Title: PSI 17/18 - Statistical Mechanics - Lecture 5

Date: Oct 16, 2017 10:45 AM

URL: http://pirsa.org/17100038

Abstract:

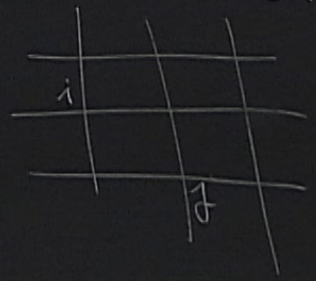
LAST TIME: MICROSCOPIC THEORY OF PHASE TRANSITIONS

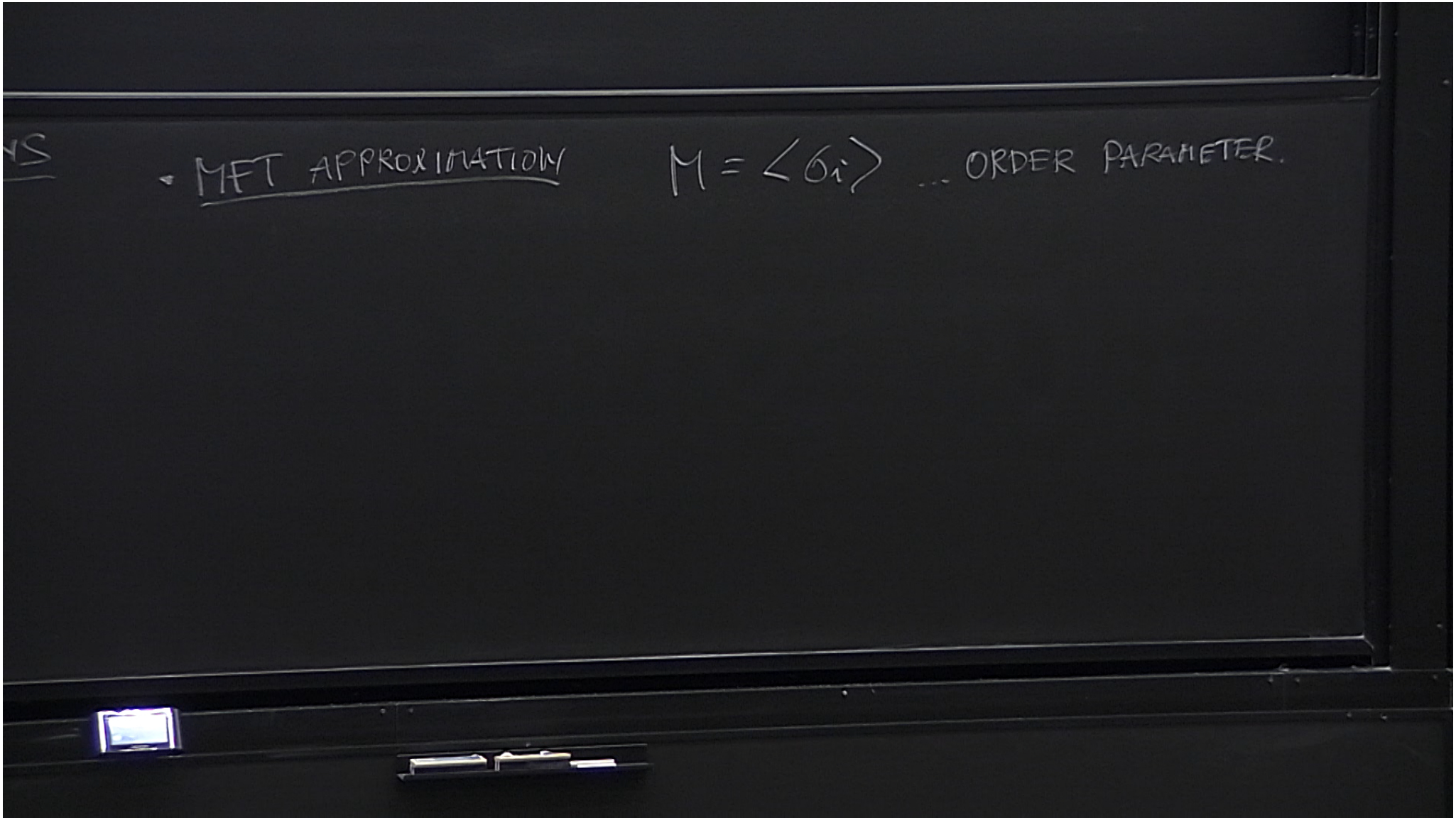
- **ISING MODEL** (OF FERROMAGNETISM)

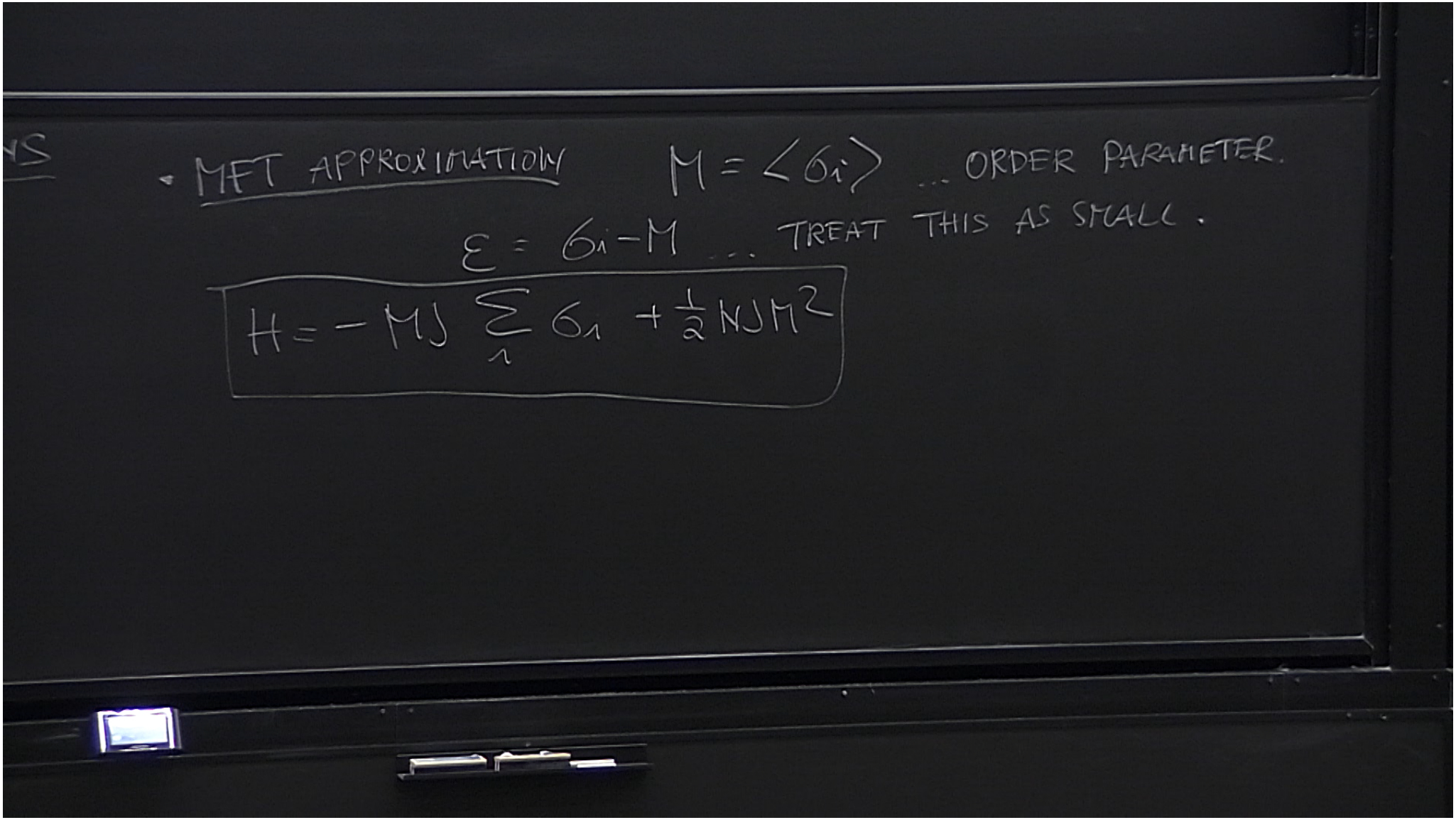
$$H = -\frac{1}{2} \sum_{i,j} J_{ij} \sigma_i \sigma_j$$

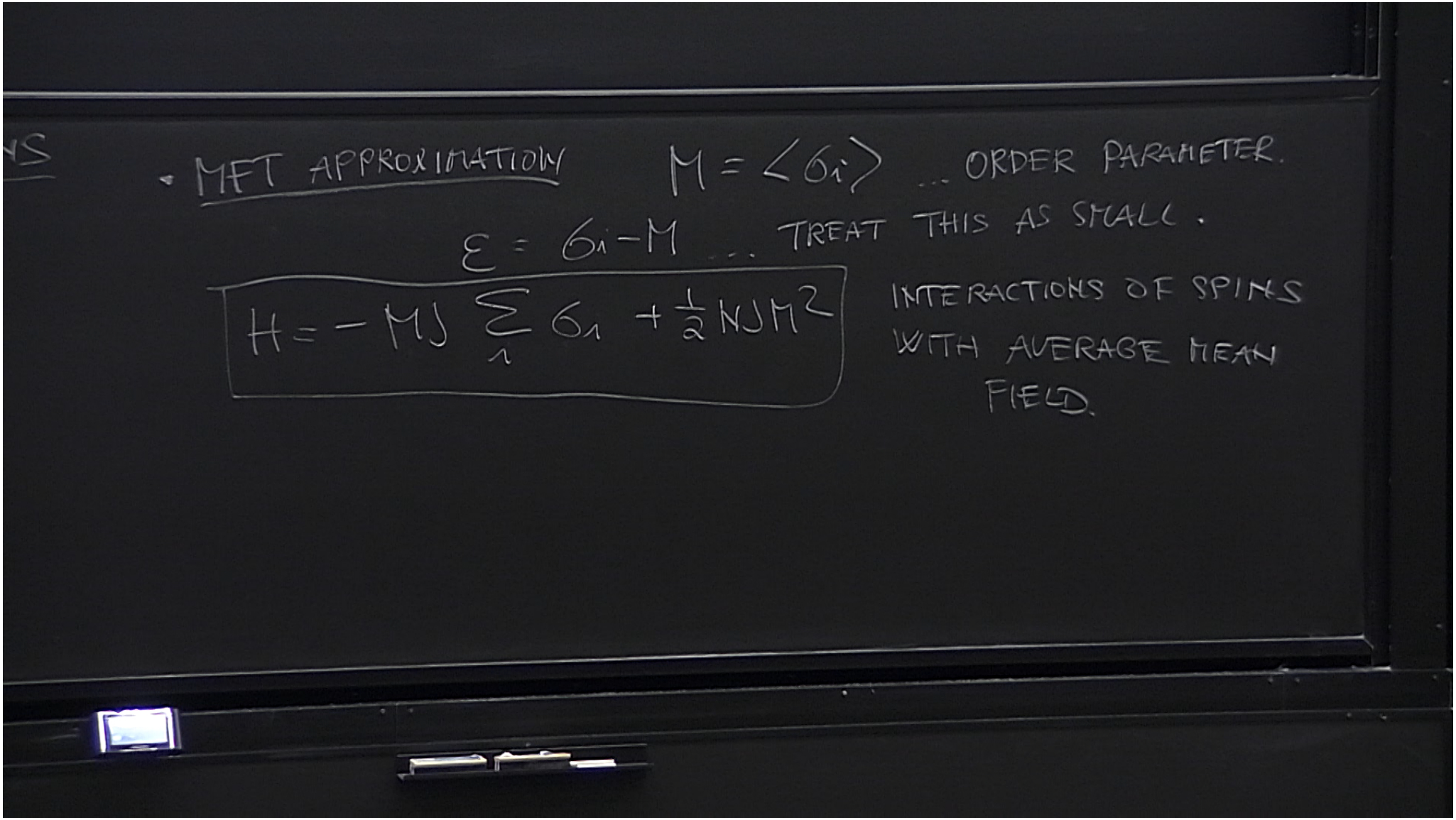
"CLASSICAL SPINS" $\sigma_i = \pm 1$

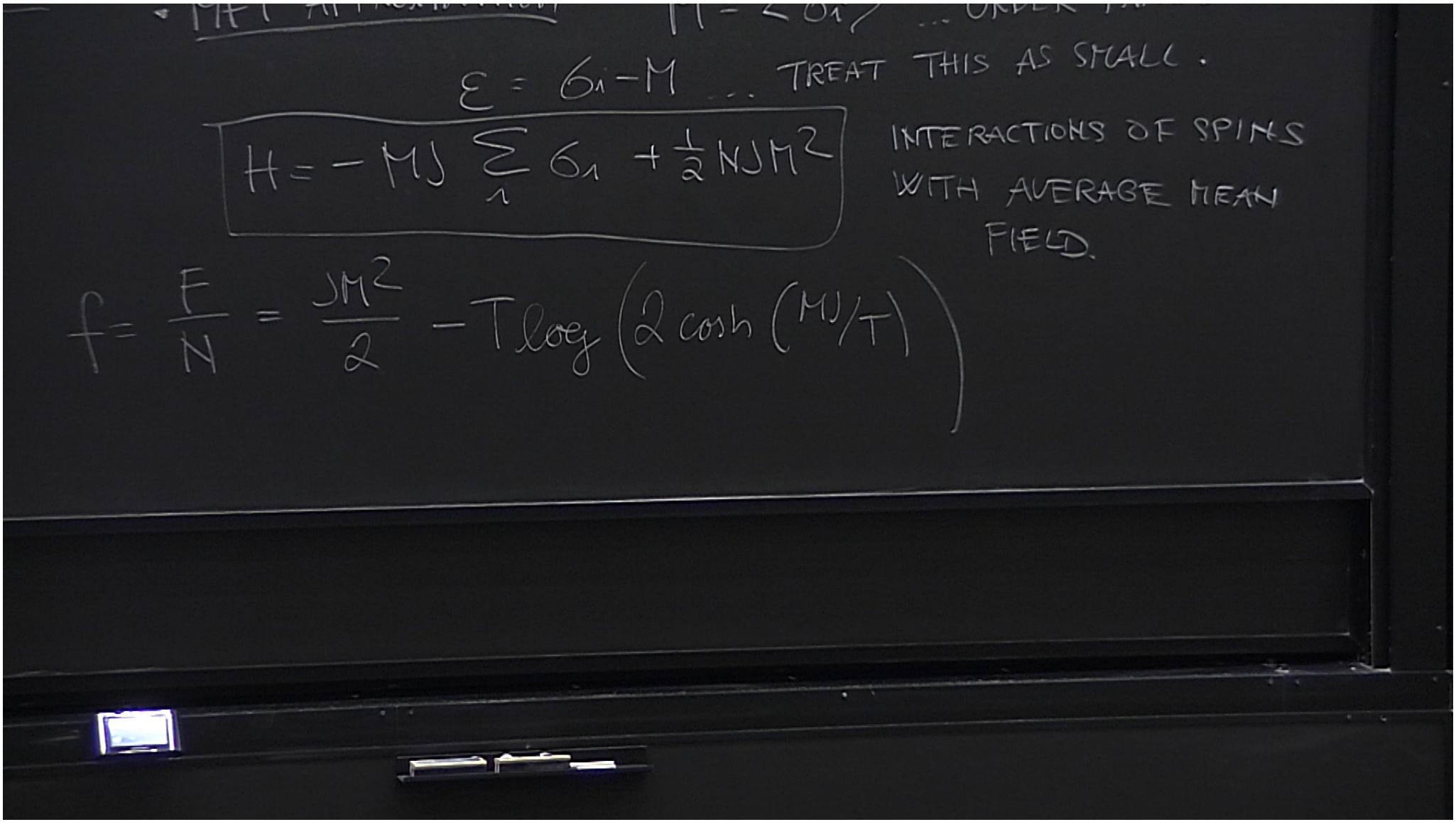
IN d-DIMENSIONAL CUBIC LATTICE

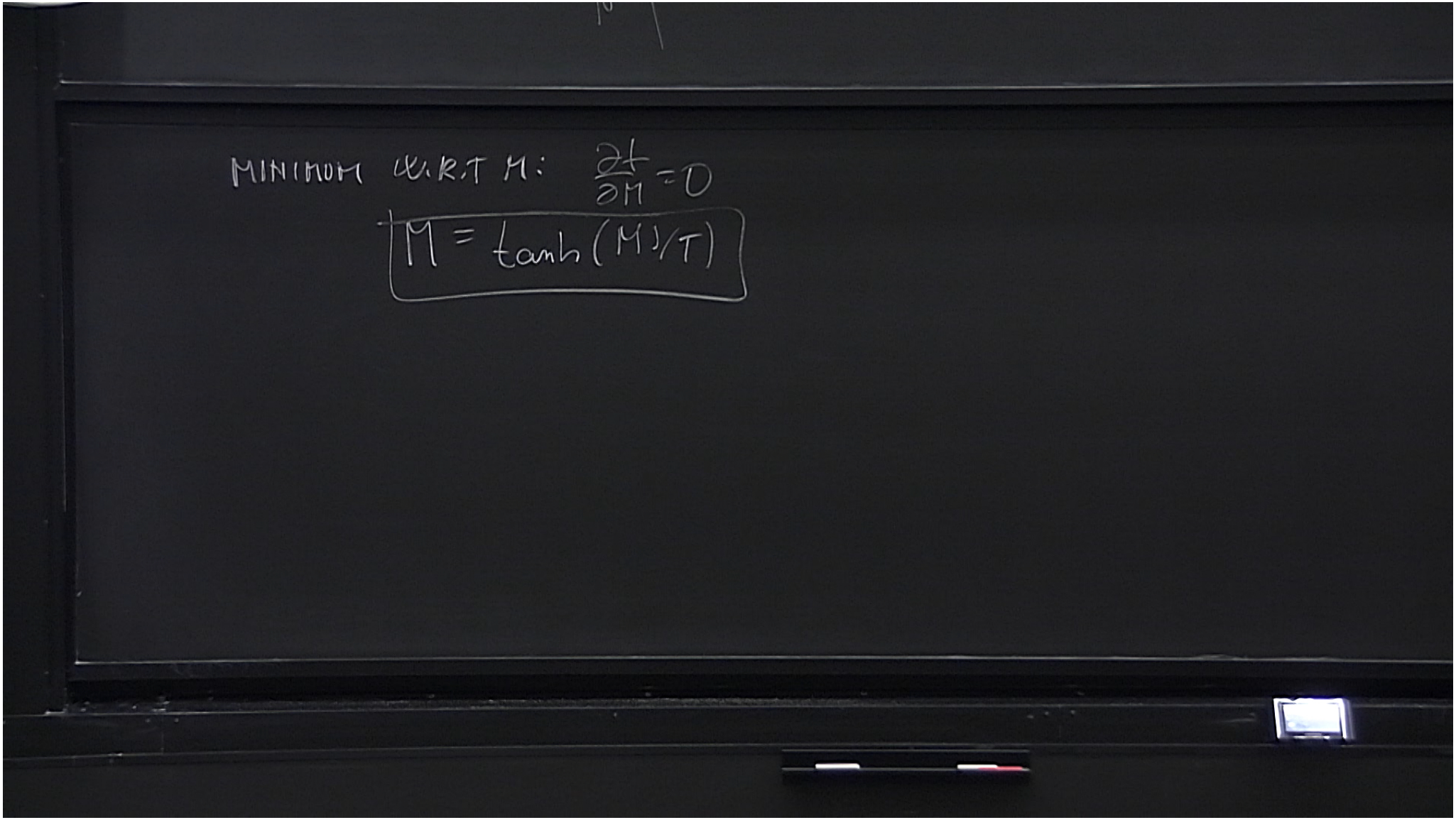



• MFT APPROXIMATION
$M = \langle \sigma_i \rangle$ ... ORDER PARAMETER.


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$\varepsilon = \sigma_i - M$ ... TREAT THIS AS SMALL.
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INTERACTIONS OF SPINS WITH AVERAGE MEAN FIELD.

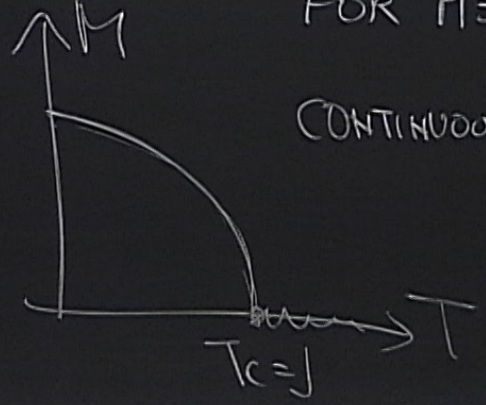

$\varepsilon = \sigma_i - M$ ... TREAT THIS AS SMALL.
$H = -MJ \sum_i \sigma_i + \frac{1}{2} NJM^2$
INTERACTIONS OF SPINS WITH AVERAGE MEAN FIELD.
$f = \frac{F}{N} = \frac{JM^2}{2} - T \log\left(2\cosh\left(MJ/T\right)\right)$


MINIMUM W.R.T M: $\frac{\partial f}{\partial M} = 0$
$M = \tanh(MJ/T)$

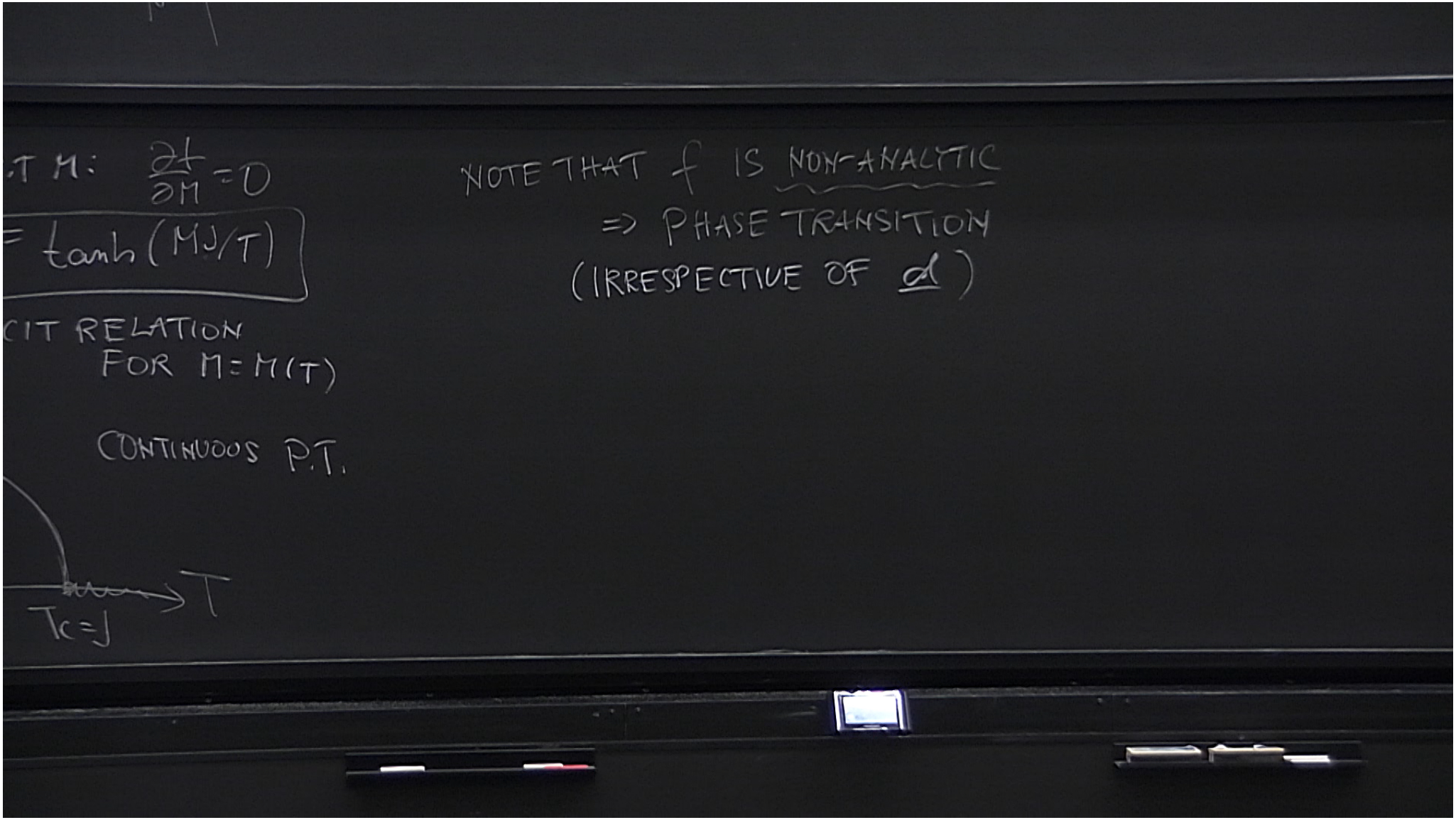
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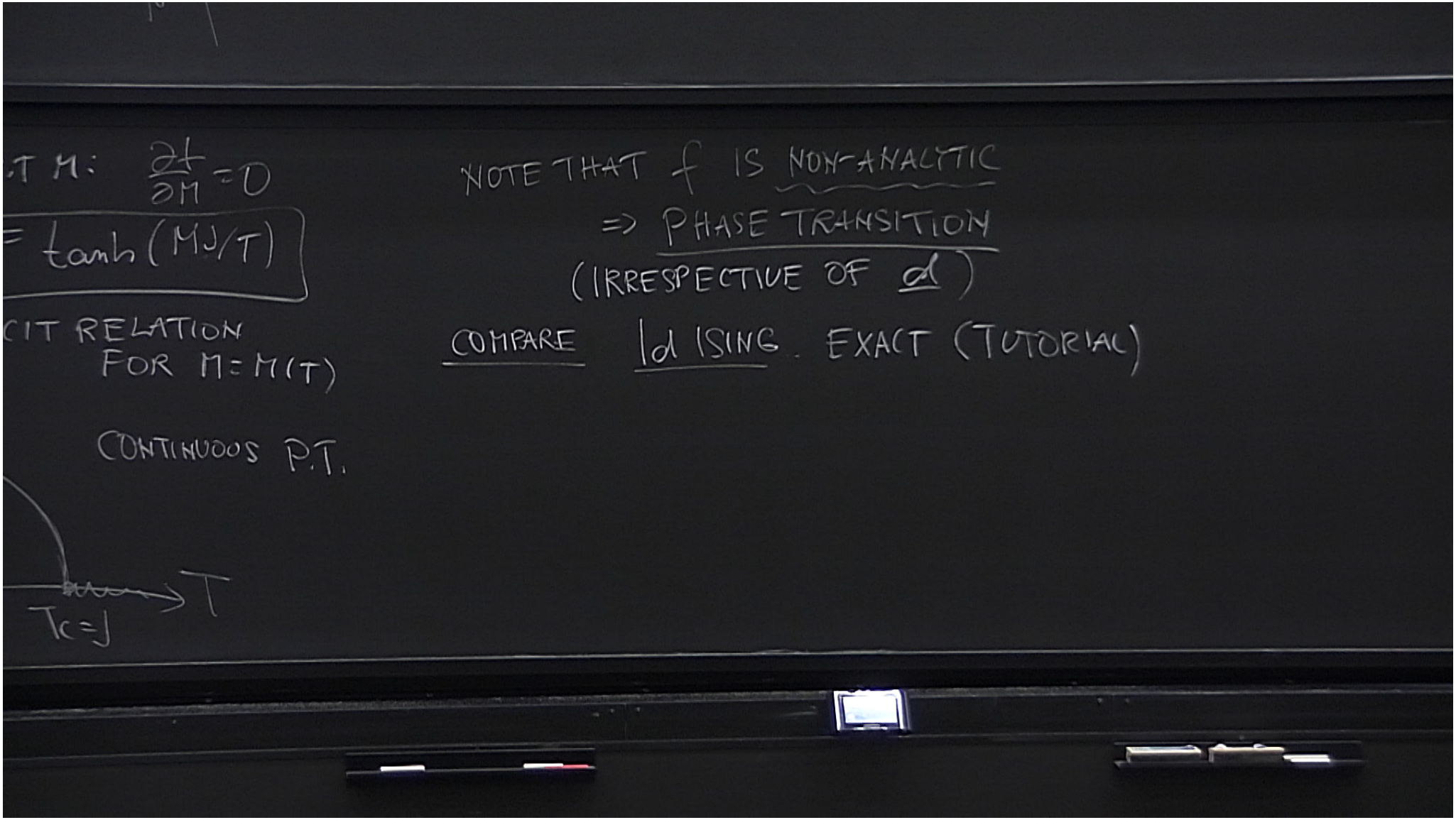
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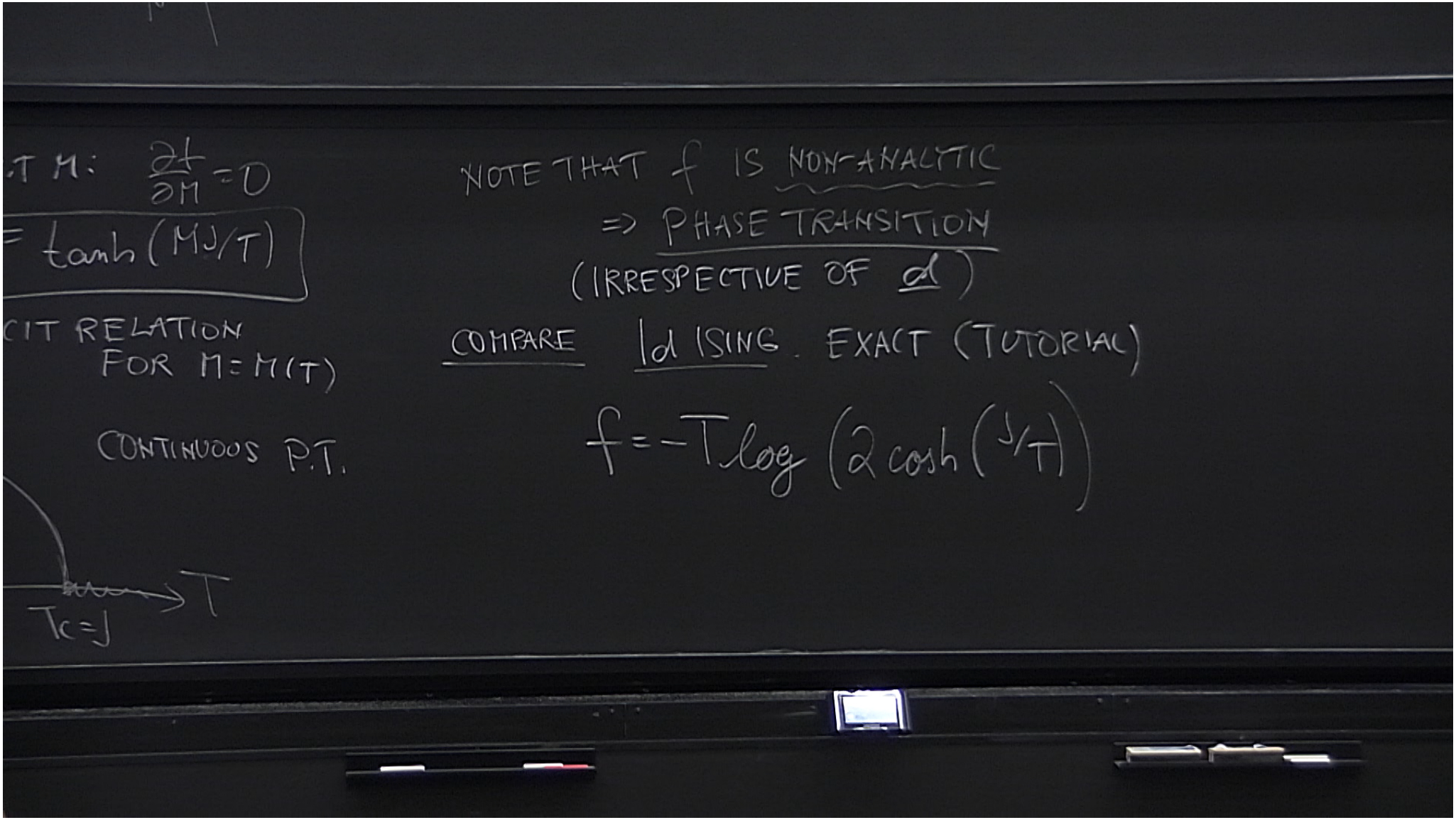
⇐ IMPLICIT RELATION FOR $M = M(T)$

