

Title: PSI 17/18 - Statistical Mechanics - Lecture 6

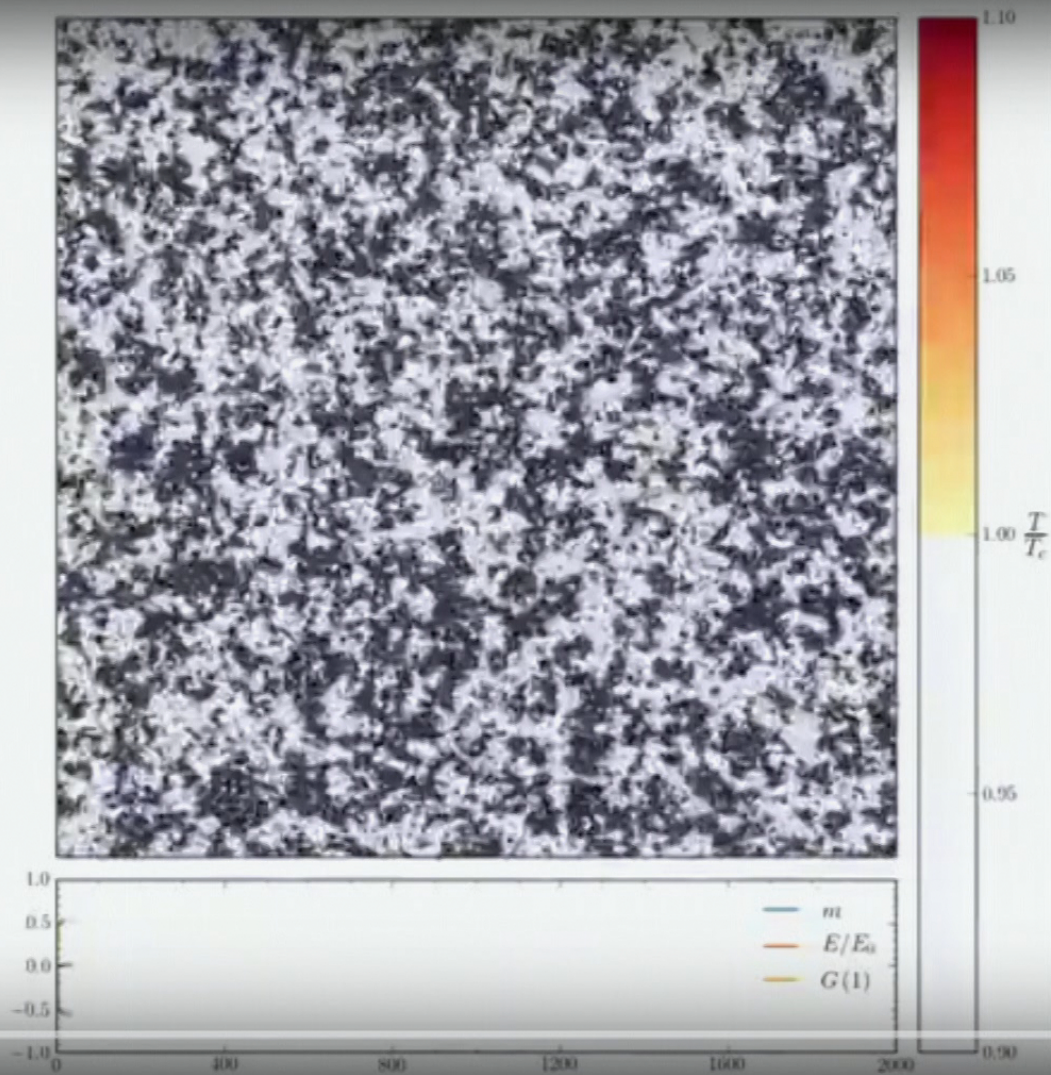
Date: Oct 17, 2017 10:45 AM

URL: <http://pirsa.org/17100039>

Abstract:



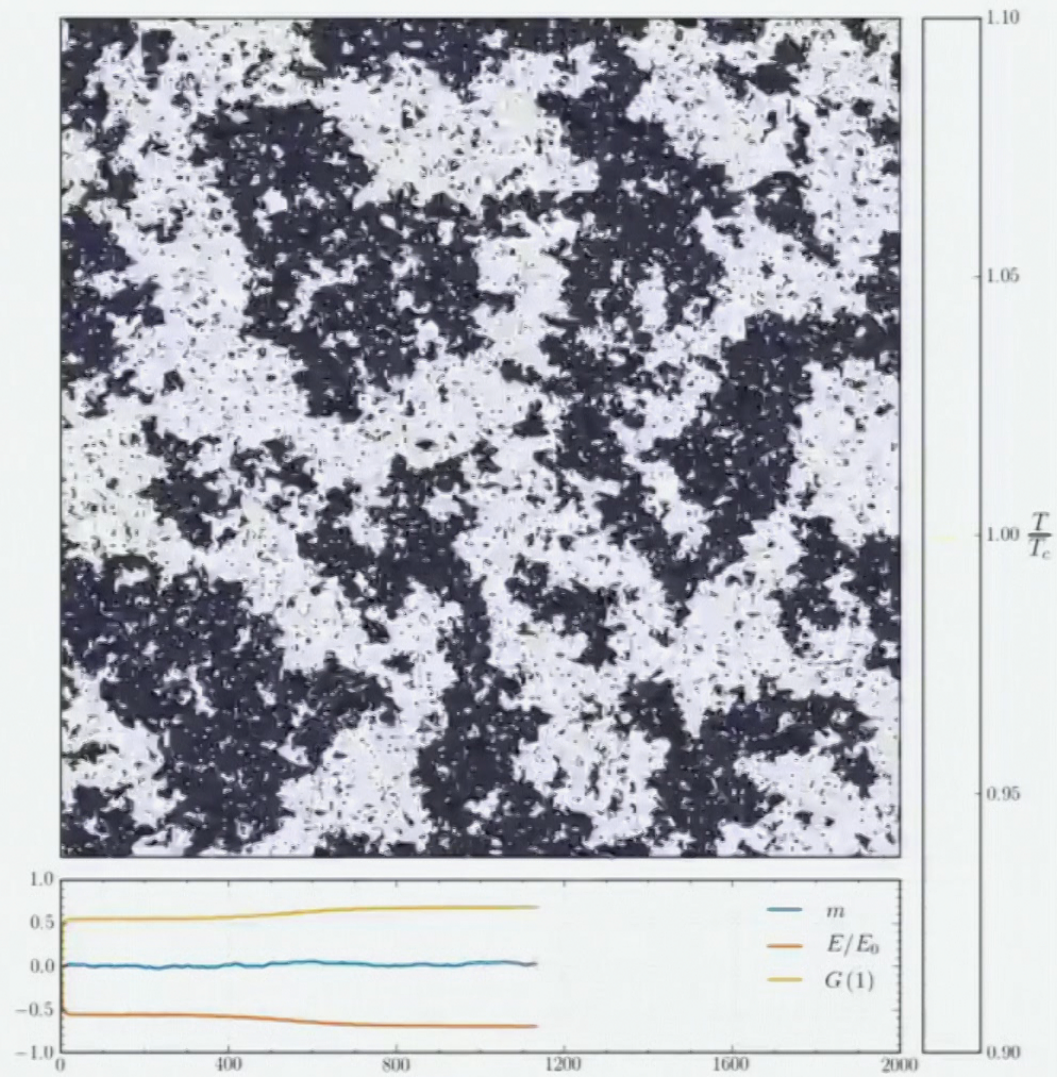
## Ising Model Simulation



0:01 / 1:40









Going with the flow | plus.maths.org - Mozilla Firefox


https://plus.maths.org/content/going-flow-0

about Plus support Plus Plus sponsors subscribe to Plus terms of use

+ plus magazine ...living mathematics

Home Articles News Packages Podcasts Puzzles Reviews Videos Login

Search

 **Going with the flow**  
By Marianne Freiberger and Rachel Thomas

Submitted by Marianne on March 27, 2014

*In the [last article](#) we saw that asymptotic freedom allowed the strong force that binds nuclei together to be described by a quantum field theory. But the perturbative calculations only worked at high energies when the strong coupling constant becomes small. Similarly, it seemed that quantum electrodynamics, the theory that described the interaction of light and matter, only worked at sufficiently low energies. If they did not work at all energy scales, how could these be thought of as valid theories? What is a valid theory, anyway? It was clear that quantum field theory needed some new ideas.*


## Fractal physics

We are used to thinking of physics as happening at very distinct scales. Many of our most successful theories give accurate predictions for a certain range of parameters, which equate to a particular length scale at which we are observing the physical phenomenon. Newton's theory of gravity, and more generally, Newtonian mechanics, is incredibly successful at describing the everyday world around us, according to any observation in a range centred roughly on a human scale. However, Newton's theories fail at the cosmological and relativistic scales, for very massive objects and those moving close to the speed of light. Here Einstein's theory of general relativity provides an accurate description of the phenomenon. And as for the smallest scales, at the subatomic level of particle physics, we are yet to find a good theory for describing the effects of gravity. Not least because at this level the effects of gravity are minuscule and the data is hard to get.


Although the same fundamental force is at play in all these scales, the phenomena we observe appear very different. We are comfortable with using the theory that is most effective at a particular scale, ignoring the contributions that are significantly at bigger or smaller scales.

Indeed the success of many of our theories depends on this isolation of the physics to just a

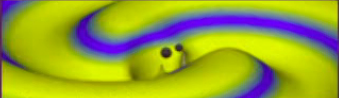
**Maths in a minute**

  
Want facts and want them fast? Our Maths in a minute series explores key mathematical concepts in just a few words.

**Maths in a minute: Euler's identity**

  
Here's a quick introduction to the beauty queen amongst mathematical formulas.

**Gravitational waves get Nobel Prize**

  
This year's Nobel Prize in Physics has gone to Rainer Weiss, Barry Barish and Kip Thorne for their role in the detection of gravitational waves.

**Maths on a beat: Young**




Going with the flow | plus.maths.org - Mozilla Firefox

https://plus.maths.org/content/going-flow-0

Most Visited 0 - 1511.02536.pdf Gmail HEP - INSPIRE-HEP Home | PSI Student P... Author Query Results Welcome to the Perim... epost - Your ONE onli... The International Soci... Entropy | Special Issue... Home | Perimeter Insti...

We are comfortable with using the theory that is most effective at a particular scale, ignoring the contributions that are significantly at bigger or smaller scales.


Indeed the success of many of our theories depends on this isolation of the physics to just a limited range of lengths. Usually, when you describe the motion of water at one scale, you can ignore contributions from physics at much larger or smaller scales. "The interaction of two adjacent water molecules is much the same whether the molecules are in the Pacific Ocean or in a teapot," said Physics Nobel laureate Kenneth Wilson in a [1979 Scientific American article](#). "Equally important, an ocean wave can be described quite accurately as a disturbance of a continuous fluid, ignoring entirely the molecular structure of the liquid." If you had to take into account the motion of every water molecule in the ocean, a theory of ocean waves would be impossible.



Kenneth Wilson.


However there are some phenomena where you have to remove such blinkers and consider the physics from many scales at the same time. Indeed, these phenomena are exactly characterised by behaviour occurring at all these scales simultaneously. One example is what happens at a second order phase transition. The phase transitions we are used to — ice melting to liquid water, water boiling to steam — are all first order phase transitions. There is a dramatic difference in the density of the different phases: in our example of water, any of us can easily tell the difference between ice, water and steam.

But when water and steam are at 374 degrees Celsius and at 218 atmospheres of pressure the distinction between the phases becomes murky. "At this critical point the distinction between water and steam disappears, the whole boiling phenomenon vanishes," said Wilson in his [Nobel Prize lecture in 1982](#). "One finds bubbles of steam and drops of water intermixed at all size scales from macroscopic, visible sizes down to atomic scales." The presence of drops and bubbles near the micron sizes causes the water and steam to appear milky, leading to this phenomenon being called critical opalescence. (You can see why in [this movie](#).) At this point, larger bubbles of gas contain smaller drops of water, that contain smaller bubbles of gas, that contain smaller drops of water, and so on. This self-similarity occurs no matter how much you zoom in on the mixture (until you finally hit the atomic level) in an analogous way to the similarity you find no matter how far you zoom in on a fractal image, like the Mandelbrot set below.



gravitational waves.


**Maths on a boat: Young researchers at the Heidelberg Laureate Forum**



Young researchers at this year's Heidelberg Laureate Forum tell us about their work.


Photo © Heidelberg Laureate Forum Foundation / Mueck - 2017.

**Maths in a minute: The two envelopes problem**



What's wrong with this probability argument?

**Happy Ada Lovelace day!**



Celebrate female mathematicians with these articles and videos.

**Tags**

- particle physics
- fundamental forces
- quantum field theory
- quantum electrodynamics
- asymptotic freedom

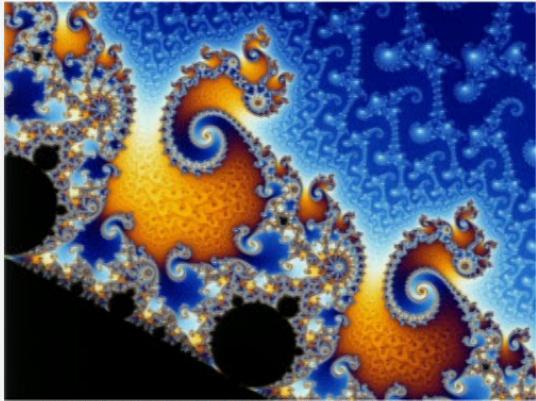


Going with the flow | plus.maths.org - Mozilla Firefox

https://plus.maths.org/content/going-flow-0

boiling to steam — are all first order phase transitions. There is a dramatic difference in the density of the different phases: in our example of water, any of us can easily tell the difference between ice, water and steam.

But when water and steam are at 374 degrees Celsius and at 218 atmospheres of pressure the distinction between the phases becomes murky. "At this critical point the distinction between water and steam disappears, the whole boiling phenomenon vanishes," said Wilson in his [Nobel Prize lecture in 1982](#). "One finds bubbles of steam and drops of water intermixed at all size scales from macroscopic, visible sizes down to atomic scales." The presence of drops and bubbles near the micron sizes causes the water and steam to appear milky, leading to this phenomenon being called critical opalescence. (You can see why in [this movie](#).) At this point, larger bubbles of gas contain smaller drops of water, that contain smaller bubbles of gas, that contain smaller drops of water, and so on. This self-similarity occurs no matter how much you zoom in on the mixture (until you finally hit the atomic level) in an analogous way to the similarity you find no matter how far you zoom in on a fractal image, like the Mandelbrot set below.



The self-similarity at all scales of the water at the critical point is analogous to that seen in fractals, such as this Mandelbrot set, created by [Wolfgang Beyer](#). Of course the positions of the features of the Mandelbrot set are entirely determined by the mathematical function that generates it. Whereas the exact positions of the water droplets and bubbles of steam are down to chance and statistical mechanics. You can read more in the article [Unveiling the Mandelbrot set](#).

## The scales fall from our eyes

commons.wikimedia.org/wiki/User:Wolfgangbeyer

behaviour can be seen in materials such as iron, where the second order phase transition occurs at the

What's wrong with this probability argument?

**Happy Ada Lovelace day!**

Celebrate female mathematicians with these articles and videos.

**Tags**

- particle physics
- fundamental forces
- quantum field theory
- quantum electrodynamics
- asymptotic freedom
- strong nuclear force
- renormalisation

**Tweets by @plusmathsorg**

**Plus Magazine**  
@plusmathsorg

Meet one of our favourite mathematicians, [@ch\\_nira](#), and celebrate Black Mathematician Month!  
[plus.maths.org/content/celebr...](#)

Oct 13, 2017

**Plus Magazine**  
@plusmathsorg

This will be a great talk and is a wonderful



Going with the flow | plus.maths.org - Mozilla Firefox

https://plus.maths.org/content/going-flow-0

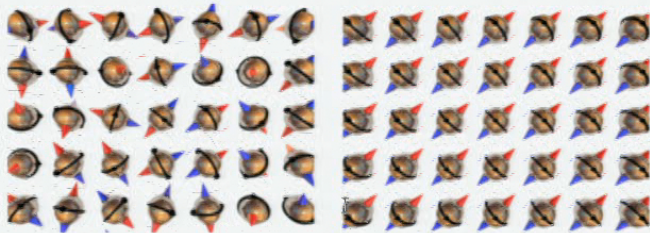
Most Visited 0 - 1511.02536.pdf Gmail HEP - INSPIRE-HEP Home | PSI Student P... Author Query Results PI Welcome to the Perim... epost - Your ONE onli... The International Soci... Entropy | Special Issue... Home | Perimeter Insti...

by Wolfgang Beyer. Of course the positions of the features of the Mandelbrot set are entirely determined by the mathematical function that generates it. Whereas the exact positions of the water droplets and bubbles of steam are down to chance and statistical mechanics. You can read more in the article [Unveiling the Mandelbrot set](#).

