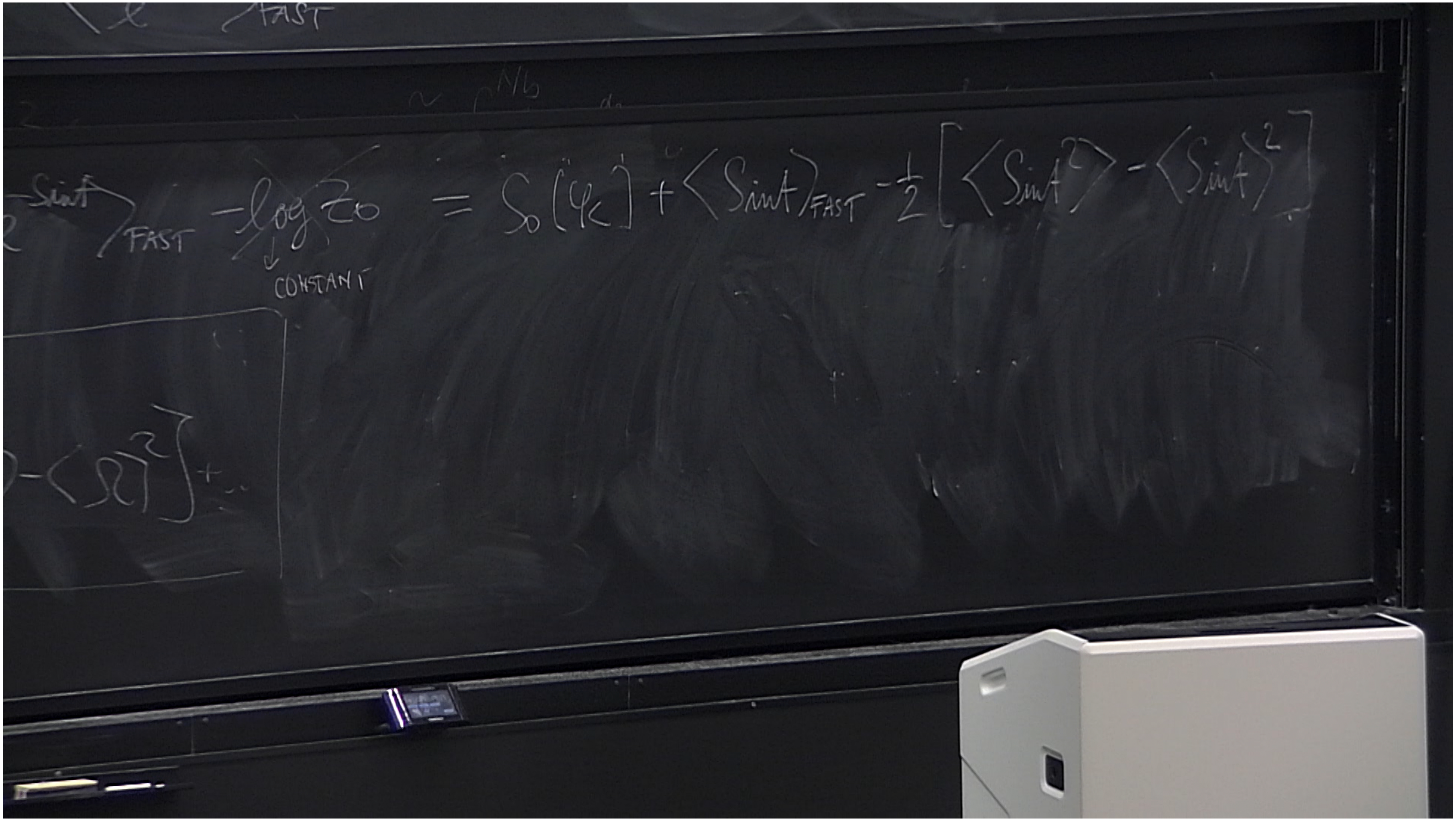
$$\boxed{\langle e^{\Omega} \rangle = e^{\langle \Omega \rangle + \frac{1}{2}\left[\langle \Omega^2 \rangle - \langle \Omega \rangle^2\right] + \dots}}$$

$$\tilde{S}[\varphi_<] = S_0[\varphi_<] - \log \left\langle e^{-S_{int}} \right\rangle_{FAST} \underbrace{\cancel{-\log Z_0}}_{\text{CONSTANT}} = S_0[\varphi_<] + \langle S_{i}$$

CUMULANT EXPANSION

Ω ... RANDOM VARIABLE

$$\boxed{\left\langle e^{\Omega} \right\rangle = e^{\langle \Omega \rangle + \frac{1}{2}\left[\langle \Omega^2 \rangle - \langle \Omega \rangle^2\right] + \dots}}$$


$-\log Z_0 = S_0[\varphi_<] + \langle S_{int} \rangle_{FAST} - \frac{1}{2}\left[\langle S_{int}^2 \rangle - \langle S_{int} \rangle^2\right]$
CONSTANT