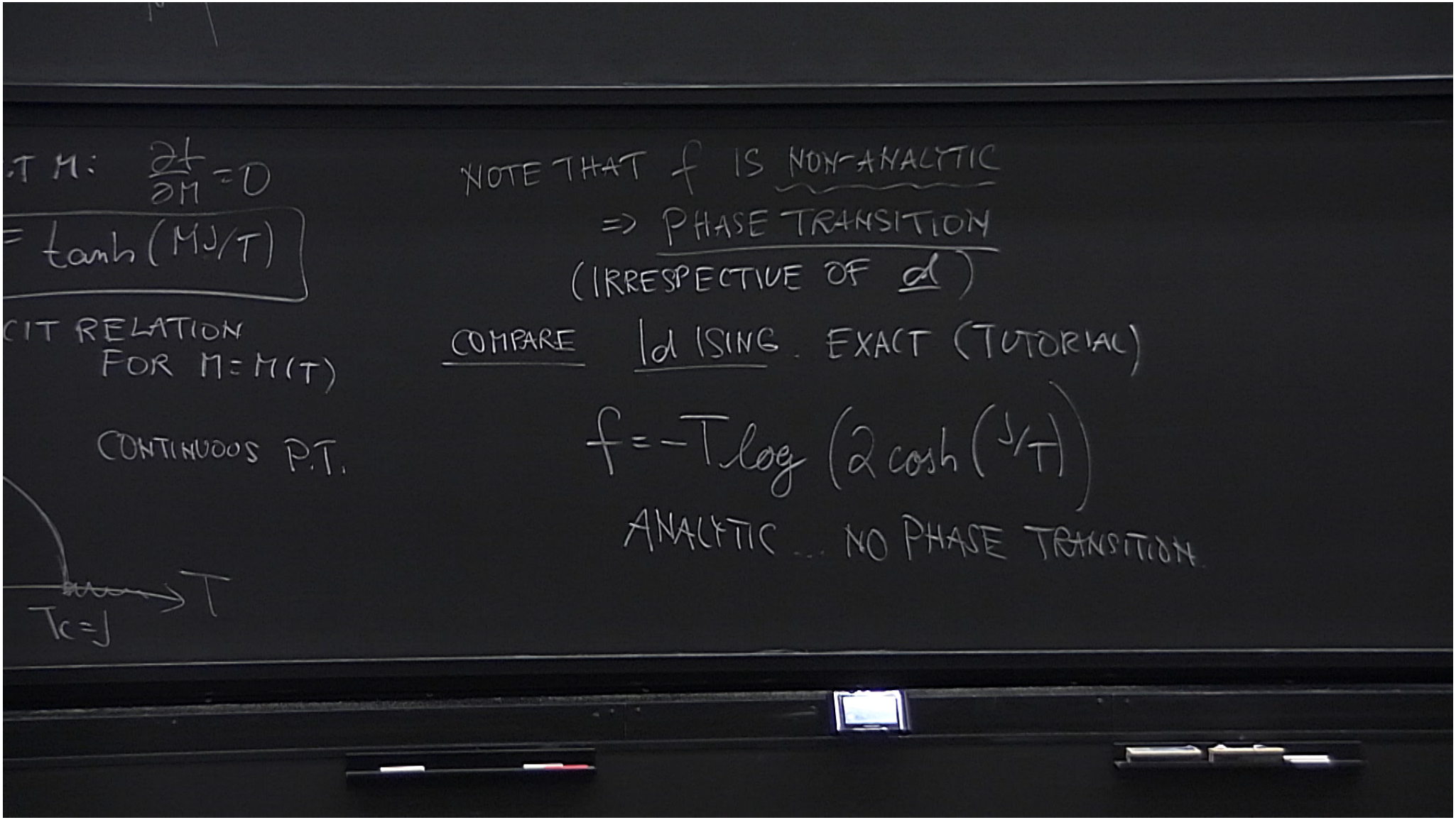


CONTINUOUS P.T.


T M: $\frac{\partial f}{\partial M} = 0$
$= \tanh(MJ/T)$
CIT RELATION
FOR $M = M(T)$
CONTINUOUS P.T.
NOTE THAT f IS NON-ANALYTIC
$\Rightarrow$ PHASE TRANSITION
(IRRESPECTIVE OF d)
$T_c = J$
T


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COMPARE 1d ISING. EXACT (TUTORIAL)


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ANALYTIC ... NO PHASE TRANSITION

NON-ANALYTIC

E TRANSITION

UE OF α)

G. EXACT (TUTORIAL)

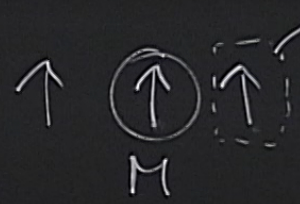
$\log\left(2\cosh\left(J/T\right)\right)$

C ... NO PHASE TRANSITION.

REASON ASTONISHY MFT BREAKS IN 1d

IS THAT FLUCTUATIONS ARE HUGE.

1d:



NON-ANALYTIC

E TRANSITION

UE OF d)

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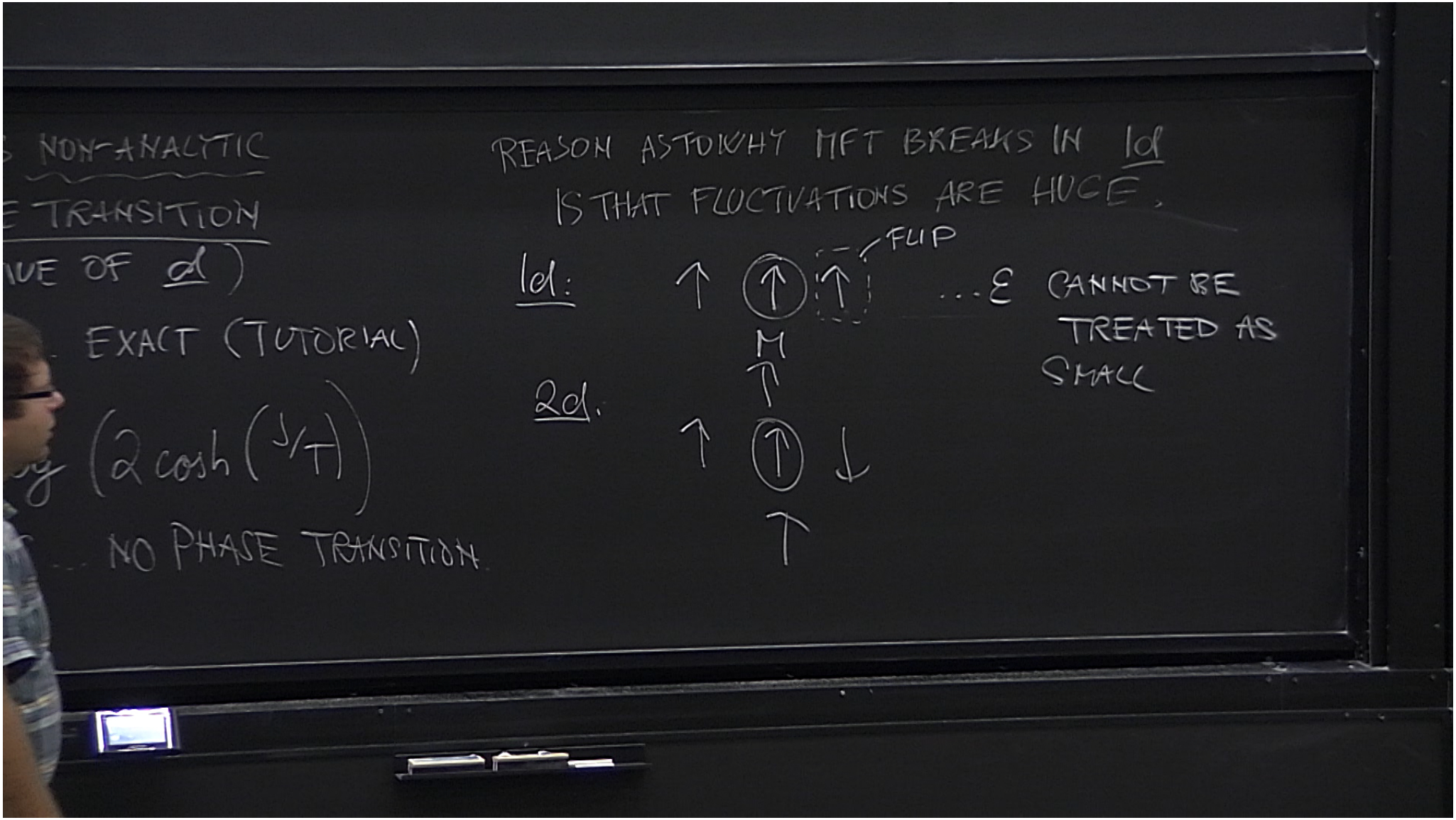
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REASON ASTOUNDING MFT BREAKS IN $1d$
IS THAT FLUCTUATIONS ARE HUGE.

$1d$: ↑ (↑) (↑) ... ε CANNOT BE TREATED AS SMALL

M

FLIP


NON-ANALYTIC
E TRANSITION
UE OF d)
EXACT (TUTORIAL)
(2 cosh (J/T))
NO PHASE TRANSITION
REASON ASTOUNDY MFT BREAKS IN 1d
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1d:
FLIP
... ε CANNOT BE TREATED AS SMALL
M
2d.

NON-ANALYTIC

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UE OF d)

G. EXACT (TUTORIAL)

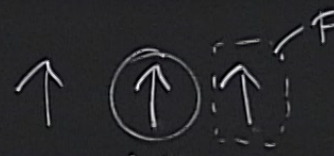
$\log\left(2\cosh\left(J/T\right)\right)$

C ... NO PHASE TRANSITION

REASON ASTOUNDING MFT BREAKS IN $1d$

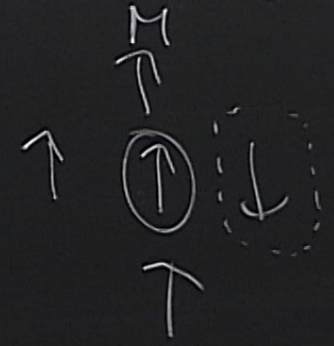
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$1d$:



... $\mathcal{E}$ CANNOT BE TREATED AS SMALL

$2d$.



NON-ANALYTIC

E TRANSITION

UE OF d)

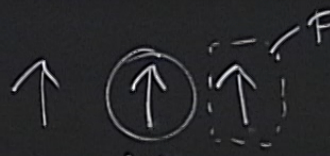
G. EXACT (TUTORIAL)

$\log\left(2\cosh\left(J/T\right)\right)$

C ... NO PHASE TRANSITION.

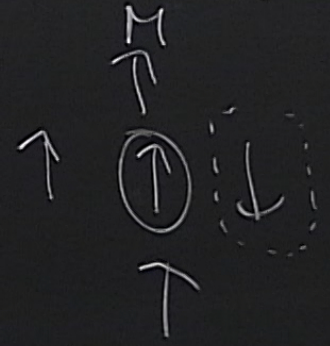
REASON ASTOUNDING MFT BREAKS IN 1d IS THAT FLUCTUATIONS ARE HUGE.

1d:

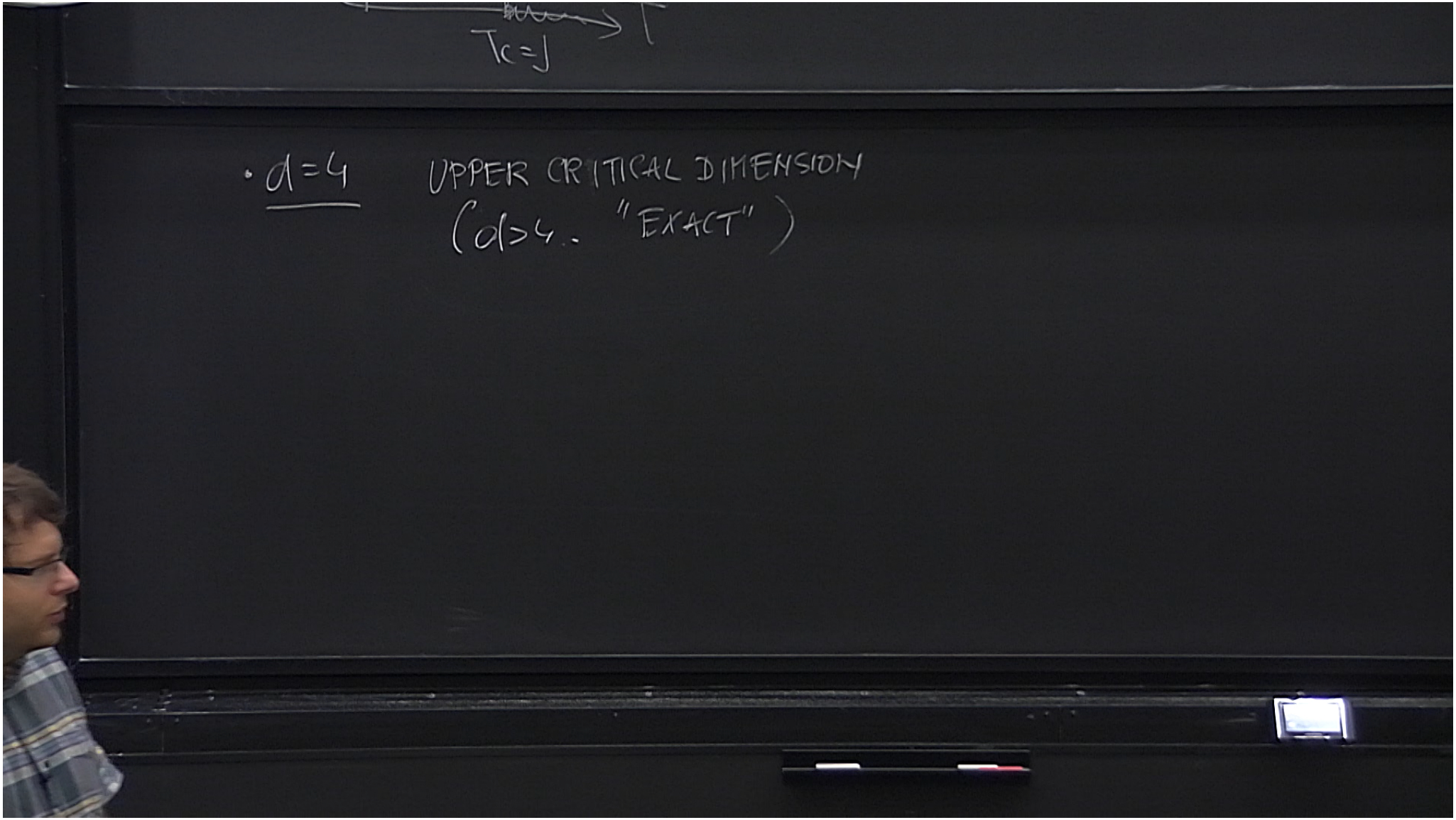


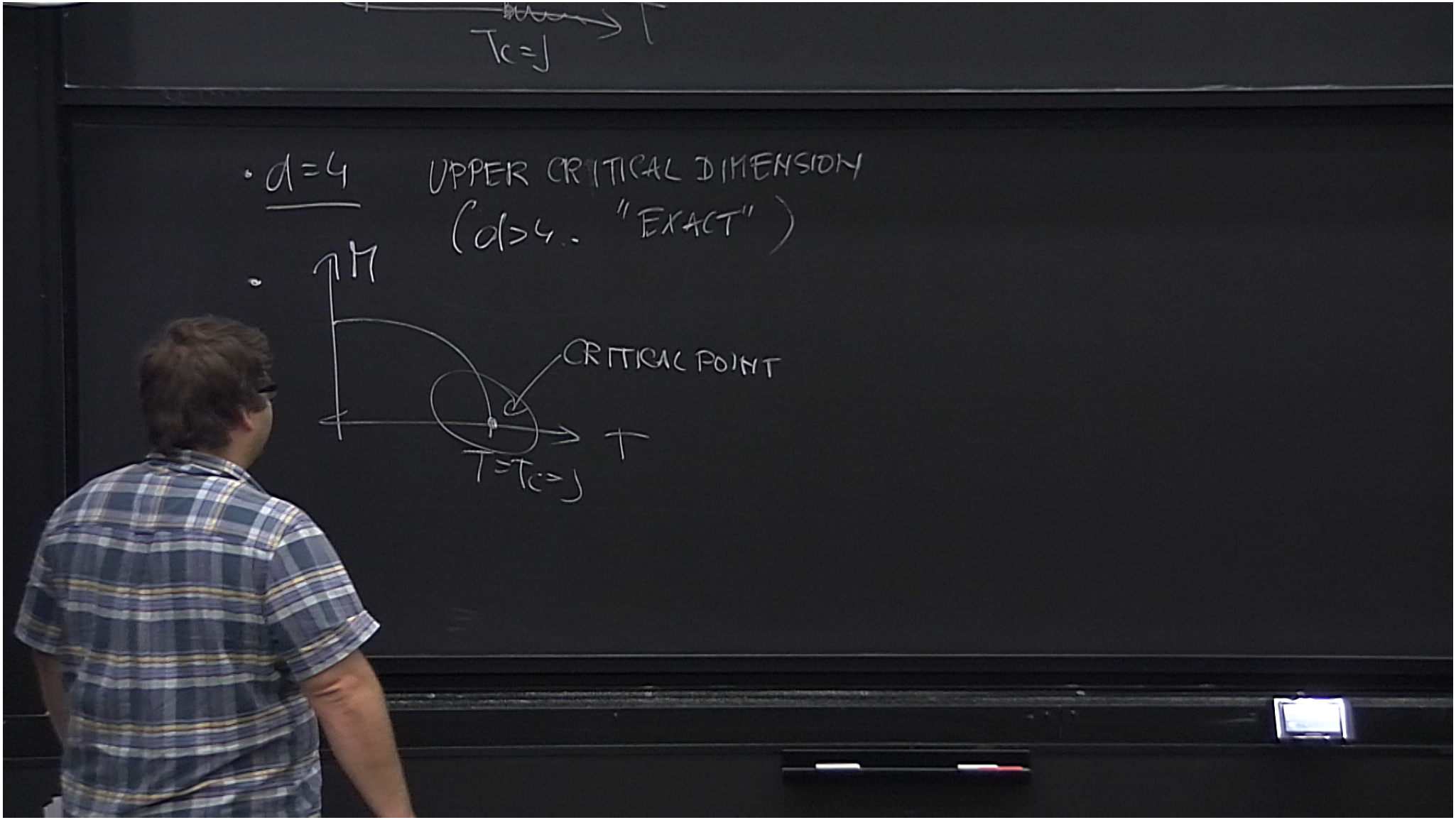
... ε CANNOT BE TREATED AS SMALL

2d.



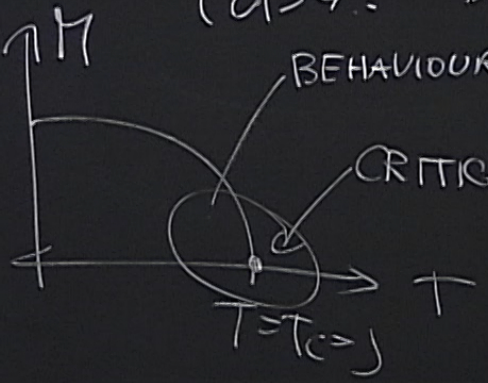
FLUCTUATIONS ARE LESS IMPORTANT.


Tc=J
• d=4 UPPER CRITICAL DIMENSION
(d>4. "EXACT")


$T_c = J$
$d = 4$ UPPER CRITICAL DIMENSION
($d > 4$. "EXACT")
M
CRITICAL POINT
T
$T = T_c = J$

$T_c = J$

- $d = 4$ UPPER CRITICAL DIMENSION ($d > 4$. "EXACT")
-

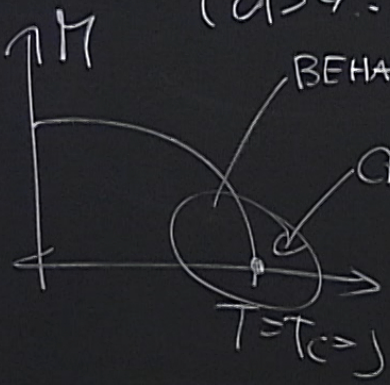


BEHAVIOUR OF PHYSICAL QUANTITIES DESCRIBED BY CRITICAL EXPONENTS

CRITICAL POINT

$T_c = J$

- $d = 4$ UPPER CRITICAL DIMENSION
  ($d > 4$. "EXACT")
-



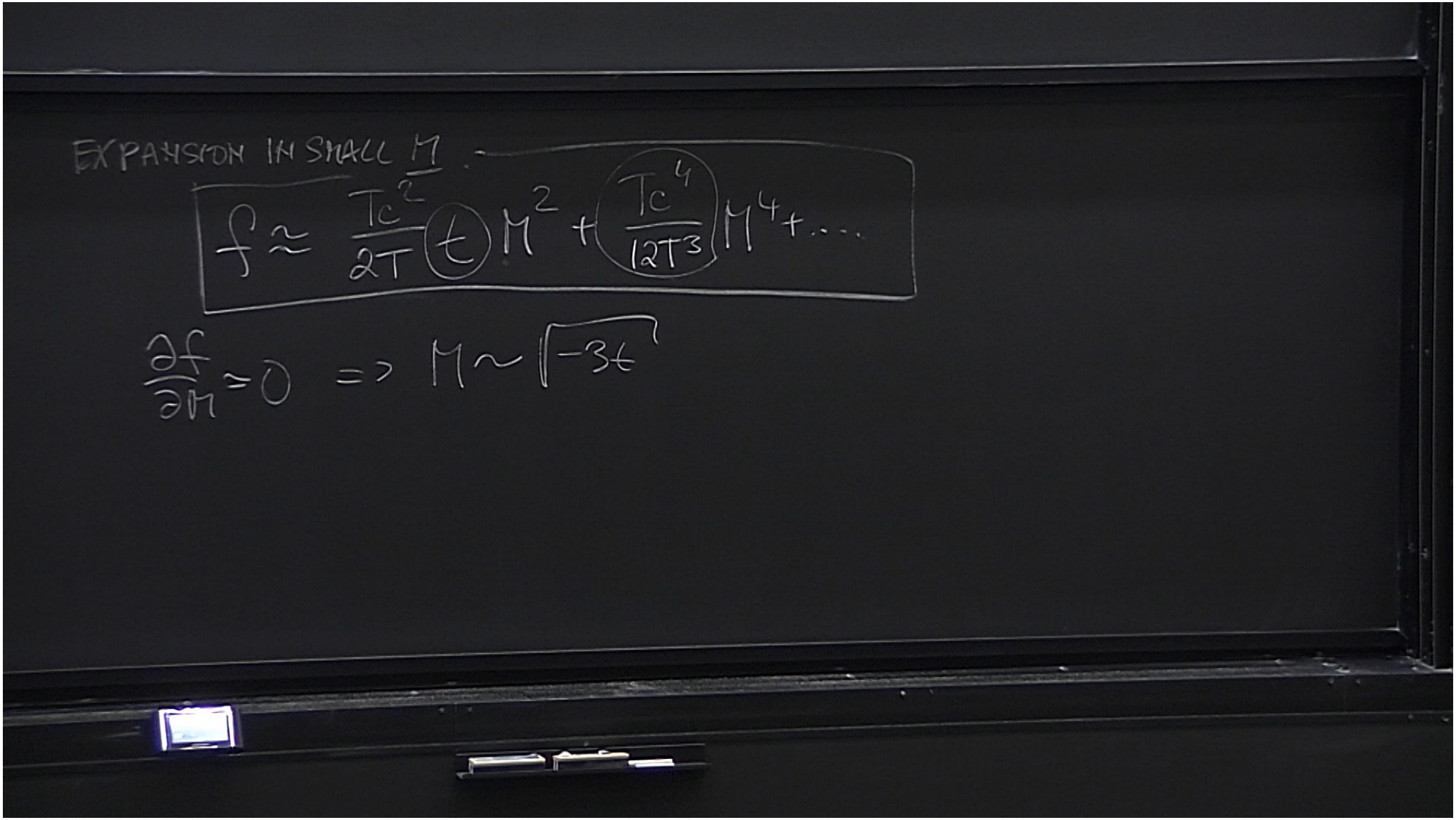
BEHAVIOUR OF PHYSICAL QUANTITIES DESCRIBED BY CRITICAL EXPONENTS

CRITICAL POINT

$$t = \frac{T - T_c}{T_c}$$

EXPANSION IN SMALL M

$$f \approx \frac{T_c^2}{2T} t M^2 + \frac{T_c^4}{12T^3} M^4 + \dots$$


EXPANSION IN SMALL M
$f \simeq \frac{T_c^2}{2T}(t)M^2 + \frac{T_c^4}{12T^3}M^4 + \ldots$
$\frac{\partial f}{\partial M} = 0 \Rightarrow M \sim \sqrt{-3t}$