## The scales fall from our eyes

Very similar behaviour can be seen in materials such as iron, where the second order phase transition occurs at the *Curie temperature*. Above this temperature, 771-degrees Celsius, iron loses its inherent magnetism. As the iron cools below this temperature it will spontaneously become magnetic, the magnetic field it creates becoming stronger the cooler the material.

Magnetic behaviour is due to the tiny magnetic fields produced by individual electrons in the material. It can be modelled as follows. If you think of each electron in iron as arranged in a lattice, then the material is magnetic if more than half of the electron spins pointed the same way, either up or down. The more electrons point the same way, the stronger the magnetic field. Electrons in iron tend to align themselves with their neighbours, creating a magnet, as being similarly aligned takes less energy. As the temperature of the material increases, the electrons have more energy to overcome this natural tendency. They jiggle about more, making them more likely to randomly flip in direction, explaining why the magnetism decreases as the temperature rises. (This trade-off between the low-energy ordered state and the random jiggling at higher energies lies at the heart of statistical mechanics. You can read more in [Secret symmetry and the Higgs boson](#).)



Left: Schematic depiction of the atoms in a ferromagnet. Left: The symmetrical phase above the Curie temperature where all the magnetic fields of the atoms are randomly oriented. Right: The asymmetrical phase below the Curie temperature where the magnetic fields of all the atoms are aligned in the same direction. Image: Nick Mee, taken from his [Plus](#) article [about the Higgs boson](#).

https://plus.maths.org/content/secret-symmetry-and-higgs-boson-part-1

This explains the gradual loss of magnetism with increased temperature. However, to understand the sudden loss of

Mathematician Month!  
plus.maths.org/content/celebr...  
Oct 13, 2017

Plus Magazine  
@plusmathsorg  
This will be a great talk and is a wonderful book!  
Embed View on Twitter

UNIVERSITY OF CAMBRIDGE

ENG US 10:53 AM 2017-10-17




Going with the flow | plus.maths.org - Mozilla Firefox

https://plus.maths.org/content/going-flow-0

majority pointing the other way, and this swapping of polarity continues for smaller and smaller blocks all the way down to individual electrons on the lattice. "Thus an ocean of spins that are mostly up may have within it an island of spins that are mostly down, which in turn surrounds a lake of up spins with an islet of down spins," as [Wilson](#) poetically put it. "The progression continues to the smallest possible scale: a single spin."

The need to account for behaviour at so many different scales made a mathematical description of the physics in second order phase transitions, also called *critical points* (such as critical opalescence and a ferromagnet's Curie temperature) incredibly complex, requiring a huge number of variables to characterise the state of the material. For example, to describe the behaviour of iron at the Curie temperature, you apparently would need to know the direction of the spin of every electron in the material.

In 1966 [Leo Kadanoff](#), an American condensed matter physicist, suggested a very intuitive averaging process as a way to overcome this difficulty. Kadanoff started with the lattice of electrons described above, with a spacing of, say,  $L=1$  unit between the electrons. He then reduced the complexity of the problem by dividing the lattice into  $3 \times 3$  blocks by treating each group of 9 blocks as a single element. The spin of this element was defined to be the same as the majority of spins of the electrons it contained. If the spin of the majority of the electrons in this block pointed up, then the block would be represented by single spin also pointing up, and vice versa. This defined a new lattice of spins, with elements (the blocks) spaced  $L$  units apart. Then he defined larger blocks containing  $3 \times 3$  of each of these elements in the same way, producing a lattice of spins with elements  $L$  units apart. This process of using an increasingly coarse-grained picture of the system could be repeated. If you think of the spacing of the lattice as analogous to the "degree of resolution" on the microscope that you are using to observe the material, then Kadanoff's block spin process was like zooming out, again and again.

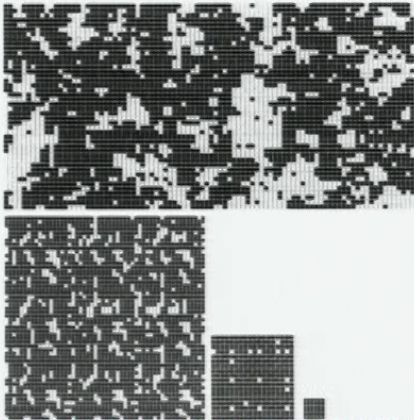





Going with the flow | plus.maths.org - Mozilla Firefox

https://plus.maths.org/content/going-flow-0

elements (the blocks) spaced  $L$  units apart. Then he defined larger blocks containing  $3 \times 3$  of each of these elements in the same way, producing a lattice of spins with elements  $L$  units apart. This process of using an increasingly coarse-grained picture of the system could be repeated. If you think of the spacing of the lattice as analogous to the "degree of resolution" on the microscope that you are using to observe the material, then Kadanoff's block spin process was like zooming out, again and again.



The progression Kadanoff's block spin method for a metal below the Curie temperature. The top lattice is the original metal below the Curie temperature with black indicating the spin of the electron pointing up, white indicating the spin is pointing down. The lower lattices showing the subsequent steps in the averaging process, with the up spins (black squares) becoming dominant at larger scales.



The progression of the block spin method for a metal above the Curie temperature. The top lattice is the original metal, with the lower lattices showing the subsequent steps in the averaging process. Above the Curie temperature no direction of spin is dominant on any scale.

For example, suppose we applied Kadanoff's approach to a piece of Iron below the Curie temperature (the images above on the left). Then, although there might be some differences in alignment at small scales (the top lattice), as we zoom out (and  $L$ , the distance between the units in the lattice, grows) these differences will average out (the lower lattices). For large values of  $L$  the spins of all these larger blocks will point in the same directions. If we instead use this approach for a piece of Iron that is above the Curie temperature (the images on the right), the differences in alignment of the blocks will remain at every length scale, from individual electrons to the largest blocks. Kadanoff's approach simplified the mathematics, so that it was finally possible to predict the behaviour of these materials, accurately matching the data measured in experiments.

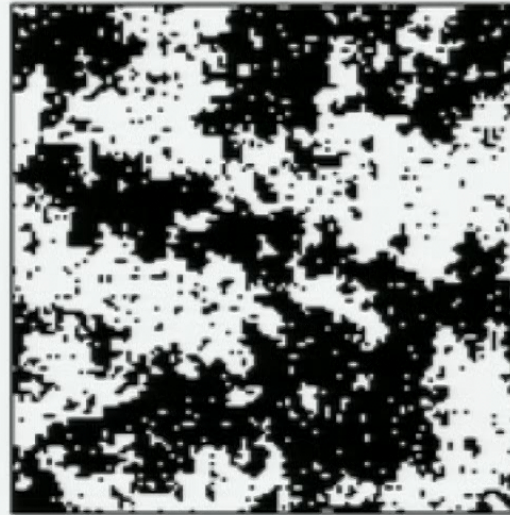


# ISING MODEL: ordering

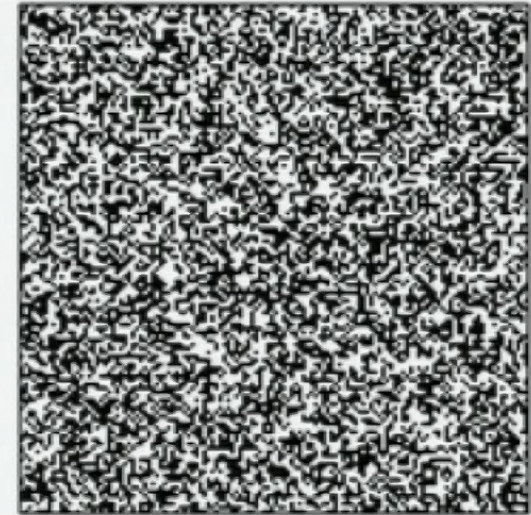
At hot temperature the spins are **randomly configured**, at low temperature they are close to an entirely **ordered** state, and at critical temperature they have a **fractal configuration**.



$T < T_c$



$T = T_c$

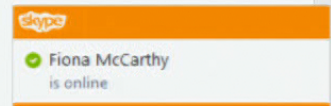


$T > T_c$

At  $T_c$ : scale invariant (fractal) structure

<https://brainnetworkdynamics.wordpress.com/research/criticality/>

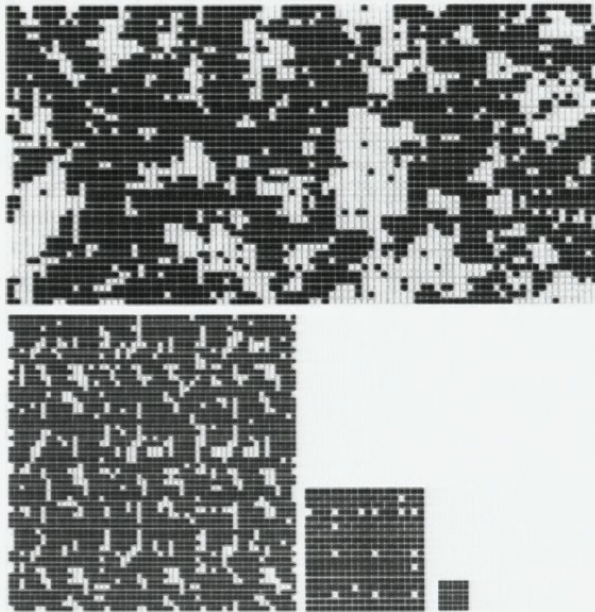
See also: <https://www.youtube.com/watch?v=kjwKgpQ-l1s>





# Block Spin RG

**Below  $T_c$  ( $M=M_0$ )**



**Above  $T_c$  ( $M=0$ )**



Block spin transformations for just under the Curie temperature

<https://plus.maths.org/content/going-flow-0>



## SPIN-SPIN CORRELATION FUNCTION

$$\begin{aligned} G_{ij} &= (\sigma_i - \langle \sigma_i \rangle) (\sigma_j - \langle \sigma_j \rangle) \\ &= \langle \sigma_i \sigma_j \rangle - \langle \sigma_i \rangle \langle \sigma_j \rangle \end{aligned}$$

## SPIN-SPIN CORRELATION FUNCTION

$$\begin{aligned} \bullet G_{ij} &= (\sigma_i - \langle \sigma_i \rangle) (\sigma_j - \langle \sigma_j \rangle) \\ &= \langle \sigma_i \sigma_j \rangle - \langle \sigma_i \rangle \langle \sigma_j \rangle \end{aligned}$$

• TO CALCULATE

$$H = -\frac{1}{2} \sum_{i,j} J_{ij} \sigma_i \sigma_j$$



$$\begin{aligned} G_{ij} &= (\sigma_i - \langle \sigma_i \rangle) (\sigma_j - \langle \sigma_j \rangle) \\ &= \langle \sigma_i \sigma_j \rangle - \langle \sigma_i \rangle \langle \sigma_j \rangle \end{aligned}$$

• TO CALCULATE

$$H = -\frac{1}{2} \sum_{i,j} J_{ij} \sigma_i \sigma_j - \sum_i B_i \sigma_i$$

$$G_{ij} = -T \frac{\partial^2 F}{\partial B_i \partial B_j} \bigg|_{B=0}$$

$$G_{ij} = \dots = \dots = \frac{a^3 T}{2\pi\theta e} \frac{e^{-|\vec{n}_i - \vec{n}_j|/\xi}}{|\vec{n}_i - \vec{n}_j|^{d-2}}$$



$$G_{ij} = \dots = \dots \stackrel{\text{MFT}}{=} \frac{a^3 T}{2\pi\theta\epsilon} \frac{e^{-|\vec{n}_i - \vec{n}_j|/\xi}}{|\vec{n}_i - \vec{n}_j|^{d-2}}$$

$$G_{ij} = \dots = \dots \stackrel{\text{MFT}}{=} \frac{a^3 T}{2\pi\theta\epsilon} \frac{e^{-|\vec{r}_i - \vec{r}_j|/\xi}}{|\vec{r}_i - \vec{r}_j|^{d-2}}$$

$$\xi = \sqrt{\frac{\partial \epsilon}{2(T-T_c)}}$$

CORRELATION LENGTH



$$G_{ij} = \dots = \dots \stackrel{\text{HFT}}{=} \frac{a^3 T}{2\pi\theta\epsilon} \frac{e^{-|\vec{r}_i - \vec{r}_j|/\xi}}{|\vec{r}_i - \vec{r}_j|^{d-2}}$$

$$\xi = \sqrt{\frac{\chi}{2(T-T_c)}}$$

CORRELATION LENGTH

MORE GENERALLY

$$G(r) \propto \frac{e^{-r/\xi}}{r^{d-2+\eta}}$$

$$G_{ij} = \dots = \dots \stackrel{\text{HFT}}{=} \frac{a^3 T}{2\pi\theta\epsilon} \frac{e^{-|\vec{r}_i - \vec{r}_j|/\xi}}{|\vec{r}_i - \vec{r}_j|^{d-2}}$$

$$\xi = \sqrt{\frac{\partial}{\partial(T-T_c)}}$$

CORRELATION LENGTH

MORE GENERALLY

$$G(r) \propto \frac{e^{-r/\xi}}{r^{d-2+\eta}}$$

$$\xi \propto |t|^{-\nu}$$



$$G_{ij} = \dots = \dots \stackrel{\text{HFT}}{=} \frac{a^3 T}{2\pi\theta\epsilon} \frac{e^{-|\vec{r}_i - \vec{r}_j|/\xi}}{|\vec{r}_i - \vec{r}_j|^{d-2}}$$

$$\xi = \sqrt{\frac{\partial}{\partial(T-T_c)}}$$

MORE GENERALLY

$$G(r) \propto \frac{e^{-r/\xi}}{r^{d-2+\eta}}$$

$$\xi \propto |t|^{-\nu}$$

V... CRIT. EXP FOR CORRELATION LENGTH

CORRELATION LENGTH

$$G_{ij} = \dots = \dots \stackrel{\text{HFT}}{=} \left( \frac{a^3 T}{2\pi\theta\epsilon} \right) \frac{e^{-|\vec{r}_i - \vec{r}_j|/\xi}}{|\vec{r}_i - \vec{r}_j|^{d-2}} \quad \xi = \sqrt{\frac{\partial}{2(T-T_c)}}$$

MORE GENERALLY

$$G(r) \propto \frac{e^{-r/\xi}}{r^{d-2+\eta}}$$

$$\xi \propto |t|^{-\nu}$$

CORRELATION LENGTH

V. CRIT. EXP FOR CORRELATION LENGTH

$$\text{HFT: } \nu = \frac{1}{2}$$



$$G_{ij} = \dots = \text{MFT} = \frac{a^3 T}{2\pi\theta\epsilon} \frac{e^{-|\vec{r}_i - \vec{r}_j|/\xi}}{|\vec{r}_i - \vec{r}_j|^{d-2}} \quad \xi = \sqrt{\frac{\chi}{2(T-T_c)}}$$

MORE GENERALLY

$$G(r) \propto \frac{e^{-r/\xi}}{r^{d-2+\gamma}}$$

$$\xi \propto |t|^{-\nu}$$

$\nu$ ... CRIT. EXP FOR CORRELATION LENGTH

$$\text{MFT: } \nu = \frac{1}{2}$$

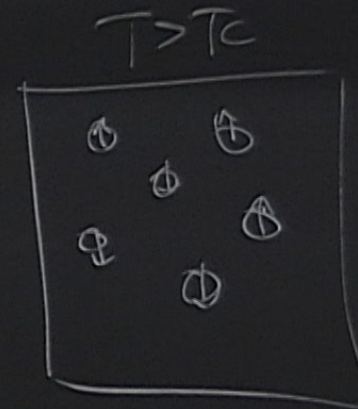
$\gamma$ ... ANOMALOUS DIMENSION

CORRELATION LENGTH

AT  $T=T_c$   $\xi$  DIVERGES  
FRACTAL STRUCTURE  
SYSTEM DESCRIBED  
BY CFT

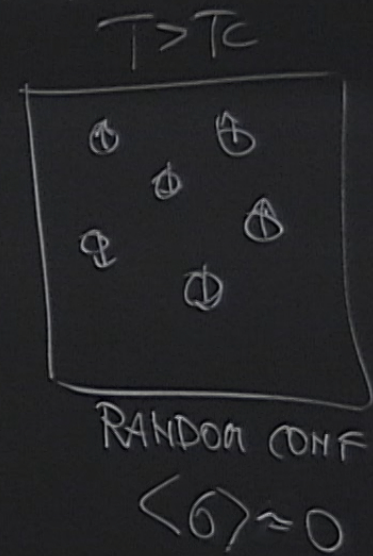


AT  $T=T_c$   $\xi$  DIVERGES  
FRACTAL STRUCTURE  
SYSTEM DESCRIBED  
BY CFT.

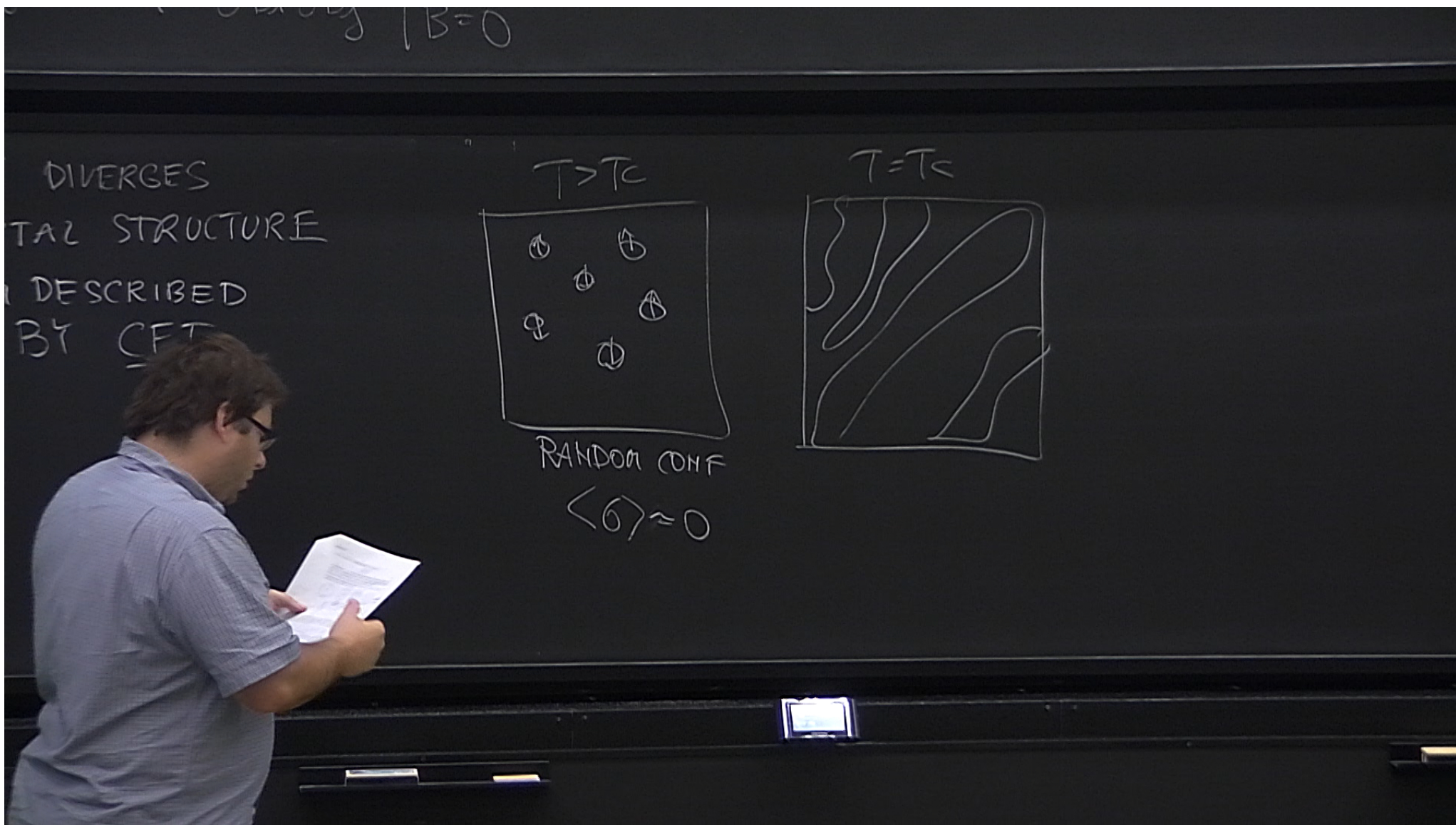




AT  $T=T_c$   $\xi$  DIVERGES  
FRACTAL STRUCTURE  
SYSTEM DESCRIBED  
BY CFT



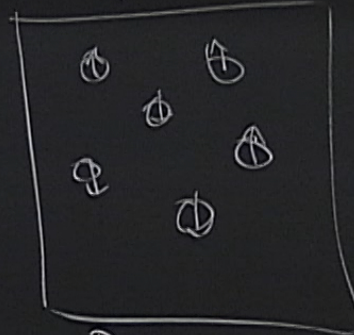






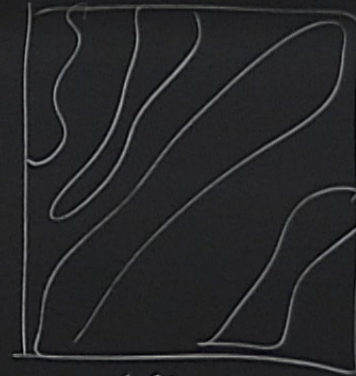
DIVERGES  
TAL STRUCTURE  
DESCRIBED  
BY CFT.

$T > T_c$



RANDOM CONF  
 $\langle \phi \rangle \approx 0$

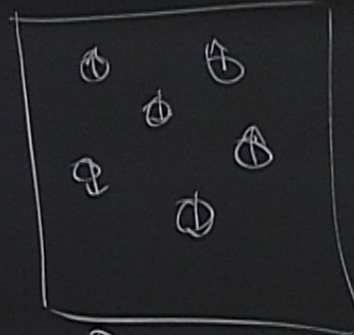
$T = T_c$



$\langle \phi \rangle = 0$   
FRACTAL STRUCTURE

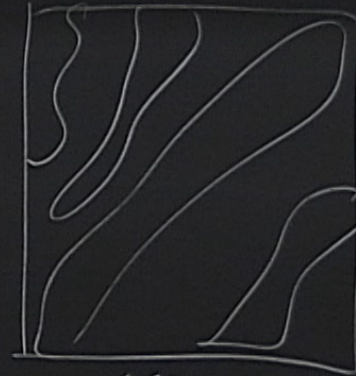
DIVERGES  
TAL STRUCTURE  
DESCRIBED  
BY CFT.

$$T > T_c$$



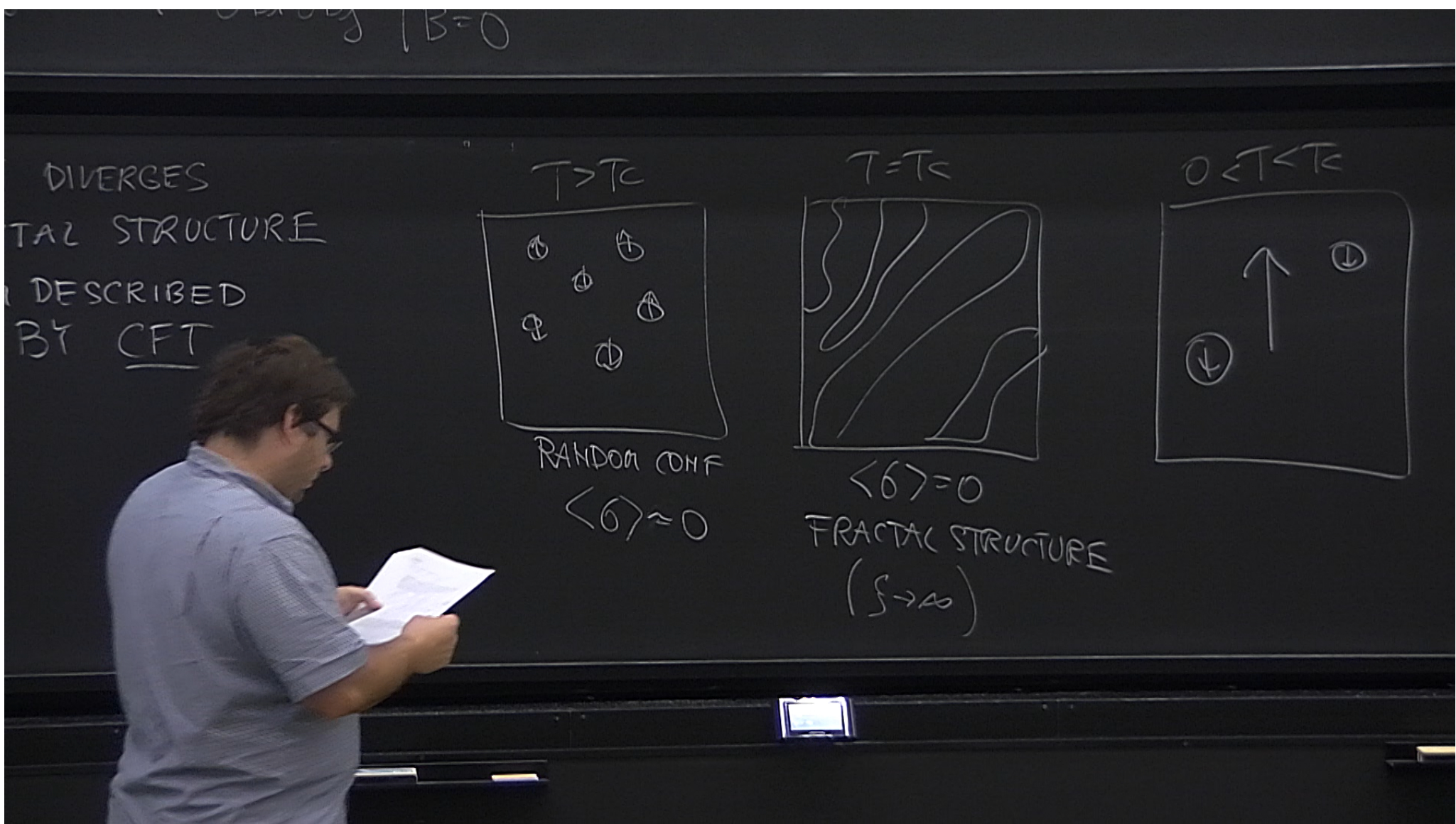
RANDOM CONF  
 $\langle \phi \rangle \approx 0$

$$T = T_c$$

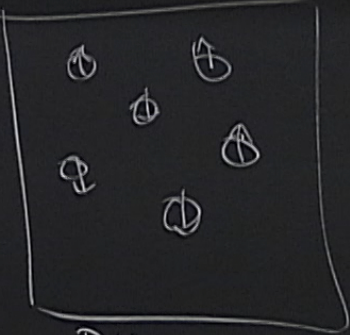


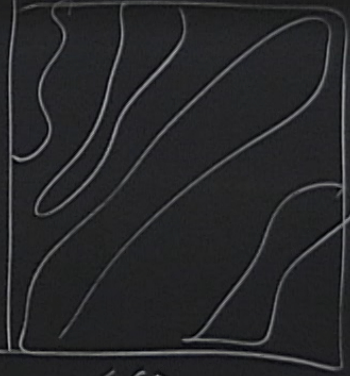
$\langle \phi \rangle = 0$   
FRACTAL STRUCTURE  
( $\xi \rightarrow \infty$ )

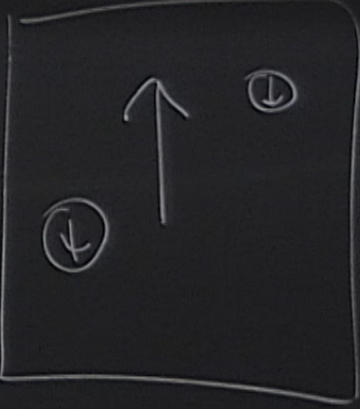




DIVERGES  
TAL STRUCTURE  
DESCRIBED  
BY CFT

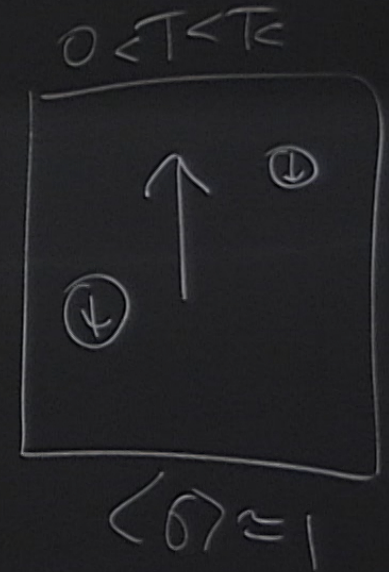
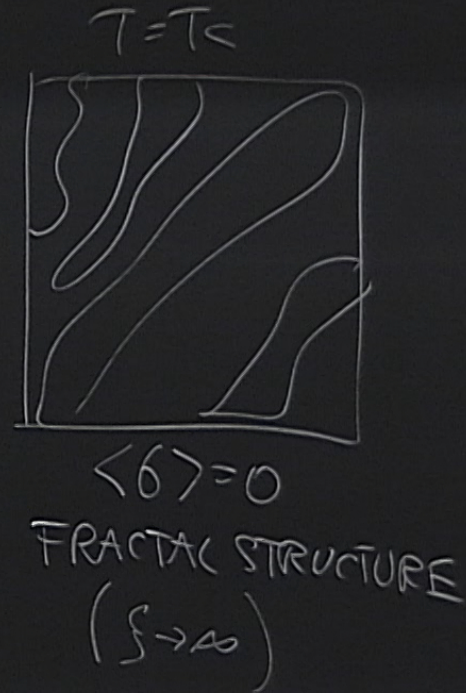
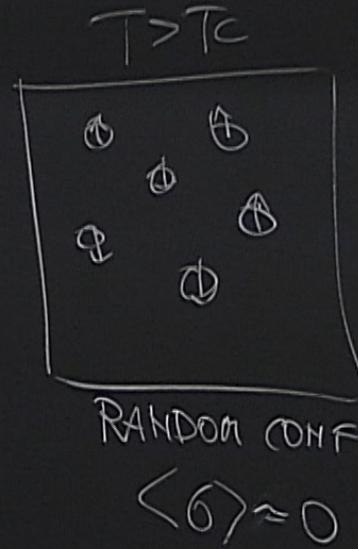
$T > T_c$   
  
RANDOM CONF  
 $\langle \phi \rangle = 0$

$T = T_c$   
  
 $\langle \phi \rangle = 0$   
FRACTAL STRUCTURE  
( $S \rightarrow \infty$ )

$0 < T < T_c$   


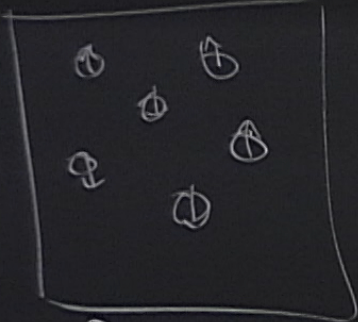


DIVERGES  
 TAL STRUCTURE  
 DESCRIBED  
 BY CFT.