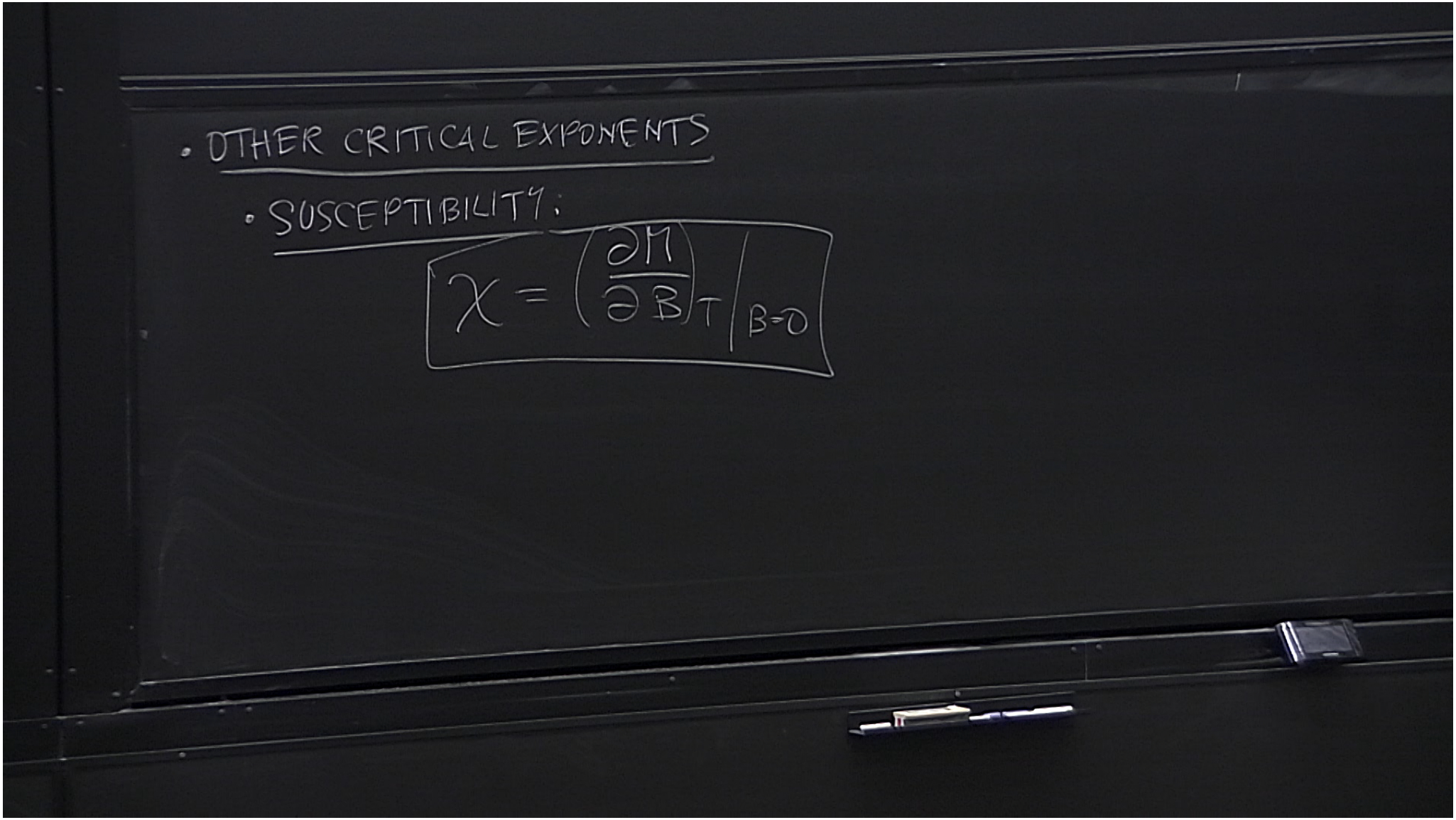
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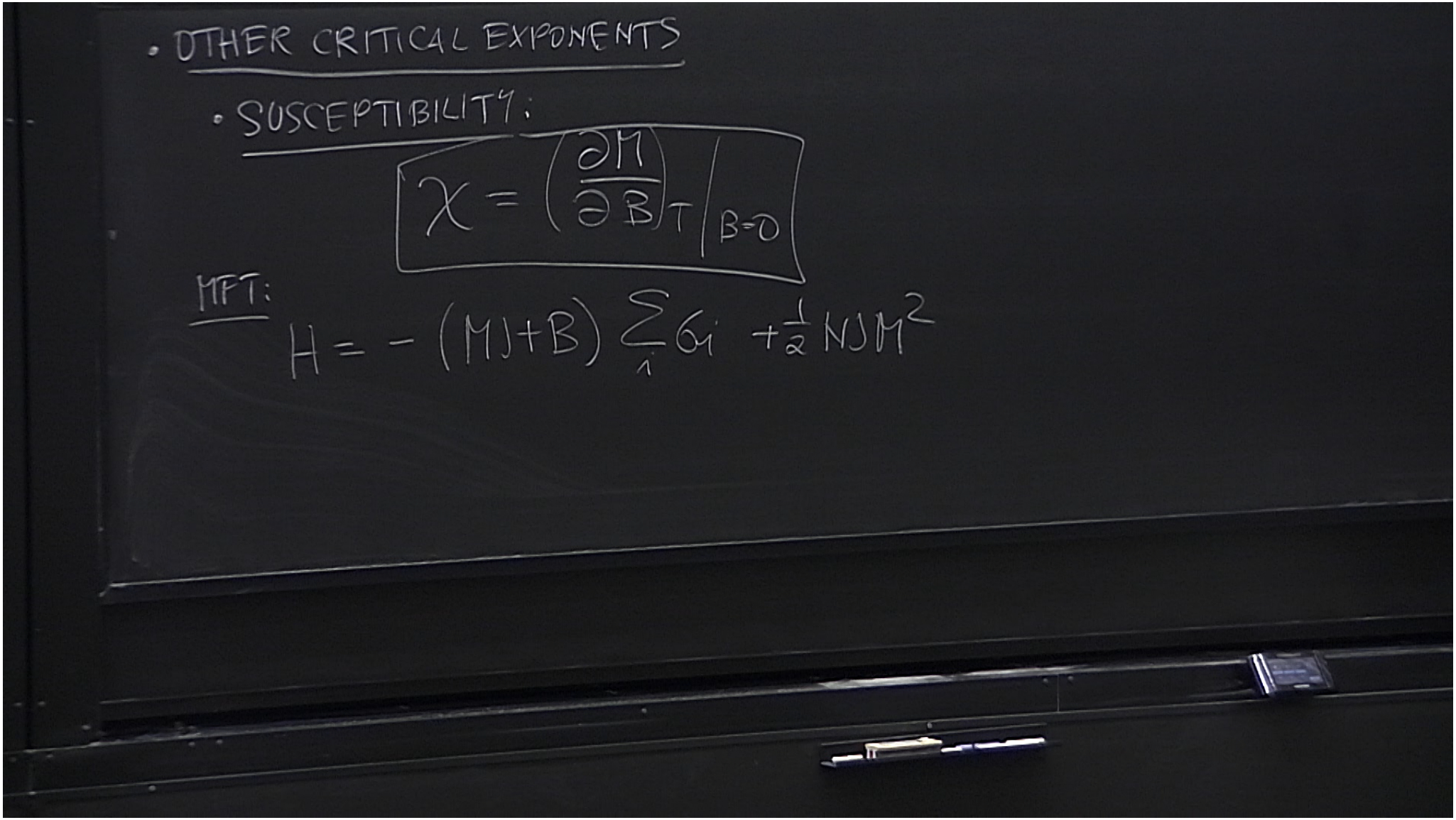
$$f \approx \frac{T_c^2}{2T}\, t\, M^2 + \frac{T_c^4}{12T^3} M^4 + \dots$$

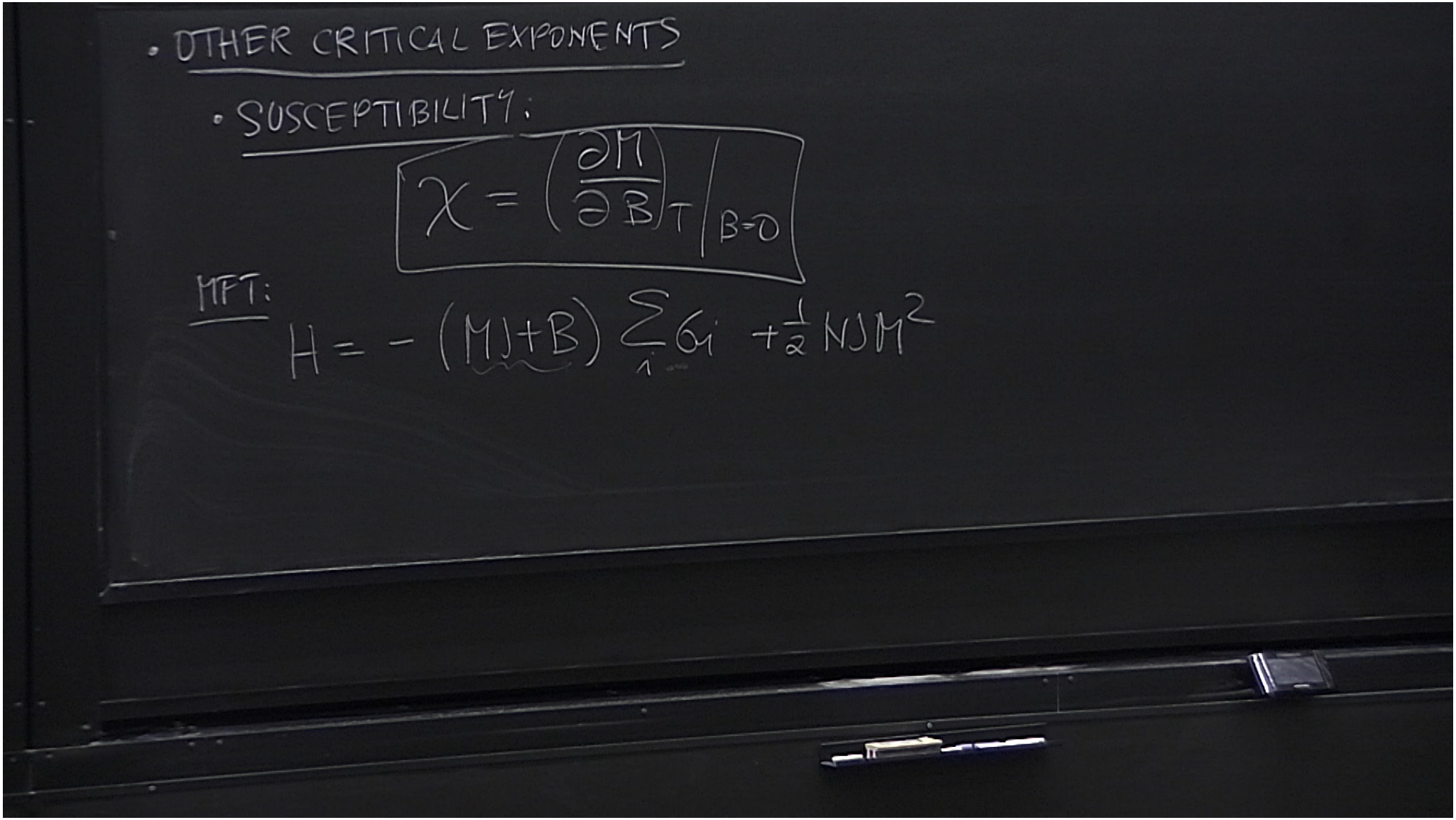
$$\frac{\partial f}{\partial M} = 0 \Rightarrow M \sim \sqrt{-3t} \sim (-t)^{\beta}$$

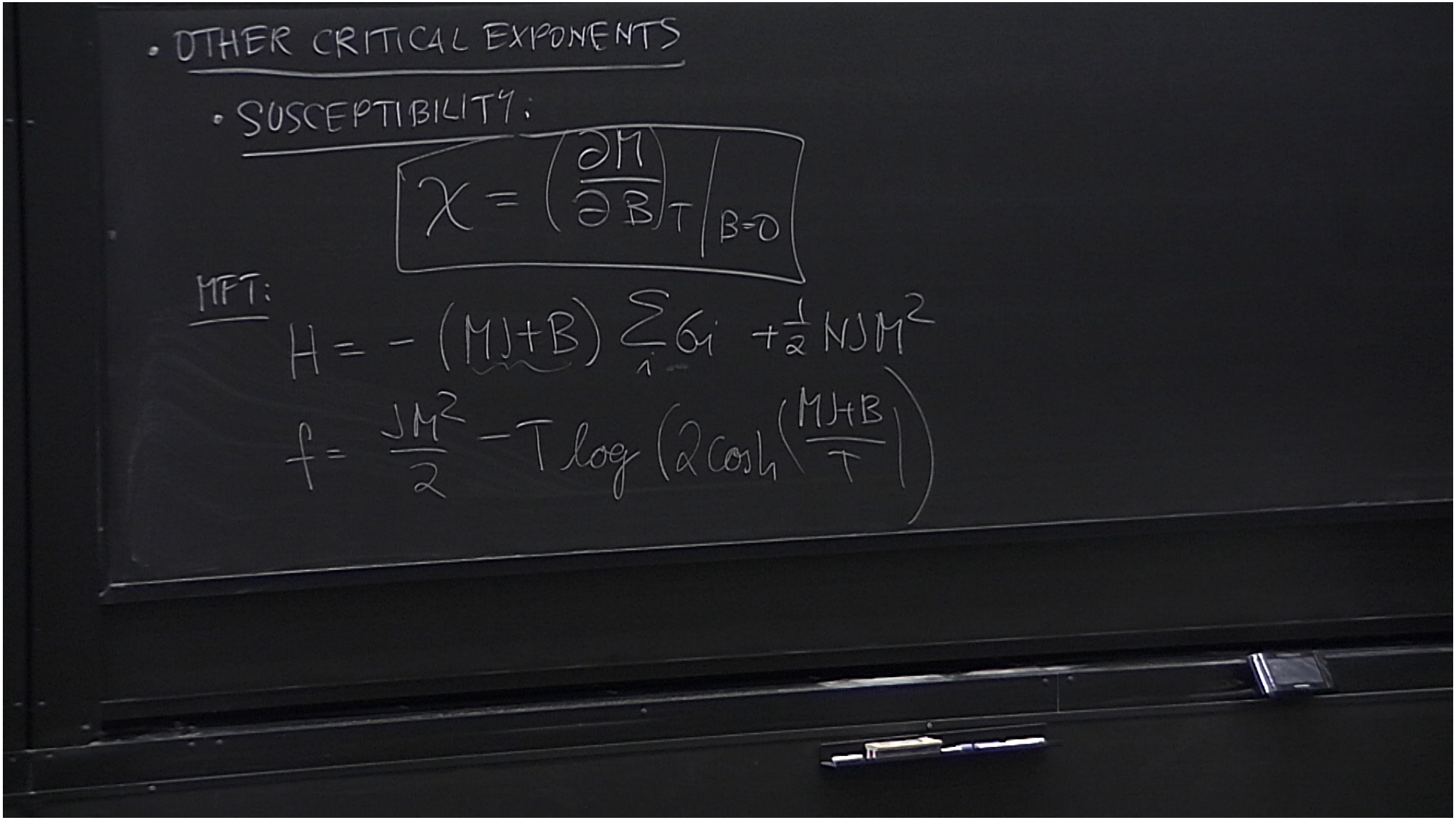
β ... CRITICAL EXPONENT FOR ORDER PARAM.

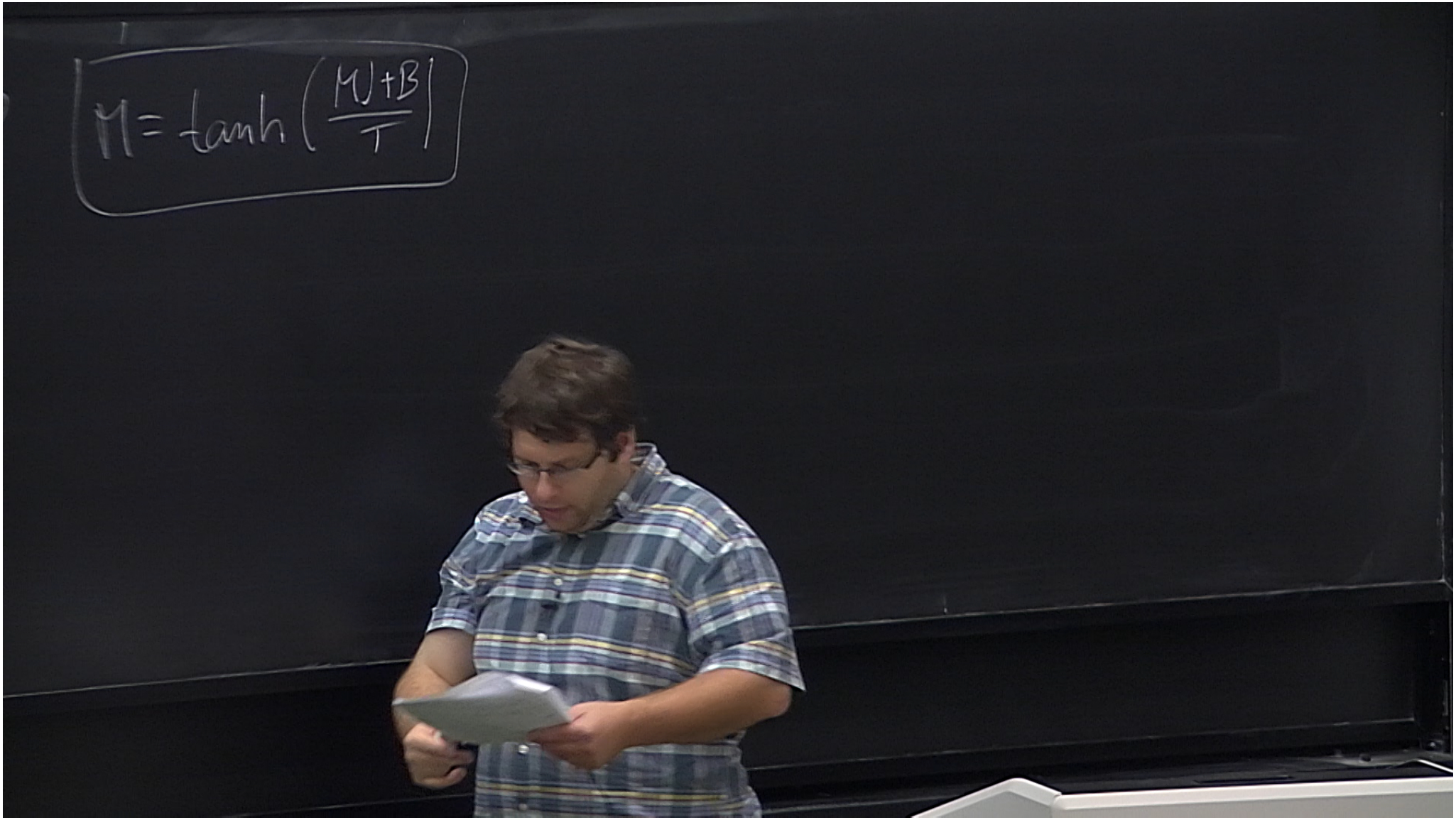
$$\beta = \tfrac{1}{2}$$


• OTHER CRITICAL EXPONENTS
• SUSCEPTIBILITY:
$\chi = \left(\frac{\partial M}{\partial B}\right)_T \Big|_{B=0}$


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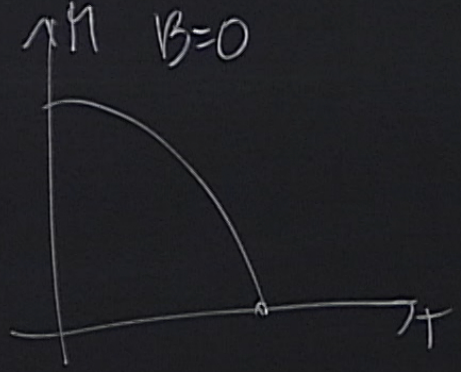

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$H = -(MJ+B)\sum_i \sigma_i + \frac{1}{2}NJM^2$
$f = \frac{JM^2}{2} - T\log\left(2\cosh\left(\frac{MJ+B}{T}\right)\right)$


M = tanh(MJ+B/T)

$\frac{\partial f}{\partial m} = 0$

$$m = \tanh\left(\frac{mJ+B}{T}\right)$$

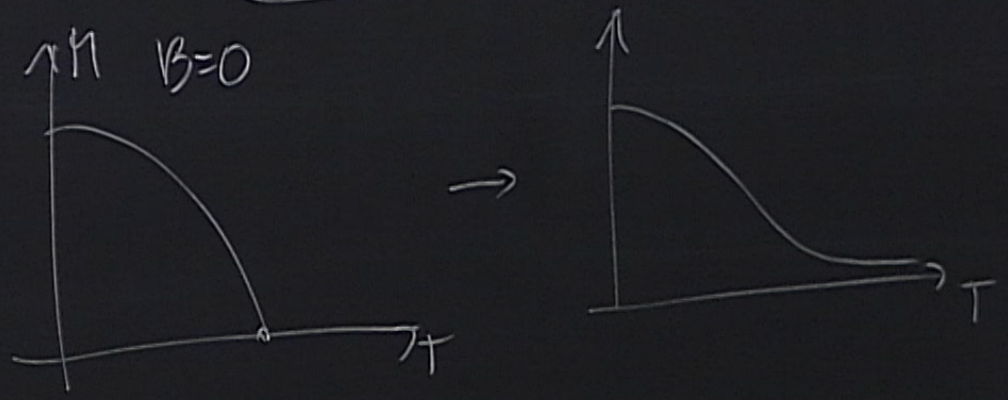


NJm^2

$\frac{mJ+B}{T}$

$\frac{\partial f}{\partial M} = 0$

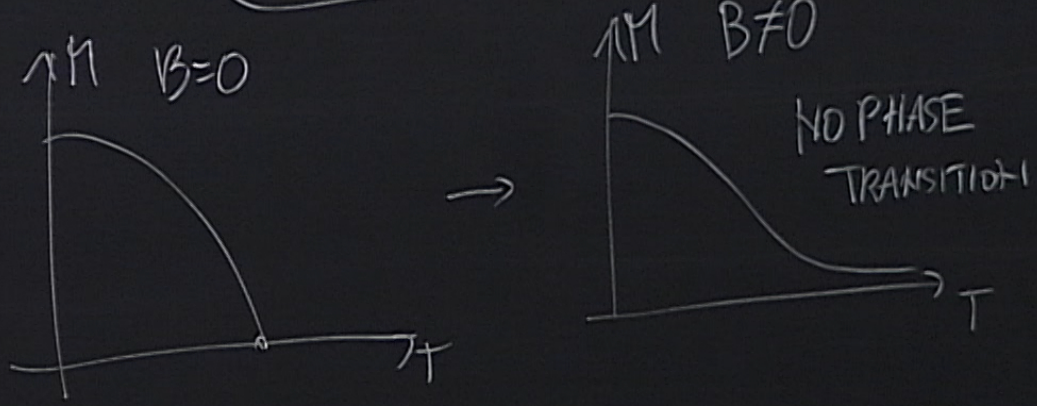
$$M = \tanh\left(\frac{MJ+B}{T}\right)$$



NJM^2

$\left(\frac{MJ+B}{T}\right)$

$\cdot \frac{\partial f}{\partial M} = 0$ $\boxed{M = \tanh\left(\frac{MJ+B}{T}\right)}$



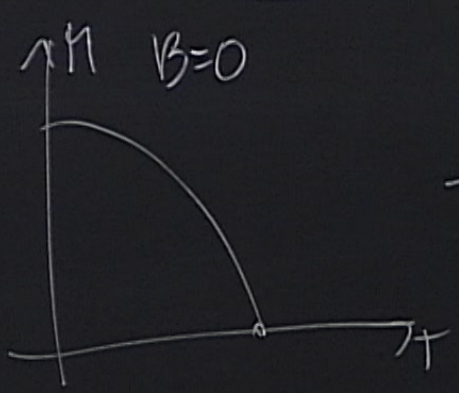
NJM^2

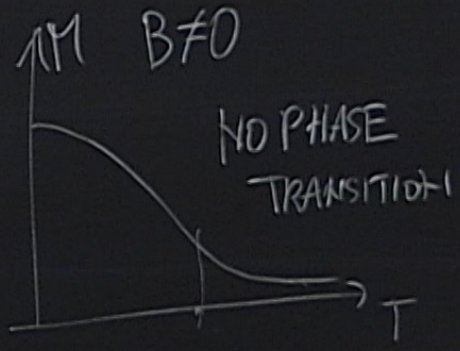
$\left(\frac{MJ+B}{T}\right)$

$\frac{\partial f}{\partial M} = 0$

$$M = \tanh\left(\frac{MJ+B}{T}\right)$$

B SMALL =>





NJM^2

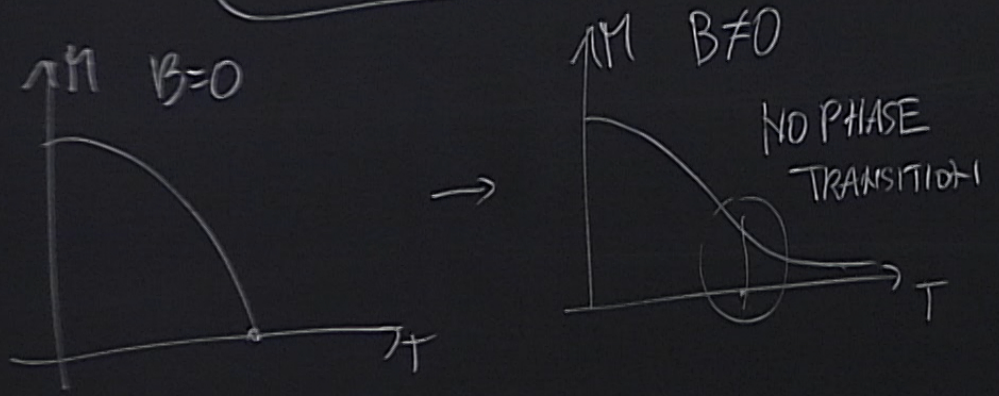
$\left(\frac{MJ+B}{T}\right)$

$\frac{\partial f}{\partial m} = 0$

$$M = \tanh\left(\frac{MJ + B}{T}\right)$$

B SMALL $\Rightarrow$ M SMALL AS WELL.

$\tanh x = x - x^3/3 + \ldots$



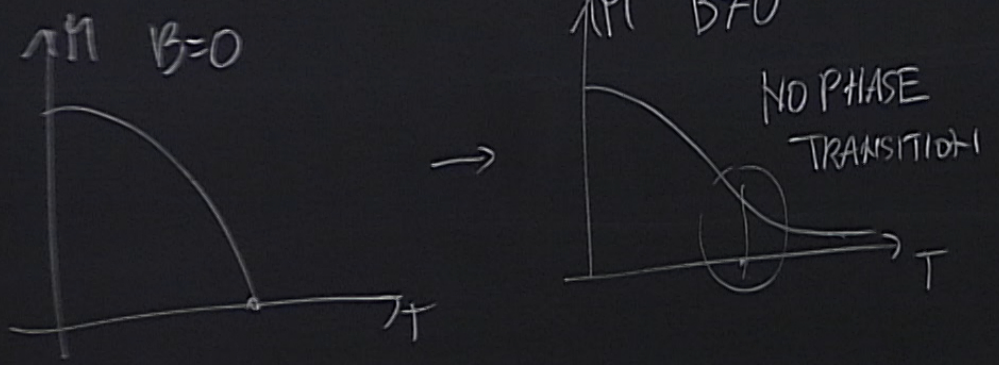
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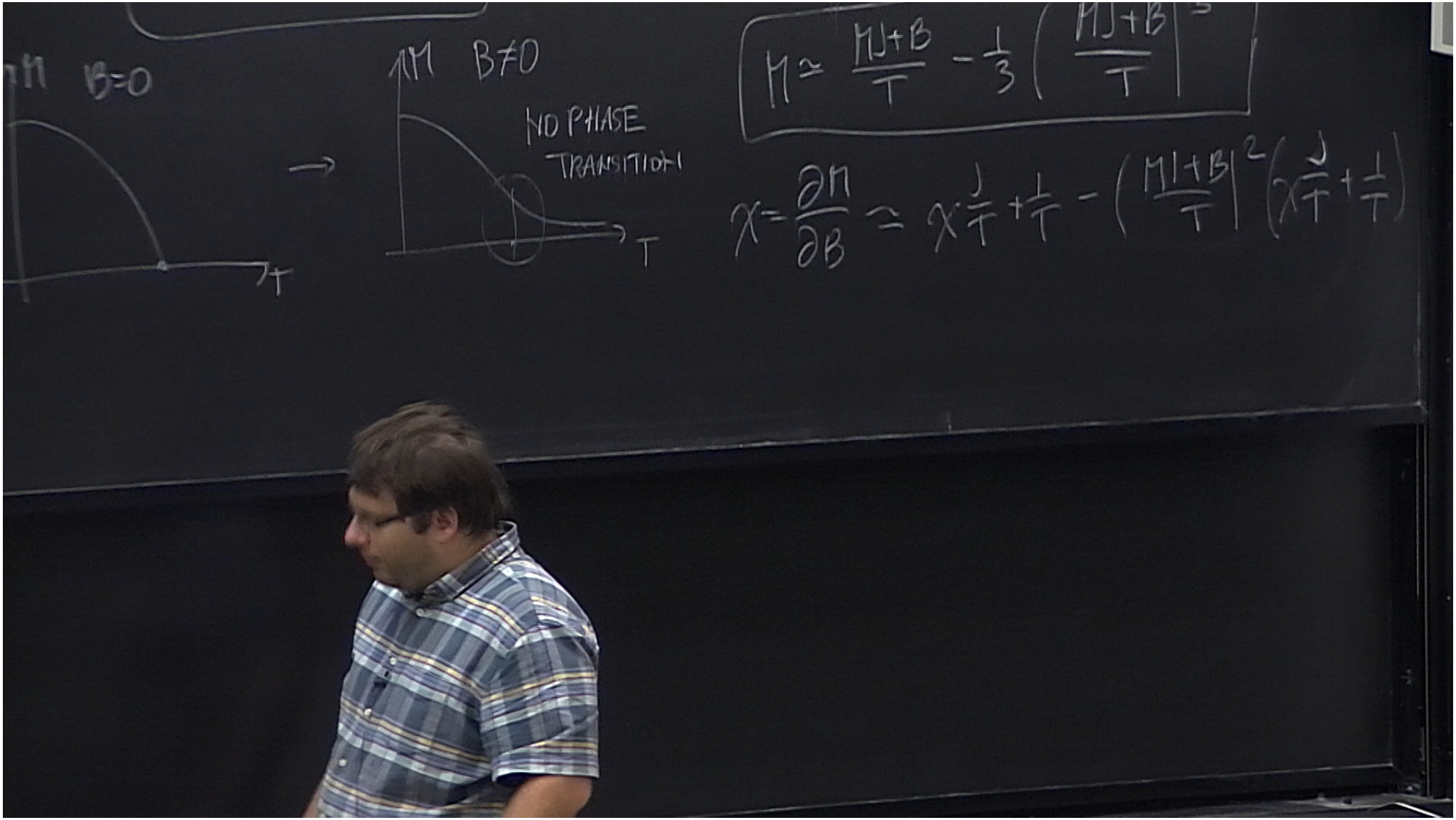
$$M = \tanh\left(\frac{MJ+B}{T}\right)$$

B SMALL $\Rightarrow$ M SMALL AS WELL.