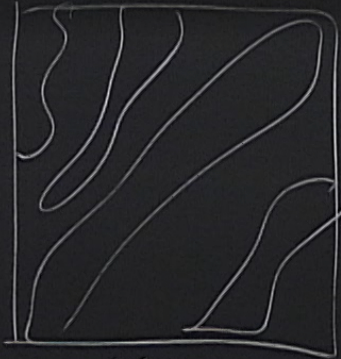
$$B=0$$

$$T > T_c$$



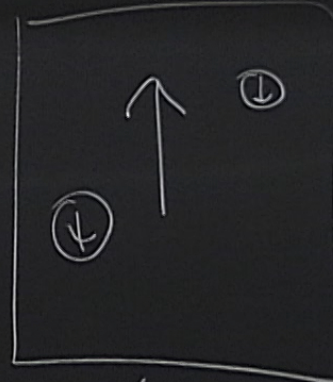
RANDOM CONF  
 $\langle \sigma \rangle \approx 0$

$$T = T_c$$



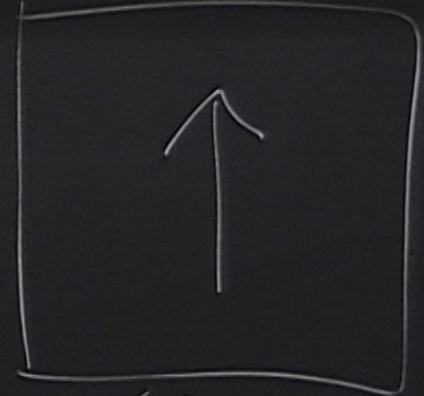
$\langle \sigma \rangle = 0$   
FRACTAL STRUCTURE  
( $\xi \rightarrow \infty$ )

$$0 < T < T_c$$



$\langle \sigma \rangle \approx 1$

$$T = 0$$

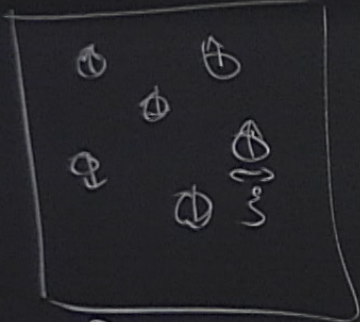


$\langle \sigma \rangle = 1$



$$B=0$$

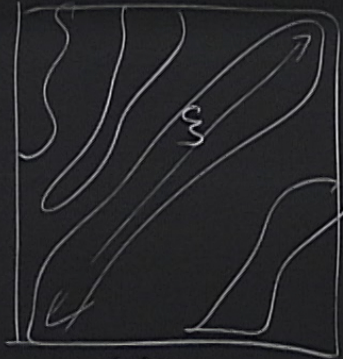
$$T > T_c$$



RANDOM CONF

$$\langle \sigma \rangle \approx 0$$

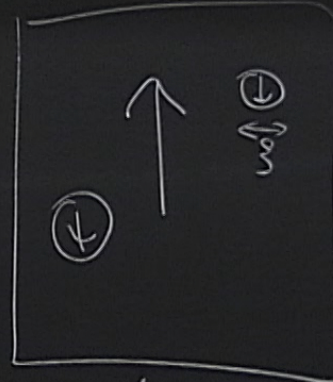
$$T = T_c$$



$$\langle \sigma \rangle = 0$$

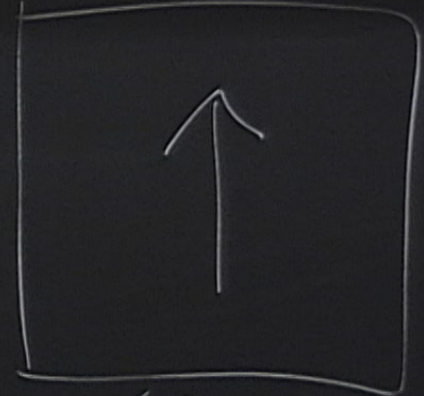
FRACTAL STRUCTURE  
( $\xi \rightarrow \infty$ )

$$0 < T < T_c$$



$$\langle \sigma \rangle \approx 1$$

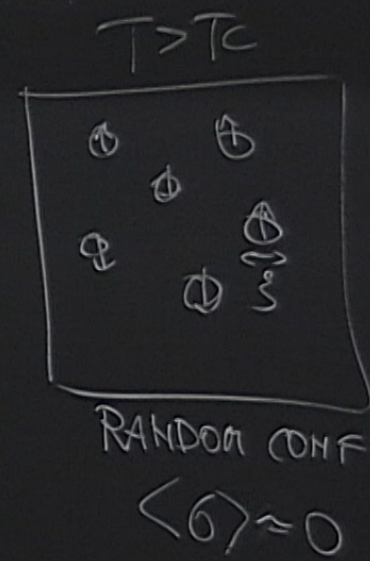
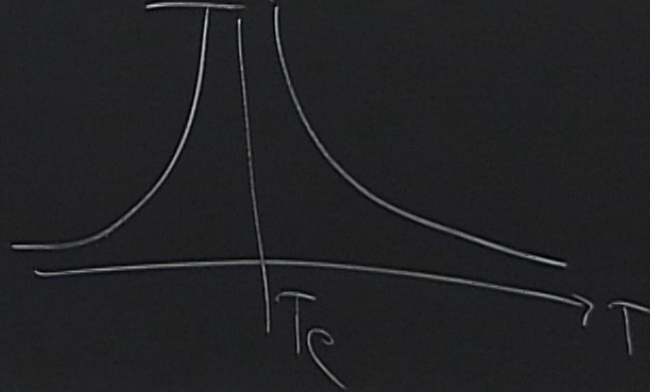
$$T = 0$$



$$\langle \sigma \rangle = 1$$



AT  $T=T_c$   $\xi$  DIVERGES  
 FRACTAL STRUCTURE  
 SYSTEM DESCRIBED  
 BY CFT





$$n^{d-2+\gamma}$$

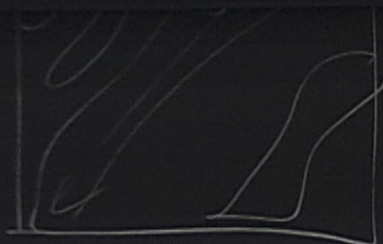
$$\xi \propto |t|^{-\nu}$$

$\nu$ ... CRIT. EXP FOR CORRELATION LENGTH

$$\text{MFT: } \nu = \frac{1}{2}$$

$\gamma$ ... ANOMALOUS DIMENSION

$$\text{MFT: } \boxed{\gamma = 0}$$



$$\langle \phi \rangle = 0$$

FRACTAL STRUCTURE  
( $\xi \rightarrow \infty$ )



$$\langle \phi \rangle \approx 1$$



$T_c \rightarrow T$

( $S \rightarrow \infty$ )

MFT:  $\alpha=0, \beta=\frac{1}{2}, \eta=1, \delta=3, \gamma=0, \nu=\frac{1}{2}$ .



MFT:  $\alpha=0, \beta=\frac{1}{2}, \eta=1, \delta=3, \gamma=0, \nu=\frac{1}{2}$ .  
ONLY 2 ARE INDEPENDENT.



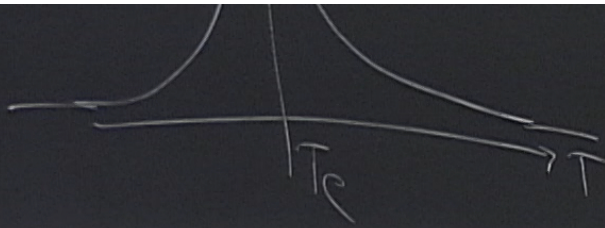
$T_c \rightarrow T$        $(S \rightarrow A)$

MFT:  $\alpha=0, \beta=\frac{1}{2}, \eta=1, \bar{\sigma}=3, \eta=0, \nu=\frac{1}{2}$ .  
 ONLY 2 ARE INDEPENDENT.

• YESTERDAY.

$$Z = \sum_{\{\sigma_i\}} e^{\frac{1}{2T} \sum_{ij} J_{ij} \sigma_i \sigma_j}$$





RANDOM CONF  
 $\langle \phi \rangle \approx 0$

$\langle \phi \rangle = 0$   
 FRACTAL STRUCTURE  
 $(\xi \rightarrow \infty)$

$\langle \phi \rangle \approx 1$

$\langle \phi \rangle$

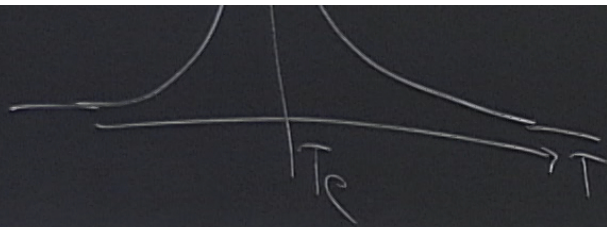
MFT:  $\alpha = 0, \beta = \frac{1}{2}, \gamma = 1, \delta = 3, \eta = 0, \nu = \frac{1}{2}$ .

ONLY 2 ARE INDEPENDENT.

TERDAY.

$$Z = \sum_{\{\phi_i\}} e^{\frac{1}{2T} \sum_{ij} J_{ij} \phi_i \phi_j} = \frac{1}{\sqrt{\det J}} \left( \frac{1}{2\pi T} \right)^{N/2} \sum_{\{\phi_i\}} \int_{-\infty}^{\infty} d\vec{\phi} e^{-\frac{1}{2T} \vec{\phi}_i J_{ij}^{-1} \vec{\phi}_j + \frac{1}{T} \vec{\phi}_i b_i}$$





RANDOM CONF  
 $\langle \phi \rangle \approx 0$

$\langle \phi \rangle = 0$   
 FRACTAL STRUCTURE  
 $(\xi \rightarrow \infty)$

$\langle \phi \rangle \approx 1$

$\langle \phi \rangle$

MFT:  $\alpha = 0, \beta = \frac{1}{2}, \eta = 1, \delta = 3, \gamma = 0, \nu = \frac{1}{2}$ .

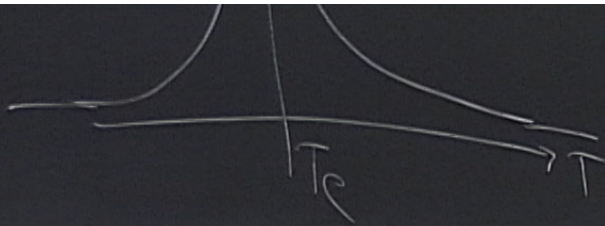
ONLY 2 ARE INDEPENDENT.

TERDAY.

$$Z = \sum_{\{\phi_i\}} e^{\frac{1}{2T} \sum_{ij} J_{ij} \phi_i \phi_j} = \frac{1}{\sqrt{\det J}} (2\pi T)^{N/2} \sum_{\{\phi_i\}} \int_{-\infty}^{\infty} d\phi \ e^{-\frac{1}{2T} \phi_i J_{ij}^{-1} \phi_j + \frac{1}{T} \phi_i b_i}$$

$\checkmark$   
 2-SPIN  
 INTERACTIONS





RANDOM CONF  
 $\langle \phi \rangle \approx 0$

$\langle \phi \rangle = 0$   
 FRACTAL STRUCTURE  
 $(\xi \rightarrow \infty)$

$\langle \phi \rangle \approx 1$

$\langle \phi \rangle$

MFT:  $\alpha = 0, \beta = \frac{1}{2}, \eta = 1, \delta = 3, \gamma = 0, \nu = \frac{1}{2}$ .

ONLY 2 ARE INDEPENDENT.

TERDAY.

$$Z = \sum_{\{\phi_i\}} e^{\frac{1}{2T} \sum_{ij} J_{ij} \phi_i \phi_j} = \frac{1}{\sqrt{\det J}} \frac{1}{(2\pi T)^{N/2}} \sum_{\{\phi_i\}} \int_{-\infty}^{\infty} d\phi \ e^{-\frac{1}{2T} \phi_i J_{ij}^{-1} \phi_j + \frac{1}{T} \phi_i b_j}$$

2-SPIN  
INTERACTIONS



STRUCTURE  
( $\mathcal{S} \rightarrow \mathcal{A}_0$ )

$$\delta = 3, \eta = 0, \nu = \frac{1}{2}.$$

IDENT.

$$= \frac{1}{\sqrt{\det J}} \frac{1}{(2\pi T)^{N/2}} \sum_{\{\phi_i\}} \int_{-\infty}^{\infty} d\vec{\phi} e^{-\frac{1}{2T} \phi_i J_{ij}^{-1} \phi_j + \frac{1}{T} \phi_i \bar{p}_i}$$

INTERACTION OF  
SPINS WITH  
FIELD  $\bar{p}$

ACTIONS



STRUCTURE  
( $\xi \rightarrow \infty$ )

$$\delta = 3, \eta = 0, \nu = \frac{1}{2}.$$

IDENT.

$$= \frac{1}{\sqrt{\det J}} \left( \frac{1}{2\pi T} \right)^{N/2} \sum_{\{\phi_i\}} \int_{-\infty}^{\infty} d\vec{\phi} e^{-\frac{1}{2T} \phi_i J_{ij}^{-1} \phi_j + \frac{1}{T} \phi_i k_i}$$

INTERACTION OF  
SPINS WITH  
FIELD  $\phi$

✓ SUGGESTS  
 $\phi \approx M$

ACTIONS



STRUCTURE  
( $\xi \rightarrow \infty$ )

$$\delta = 3, \eta = 0, \nu = \frac{1}{2}.$$

IDENT.

$$= \frac{1}{\sqrt{\det J}} \frac{1}{(2\pi T)^{N/2}} \sum_{\{\phi_i\}} \int_{-\infty}^{\infty} d\phi \quad e^{-\frac{1}{2T} \phi_i J_{ij}^{-1} \phi_j + \frac{1}{T} \phi_i k_i}$$

INTERACTION OF  
SPINS WITH  
FIELD  $\phi$

SUGGESTS

VARIABLE FIELD.  $\phi \approx M$  CONST

ACTIONS



STRUCTURE  
( $\xi \rightarrow \infty$ )

$$\delta = 3, \eta = 0, \nu = \frac{1}{2}.$$

IDENT.

$$= \left( \frac{1}{\det J} \right)^{1/2} (2\pi T)^{N/2} \sum_{\{G_i\}} \int_{-\infty}^{\infty} d\phi \, e^{-\frac{1}{2T} \phi_i J_{ij}^{-1} \phi_j + \frac{1}{T} G_i \phi_i}$$

INTERACTION OF SPINS WITH FIELD  $\phi$

SUGGESTS

VARIABLE FIELD.  $\phi \approx \mu$  CONST

ACTIONS



$$Z = \int D\varphi e^{-S[\varphi]}, \quad S[\varphi] = \frac{1}{2T} \varphi_i J_{ij}^{-1} \varphi_j - \sum_i \log(2 \cosh(\varphi_i/T))$$



$$Z = \int D\varphi e^{-S[\varphi]}, \quad S[\varphi] = \frac{1}{2T} \varphi_i J_{ij}^{-1} \varphi_j - \sum_i \log(2 \cosh(\varphi_i/T))$$

EXACT.

• SADDLE POINT (WKB) APPROX



$$Z = \int D\psi \ell$$

EXACT.

- SADDLE POINT (WKB) APPROX  
WANT TO MINIMIZE  $S$ .



$$Z = \int D\varphi \mathcal{L}$$

EXACT.

• SADDLE POINT (WKB) APPROX.

WANT TO MINIMIZE  $S$ .

$$\frac{\partial S}{\partial \varphi_i} \Big|_{\varphi_i = \bar{\varphi}_i} = 0$$



$$= \int D\varphi \mathcal{L} \quad \text{EXACT.}$$

WKB POINT (WKB) APPROX.

WANT TO MINIMIZE  $S$ .

$$\frac{\partial S}{\partial \varphi_i} \Big|_{\varphi_i = \bar{\varphi}_i} = 0 = \frac{1}{T} J_{ij}^{-1} \bar{\varphi}_j - \frac{1}{T} \tanh \bar{\varphi}_i / T$$



EXACT.

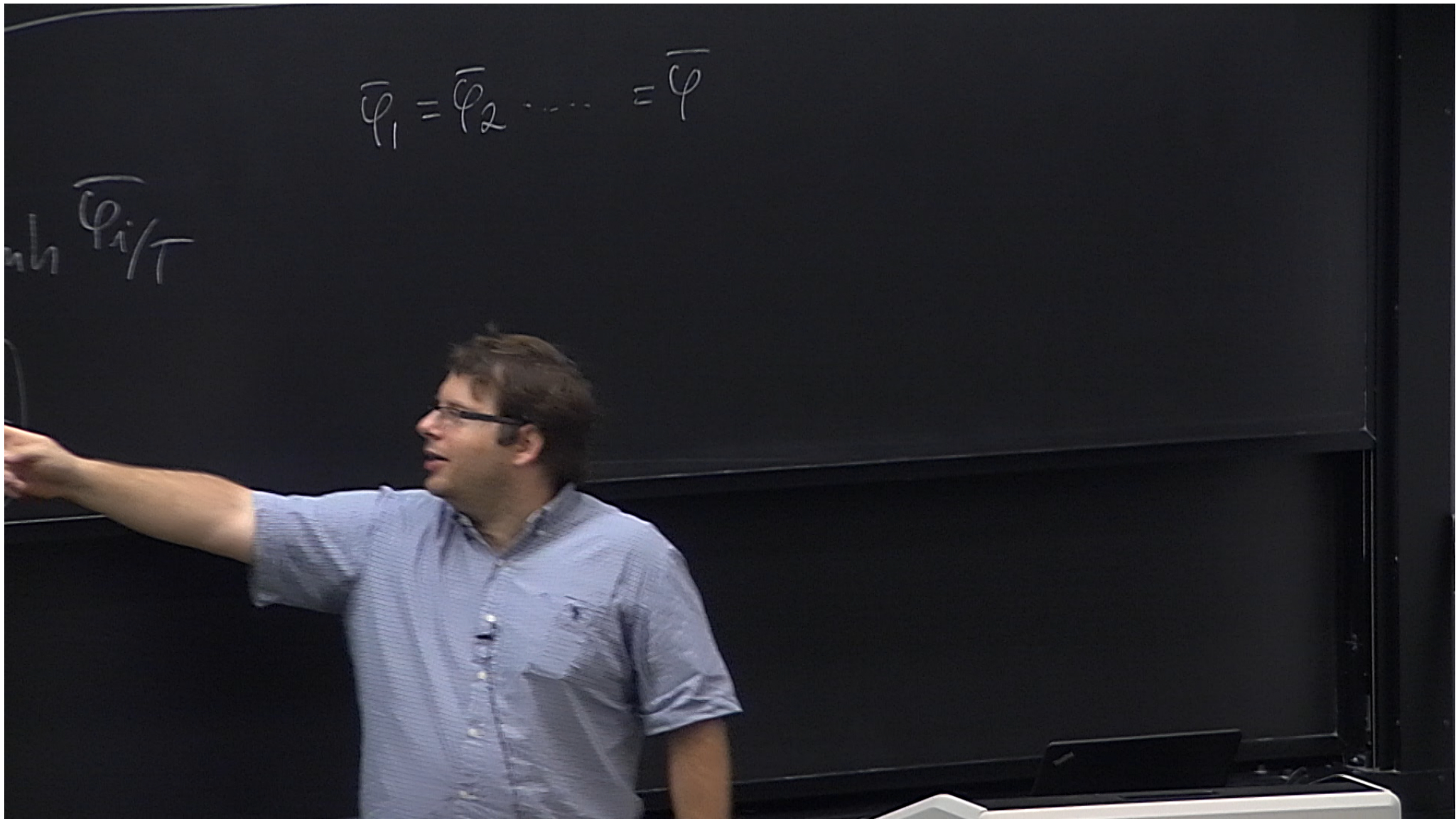
WKB POINT (WKB) APPROX.

WANT TO MINIMIZE  $S$ .

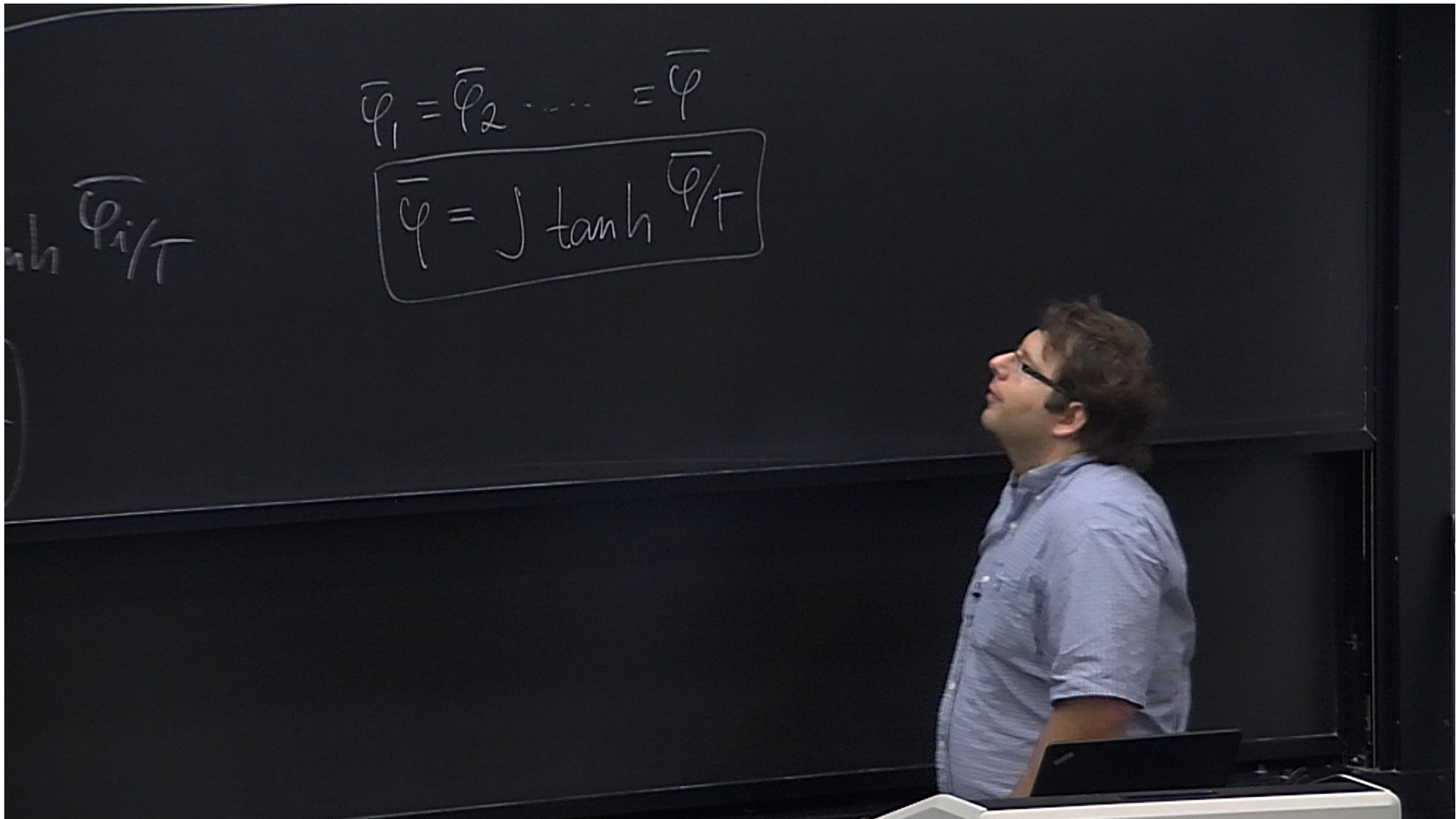
$$\frac{\partial S}{\partial \varphi_i} \bigg|_{\varphi_i = \bar{\varphi}_i} = 0 = \frac{1}{T} J_{ij} \bar{\varphi}_j - \frac{1}{T} \tanh \bar{\varphi}_i / T$$

$$\Rightarrow \boxed{\bar{\varphi}_i = \sum_j J_{ij} \tanh \bar{\varphi}_j / T}$$











$$\bar{\varphi}_1 = \bar{\varphi}_2 \dots = \bar{\varphi}$$

$$\bar{\varphi} = J \tanh \bar{\varphi}/T$$

$$M = \tanh(M/T)$$



$$\bar{\varphi}_1 = \bar{\varphi}_2 \dots = \bar{\varphi}$$

$$\bar{\varphi} = J \tanh \bar{\varphi}/T$$

$$\bar{\varphi} = MJ \dots MFT$$

$$M = \tanh(M/T)$$



$$\bar{\varphi}_1 = \bar{\varphi}_2 \dots = \bar{\varphi}$$

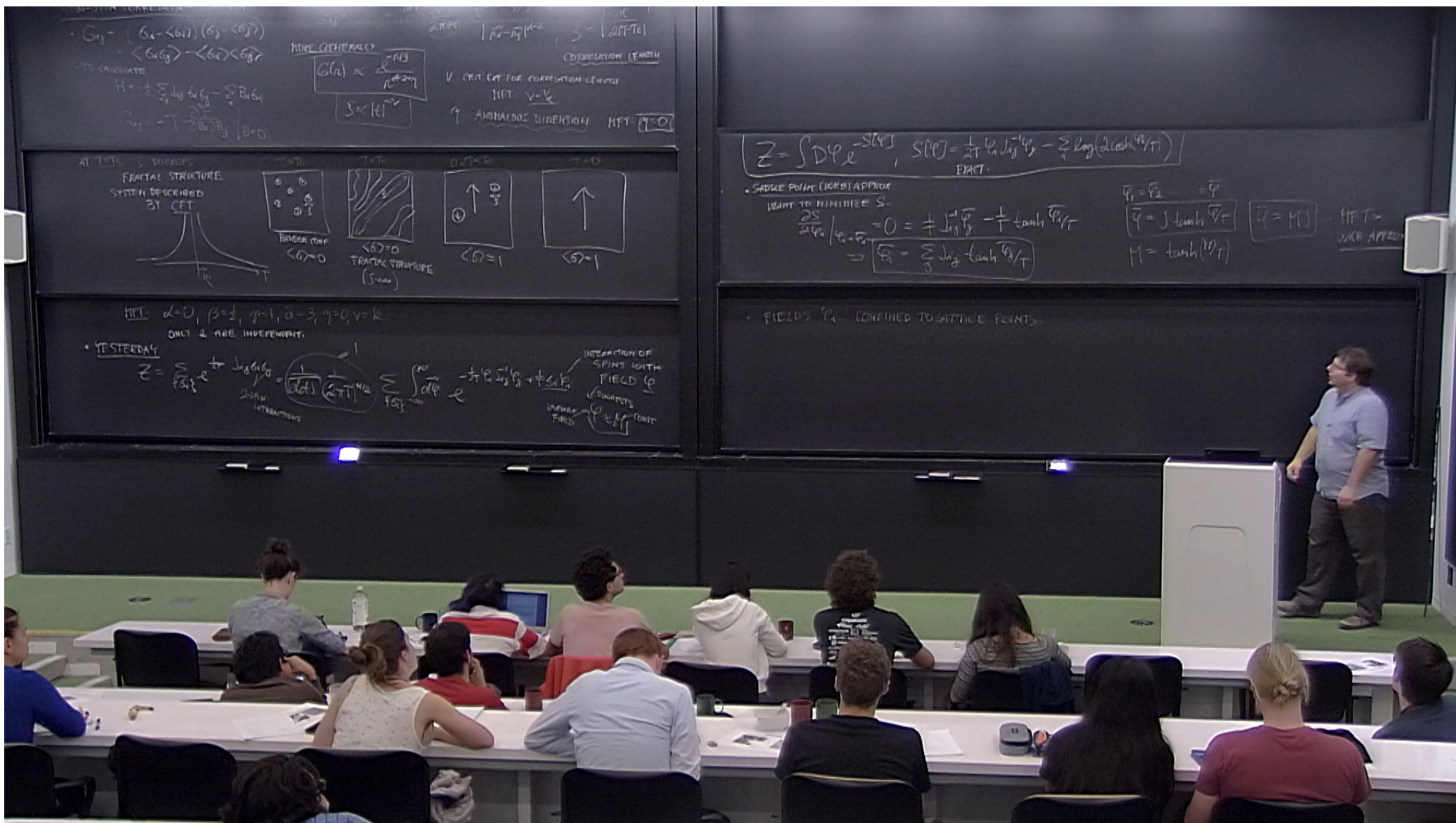
$$\bar{\varphi} = J \tanh \bar{\varphi}/T$$

$$\bar{\varphi} = MJ$$

... MFT  $\approx$   
WKB APPROX.

$$M = \tanh(M/T)$$







- FIELDS  $\mathbb{P}_i$ .. CONFINED TO LATTICE POINTS.



- FIELDS  $\varphi_i$ .. CONFINED TO LATTICE POINTS.
- CLOSE TO CRITICAL TEMPERATURE, WE CAN ZOOM OUT OF THE SYSTEM  
→ CONTINUOUS LIMIT. (LATTICE DISAPPEARS)



• CLOSE TO CRITICAL TEMPERATURE, WE CAN  
ZOOM OUT OF THE SYSTEM

→ CONTINUOUS LIMIT. (LATTICE DISAPPEARS)

$$\bullet \log(2 \cosh(\varphi/T)) \approx \log 2 + \frac{1}{2} \left( \frac{\varphi_i}{T} \right)^2 - \frac{1}{12} \left( \frac{\varphi_i}{T} \right)^4 + \dots$$



• CLOSE TO CRITICAL TEMPERATURE, WE CAN  
ZOOM OUT OF THE SYSTEM

→ CONTINUOUS LIMIT. (LATTICE DISAPPEARS)

$$\bullet \log(2 \cosh(\phi/T)) \approx \cancel{\log 2} + \frac{1}{2} \left( \frac{\phi_i}{T} \right)^2 - \frac{1}{12} \left( \frac{\phi_i}{T} \right)^4 + \dots$$



• CLOSE TO CRITICAL TEMPERATURE, WE CAN  
ZOOM OUT OF THE SYSTEM

→ CONTINUOUS LIMIT. (LATTICE DISAPPEARS)

$$\bullet \log(2 \cosh(\phi/T)) \approx \cancel{\log 2} + \frac{1}{2} \left( \frac{\phi_i}{T} \right)^2 - \frac{1}{12} \left( \frac{\phi_i}{T} \right)^4 + \dots$$

IMPORTANT.



## LATTICE FOURIER TRANSFORM

$$\varphi_i = \frac{1}{N} \sum_{\vec{q}} \varphi(\vec{q}) e^{i\vec{q} \cdot \vec{r}_i}$$

$$J_{ij} = \frac{1}{N} \sum_{\vec{q}}$$



## LATTICE FOURIER TRANSFORM

$$\varphi_i = \frac{1}{N} \sum_{\vec{q}} \varphi(\vec{q}) e^{i\vec{q} \cdot \vec{r}_i}$$

$$J_{ij} = \frac{1}{N} \sum_{\vec{q}} J(\vec{q}) e^{i\vec{q} \cdot (\vec{r}_i - \vec{r}_j)}$$



## LATTICE FOURIER TRANSFORM

$$\varphi_i = \frac{1}{N} \sum_{\vec{q}} \varphi(\vec{q}) e^{i\vec{q} \cdot \vec{r}_i}$$

$$J_{ij} = \frac{1}{N} \sum_{\vec{q}} J(\vec{q}) e^{i\vec{q} \cdot (\vec{r}_i - \vec{r}_j)}$$

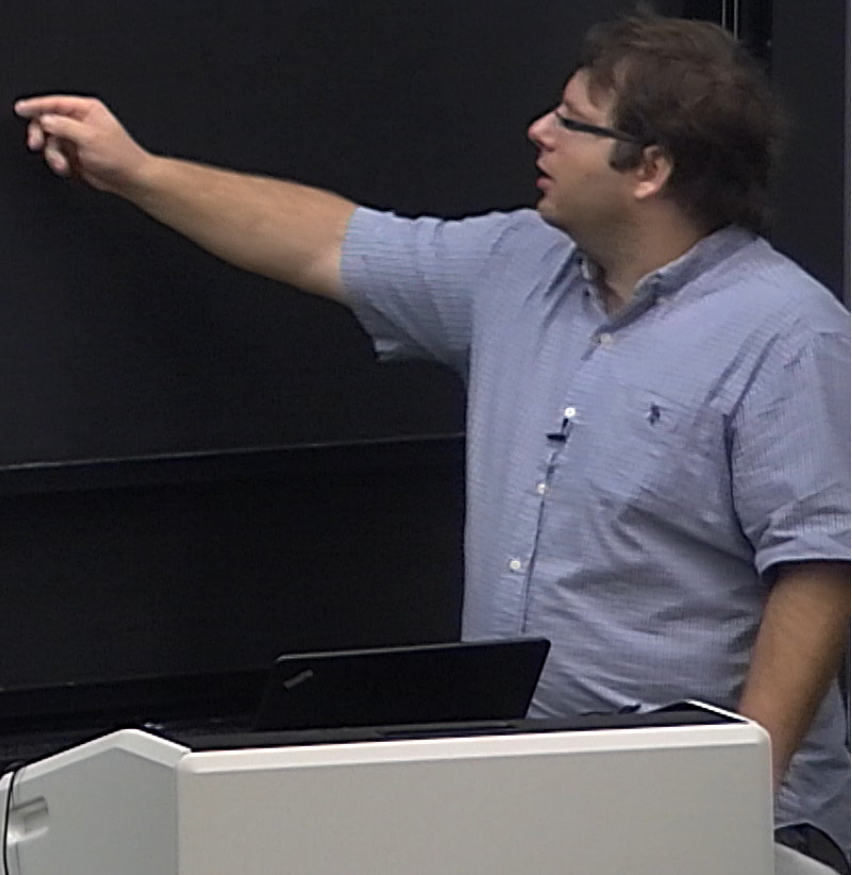


$$\varphi_i = \frac{1}{N} \sum_{\vec{q}} \varphi(\vec{q}) e^{i\vec{q} \cdot \vec{r}_i}$$

$$J_{ij} = \frac{1}{N} \sum_{\vec{q}} J(\vec{q}) e^{i\vec{q} \cdot (\vec{r}_i - \vec{r}_j)}$$

ORTHOGONALITY RELATIONS:

$$\frac{1}{N} \sum_j e^{-i(\vec{k} - \vec{k}') \cdot \vec{r}_j} = \delta_{\vec{k}, \vec{k}'}$$





## LATTICE FOURIER TRANSFORM

$$\varphi_i = \frac{1}{N} \sum_{\vec{q}} \varphi(\vec{q}) e^{i\vec{q} \cdot \vec{r}_i}$$

$$J_{ij} = \frac{1}{N} \sum_{\vec{q}} J(\vec{q}) e^{i\vec{q}(\vec{r}_i - \vec{r}_j)}$$

ORTHOGONALITY RELATIONS:

$$\frac{1}{N} \sum_j e^{-i(\vec{k} - \vec{k}') \cdot \vec{r}_j} = \delta_{\vec{k}, \vec{k}'},$$

$$\frac{1}{N} \sum_{\vec{q}} e^{-i\vec{q}(\vec{r}_i - \vec{r}_j)} = \delta_{ij}.$$