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$$M \simeq \frac{MJ+B}{T} - \frac{1}{3}\left(\frac{MJ+B}{T}\right)^3$$



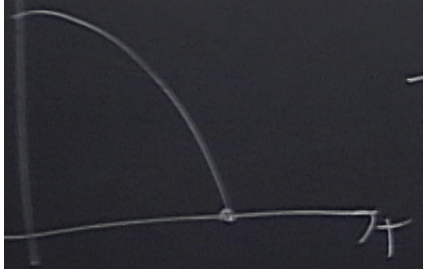

B=0
T
M B≠0
NO PHASE TRANSITION
T
$M \simeq \frac{MJ+B}{T} - \frac{1}{3}\left(\frac{MJ+B}{T}\right)^3$
$\chi = \frac{\partial M}{\partial B} \simeq \chi\frac{J}{T} + \frac{1}{T} - \left(\frac{MJ+B}{T}\right)^2\left(\chi\frac{J}{T} + \frac{1}{T}\right)$

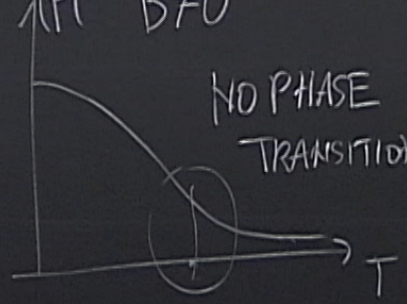
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B SMALL => M SMALL AS WELL.

$\tanh x = x - x^3/3 + \ldots$

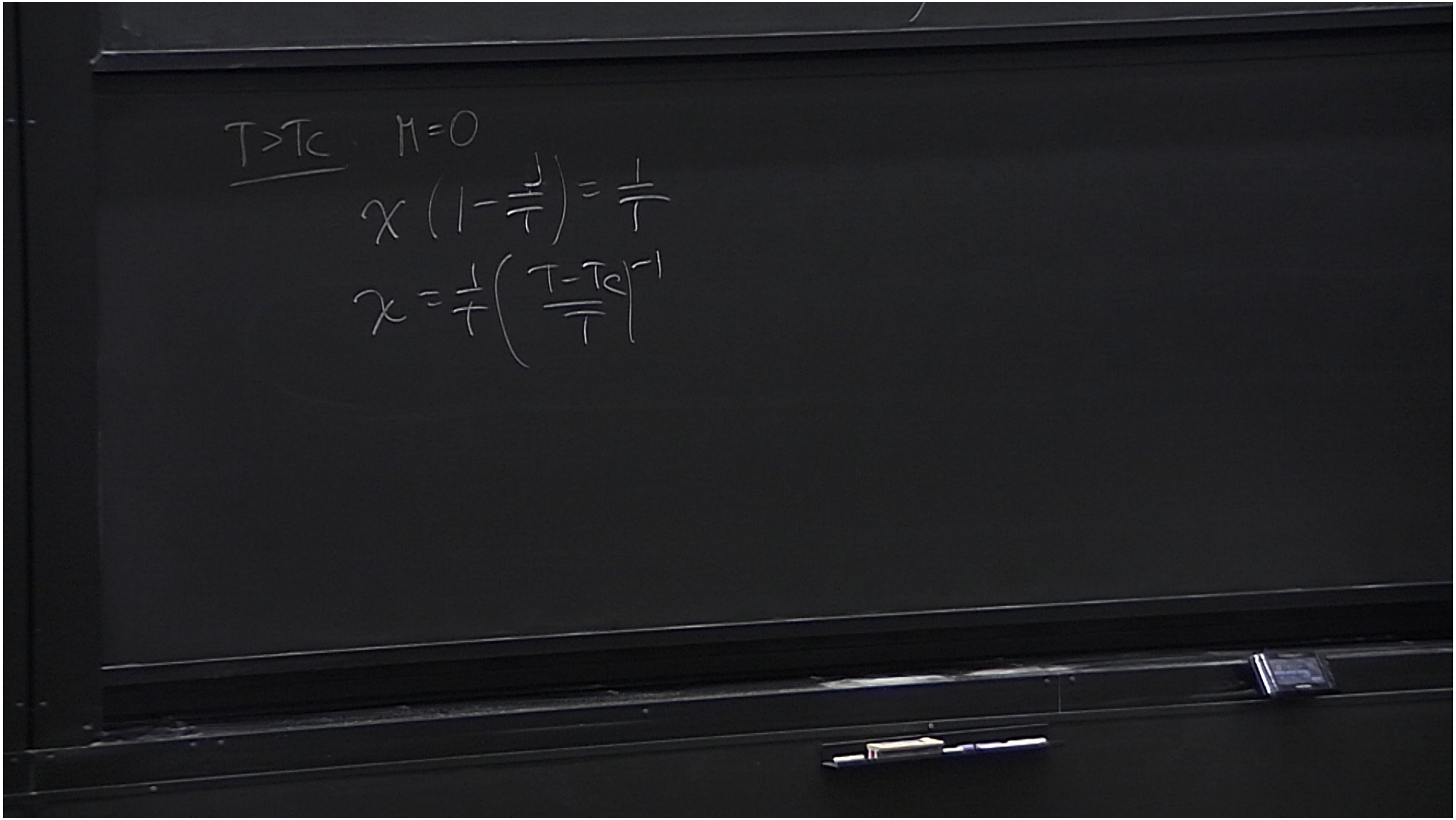
$M \simeq \frac{MJ+B}{T} - \frac{1}{3}\left(\frac{MJ+B}{T}\right)^3$

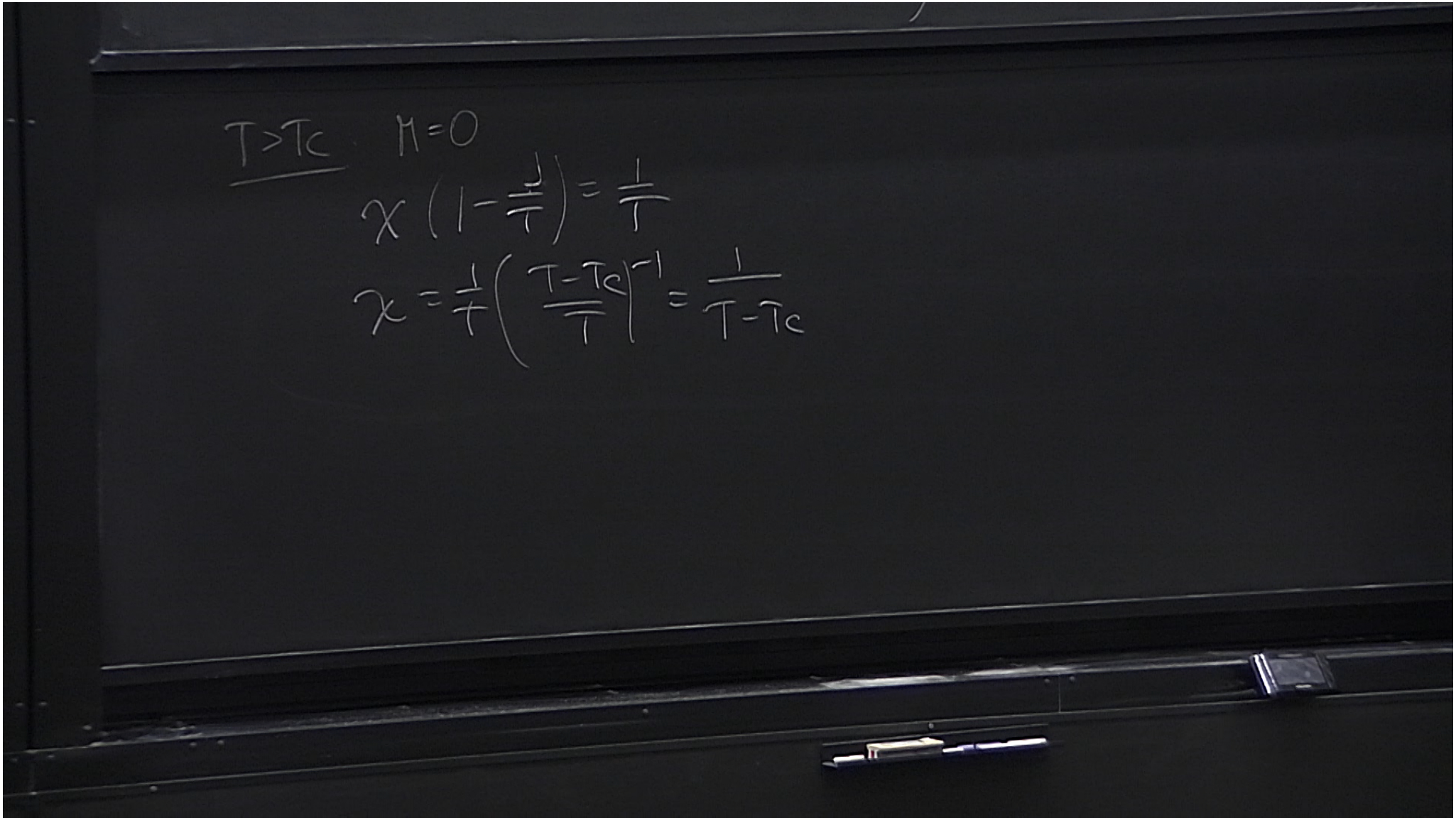


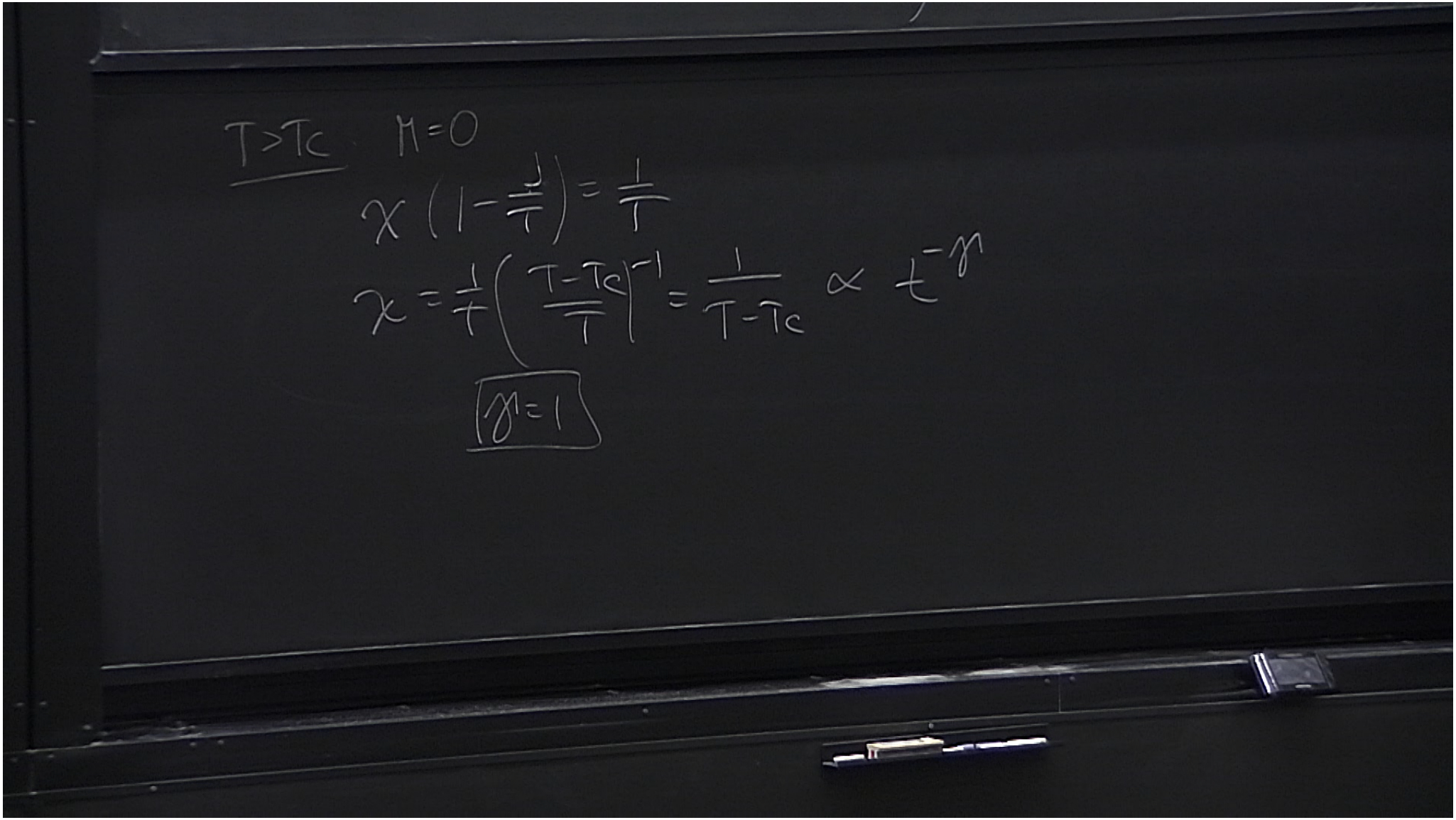


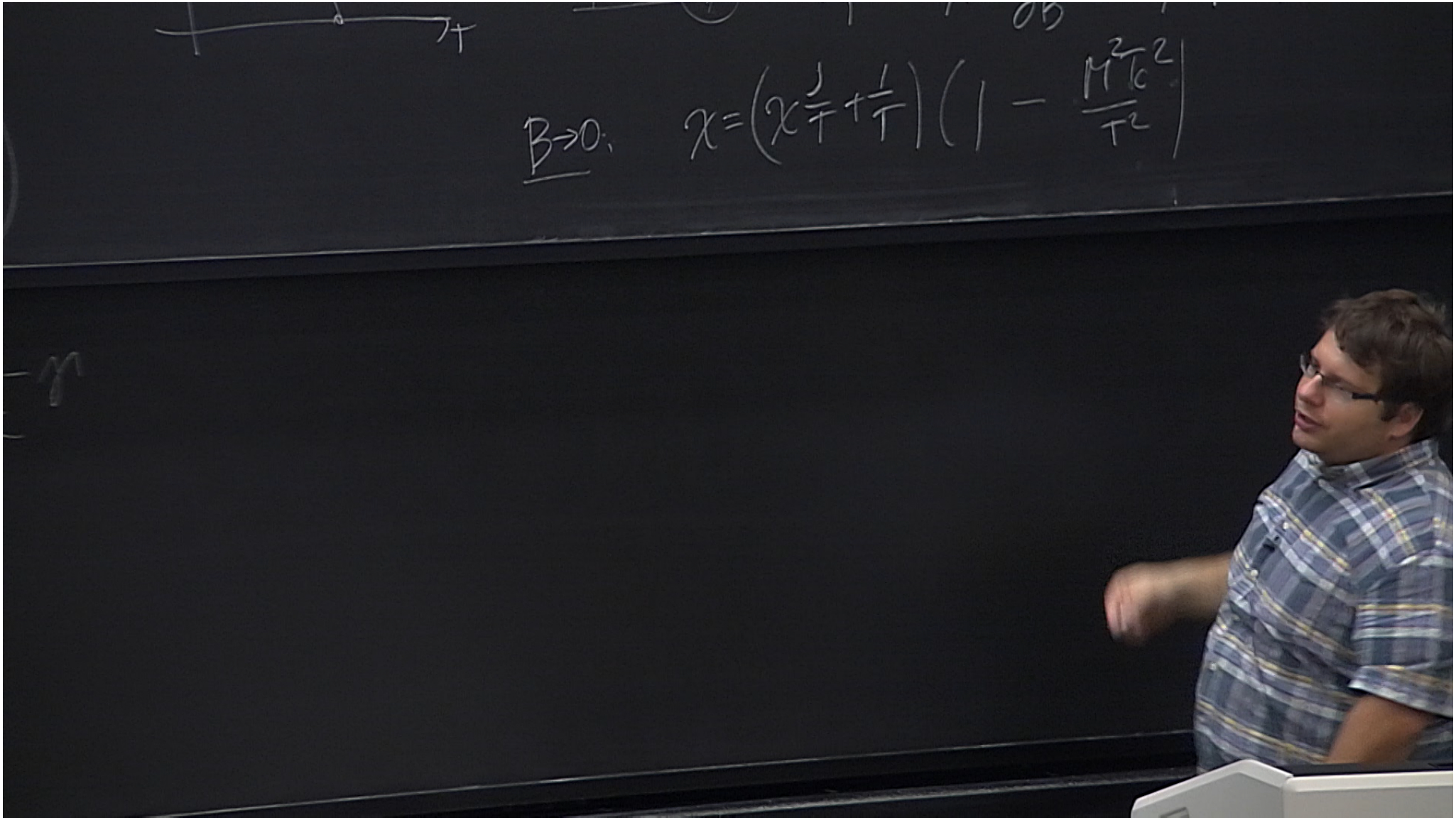
$\chi = \frac{\partial M}{\partial B} \simeq \chi\frac{J}{T} + \frac{1}{T} - \left(\frac{MJ+B}{T}\right)^2\left(\chi\frac{J}{T} + \frac{1}{T}\right)$

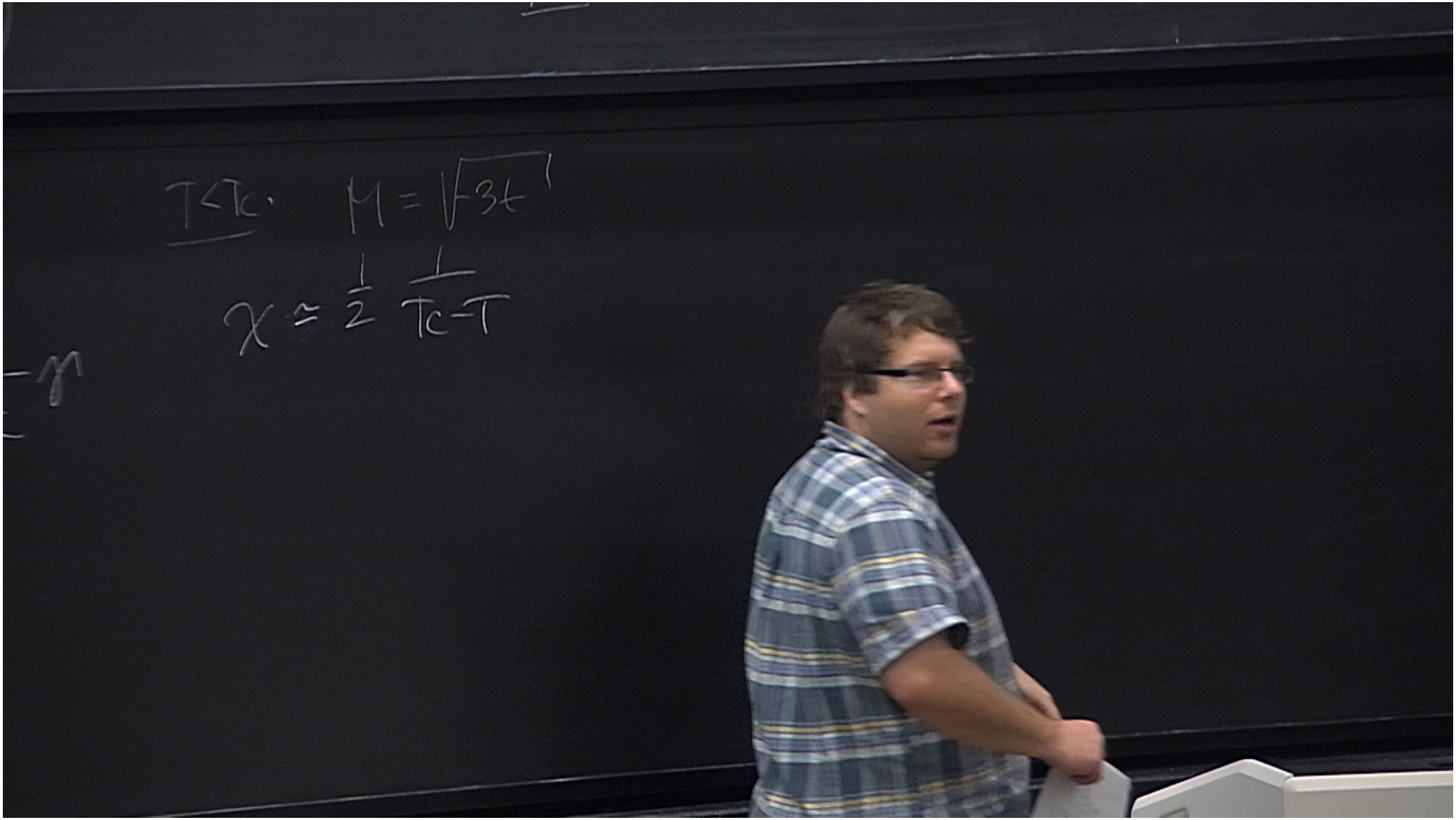
$B \to 0: \quad \chi = \left(\chi\frac{J}{T} + \frac{1}{T}\right)\left(1 - \frac{MT_c^2}{T^2}\right)$

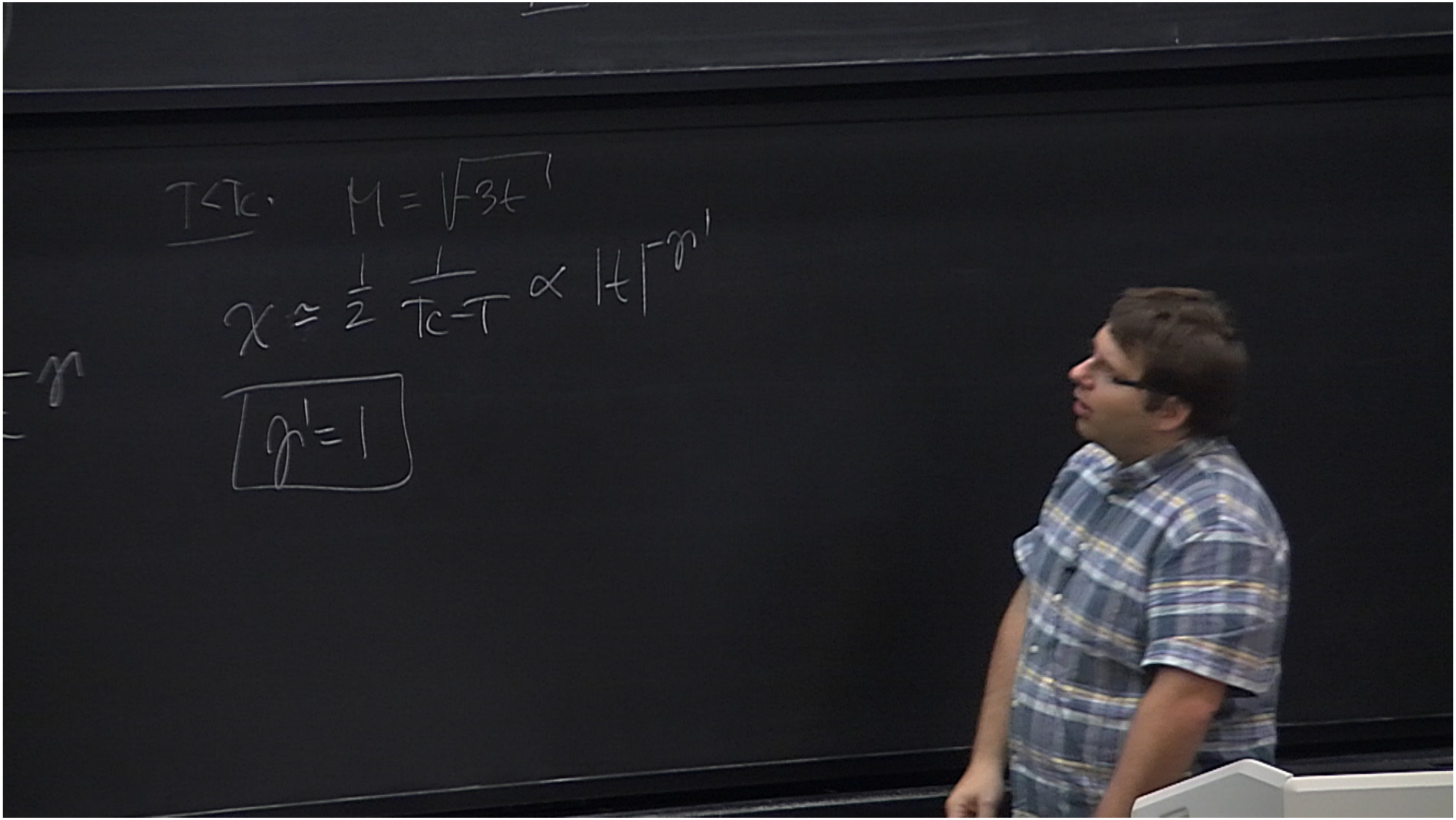

T>Tc M=0
χ(1−J/T) = 1/T
χ = 1/T (T−Tc)^−1


T>Tc M=0
χ(1 − J/T) = 1/T
χ = 1/T (T−Tc/T)^−1 = 1/(T−Tc)


T>Tc M=0
χ(1 − J/T) = 1/T
χ = 1/T (T−Tc/T)^−1 = 1/(T−Tc) ∝ t^−γ
γ=1


B→0:
$\chi = \left(\chi \frac{J}{T} + \frac{1}{T}\right)\left(1 - \frac{M^2 T_c^2}{T^2}\right)$