• LET'S DEAL WITH

$$\sum_{i,j} \varphi_i J_{ij}^{-1} \varphi_j$$



• LET'S DEAL WITH

$$\sum_{i,j} \varphi_i J_{ij}^{-1} \varphi_j = \sum_{ij} \underbrace{\frac{1}{N} \sum_q \varphi(q)}_{\varphi_1} e^{i\vec{q} \cdot \vec{r}_i}$$



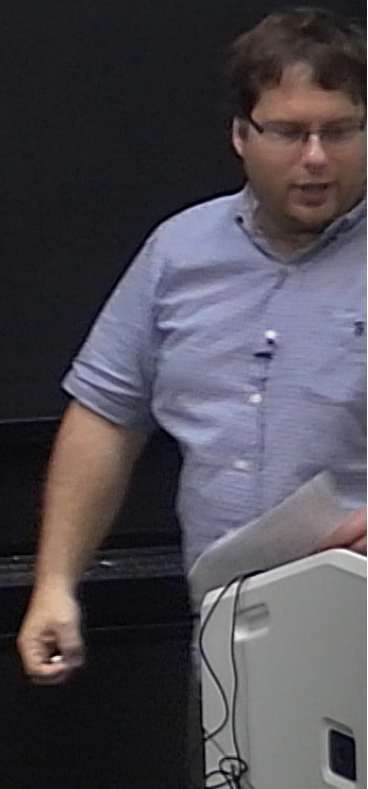
$$T_H = \sum_{ij} \underbrace{\frac{1}{H} \sum_q \varphi(q)}_{\varphi_1} e^{i\vec{q} \cdot \vec{r}_{ij}} + \frac{1}{H} \sum_{q'} J'(q') e^{iq'(\vec{r}_i - \vec{r}_j)}$$



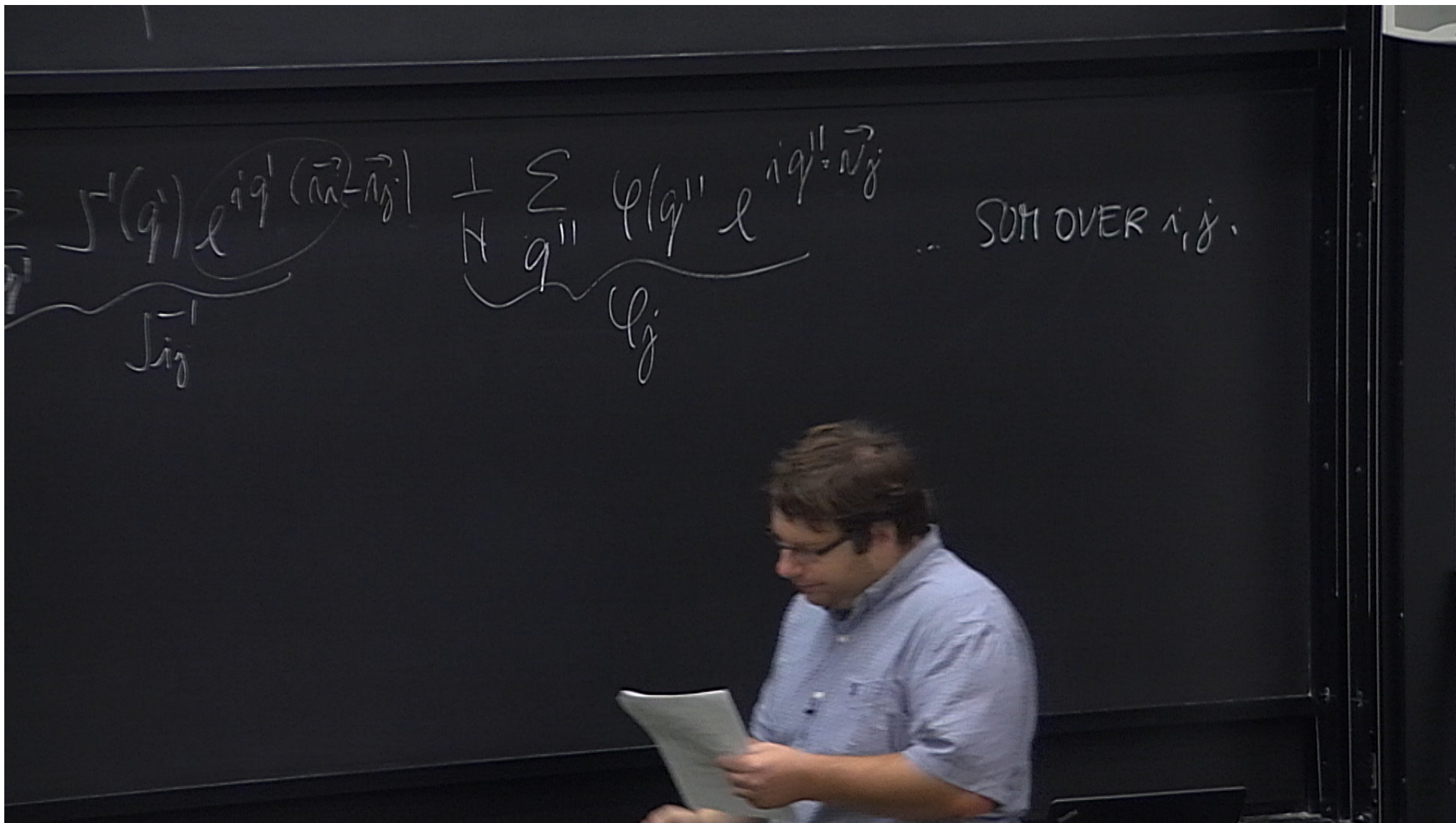
$$\begin{aligned}
 T_H &= \sum_{ij} \underbrace{\frac{1}{H} \sum_q \varphi(q)}_{\varphi_i} e^{i\vec{q} \cdot \vec{r}_i} \underbrace{\frac{1}{H} \sum_{q'} J'(q') e^{iq'(\vec{r}_i - \vec{r}_j)}}_{J_{ij}^{-1}} \underbrace{\frac{1}{H} \sum_{q''} \varphi(q'') e^{iq'' \cdot \vec{r}_j}}_{\varphi_j}
 \end{aligned}$$



$$\begin{aligned}
 &= \sum_{\vec{n}_j} \underbrace{\frac{1}{N} \sum_{\vec{q}} \psi(\vec{q})}_{\psi_1} \underbrace{e^{i\vec{q} \cdot \vec{n}_i}}_{e^{i\vec{q}'(\vec{n}_i - \vec{n}_j)}} \underbrace{\frac{1}{N} \sum_{\vec{q}'} \psi'(\vec{q}')}_{\psi_2'} \underbrace{e^{i\vec{q}' \cdot \vec{n}_j}}_{\psi_j}
 \end{aligned}$$







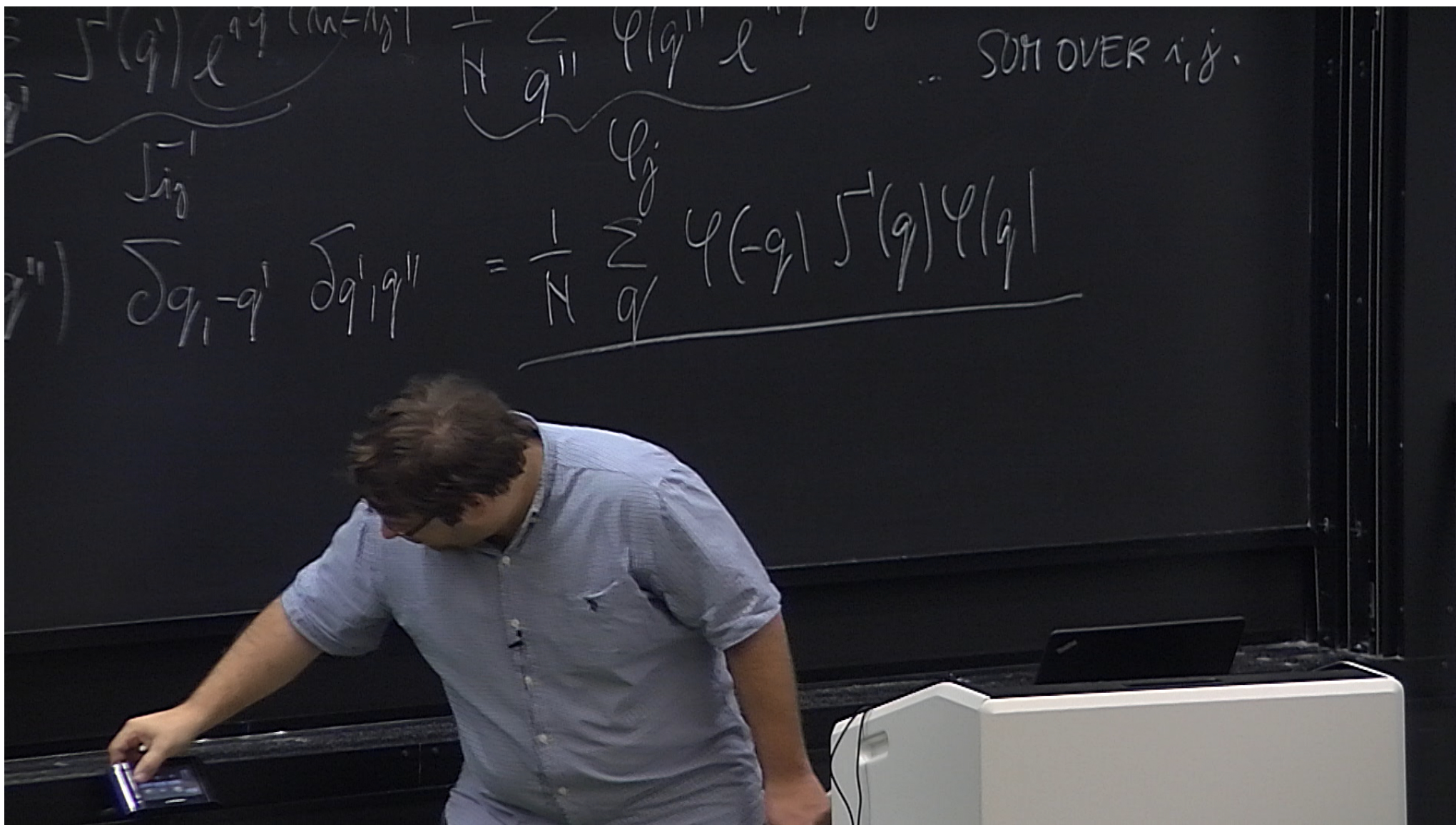


$$\begin{aligned}
 T_H &= \sum_{ij} \underbrace{\frac{1}{N} \sum_q \varphi(q)}_{\varphi_1} \underbrace{e^{i\vec{q} \cdot \vec{r}_{ij}}}_{J_{ij}} \underbrace{\frac{1}{N} \sum_{q'} J'(q') e^{iq'(\vec{r}_i - \vec{r}_j)}}_{J_{ij}'} \underbrace{\frac{1}{N} \sum_{q''} \varphi(q'') e^{iq'' \cdot \vec{r}_{ij}}}_{\varphi_j} \\
 &= \frac{1}{N} \sum_{q, q', q''} \varphi(q) J'(q') \varphi(q'') \delta_{q, -q'} \delta_{q', q''}
 \end{aligned}$$



$$\begin{aligned}
 & \text{TH} \\
 & = \sum_{ij} \underbrace{\frac{1}{N} \sum_q \varphi(q)}_{\varphi_i} \underbrace{e^{i\vec{q} \cdot \vec{r}_{ij}}}_{J_{ij}} \underbrace{\frac{1}{N} \sum_{q'} J'(q') e^{iq'(\vec{r}_i - \vec{r}_j)}}_{J_{ij}^{-1}} \underbrace{\frac{1}{N} \sum_{q''} \varphi(q'') e^{iq'' \cdot \vec{r}_j}}_{\varphi_j} \\
 & = \frac{1}{N} \sum_{q, q', q''} \varphi(q) J'(q') \varphi(q'') \delta_{q, -q'} \delta_{q', q''} = \frac{1}{N} \sum_q \varphi
 \end{aligned}$$







$$\sum_{i,j} \varphi_i J_{ij} \varphi_j = \sum_{i,j} \varphi_i \underbrace{J_{ij}}_{\varphi_i} \underbrace{\varphi_j}_{J_{ij}}$$

$$= \frac{1}{N} \sum_{q, q', q''} \varphi(q) J'(q') \varphi(q'') \delta_{q, q'} \delta_{q, q''}$$

WHAT IS  $J(q)$ ?

$$J(q) = \frac{1}{N} \sum_{ij} J_{ij} e^{i\vec{q} \cdot (\vec{r}_i - \vec{r}_j)}$$



$$\sum_{i,j} \varphi_i J_{ij} \varphi_j = \sum_{i,j} \varphi_i \underbrace{J_{ij}}_{\varphi_i} \underbrace{\varphi_j}_{J_{ij}}$$

$$= \frac{1}{N} \sum_{q, q', q''} \varphi(q) J'(q') \varphi(q'') \delta_{q, q'} \delta_{q, q''}$$

WHAT IS  $J(q)$ ?

$$J(q) = \frac{1}{N} \sum_{ij} J_{ij} e^{i \vec{q} \cdot (\vec{r}_i - \vec{r}_j)}$$

↑  
STILL... CONTINUUM  
LIMIT.



$$\begin{aligned}
 &= \frac{1}{N} \sum_{q, q', q''} \varphi(q) J'(q') \varphi(q'') \delta_{q, -q'} \delta_{q', q''} = \frac{1}{N} \sum_{q, q'} \varphi(q) J'(q') \delta_{q, -q'} \\
 &= \frac{1}{N} \sum_{ij} J_{ij} e^{-i \vec{q}_k (\vec{r}_i - \vec{r}_j)} = \frac{1}{N} \sum_{ij} J_{ij} \left( 1 - i \vec{q}_k (\vec{r}_i - \vec{r}_j) \right)
 \end{aligned}$$

SMALL.. CONTINUUM  
LIMIT.



$$\begin{aligned}
 \varphi_j &= \sum_{ij} \underbrace{\frac{1}{N} \sum_{q'} \varphi(q') e^{i q' \cdot \vec{r}_j}}_{\varphi_1} \underbrace{\frac{1}{N} \sum_{q''} J(q'') e^{i q'' \cdot \vec{r}_j}}_{J_{ij}} \varphi_j \\
 &= \frac{1}{N} \sum_{q, q', q''} \varphi(q) J'(q') \varphi(q'') \delta_{q, -q'} \delta_{q', q''} = \frac{1}{N} \sum_{q'} \varphi(-q') J'(q') \varphi(q') \\
 \varphi(q) &= \frac{1}{N} \sum_{ij} J_{ij} e^{-i \vec{q} \cdot (\vec{r}_i - \vec{r}_j)} = \frac{1}{N} \sum_{ij} J_{ij} \left( 1 - i \vec{q} \cdot (\vec{r}_i - \vec{r}_j) - \frac{1}{2} |\vec{q} \cdot (\vec{r}_i - \vec{r}_j)|^2 + \dots \right)
 \end{aligned}$$

SMALL... CONTINUUM LIMIT.