T<Tc
M = √3t
χ ≃ 1/2 1/Tc−T

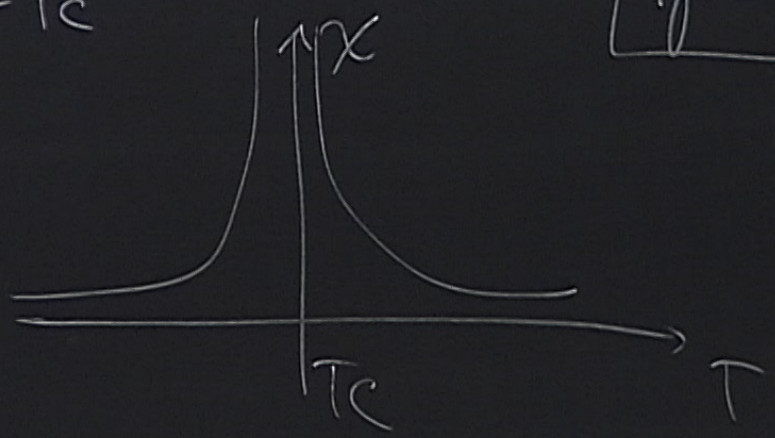

T<Tc. M = √3t^1
χ ≃ 1/2 1/(Tc−T) ∝ |t|^−γ'
γ' = 1

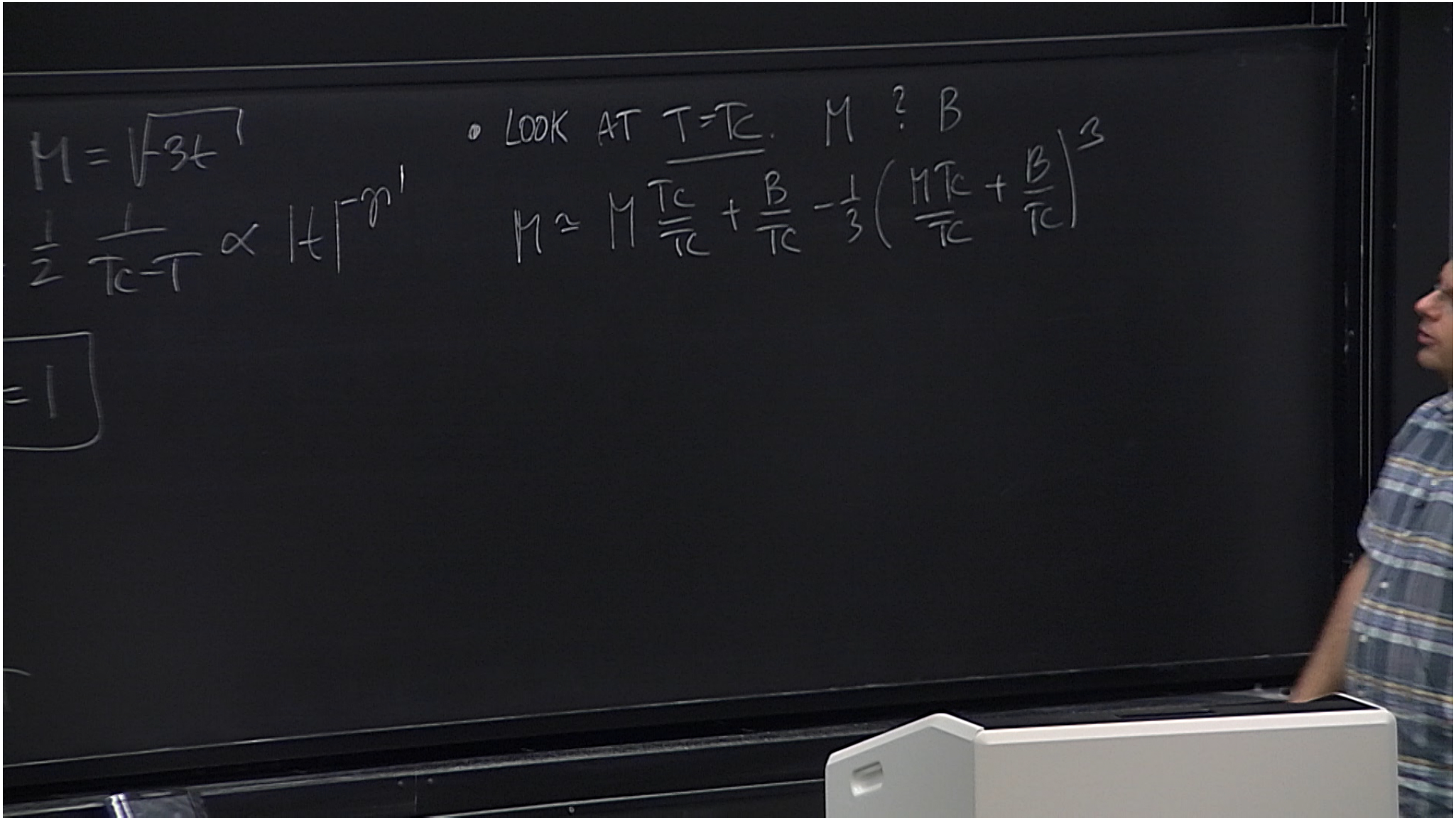
$= \frac{1}{T}$

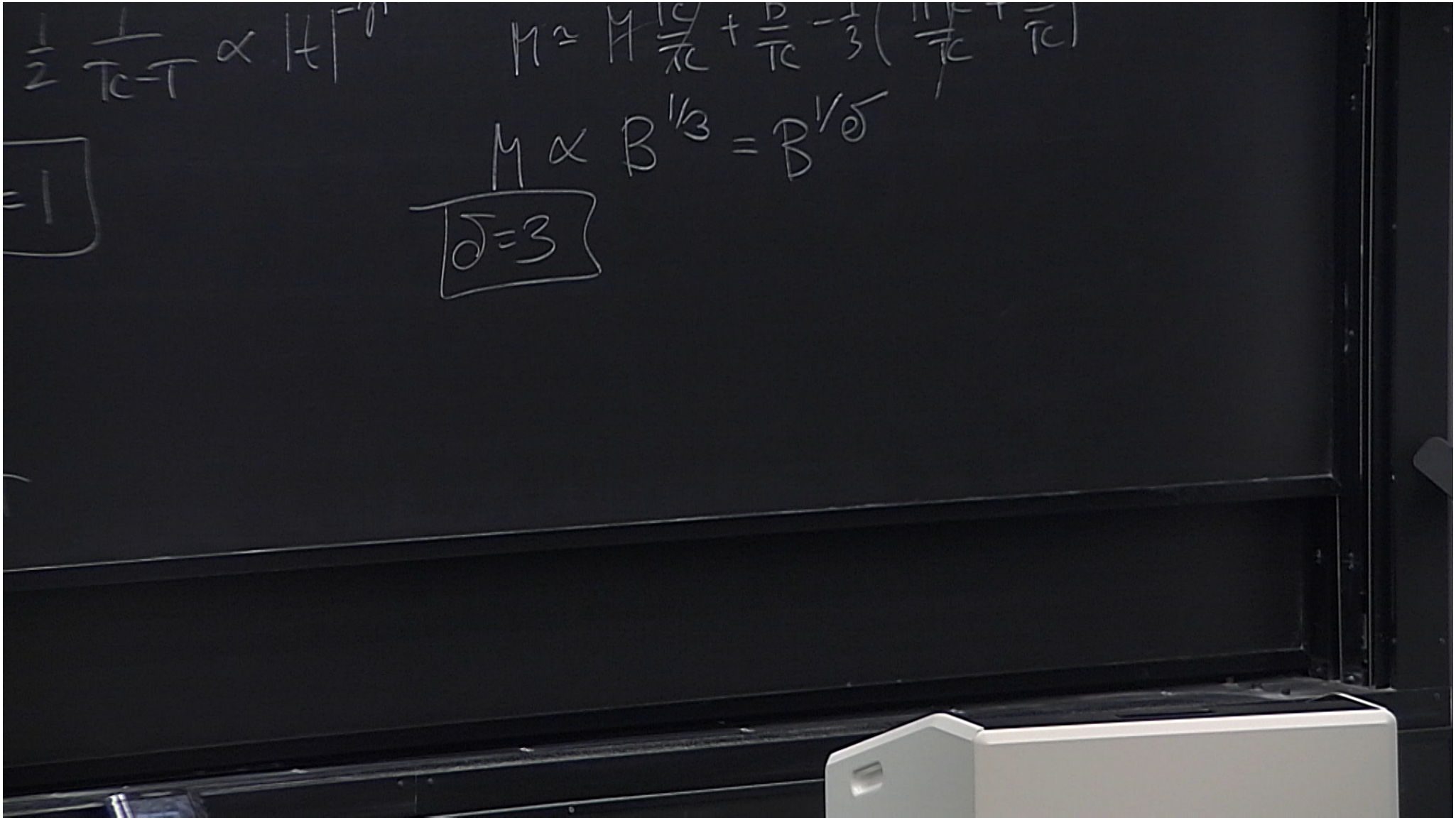
$\frac{T-T_c}{T}\Big|^{-1} = \frac{1}{T-T_c} \propto t^{-\gamma}$

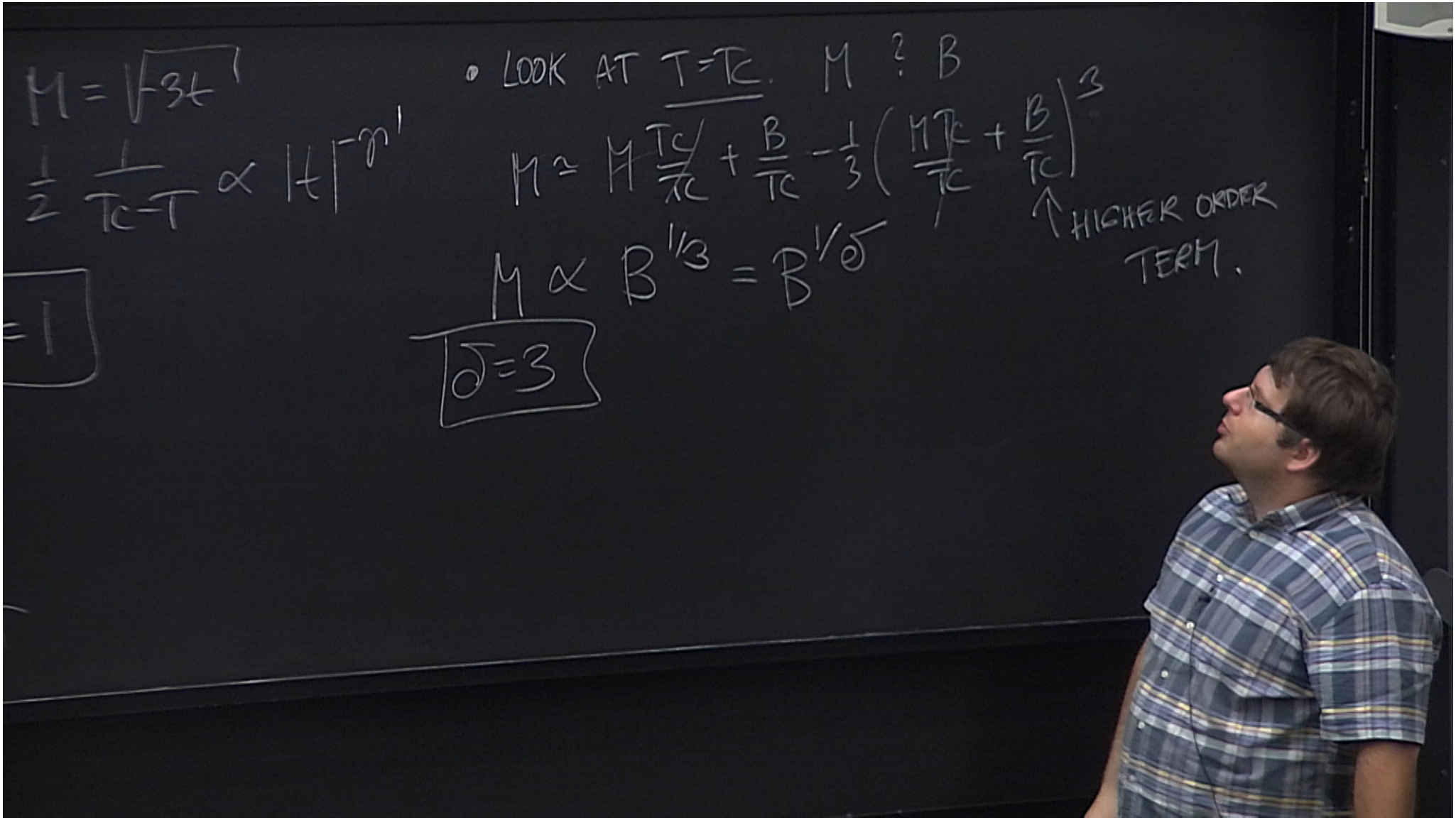
$\chi \simeq \frac{1}{2} \frac{1}{T_c - T} \propto |t|^{-\gamma'}$

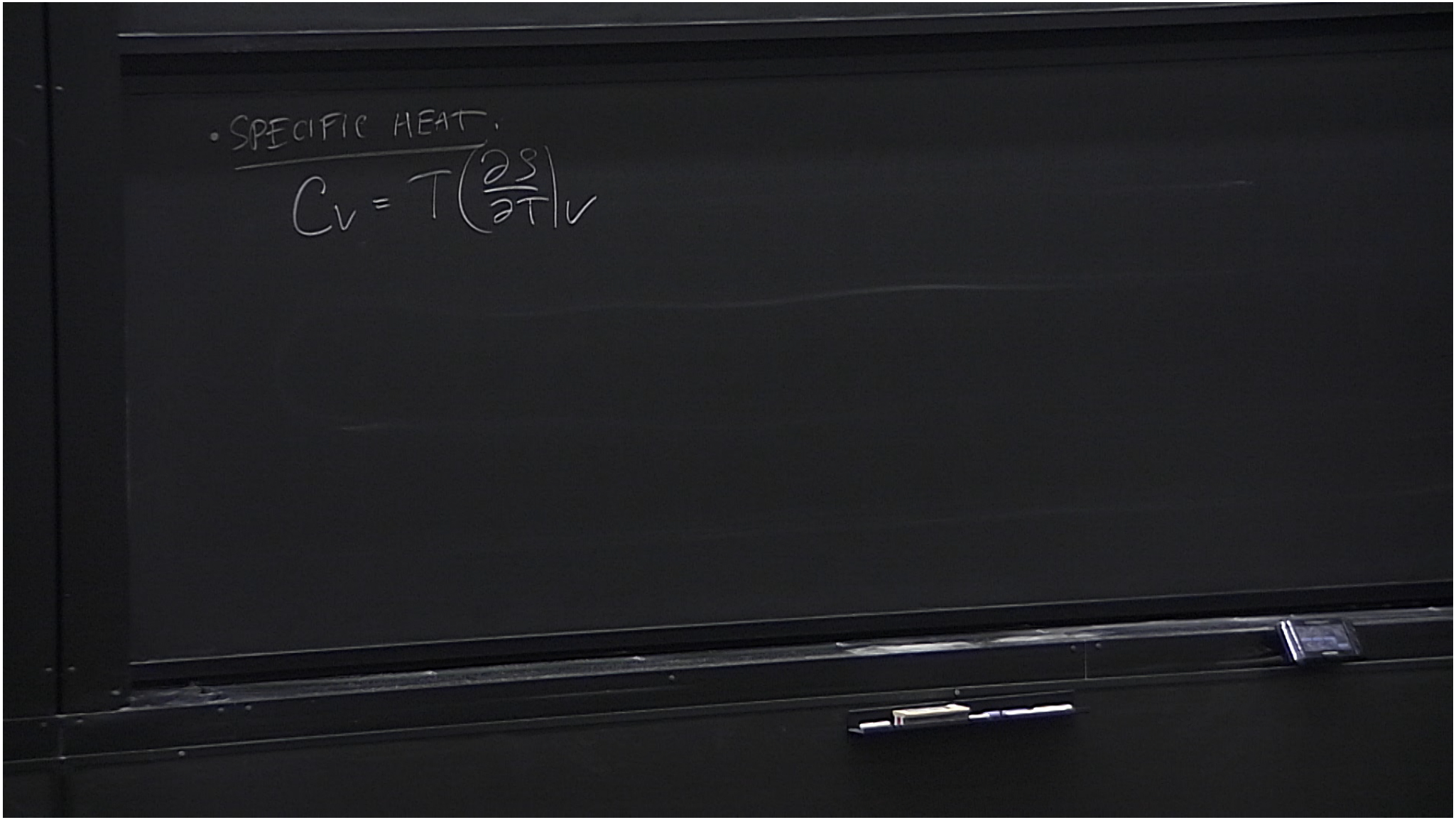
$\boxed{\gamma' = 1}$

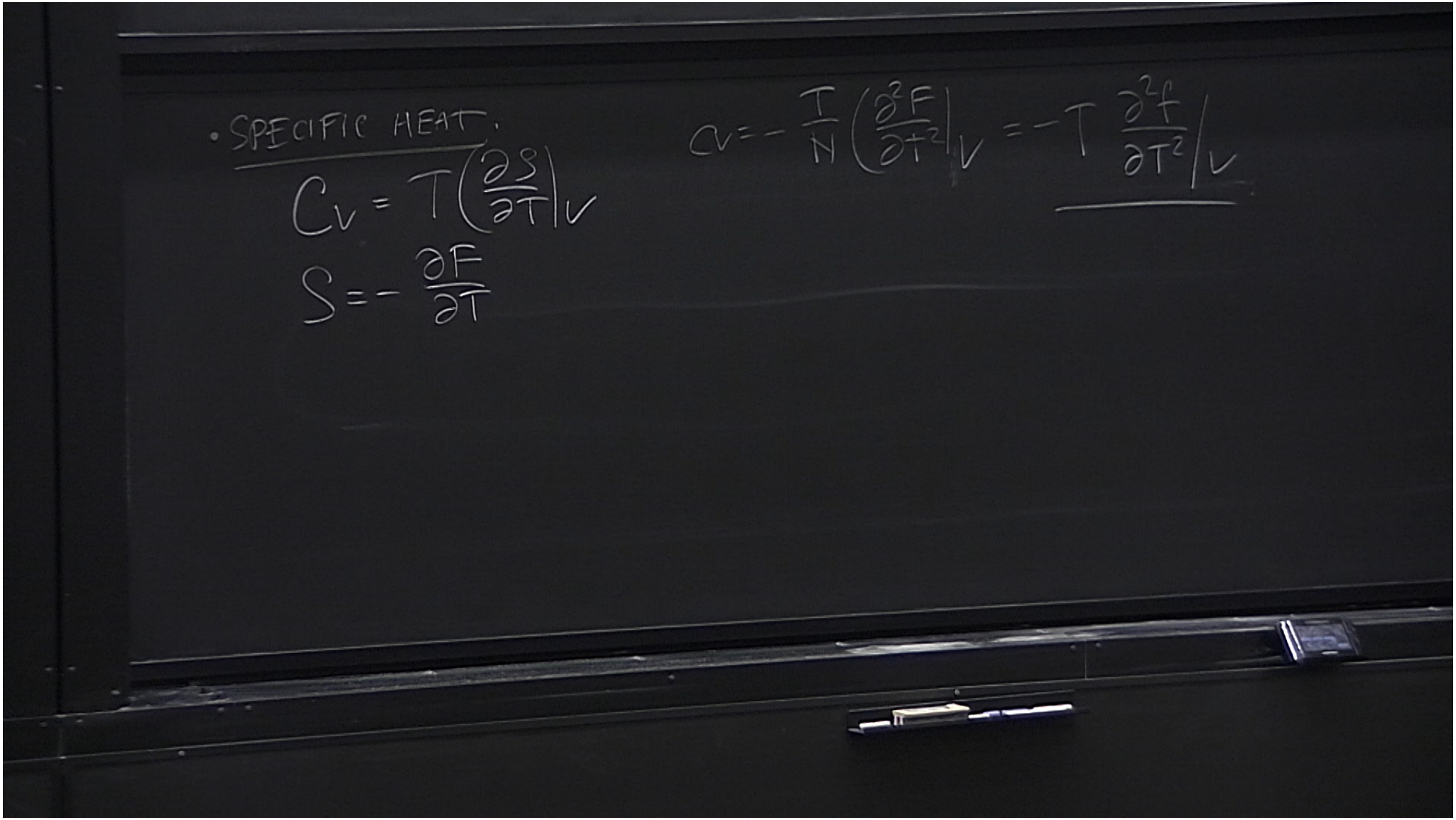


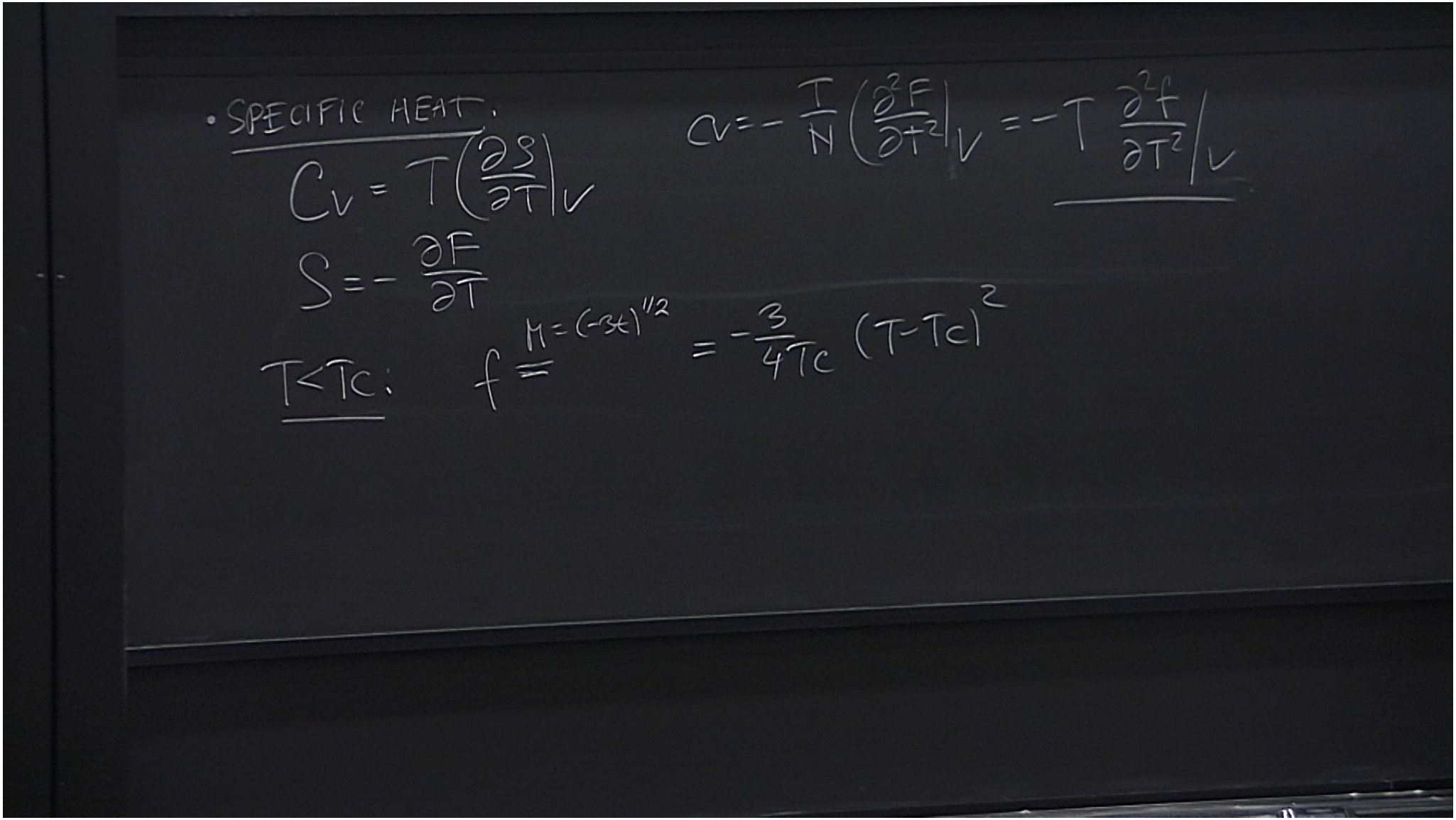

$M = \sqrt{-3t}$
$\frac{1}{2}\frac{1}{T_c - T} \propto |t|^{-\alpha'}$
• LOOK AT $T = T_c$. M ? B
$M \simeq M\frac{T_c}{T_c} + \frac{B}{T_c} - \frac{1}{3}\left(\frac{MT_c}{T_c} + \frac{B}{T_c}\right)^3$
$= 1$

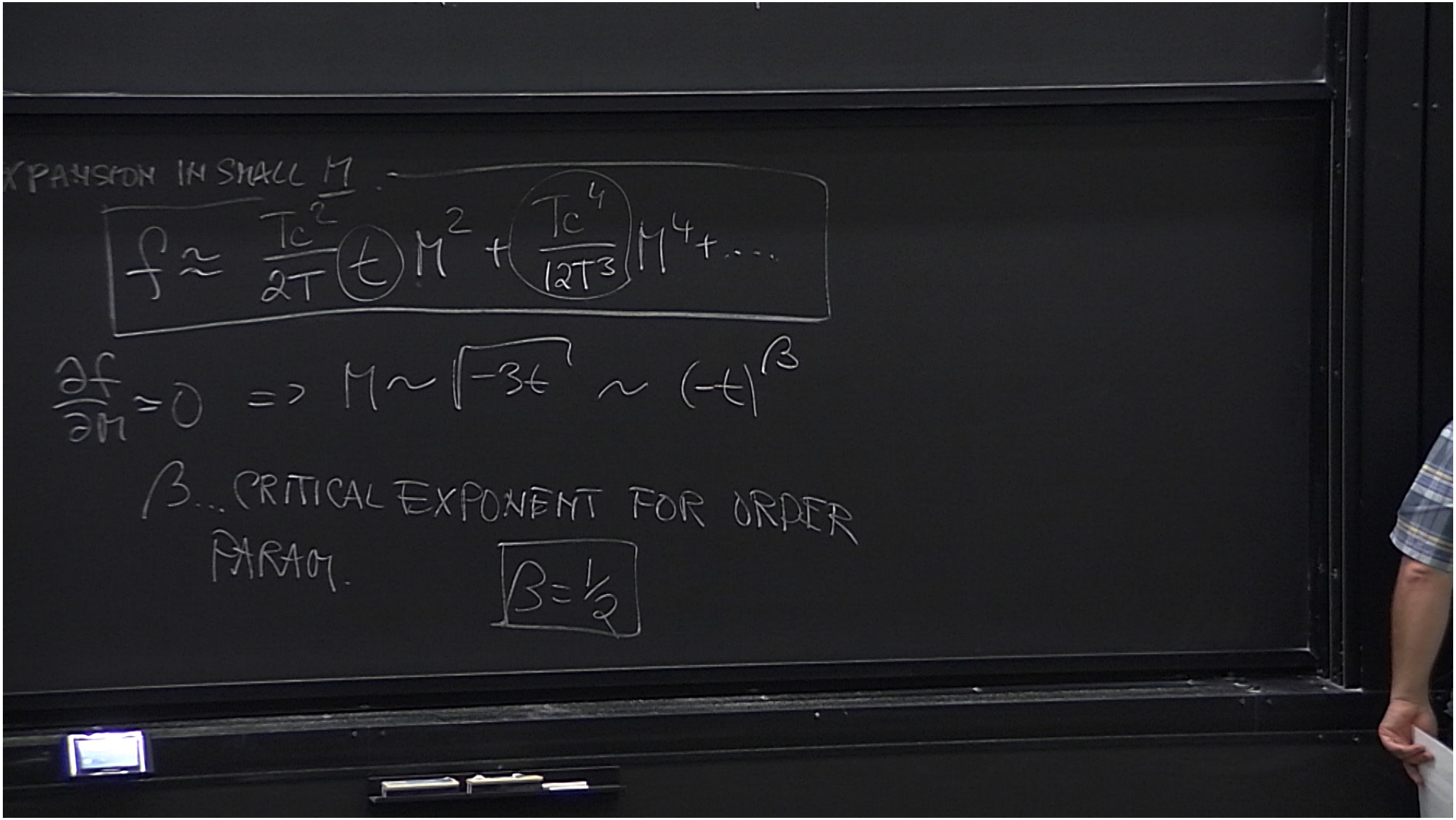

$\frac{1}{2}\frac{1}{T_c - T} \propto |t|^{-\gamma}$
$M \propto B^{1/3} = B^{1/\delta}$
$\delta = 3$

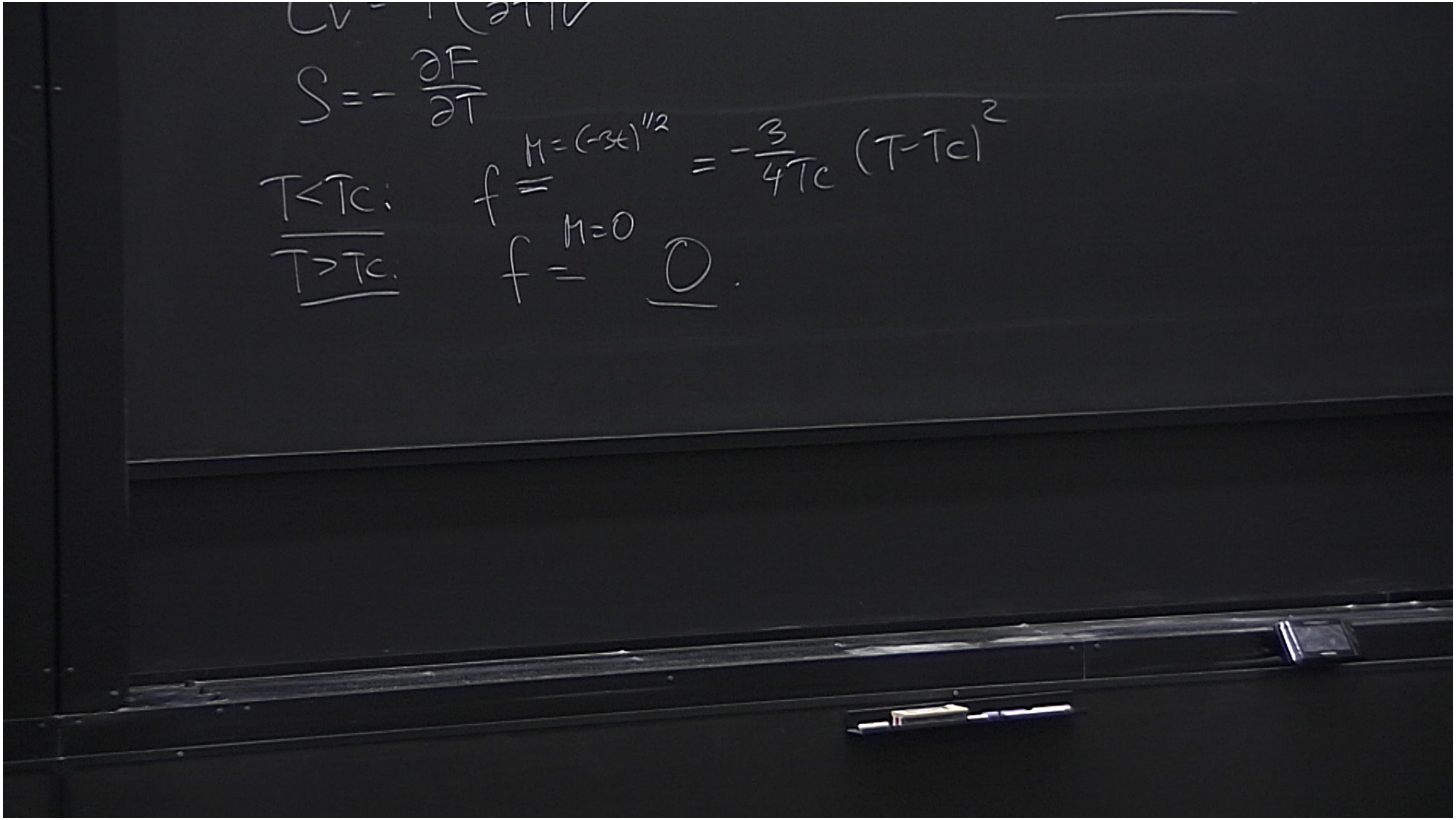

$M = \sqrt{-3t}^{1}$
$\frac{1}{2} \frac{1}{T_c - T} \propto |t|^{-\gamma'}$
• LOOK AT $T = T_c$. M ? B
$M \simeq M \frac{T_c}{T_c} + \frac{B}{T_c} - \frac{1}{3}\left(\frac{M T_c}{T_c} + \frac{B}{T_c}\right)^3$
↑ HIGHER ORDER TERM.
$M \propto B^{1/3} = B^{1/\delta}$
$\delta = 3$

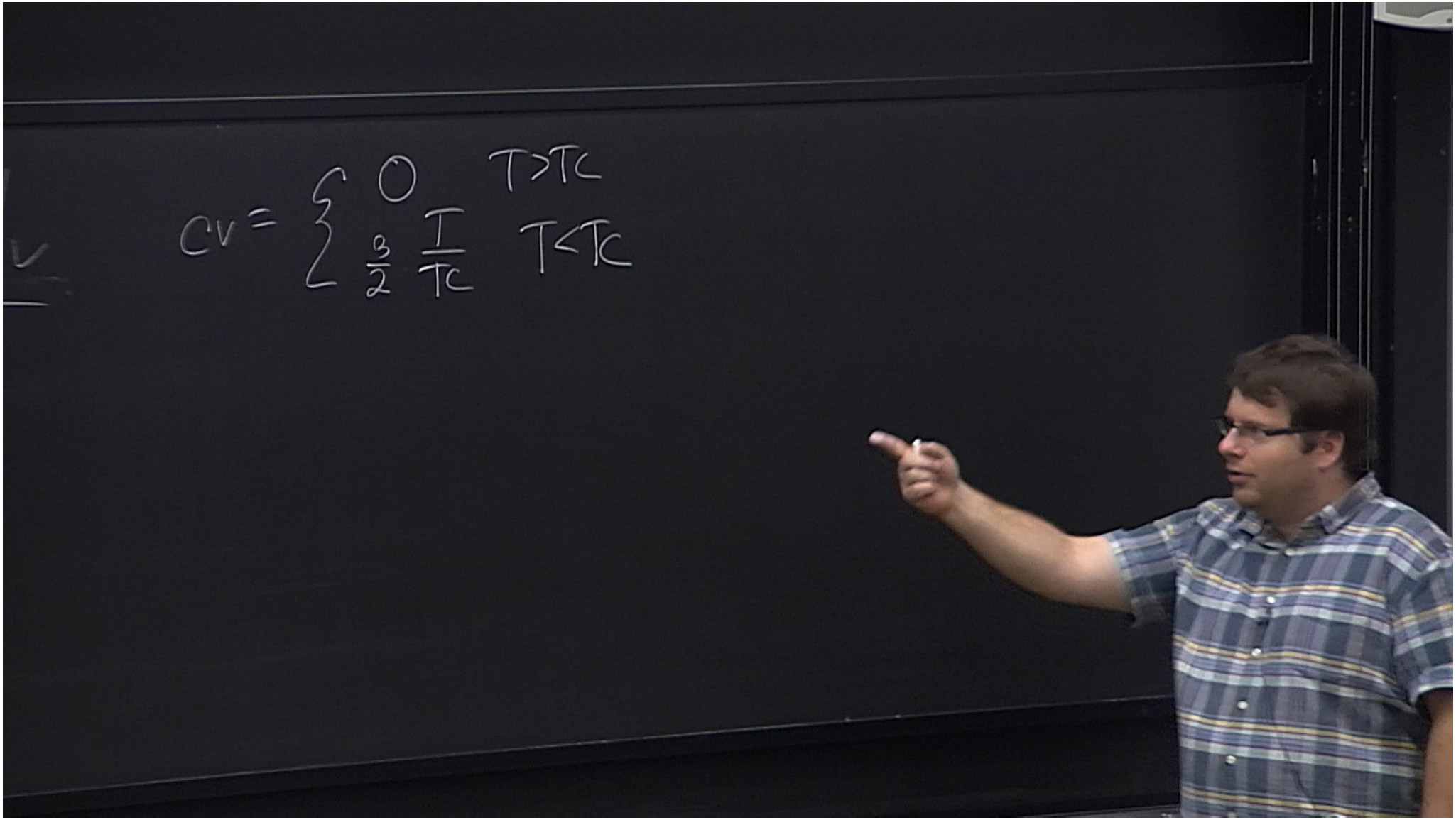

• SPECIFIC HEAT:
$C_V = T\left(\frac{\partial S}{\partial T}\right)_V$


• SPECIFIC HEAT:
$C_V = T\left(\frac{\partial S}{\partial T}\right)_V$
$S = -\frac{\partial F}{\partial T}$
$C_V = -\frac{T}{N}\left(\frac{\partial^2 F}{\partial T^2}\right)_V = -T\frac{\partial^2 f}{\partial T^2}\Big|_V$


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$T<T_c$: $f \overset{M=(-3t)^{1/2}}{=} -\frac{3}{4T_c}(T-T_c)^2$

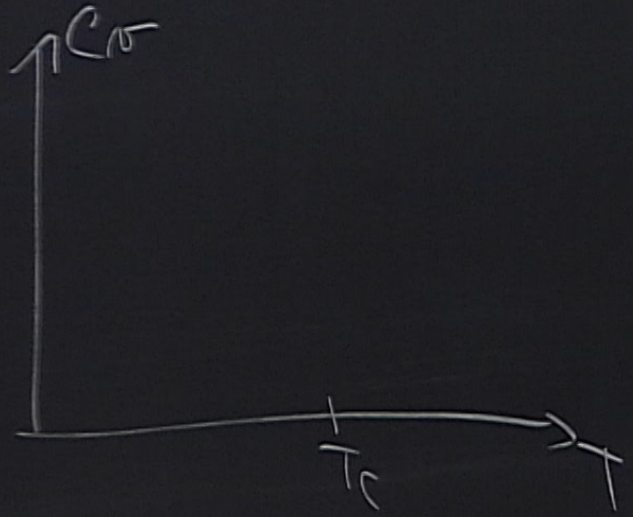

EXPANSION IN SMALL M
$f \approx \frac{T_c^2}{2T}(t)M^2 + \frac{T_c^4}{12T^3}M^4 + \ldots$
$\frac{\partial f}{\partial M} = 0 \Rightarrow M \sim \sqrt{-3t} \sim (-t)^{\beta}$
β... CRITICAL EXPONENT FOR ORDER PARAM.
$\beta = \frac{1}{2}$


$S = -\frac{\partial F}{\partial T}$
$T < T_c$: $f \overset{M=(-3t)^{1/2}}{=} -\frac{3}{4T_c}(T-T_c)^2$
$T > T_c$: $f \overset{M=0}{=} 0$.


$C_V = \begin{cases} 0 & T > T_c \\ \frac{3}{2} \frac{I}{T_c} & T < T_c \end{cases}$

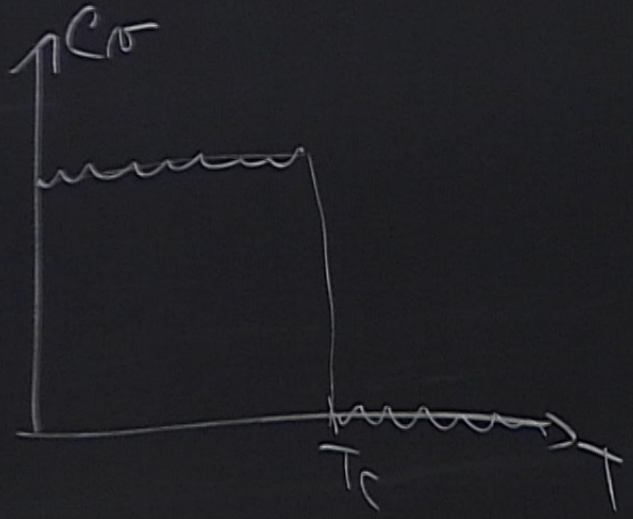
$$\left.\frac{\partial^2 f}{\partial T^2}\right|_V$$

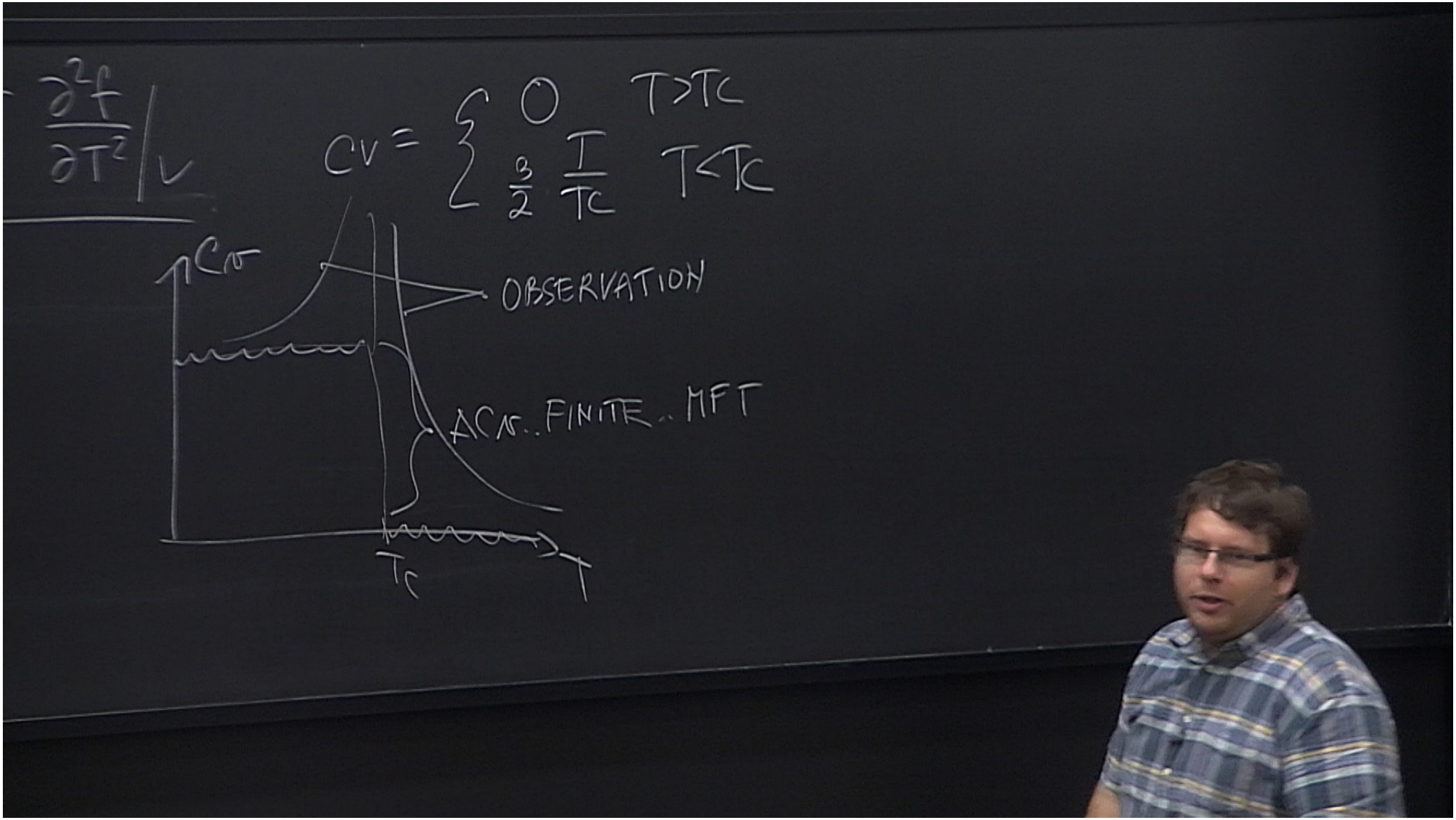
$$C_V = \begin{cases} 0 & T > T_c \\ \frac{3}{2}\frac{I}{T_c} & T < T_c \end{cases}$$



$$-\left.\frac{\partial^2 f}{\partial T^2}\right|_V$$

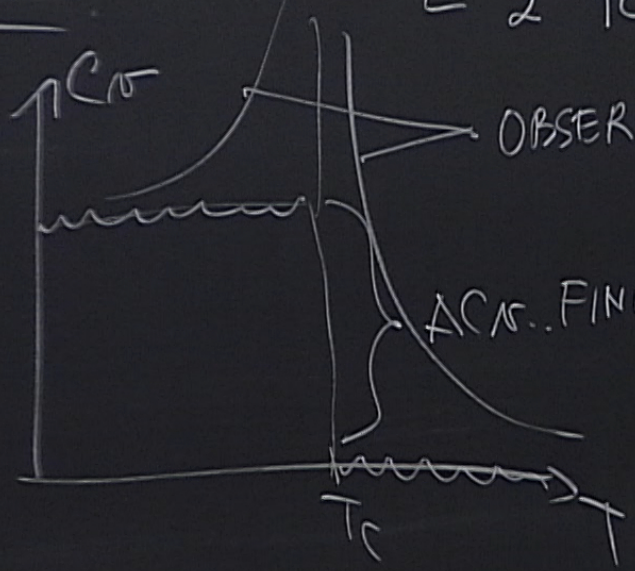
$$C_V = \begin{cases} 0 & T > T_c \\ \frac{3}{2}\frac{I}{T_c} & T < T_c \end{cases}$$




$\frac{\partial^2 f}{\partial T^2}\Big|_V$
$C_V = \begin{cases} 0 & T > T_C \\ \frac{3}{2}\frac{I}{T_C} & T < T_C \end{cases}$
C_V
OBSERVATION
ΔC_V.. FINITE .. MFT
T_C
T

$T \left.\frac{\partial^2 f}{\partial T^2}\right|_V$

$$C_V = \begin{cases} 0 & T > T_C \\ \frac{3}{2}\frac{I}{T_C} & T < T_C \end{cases}$$

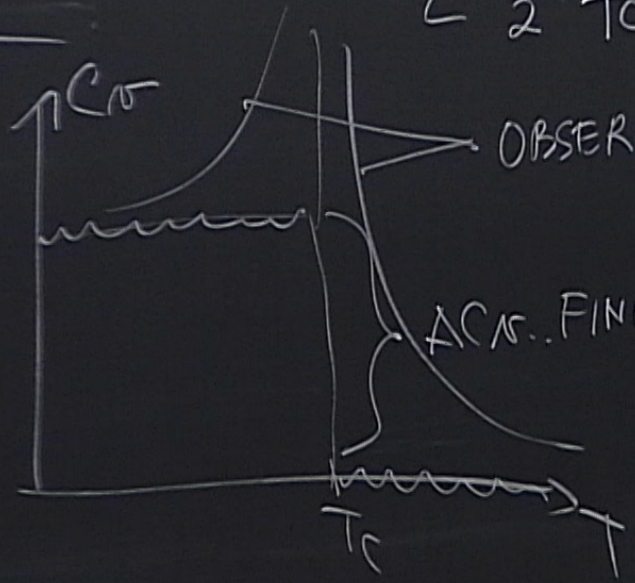


DEF: $C_V \propto |t|^{-\alpha'}$ $T < T_C$

$t^{-\alpha}$ $T > T_C$

$T \left.\frac{\partial^2 f}{\partial T^2}\right|_V$

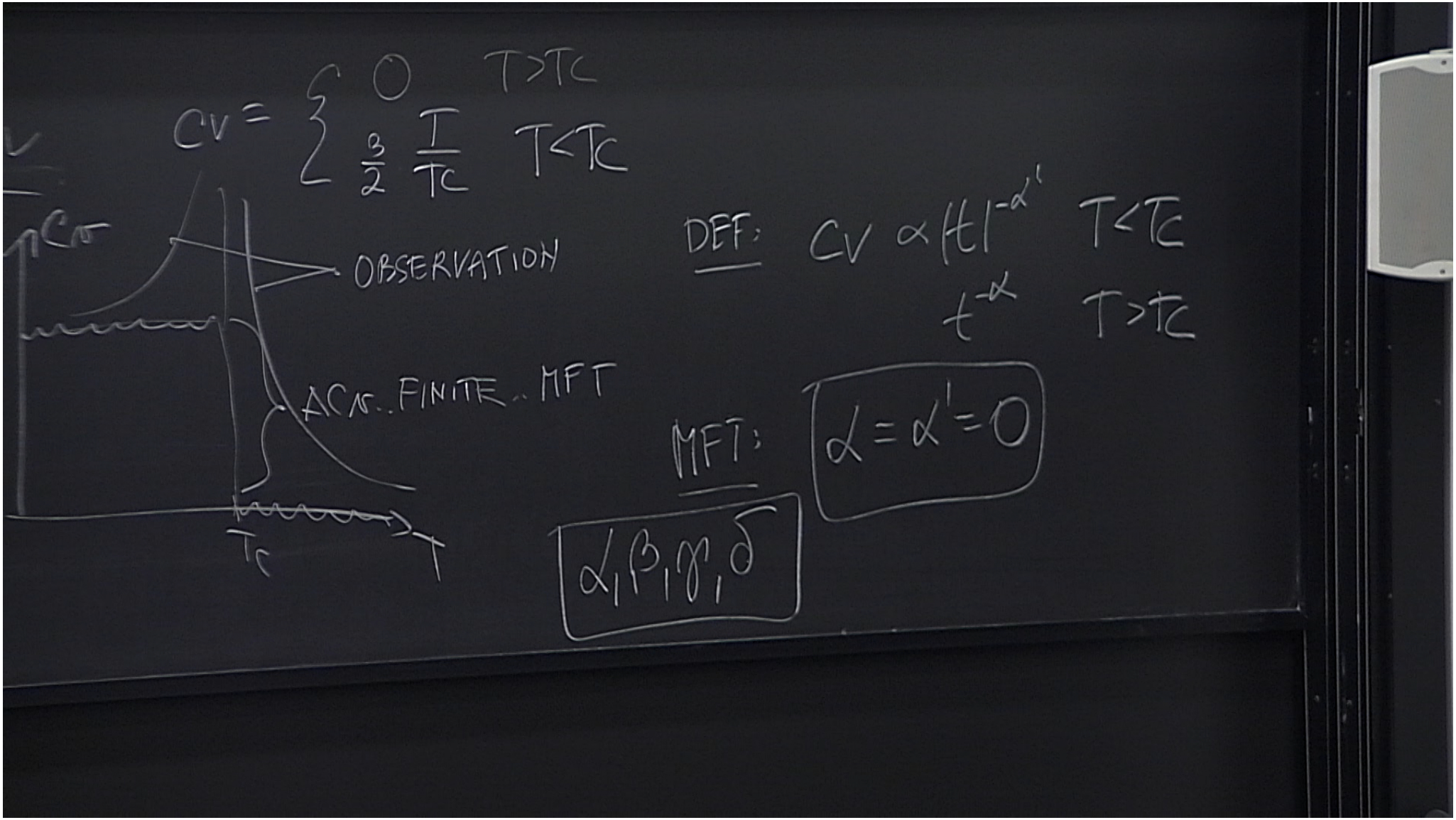
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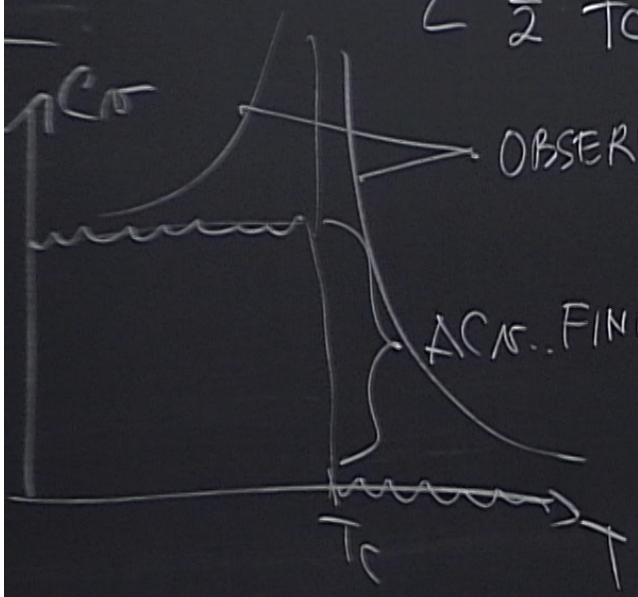
DEF: $C_V \propto |t|^{-\alpha'}$ $T < T_C$

$t^{-\alpha}$ $T > T_C$

MFT: $\boxed{\alpha = \alpha' = 0}$


$C_V = \begin{cases} 0 & T>T_C \\ \frac{3}{2}\frac{I}{T_C} & T<T_C \end{cases}$
C_V
OBSERVATION
DEF: $C_V \propto |t|^{-\alpha'}$ $T<T_C$
$t^{-\alpha}$ $T>T_C$
ΔC_V.. FINITE .. MFT
MFT: $\alpha = \alpha' = 0$
T_C
T
$\alpha, \beta, \gamma, \delta$

$C_V = \begin{cases} 0 & T > T_C \\ \frac{3}{2}\frac{I}{T_C} & T < T_C \end{cases}$



OBSERVATION

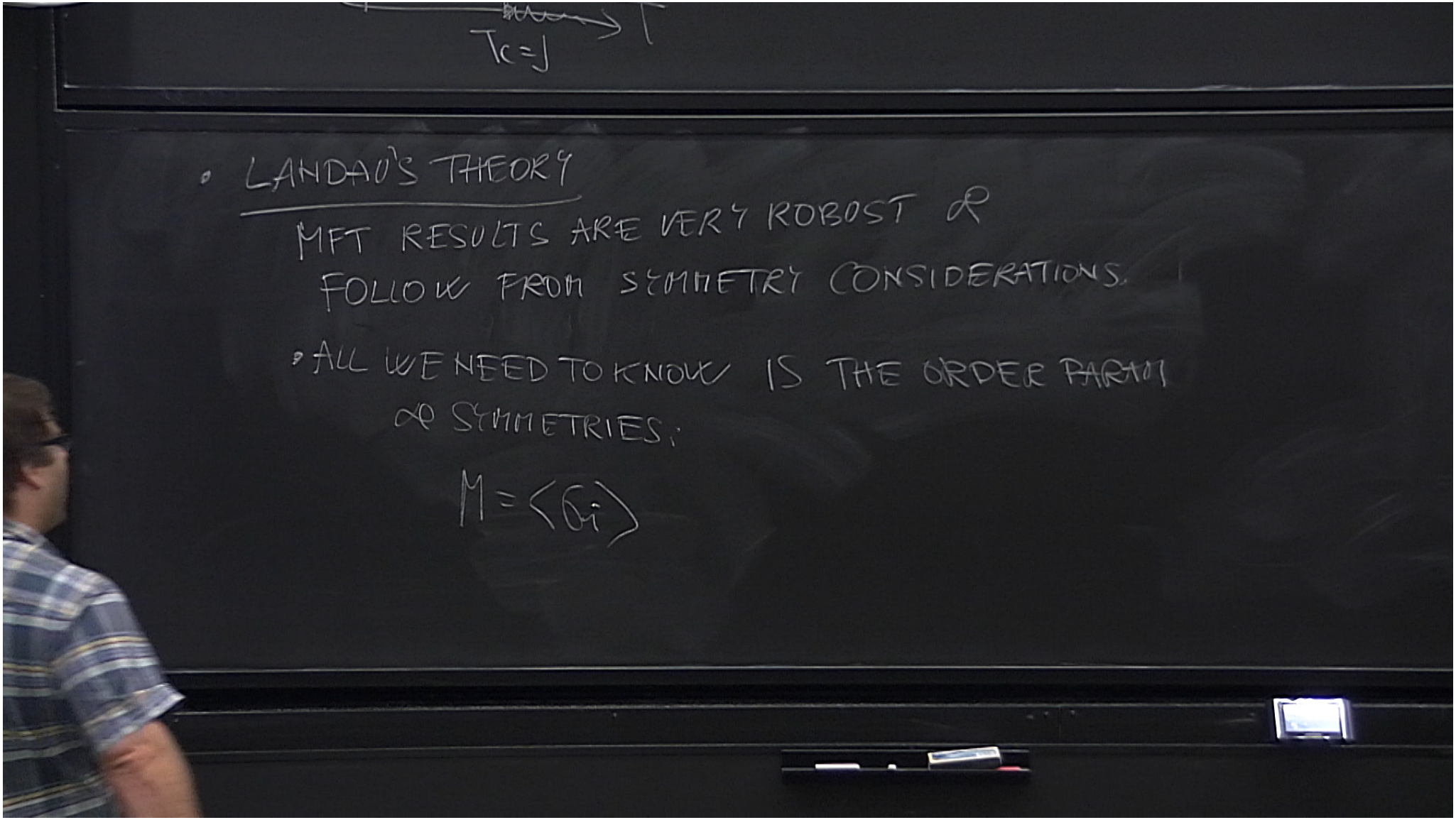
DEF: $C_V \propto |t|^{-\alpha'}$ $T < T_C$

$t^{-\alpha}$ $T > T_C$

ΔC_V .. FINITE .. MFT

MFT: $\boxed{\alpha = \alpha' = 0}$

$\boxed{\alpha, \beta, \gamma, \delta}$ .. CHARACTERIZE CONT. PHASE TRANSITION.


Tc=J
• LANDAU'S THEORY
MFT RESULTS ARE VERY ROBOST &
FOLLOW FROM SYMMETRY CONSIDERATIONS.
• ALL WE NEED TO KNOW IS THE ORDER PARAM
& SYMMETRIES:
M=<Gi>

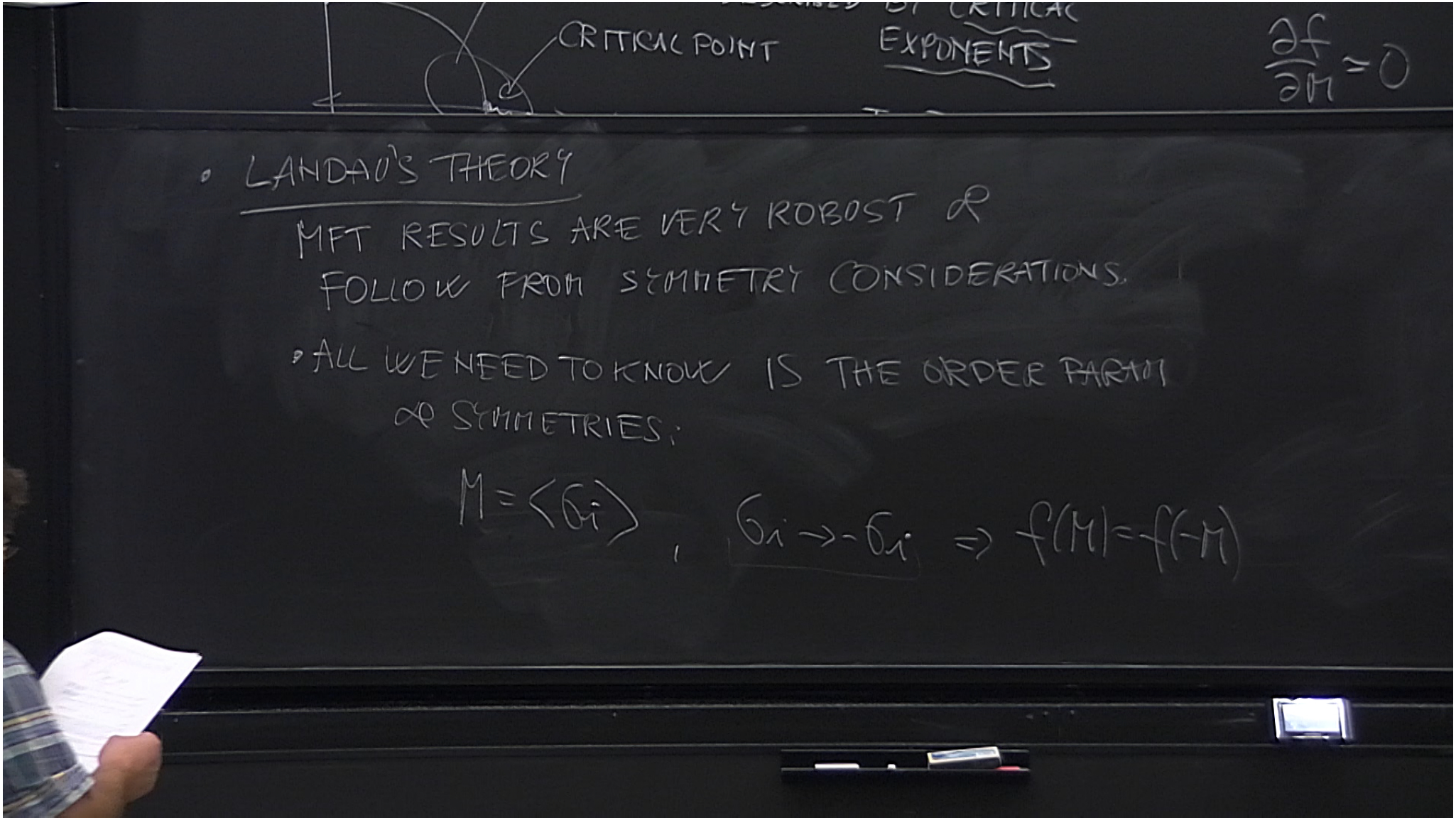
$T_c = J$

- LANDAU'S THEORY

MFT RESULTS ARE VERY ROBOST & FOLLOW FROM SYMMETRY CONSIDERATIONS.