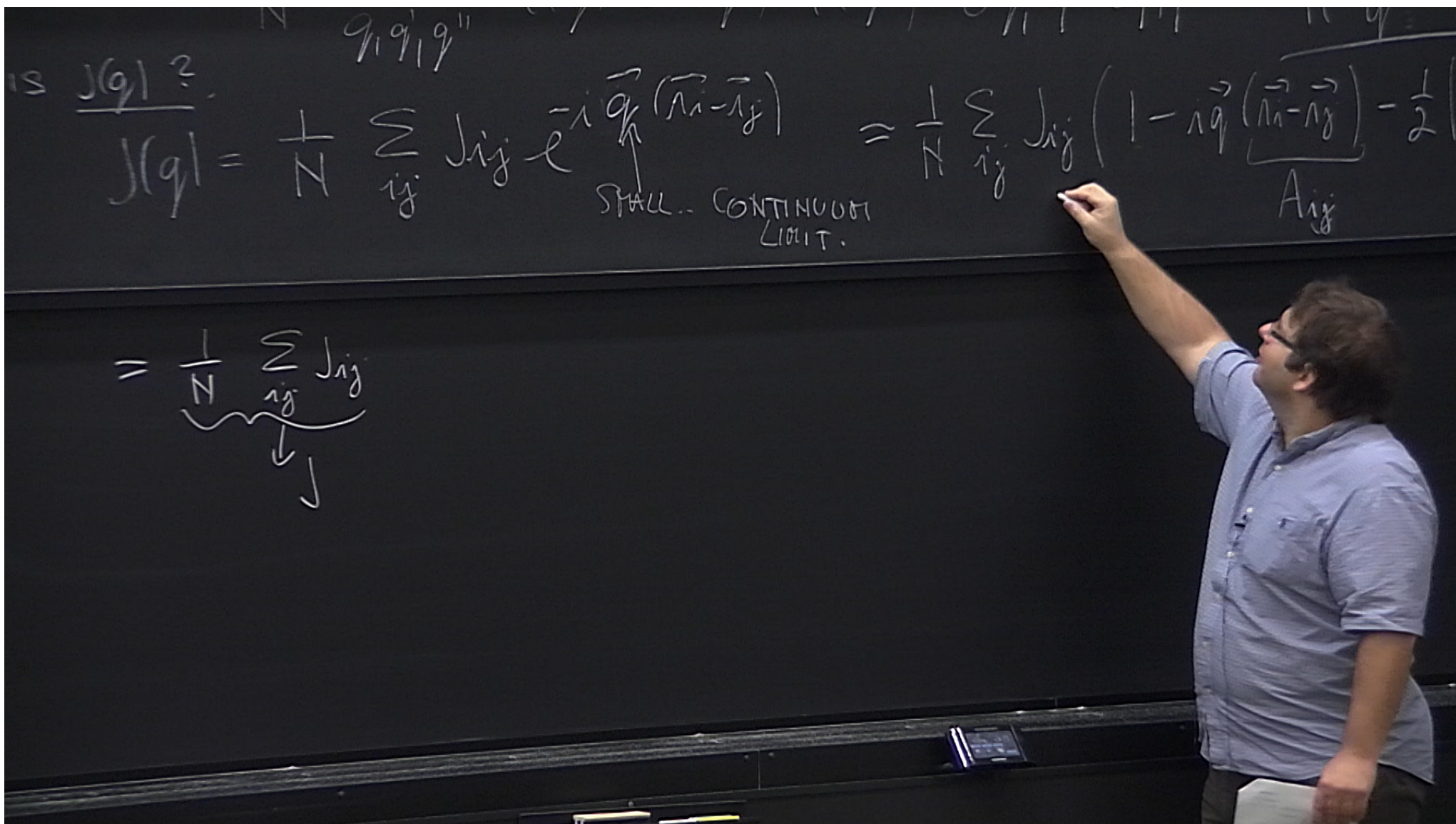
$$J(q) = \frac{1}{N} \sum_{ij} J_{ij} e^{-i q \cdot (\vec{r}_i - \vec{r}_j)} = \frac{1}{N} \sum_{ij} J_{ij} \left( 1 - \frac{i q \cdot (\vec{r}_i - \vec{r}_j)}{A_{ij}} - \frac{1}{2} \right)$$

SMALL... CONTINUUM  
LIMIT.

$$= \frac{1}{N} \sum_{ij} J_{ij}$$

↓







is  $J(q)$ ?

$$J(q) = \frac{1}{N} \sum_{ij} J_{ij} e^{-i \vec{q} \cdot (\vec{r}_i - \vec{r}_j)} = \frac{1}{N} \sum_{ij} \underbrace{J_{ij}}_{S_{ij}} \left( 1 - \underbrace{i \vec{q} \cdot (\vec{r}_i - \vec{r}_j)}_{A_{ij}} - \frac{1}{2} \right)$$

SMALL... CONTINUUM LIMIT.

$$= \frac{1}{N} \sum_{ij} J_{ij}$$

↓



$$J(q) = \frac{1}{N} \sum_{ij} J_{ij} e^{-i \vec{q} \cdot (\vec{r}_i - \vec{r}_j)} = \frac{1}{N} \sum_{ij} J_{ij} \underbrace{\left( 1 - i \vec{q} \cdot (\vec{r}_i - \vec{r}_j) - \frac{1}{2} |\vec{q} \cdot (\vec{r}_i - \vec{r}_j)|^2 \right)}_{A_{ij}}$$

SMALL... CONTINUUM LIMIT.

$$= \underbrace{\frac{1}{N} \sum_{ij} J_{ij}}_{\downarrow} + 0 - \frac{1}{2} \frac{1}{N} \sum_{ij} J_{ij} |\vec{q} \cdot (\vec{r}_i - \vec{r}_j)|^2$$



$$J(q) = \frac{1}{N} \sum_{ij} J_{ij} e^{-i \vec{q} \cdot (\vec{r}_i - \vec{r}_j)} = \frac{1}{N} \sum_{ij} J_{ij} \underbrace{\left( 1 - i \vec{q} \cdot (\vec{r}_i - \vec{r}_j) - \frac{1}{2} |\vec{q} \cdot (\vec{r}_i - \vec{r}_j)|^2 \right)}_{A_{ij}}$$

SMALL... CONTINUUM  
LIMIT.

$$= \underbrace{\frac{1}{N} \sum_{ij} J_{ij}}_{\downarrow} + 0 - \frac{1}{2} \frac{1}{N} \sum_{ij} J_{ij} |\vec{q} \cdot (\vec{r}_i - \vec{r}_j)|^2$$

$-\frac{1}{2} \propto |q|^2$



WHAT IS  $J(q)$ ?

$$J(q) = \frac{1}{N} \sum_{ij} J_{ij} e^{-i \vec{q} \cdot (\vec{r}_i - \vec{r}_j)} = \frac{1}{N} \sum_{ij} J_{ij} \underbrace{\left( 1 - i \vec{q} \cdot (\vec{r}_i - \vec{r}_j) + \dots \right)}_{A_{ij}}$$

SMALL... CONTINUUM LIMIT.

$$= \underbrace{\frac{1}{N} \sum_{ij} J_{ij}}_J + 0 - \frac{1}{2} \frac{1}{N} \sum_{ij} J_{ij} |\vec{q} \cdot (\vec{r}_i - \vec{r}_j)|^2$$

$-\frac{1}{2} \propto |q|^2$

$$J(q) = J - \frac{1}{2} \propto |q|^2 + \dots$$



WHAT IS  $J(q)$ ?

$$J(q) = \frac{1}{N} \sum_{ij} J_{ij} e^{-i \vec{q} \cdot (\vec{r}_i - \vec{r}_j)} = \frac{1}{N} \sum_{ij} J_{ij} \underbrace{\left( 1 - i \vec{q} \cdot (\vec{r}_i - \vec{r}_j) + \dots \right)}_{A_{ij}}$$

SMALL... CONTINUUM LIMIT.

$$= \underbrace{\frac{1}{N} \sum_{ij} J_{ij}}_J + 0 - \frac{1}{2} \frac{1}{N} \sum_{ij} J_{ij} |\vec{q} \cdot (\vec{r}_i - \vec{r}_j)|^2$$

$$= J - \frac{1}{2} \chi |\vec{q}|^2$$

↑ SPIN STIFFNESS

$$J(q) = J - \frac{1}{2} \chi |\vec{q}|^2 + \dots$$





WHAT IS  $J(q)$ ?

$$J(q) = \frac{1}{N} \sum_{ij} J_{ij} e^{-i \vec{q} \cdot (\vec{r}_i - \vec{r}_j)} = \frac{1}{N} \sum_{ij} J_{ij} \underbrace{\left( 1 - \underbrace{i \vec{q} \cdot (\vec{r}_i - \vec{r}_j)}_{A_{ij}} \right)}_{S_{ij}}$$

SMALL... CONTINUUM LIMIT.

$$= \underbrace{\frac{1}{N} \sum_{ij} J_{ij}}_J + 0 - \frac{1}{2} \frac{1}{N} \sum_{ij} J_{ij} |q \cdot (\vec{r}_i - \vec{r}_j)|^2$$

$-\frac{1}{2} \propto |q|^2$   $\propto R^2$

$\nwarrow$  SPIN STIFFNESS

$$J(q) = J - \frac{1}{2} \propto |q|^2 + \dots$$



WHAT IS  $J(q)$ ?

$$J(q) = \frac{1}{N} \sum_{ij} J_{ij} e^{-i \vec{q} \cdot (\vec{r}_i - \vec{r}_j)} = \frac{1}{N} \sum_{ij} J_{ij} \underbrace{\left( 1 - \underbrace{i \vec{q} \cdot (\vec{r}_i - \vec{r}_j)}_{A_{ij}} \right)}_{S_{ij}}$$

SMALL... CONTINUUM LIMIT.

$$= \underbrace{\frac{1}{N} \sum_{ij} J_{ij}}_J + 0 - \frac{1}{2} \frac{1}{N} \sum_{ij} J_{ij} |q \cdot (\vec{r}_i - \vec{r}_j)|^2$$

$$= J - \frac{1}{2} \chi |q|^2$$

$\chi = \frac{R^2 J N}{d}$

↑ SPIN STIFFNESS

$$J(q) = J - \frac{1}{2} \chi |q|^2 + \dots$$



WHAT IS  $J(q)$ ?

$$J(q) = \frac{1}{N} \sum_{ij} J_{ij} e^{-i \vec{q} \cdot (\vec{r}_i - \vec{r}_j)} = \frac{1}{N} \sum_{ij} J_{ij} \underbrace{\left( 1 - i \vec{q} \cdot (\vec{r}_i - \vec{r}_j) + \dots \right)}_{A_{ij}}$$

SMALL... CONTINUUM LIMIT.

$$= \underbrace{\frac{1}{N} \sum_{ij} J_{ij}}_J + 0 - \frac{1}{2} \frac{1}{N} \sum_{ij} J_{ij} |\vec{q} \cdot (\vec{r}_i - \vec{r}_j)|^2$$

$$= J - \frac{1}{2} \chi |\vec{q}|^2$$

↑ SPIN STIFFNESS

$\chi = \frac{R^2 J N}{d}$

$J(q) = J - \frac{1}{2} \chi |\vec{q}|^2 + \dots$



WHAT IS  $J(q)$ ?

$$J(q) = \frac{1}{N} \sum_{ij} J_{ij} e^{-i \vec{q} \cdot (\vec{r}_i - \vec{r}_j)} = \frac{1}{N} \sum_{ij} J_{ij} \underbrace{\left( 1 - i \vec{q} \cdot (\vec{r}_i - \vec{r}_j) + \dots \right)}_{A_{ij}}$$

SMALL... CONTINUUM LIMIT.

$$= \underbrace{\frac{1}{N} \sum_{ij} J_{ij}}_J + 0 - \frac{1}{2} \frac{1}{N} \sum_{ij} J_{ij} |\vec{q} \cdot (\vec{r}_i - \vec{r}_j)|^2$$

$$= J - \frac{1}{2} \chi |\vec{q}|^2$$

↑ SPIN STIFFNESS

$\chi = \frac{R^2 J N}{d N}$

$J(q) = J - \frac{1}{2} \chi |\vec{q}|^2 + \dots$



WHAT IS  $J(q)$ ?

$$J(q) = \frac{1}{N} \sum_{ij} J_{ij} e^{-i \vec{q} \cdot (\vec{r}_i - \vec{r}_j)} = \frac{1}{N} \sum_{ij} J_{ij} \underbrace{\left( 1 - i \vec{q} \cdot (\vec{r}_i - \vec{r}_j) + \dots \right)}_{A_{ij}}$$

SMALL... CONTINUUM LIMIT.

$$= \underbrace{\frac{1}{N} \sum_{ij} J_{ij}}_J + 0 - \frac{1}{2} \frac{1}{N} \sum_{ij} J_{ij} |\vec{q} \cdot (\vec{r}_i - \vec{r}_j)|^2$$

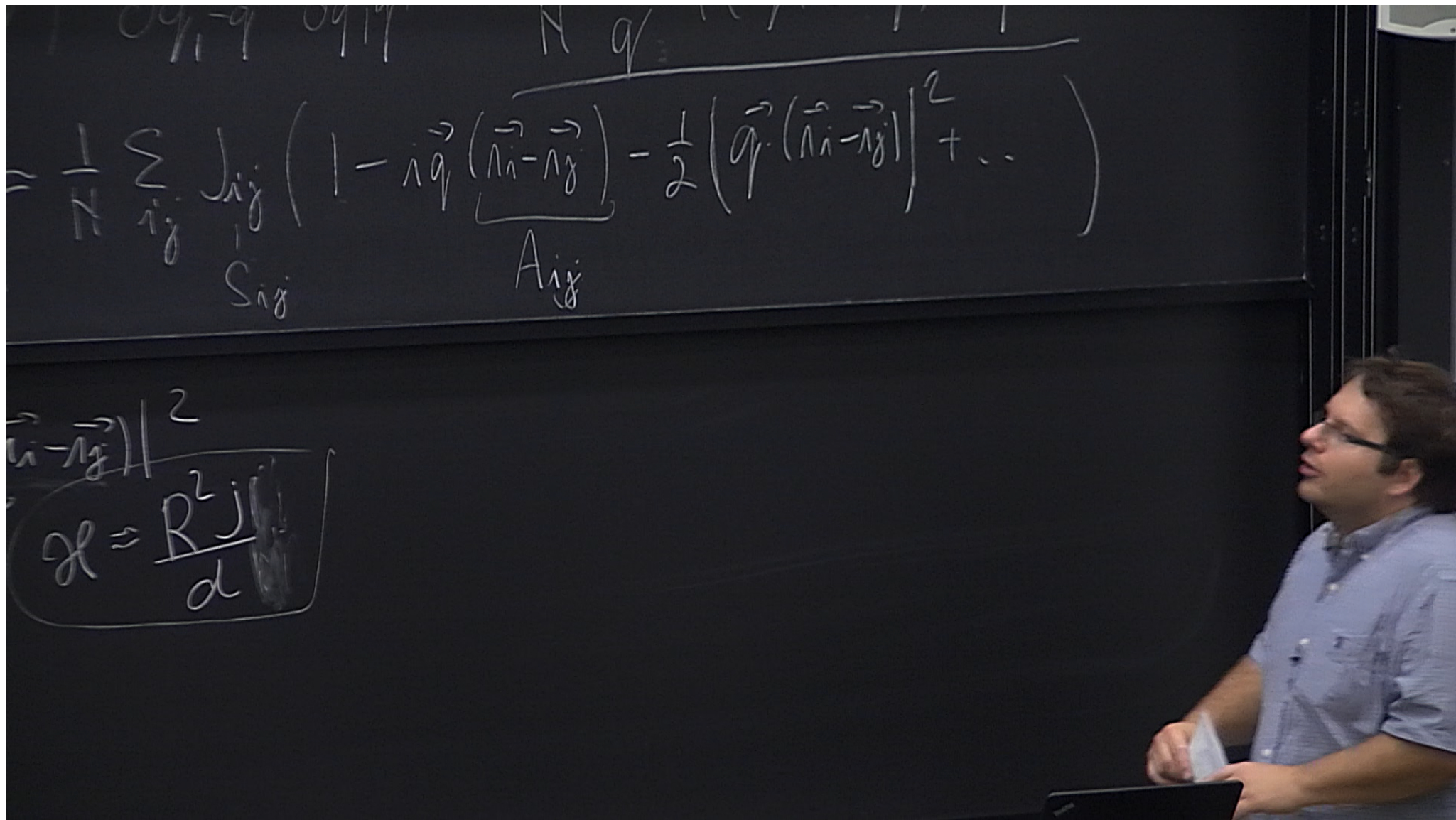
$$= J - \frac{1}{2} \chi |\vec{q}|^2$$

$$\chi = \frac{R^2 J}{d}$$

↑ SPIN STIFFNESS

$$J(q) = J - \frac{1}{2} \chi |\vec{q}|^2 + \dots$$







$T_C \rightarrow T$

$(S \rightarrow A)$

$$S = S_0 + S_{int}$$

$$S_0(\varphi) = \frac{1}{2T^2} \sum_{ij} T \varphi_i L_{ij}^{-1} \varphi_j$$



$$S = S_0 + S_{int}$$

$$S_0[\varphi] = \frac{1}{2T^2} \sum_{ij} T \varphi_i J_{ij}^{-1} \varphi_j - \frac{1}{2} \sum_i \frac{\varphi_i^2}{T^2}$$



$(f \rightarrow \infty)$

$$S_{int} = -\frac{1}{12} \sum_i \frac{(\varphi_i)^4}{T^4}$$



$$S = S_0 + S_{int}$$

$$S_0(\varphi) = \frac{1}{2T^2} \sum_{ij} T \varphi_i J_{ij}^{-1} \varphi_j - \frac{1}{2} \sum_i \frac{\varphi_i^2}{T^2}$$

$$= \frac{1}{2T^2} N \sum_q \varphi(-q)$$



$$S = S_0 + S_{int}$$

$$S_0(\varphi) = \frac{1}{2T^2} \sum_{ij} T \varphi_i J_{ij}^{-1} \varphi_j - \frac{1}{2} \sum_i \frac{\varphi_i^2}{T^2}$$

$$= \frac{1}{2T^2 N} \sum_q \varphi(-q) \left( \frac{T}{J(q)} - 1 \right) \varphi(q)$$



$$S = S_0 + S_{int}$$

$$S_0(\varphi) = \frac{1}{2T^2} \sum_{ij} T \varphi_i J_{ij}^{-1} \varphi_j - \frac{1}{2} \sum_i \frac{\varphi_i^2}{T^2}$$

$$= \frac{1}{2T^2} N \sum_q \varphi(1-q) \underbrace{\left( \frac{T}{J(q)} - 1 \right)}_{\underline{T}} \varphi(q)$$