- ALL WE NEED TO KNOW IS THE ORDER PARAM & SYMMETRIES:

$M = \langle \sigma_i \rangle$, $\sigma_i \to -\sigma_i$


CRITICAL POINT
EXPONENTS
$\frac{\partial f}{\partial M} = 0$
• LANDAU'S THEORY
MFT RESULTS ARE VERY ROBOST &
FOLLOW FROM SYMMETRY CONSIDERATIONS.
• ALL WE NEED TO KNOW IS THE ORDER PARAM
& SYMMETRIES:
$M = \langle \sigma_i \rangle$, $\sigma_i \to -\sigma_i \Rightarrow f(M) = f(-M)$

$\frac{\partial f}{\partial M} = 0 \Rightarrow M \sim \sqrt{-3t} \sim (-t)^{1/3}$

- NEAR T_c: WE MUST HAVE:

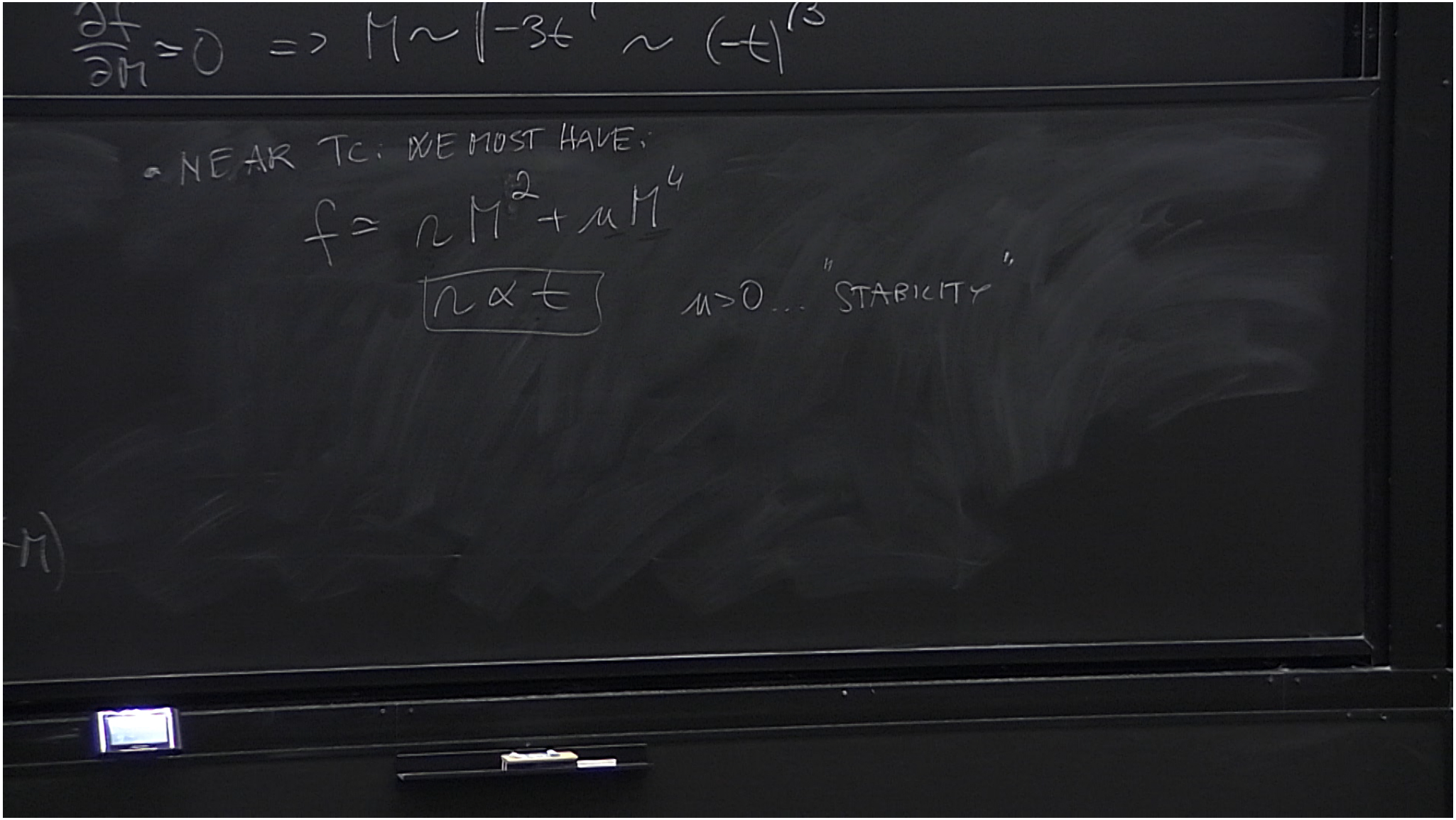
$$f = rM^2 + uM^4$$

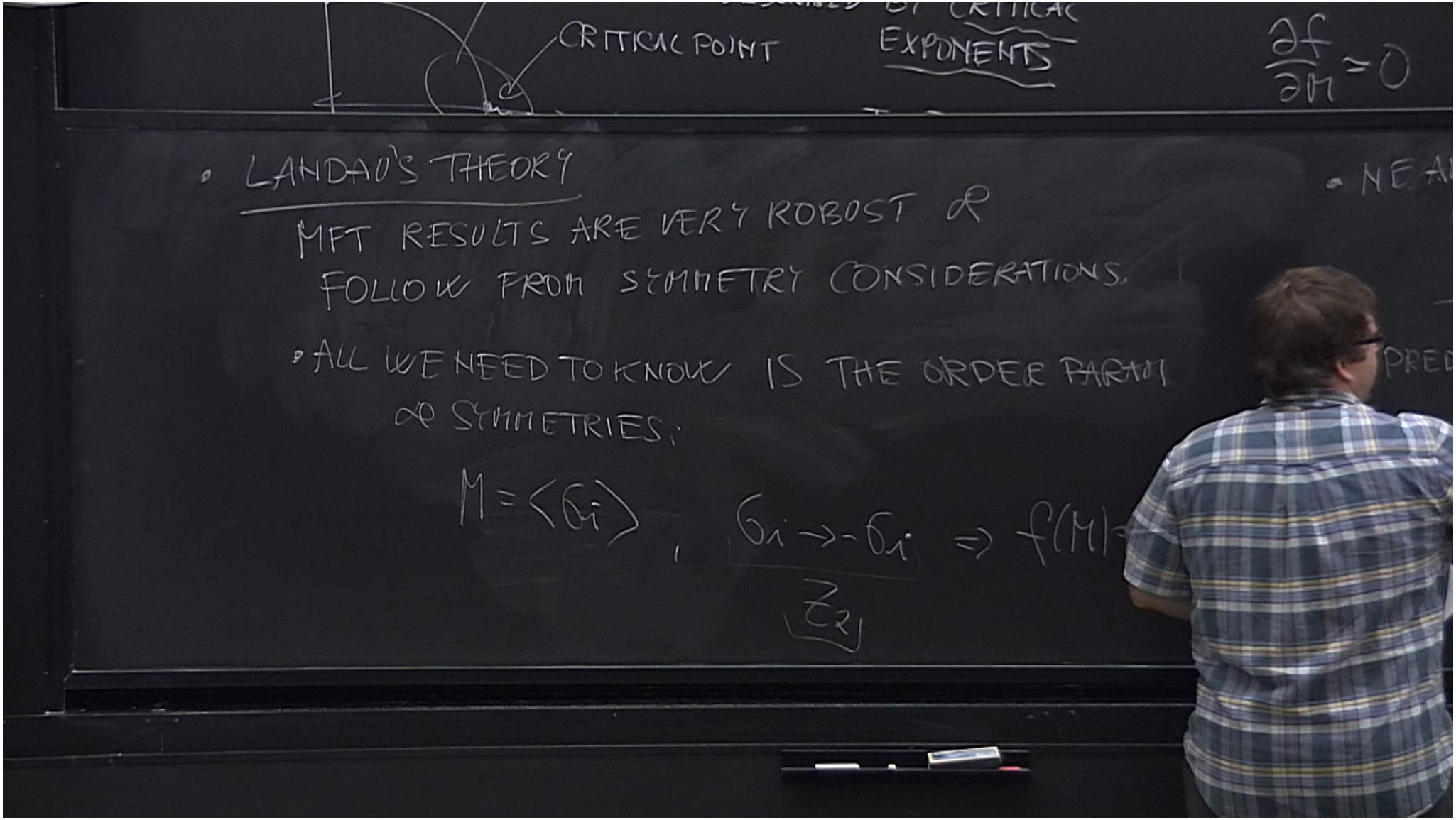
$\frac{\partial f}{\partial M} = 0 \Rightarrow M \sim \sqrt{-3t} \sim (-t)^{1/3}$

- NEAR T_c: WE MUST HAVE:

$$f = r M^2 + u M^4$$

$$\boxed{r \propto t}$$


$\frac{\partial f}{\partial M} = 0 \Rightarrow M \sim \sqrt{-3t} \sim (-t)^{1/3}$
NEAR TC: WE MUST HAVE:
$f = r M^2 + u M^4$
$r \propto t$
$u > 0$ ... "STABICITY"


CRITICAL POINT
EXPONENTS
$\frac{\partial f}{\partial M} = 0$
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MFT RESULTS ARE VERY ROBUST &
FOLLOW FROM SYMMETRY CONSIDERATIONS.
• ALL WE NEED TO KNOW IS THE ORDER PARAM
& SYMMETRIES:
$M = \langle \sigma_i \rangle$,
$\sigma_i \to -\sigma_i \Rightarrow f(M)$
Z_2

$\frac{\partial f}{\partial M} = 0 \Rightarrow M \sim \sqrt{-3t} \sim (-t)^{1/3}$

- NEAR T_c: WE MUST HAVE:

$$f = r M^2 + u M^4$$

$r \propto t$ $\quad u > 0$ ... "STABICITY"

- PREDICTION IS ROBUST ... MANY DIVERSE SYSTEMS MAY LEAD TO THE SAME "FREE ENERGY"

M)

$\frac{\partial f}{\partial M} = 0 \Rightarrow M \sim \sqrt{-3t} \sim (-t)^{1/3}$

- NEAR T_c: WE MUST HAVE:

$$f = r M^2 + u M^4$$

$r \propto t$ $\quad u > 0$ ... "STABILITY"

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M)

$\frac{\partial f}{\partial M} = 0 \Rightarrow M \sim |-3t|^{1} \sim (-t)^{1/3}$

- NEAR T_c: WE MUST HAVE:

$$f \simeq r M^2 + u M^4$$ = LANDAU'S ENERGY

$r \propto t$ $\quad u > 0$ ... "STABILITY"

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$\Rightarrow$ SHOULD HAVE THE SAME CRITICAL EXPONENTS

(BELONG TO THE SAME UNIVERSALITY CLASS)

- NEAR T_c: WE MUST HAVE:

$$f \simeq r M^2 + u M^4$$ = LANDAU'S ENERGY

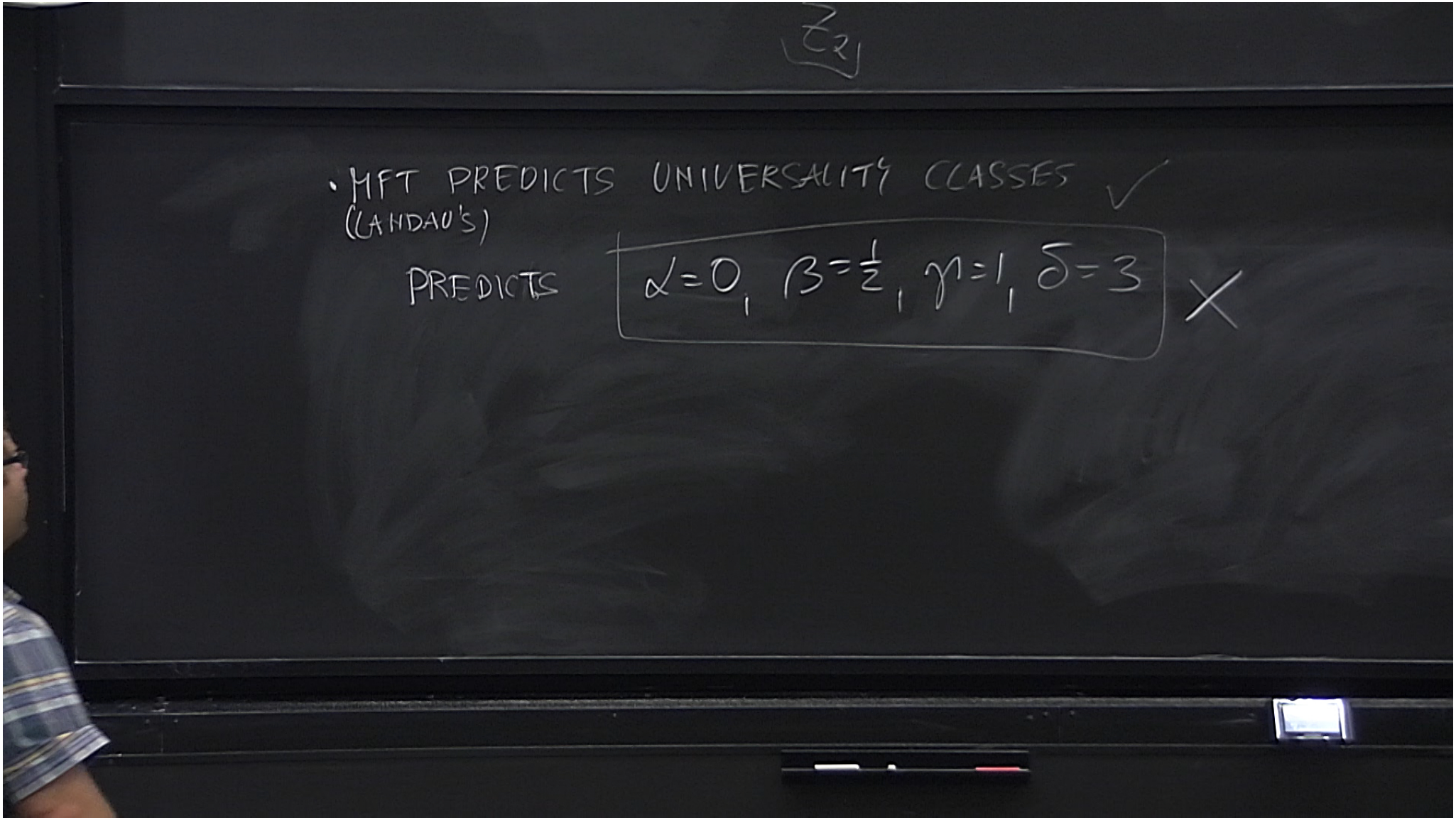
$r \propto t$ $\quad u > 0$ ... "STABILITY"

- PREDICTION IS ROBUST ... MANY DIVERSE SYSTEMS MAY LEAD TO THE SAME "FREE ENERGY"

⇒ SHOULD HAVE THE SAME CRITICAL EXPONENTS

(BELONG TO THE SAME UNIVERSALITY CLASS)

TRUE BUT CRITICAL EXPONENTS TAKE VALUES NOT PREDICTED BY LANDAU.

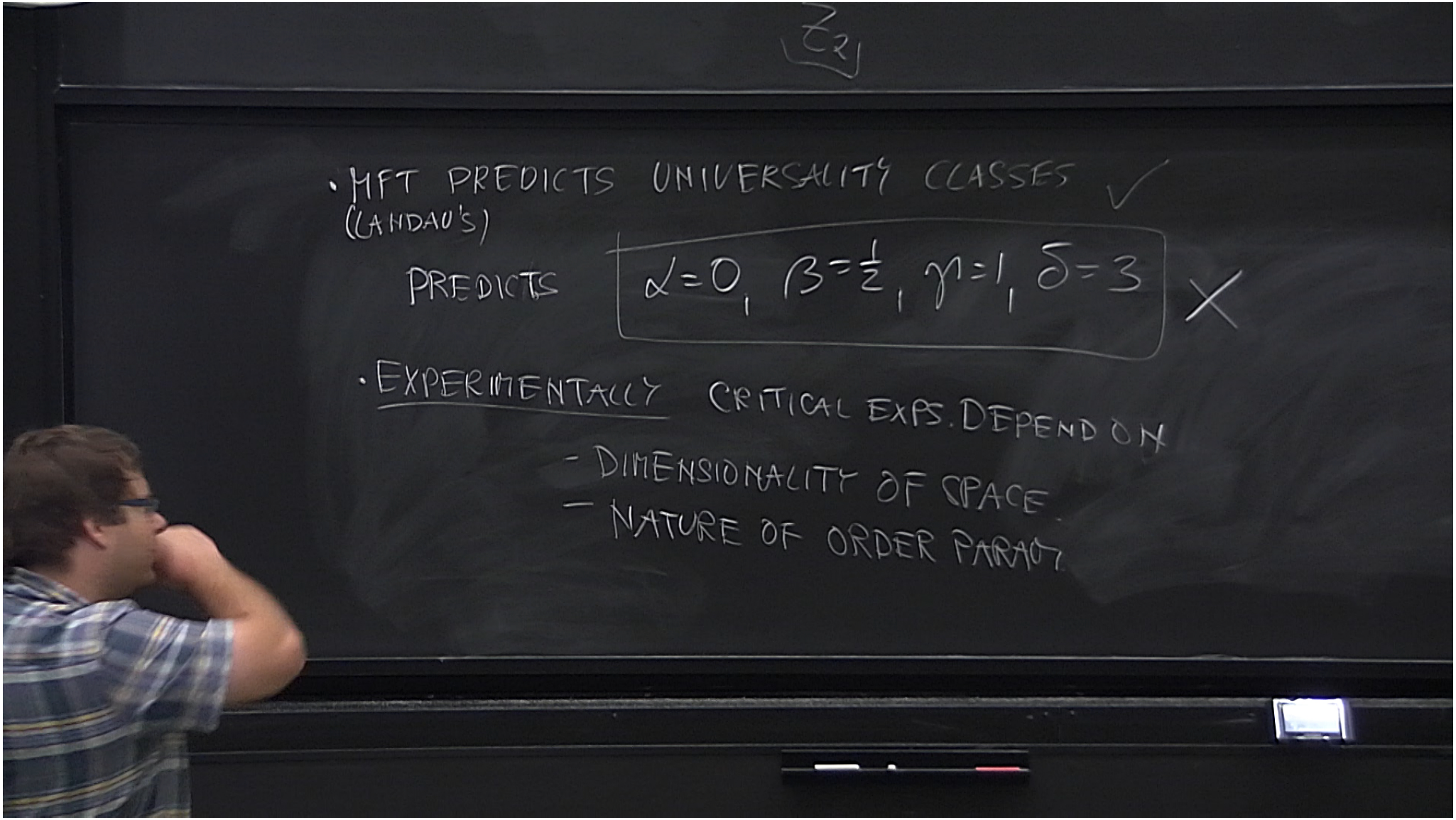

$\mathbb{Z}_2$
• MFT PREDICTS UNIVERSALITY CLASSES ✓
(LANDAU'S)
PREDICTS $\alpha = 0,\ \beta = \frac{1}{2},\ \gamma = 1,\ \delta = 3$ ✗

$\mathbb{Z}_2$

- MFT PREDICTS UNIVERSALITY CLASSES ✓
  (LANDAU'S)

  PREDICTS $\alpha = 0,\ \beta = \frac{1}{2},\ \gamma = 1,\ \delta = 3$ ✗

- EXPERIMENTALLY CRITICAL EXPS. DEPEND ON
  - DIMENSIONALITY OF SPACE.


$\mathbb{Z}_2$
• MFT PREDICTS UNIVERSALITY CLASSES (LANDAU'S) ✓
PREDICTS $\alpha = 0,\ \beta = \frac{1}{2},\ \gamma = 1,\ \delta = 3$ ✗
• EXPERIMENTALLY CRITICAL EXPS. DEPEND ON
- DIMENSIONALITY OF SPACE
- NATURE OF ORDER PARAM

Z_2

- MFT PREDICTS UNIVERSALITY CLASSES ✓ (LANDAU'S)

  PREDICTS $\alpha = 0,\ \beta = \frac{1}{2},\ \gamma = 1,\ \delta = 3$ ✗

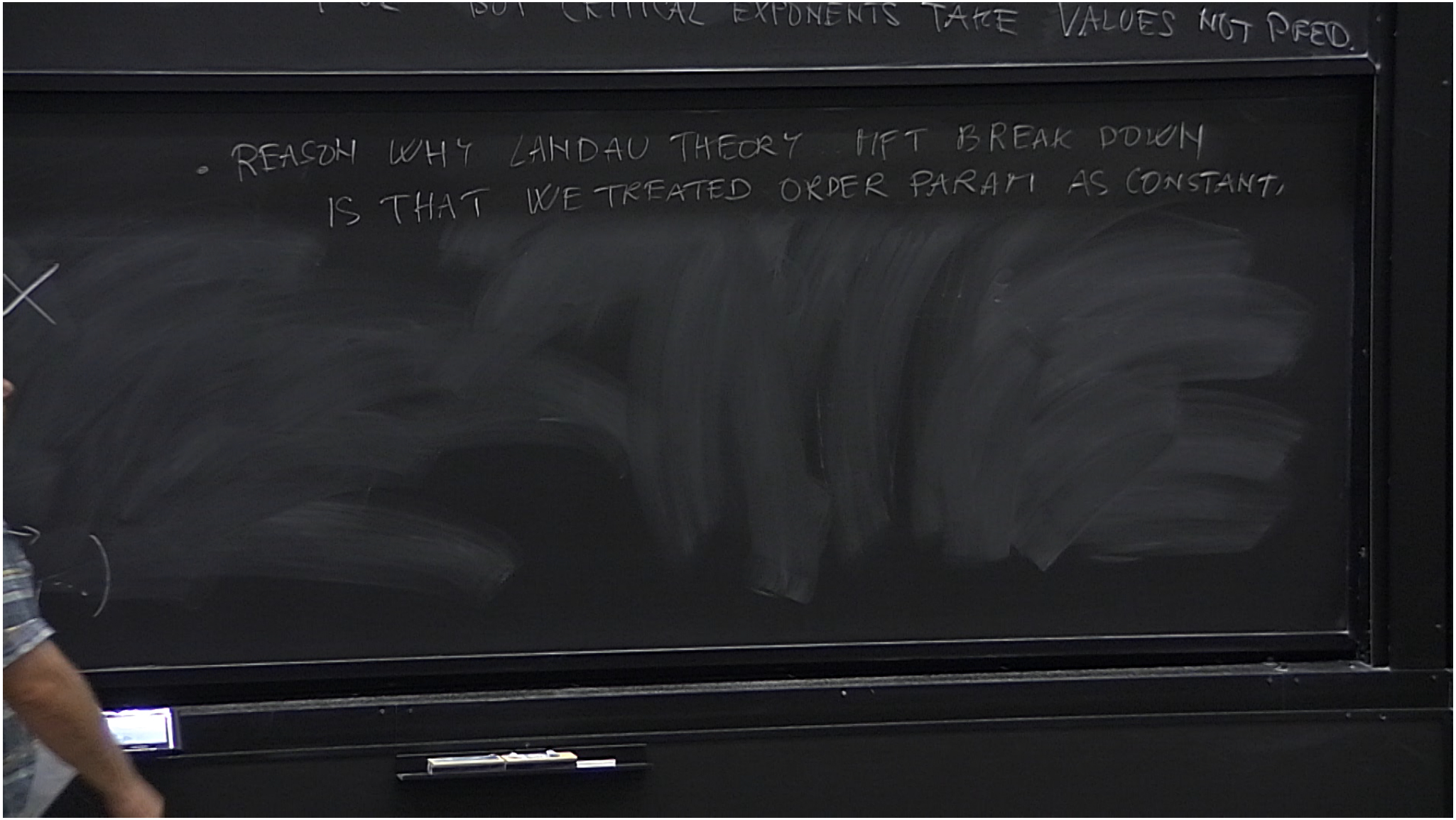
- EXPERIMENTALLY CRITICAL EXPS. DEPEND ON
  - DIMENSIONALITY OF SPACE
  - NATURE OF ORDER PARAM. $(M, \vec{n}, \ldots)$

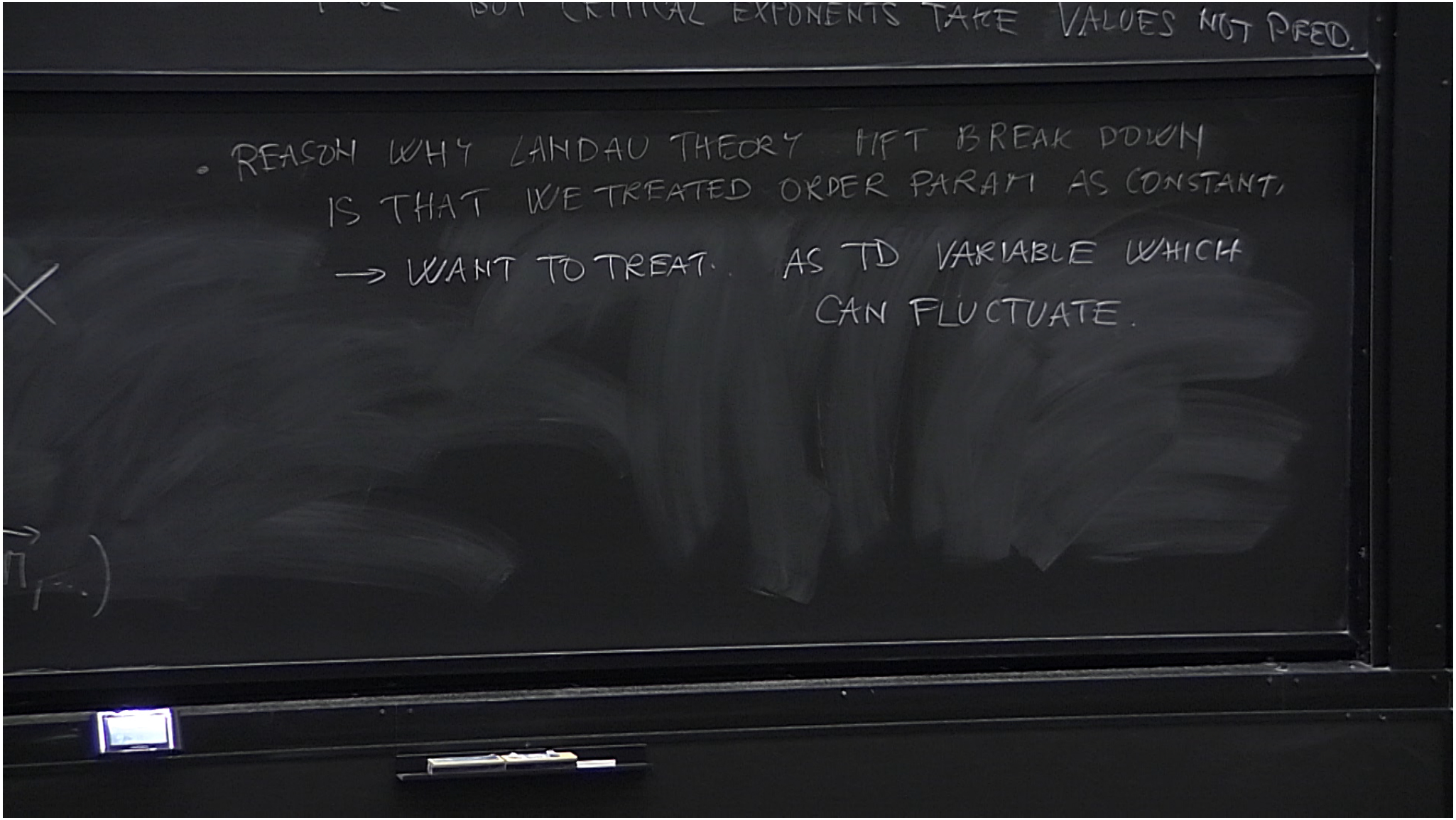
$\mathbb{Z}_2$

- MFT PREDICTS UNIVERSALITY CLASSES ✓ (LANDAU'S)

  PREDICTS $\alpha = 0,\ \beta = \frac{1}{2},\ \gamma = 1,\ \delta = 3$ ✗

- EXPERIMENTALLY CRITICAL EXPS. DEPEND ON
  - DIMENSIONALITY OF SPACE
  - NATURE OF ORDER PARAM. $(M, \vec{M}, \ldots)$
  - SYMMETRY OF H.