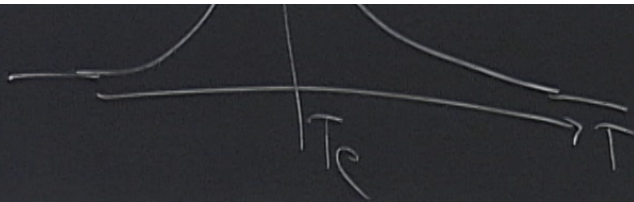
$$S = S_0 + S_{int}$$

$$S_0(\varphi) = \frac{1}{2T^2} \sum_{ij} T \varphi_i J_{ij}^{-1} \varphi_j - \frac{1}{2} \sum_i \frac{\varphi_i^2}{T^2}$$

$$= \frac{1}{2T^2} N \sum_q \varphi(1-q) \underbrace{\left( \frac{T}{J(q)} - 1 \right)}_{\frac{T}{J(1 - \frac{1}{2} \frac{|q|^2}{J}} - 1)} \varphi(q)$$

$$J(1 - \frac{1}{2} \frac{|q|^2}{J}) - 1$$





$$\langle \phi \rangle \approx 0$$

$\langle \phi \rangle = 0$   
 FRACTAL STRUCTURE  
 $(S \rightarrow \infty)$

$$\langle \phi \rangle \approx 1$$

$$S = S_0 + S_{int}$$

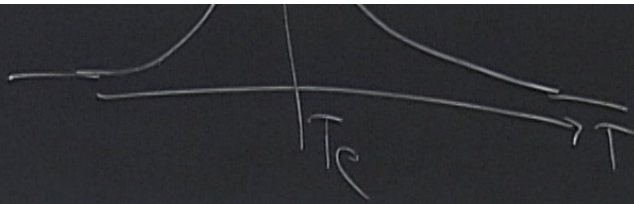
$$S_0[\varphi] = \frac{1}{2T^2} \sum_{i,j} T \varphi_i J_{ij}^{-1} \varphi_j - \frac{1}{2} \sum_i \frac{\varphi_i^2}{T^2}$$

$$= \frac{1}{2T^2} \sum_q \varphi(-q) \left( \underbrace{\frac{T}{J(q)}}_T - 1 \right) \varphi(q)$$

$$\frac{T}{J(1 - \frac{1}{2} \frac{|q|^2}{J})} - 1 \approx \frac{T}{J} \left( 1 + \frac{1}{2} \frac{q^2}{J} \right) - 1$$

$$S_{int} = -\frac{1}{12} \sum_i \frac{(\varphi_i)^4}{T^4}$$





$$\langle \phi \rangle \approx 0$$

$\langle \phi \rangle = 0$   
FRACTAL STRUCTURE  
( $\xi \rightarrow \infty$ )

$$\langle \phi \rangle \approx 1$$

$$S = S_0 + S_{int}$$

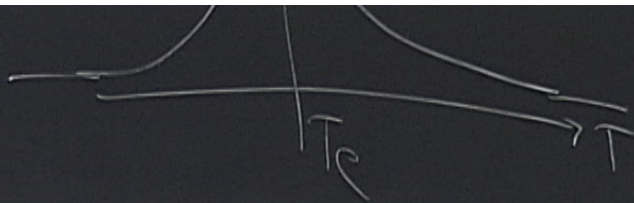
$$S_0(\varphi) = \frac{1}{2T^2} \sum_{ij} T \varphi_i J_{ij}^{-1} \varphi_j - \frac{1}{2} \sum_i \frac{\varphi_i^2}{T^2}$$

$$= \frac{1}{2} \sum_q \varphi(-q) \left( \frac{T}{J(q)} - 1 \right) \varphi(q)$$

$$\frac{T}{J(1 - \frac{1}{2} \frac{|q|^2 \lambda}{J})} - 1 \approx \frac{T}{J_c} \left( 1 + \frac{1}{2} \frac{q^2 \lambda}{J} \right) - 1$$

$$S_{int} = -\frac{1}{12} \sum_i \frac{(\varphi_i)^4}{T^4}$$





$$\langle \phi \rangle \approx 0$$

$\langle \phi \rangle = 0$   
FRACTAL STRUCTURE  
( $\xi \rightarrow \infty$ )

$$\langle \phi \rangle \approx 1$$

$$S = S_0 + S_{int}$$

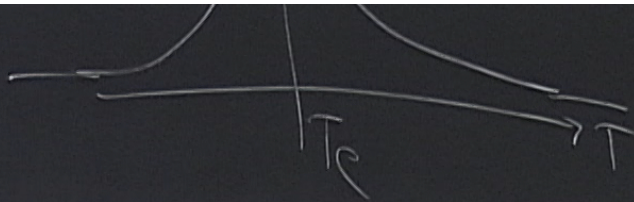
$$S_0(\varphi) = \frac{1}{2T^2} \sum_{ij} T \varphi_i J_{ij}^{-1} \varphi_j - \frac{1}{2} \sum_i \frac{\varphi_i^2}{T^2}$$

$$= \frac{1}{2T^2 N} \sum_q \varphi(-q) \left( \frac{T}{J(q)} - 1 \right) \varphi(q)$$

$$\frac{T}{J(1 - \frac{1}{2} \frac{|q|^2 \lambda}{J})} - 1 \approx \frac{T}{J_{T_c}} \left( 1 + \frac{1}{2} q^2 \frac{\lambda}{J} \right) - 1 = \frac{T - T_c}{T_c} + \frac{1}{2} q^2 \frac{\lambda T}{T_c^2}$$

$$S_{int} = -\frac{1}{12} \sum_i \frac{(\varphi_i)^4}{T^4}$$





$$\langle \phi \rangle \approx 0$$

$\langle \phi \rangle = 0$   
FRACTAL STRUCTURE  
( $\xi \rightarrow \infty$ )

$$\langle \phi \rangle \approx 1$$

$$S = S_0 + S_{int}$$

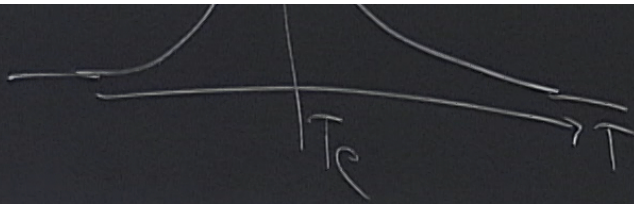
$$S_0(\varphi) = \frac{1}{2T^2} \sum_{ij} T \varphi_i J_{ij}^{-1} \varphi_j - \frac{1}{2} \sum_i \frac{\varphi_i^2}{T^2}$$

$$= \frac{1}{2T^2 N} \sum_q \varphi(-q) \left( \underbrace{\frac{T}{J(q)}}_T - 1 \right) \varphi(q)$$

$$S_{int} = -\frac{1}{12} \sum_i \frac{(\varphi_i)^4}{T^4}$$

$$\sqrt{\left(1 - \frac{1}{2} \frac{|q|^2 \chi}{J}\right)} - 1 \approx \frac{T}{J_{T_c}} \left(1 + \frac{1}{2} q^2 \frac{\chi}{J}\right) - 1 = \frac{T - T_c}{T_c} + \frac{1}{2} q^2 \frac{\chi T}{T_c^2}$$





$$\langle \phi \rangle \approx 0$$

$\langle \phi \rangle = 0$   
FRACTAL STRUCTURE  
( $\xi \rightarrow \infty$ )

$$\langle \phi \rangle \approx 1$$

$$S = S_0 + S_{int}$$

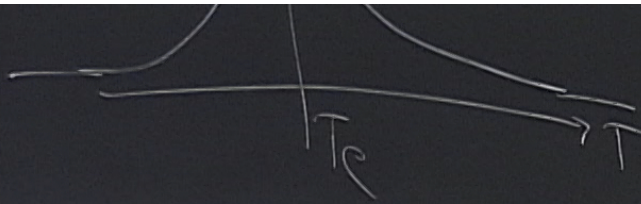
$$S_0(\varphi) = \frac{1}{2T^2} \sum_{ij} T \varphi_i J_{ij}^{-1} \varphi_j - \frac{1}{2} \sum_i \frac{\varphi_i^2}{T^2}$$

$$= \frac{1}{2T^2 N} \sum_q \varphi(-q) \left( \frac{T}{J(q)} - 1 \right) \varphi(q)$$

$$\frac{T}{J(1 - \frac{1}{2} \frac{|q|^2 \lambda}{J})} - 1 \approx \frac{T}{J_{T_c}} \left( 1 + \frac{1}{2} \frac{q^2 \lambda}{J} \right) - 1 = \frac{T - T_c}{T_c} + \frac{1}{2} q^2 \frac{\lambda T}{T_c^2}$$

$$S_{int} = -\frac{1}{12} \sum_i \frac{(\varphi_i)^4}{T^4}$$





$$\langle \phi \rangle \approx 0$$

$\langle \phi \rangle = 0$   
FRACTAL STRUCTURE  
( $\xi \rightarrow \infty$ )

$$\langle \phi \rangle \approx 1$$

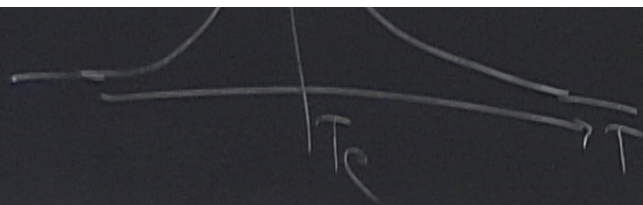
$$= S_0 + S_{int}$$

$$S_0(\varphi) = \frac{1}{2T^2} \sum_{ij} T \varphi_i h_{ij}^{-1} \varphi_j - \frac{1}{2} \sum_i \frac{\varphi_i^2}{T^2}$$

$$= \frac{1}{2T^2 N} \sum_q \varphi(-q) \left( \frac{T}{J(q)} - 1 \right) \varphi(q) = \frac{1}{2T^2 N} \sum_q \varphi(-q) \left( t + \frac{1}{2} |q|^2 \frac{\partial T}{T_c^2} \right) \varphi(q)$$

$$\frac{T}{J\left(1 - \frac{1}{2} |q|^2 \frac{\partial T}{J}\right)} - 1 \approx \frac{T}{J} \left( 1 + \frac{1}{2} |q|^2 \frac{\partial T}{J} \right) - 1 = \frac{T - T_c}{T_c} + \frac{1}{2} |q|^2 \frac{\partial T}{T_c^2}$$





$$\langle G \rangle \approx 0$$

$\langle G \rangle \approx 0$   
FRACTAL STRUCTURE  
( $\xi \rightarrow \infty$ )

$$\langle G \rangle \approx 1$$

$$= S_0 + S_{int}$$

$$S_0(\varphi) = \frac{1}{2T^2} \sum_{ij} T \varphi_i \chi_{ij}^{-1} \varphi_j - \frac{1}{2} \sum_i \frac{\varphi_i^2}{T^2}$$

$$= \frac{1}{2} \sum_q \varphi(-q) \left( \frac{T}{J(q)} - 1 \right) \varphi(q) = \frac{1}{2T^2 N} \sum_q \varphi(-q) \left( t + \frac{1}{2} |q|^2 \frac{\partial T}{T_c^2} \right) \varphi(q)$$

$$\frac{T}{J\left(1 - \frac{1}{2} |q|^2 \frac{\partial T}{J}\right)} - 1 \approx \frac{T}{J} \left( 1 + \frac{1}{2} |q|^2 \frac{\partial T}{J} \right) - 1 = \frac{T - T_c}{T_c} + \frac{1}{2} |q|^2 \frac{\partial T}{T_c^2}$$



$$S_{int} = -\frac{1}{12} \sum_i \frac{(\phi_i)^4}{T^4}$$

$$\phi \rightarrow \frac{T_c \sqrt{2d}}{R a^{d/2}} \phi$$

$$\sum_q \phi(-q) \left( t + \frac{1}{2} |q|^2 \frac{\partial T}{T_c^2} \right) \phi(q)$$

$$\mathcal{L} = \frac{2d t}{R^2}$$

$$\left. \frac{1}{2} |q|^2 \frac{\partial T}{T_c^2} \right|_{-1} = \frac{\frac{1}{2} |q|^2 \frac{\partial T}{T_c^2}}{\frac{T - T_c}{T_c}} + \frac{1}{2} |q|^2 \frac{\partial T}{T_c^2}$$

$$0 < t < 1$$

$$T = 0$$



$$S_0(\psi) = \frac{1}{2} \frac{1}{N a^d} \sum_q \psi(-q) (1 + q^2) \psi(q)$$



$$S_0(\varphi) = \frac{1}{2} \left( \frac{1}{N a^d} \sum_q \right) \varphi(-q) (1 + q^2) \varphi(q)$$

$$= \frac{1}{2} \int \frac{d^d q}{(2\pi)^d} \varphi(-q) (1 + q^2) \varphi(q)$$



$$S_0(\psi) = \frac{1}{2} \left( \frac{1}{N a^d} \sum_q \right) \psi(-q) (1 + q^2) \psi(q)$$

$$= \int \frac{d^d q}{(2\pi)^d} \psi(-q) (1 + q^2) \psi(q)$$



$$\begin{aligned}
S_0(\varphi) &= \frac{1}{2} \left( \frac{1}{N a^d} \sum_q \right) \varphi(-q) (1 + q^2) \varphi(q) \\
&= \frac{1}{2} \int \frac{d^d q}{(2\pi)^d} \varphi(-q) (1 + q^2) \varphi(q) \\
&= \frac{1}{2} \int d^d x \left( |\nabla \varphi|^2 + \varphi^2 \right)
\end{aligned}$$



$$S_{int}[\varphi] = \int d^4x \frac{\mu}{4!} \varphi^4$$



$$S_{int}[\varphi] = \int d^4x \frac{\mu}{4!} \varphi^4$$

$$S[\varphi] = \int d^d x \left( \frac{1}{2} (\nabla \varphi)^2 + \frac{\lambda}{2} \varphi^2 + \frac{\mu}{4!} \varphi^4 \right)$$



$$S_{int}[\varphi] = \int d^4x \frac{u}{4!} \varphi^4$$

$$S[\varphi] = \int d^d x \left( \frac{1}{2} (\nabla \varphi)^2 + \frac{\lambda}{2} \varphi^2 + \frac{u}{4!} \varphi^4 \right)$$

$N \propto t$



$$S_{int}[\varphi] = \int d^4x \frac{\mu}{4!} \varphi^4$$

$$S[\varphi] = \int d^4x \left( \frac{1}{2} (\nabla\varphi)^2 + \frac{\lambda}{2} \varphi^2 + \frac{\mu}{4!} \varphi^4 \right)$$

•  $\lambda \propto t$  CHANGES SIGN AS CROSS  $T_c$



$$S_{int}[\varphi] = \int d^4x \frac{\mu}{4!} \varphi^4$$

$$S[\varphi] = \int d^4x \left( \frac{1}{2} (\partial\varphi)^2 + \frac{\lambda}{2} \varphi^2 + \frac{\mu}{4!} \varphi^4 \right)$$

- $\lambda \propto t$  CHANGES SIGN AS CROSS  $T_c$
- $\mu > 0$



$$S_{int}[\varphi] = \int d^4x \frac{\mu}{4!} \varphi^4$$

$$S[\varphi] = \int d^d x \left( \frac{1}{2} (\nabla \varphi)^2 + \frac{\lambda}{2} \varphi^2 + \frac{\mu}{4!} \varphi^4 \right)$$

LANDAU-GINSBURG

- $\lambda \propto t$  CHANGES SIGN AS CROSS  $T_c$
- $\mu > 0$



$$S_{int}[\varphi] = \int d^4x \frac{\mu}{4!} \varphi^4$$

FIELD THEORY WITH IMAGINARY  
TIME

$$S[\varphi] = \int d^d x \left( \frac{1}{2} (\nabla \varphi)^2 + \frac{\lambda}{2} \varphi^2 + \frac{\mu}{4!} \varphi^4 \right)$$

LANDAU-GINSBURG

- $\lambda \propto t$  CHANGES SIGN AS CROSS  $T_c$
- $\mu > 0$