CRITICAL EXPONENTS TAKE VALUES NOT DREAD.
• REASON WHY LANDAU THEORY MFT BREAK DOWN
IS THAT WE TREATED ORDER PARAM AS CONSTANT,

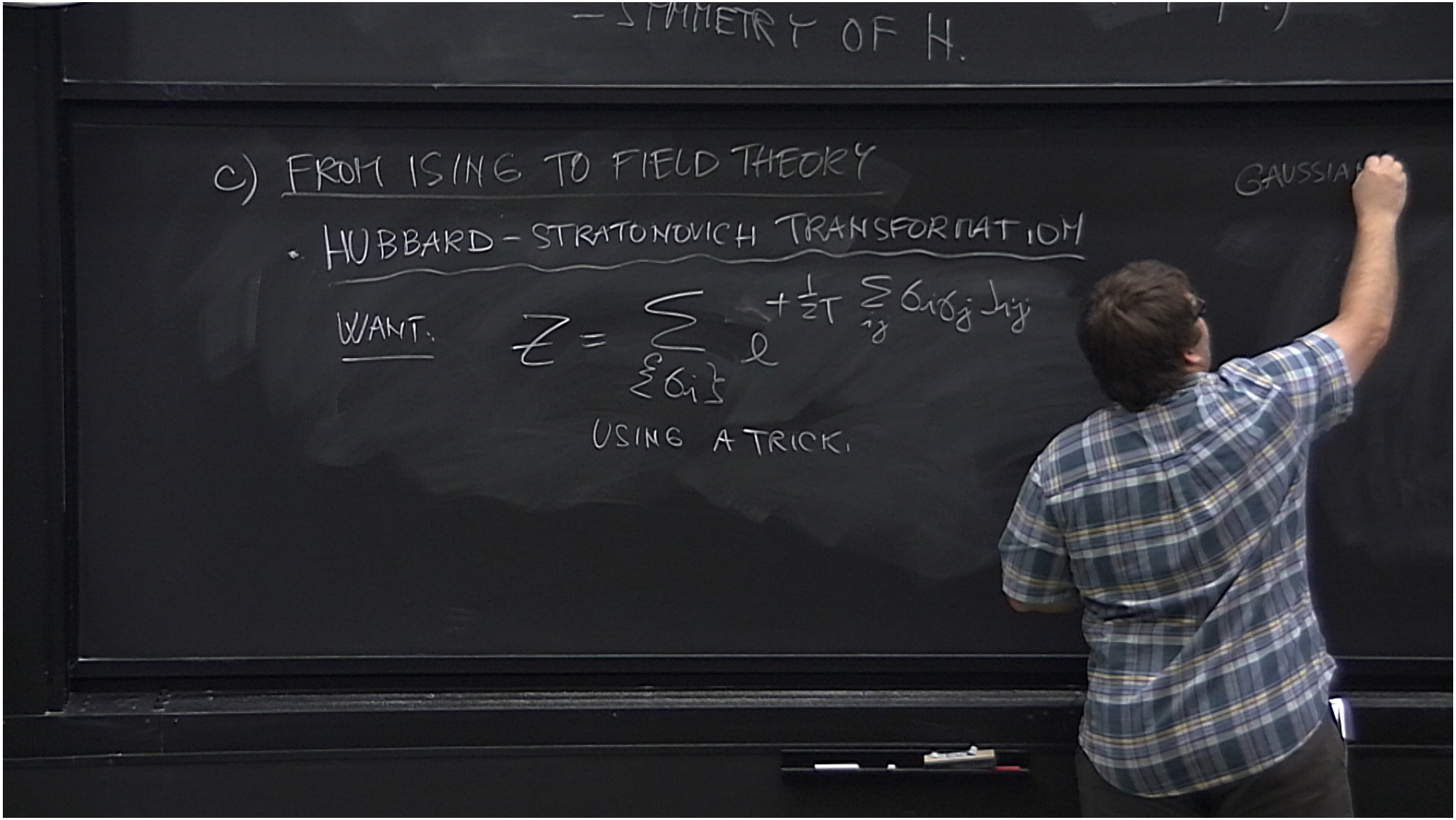

CRITICAL EXPONENTS TAKE VALUES NOT DEED.
• REASON WHY LANDAU THEORY MFT BREAK DOWN
IS THAT WE TREATED ORDER PARAM AS CONSTANT,
→ WANT TO TREAT. AS TD VARIABLE WHICH
CAN FLUCTUATE.

- SYMMETRY OF H.

c) FROM ISING TO FIELD THEORY

- HUBBARD-STRATONOVICH TRANSFORMATION

WANT: $Z = \sum_{\{\sigma_i\}} e^{+\frac{1}{2T}\sum_{ij}\sigma_i\sigma_j J_{ij}}$


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GAUSSIA
• HUBBARD - STRATONOVICH TRANSFORMATION
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USING A TRICK.

GAUSSIAN INTEGRATION

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\varphi \, e^{-\frac{1}{2}a\varphi^2 + b\varphi} = \frac{1}{\sqrt{a}} e^{\frac{b^2}{2a}} \qquad a > 0$$

GAUSSIAN INTEGRATION

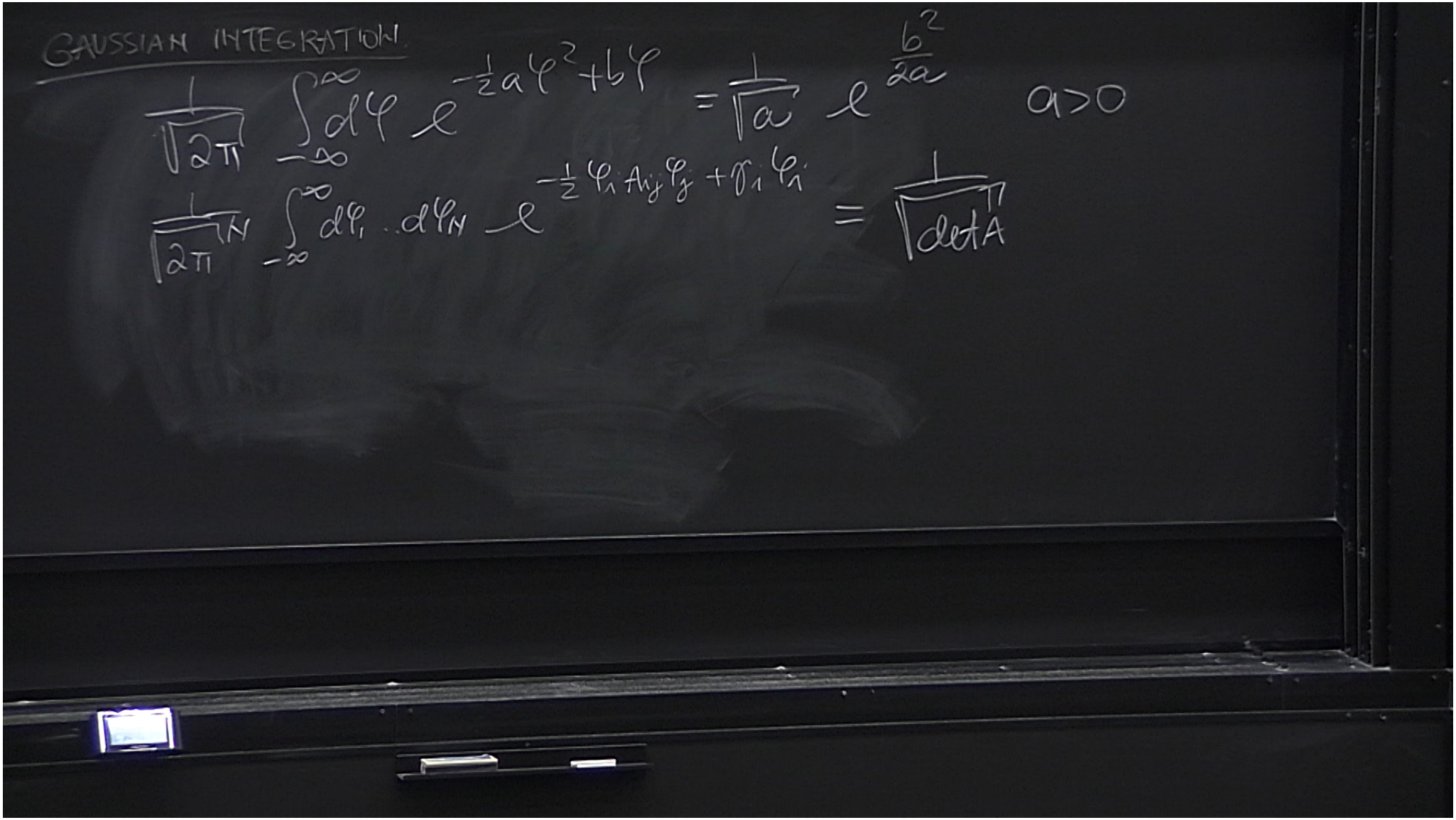
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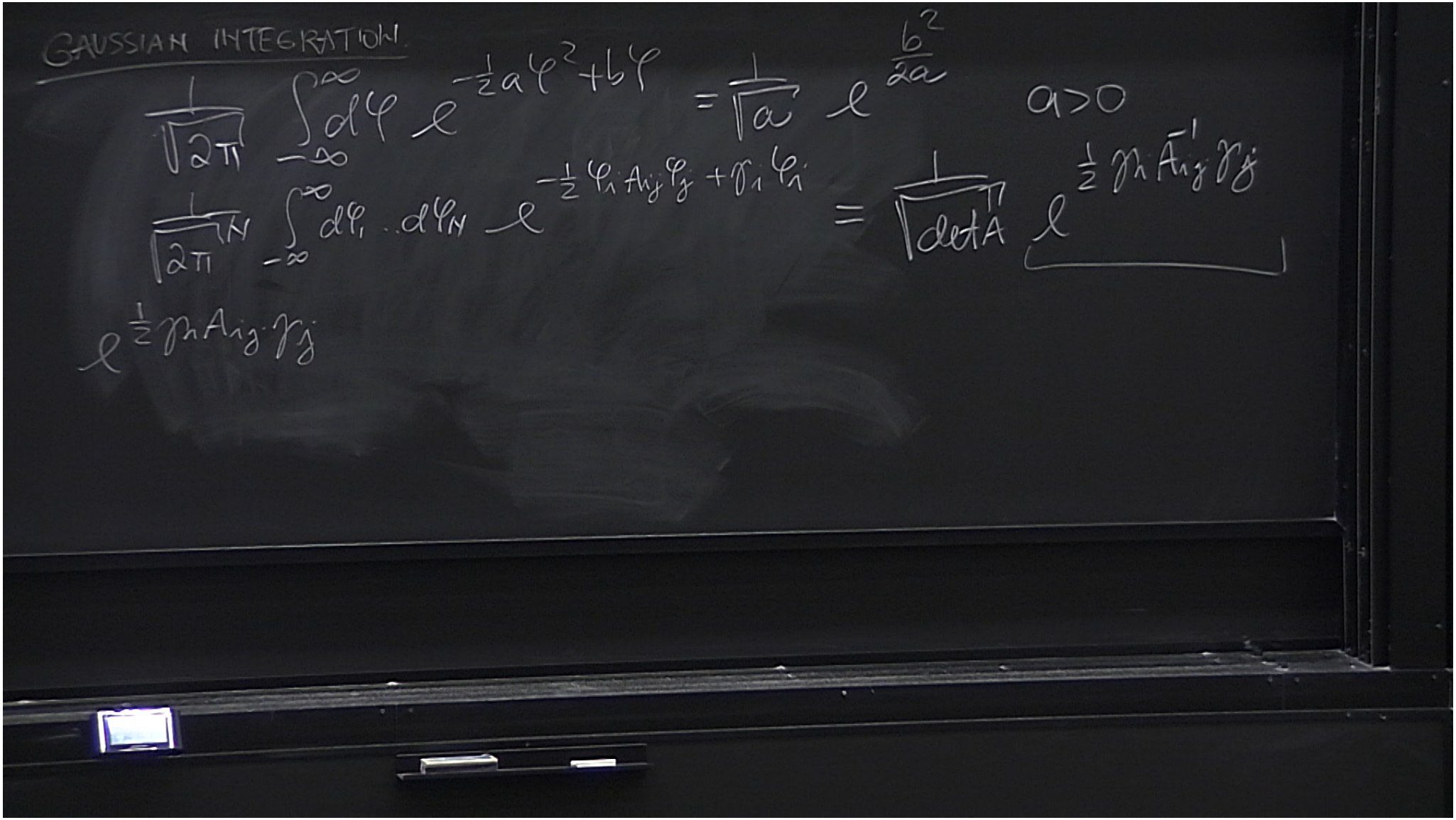
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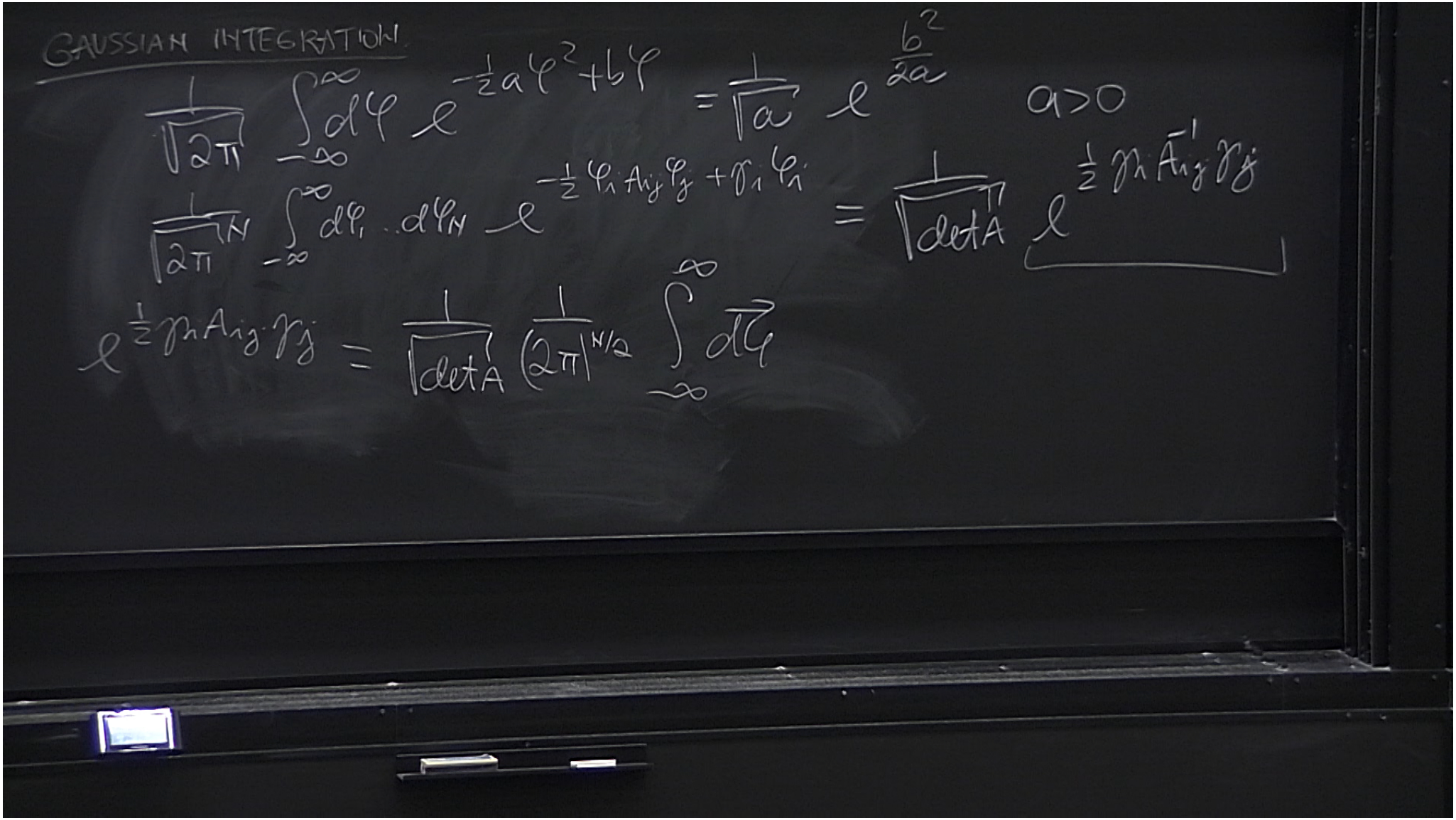

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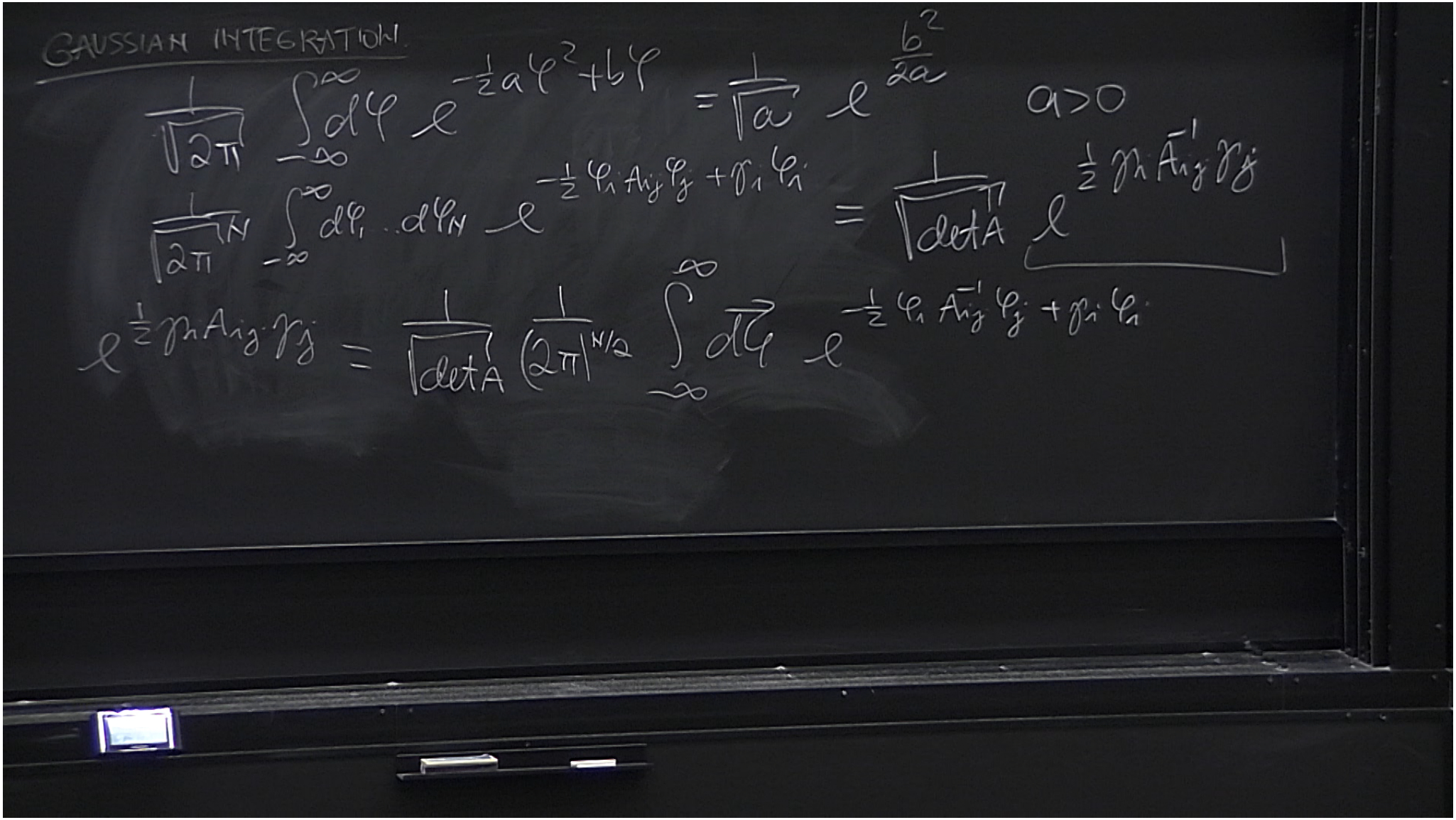
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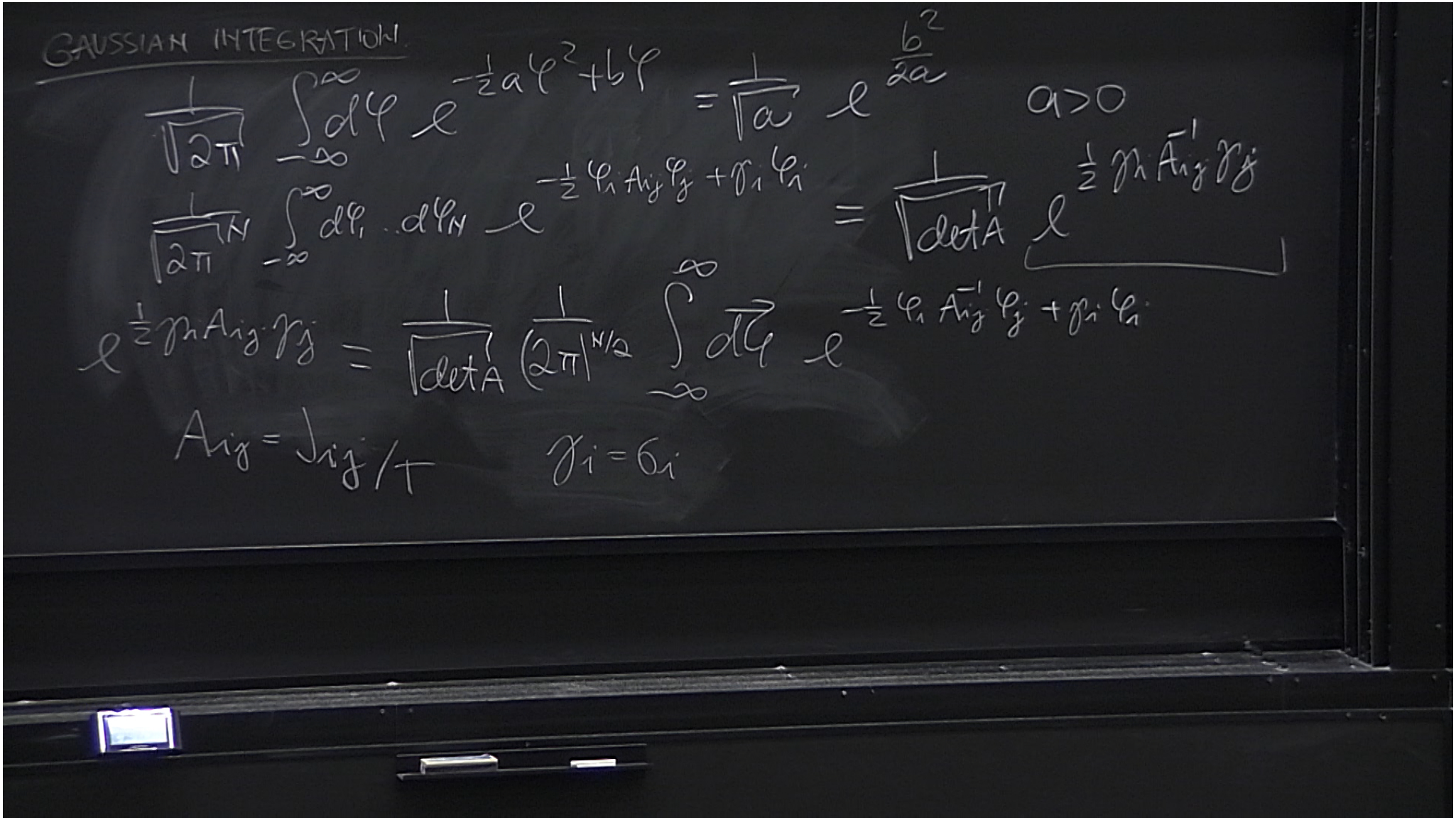
GAUSSIAN INTEGRATION

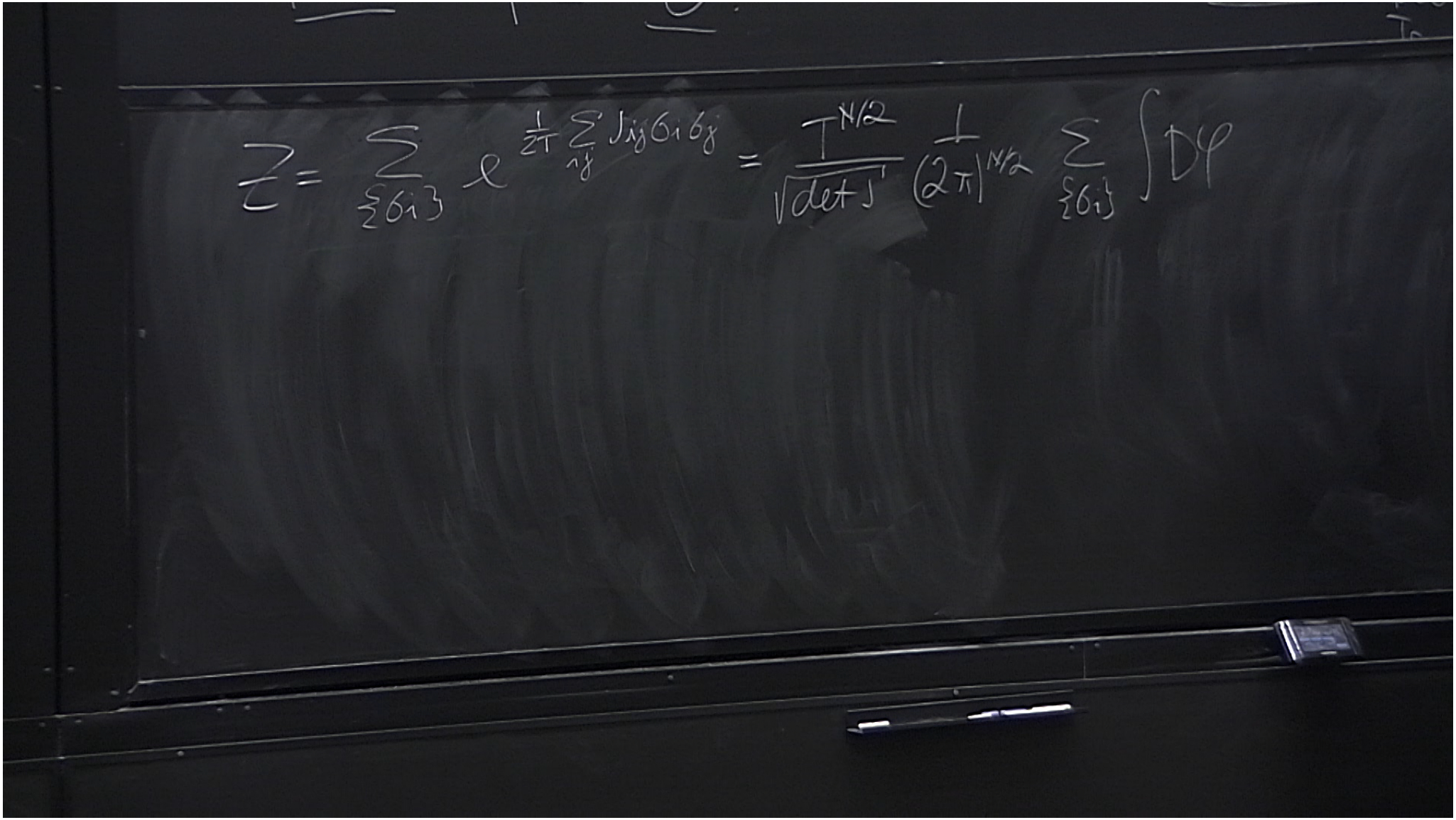
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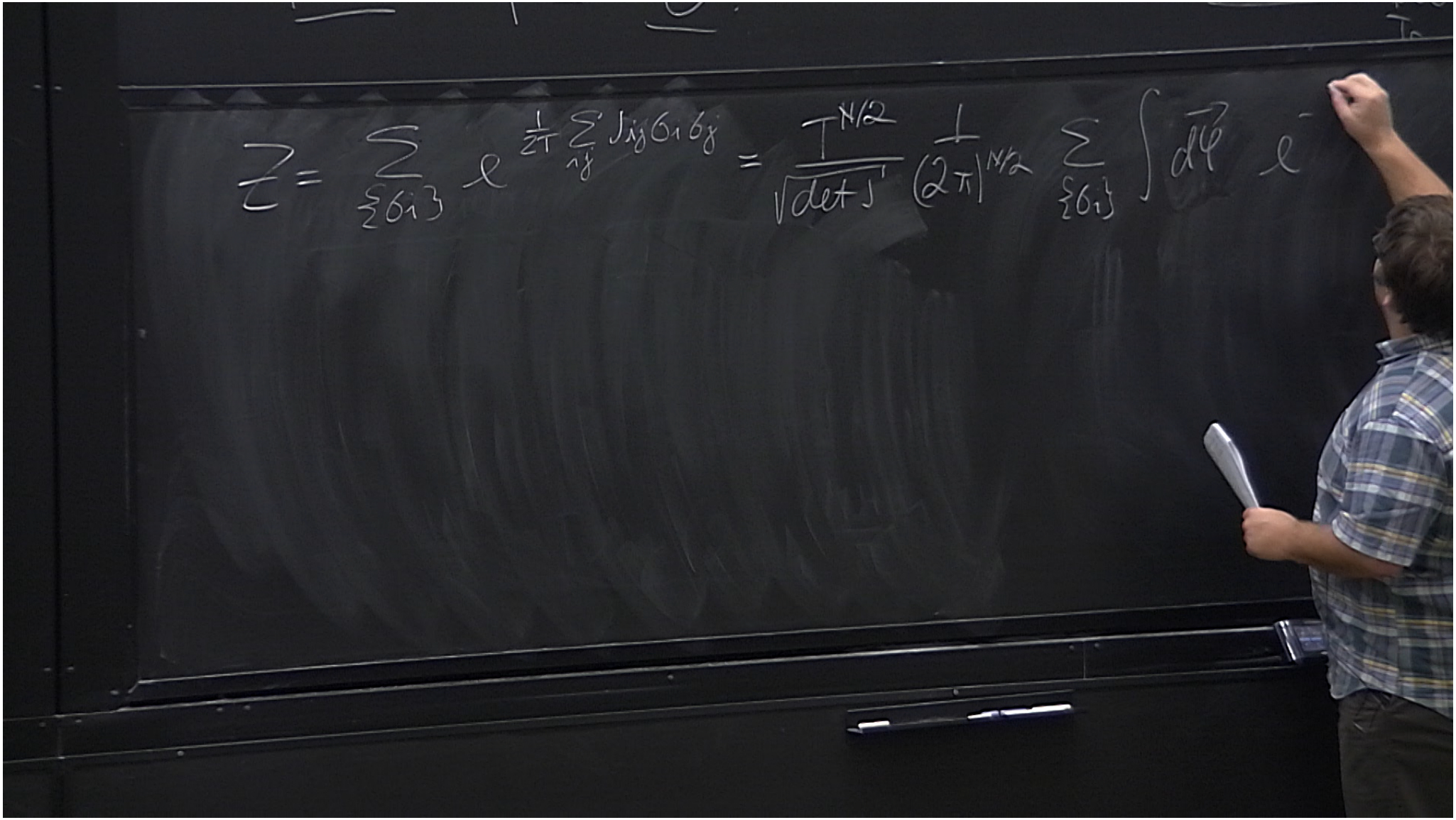
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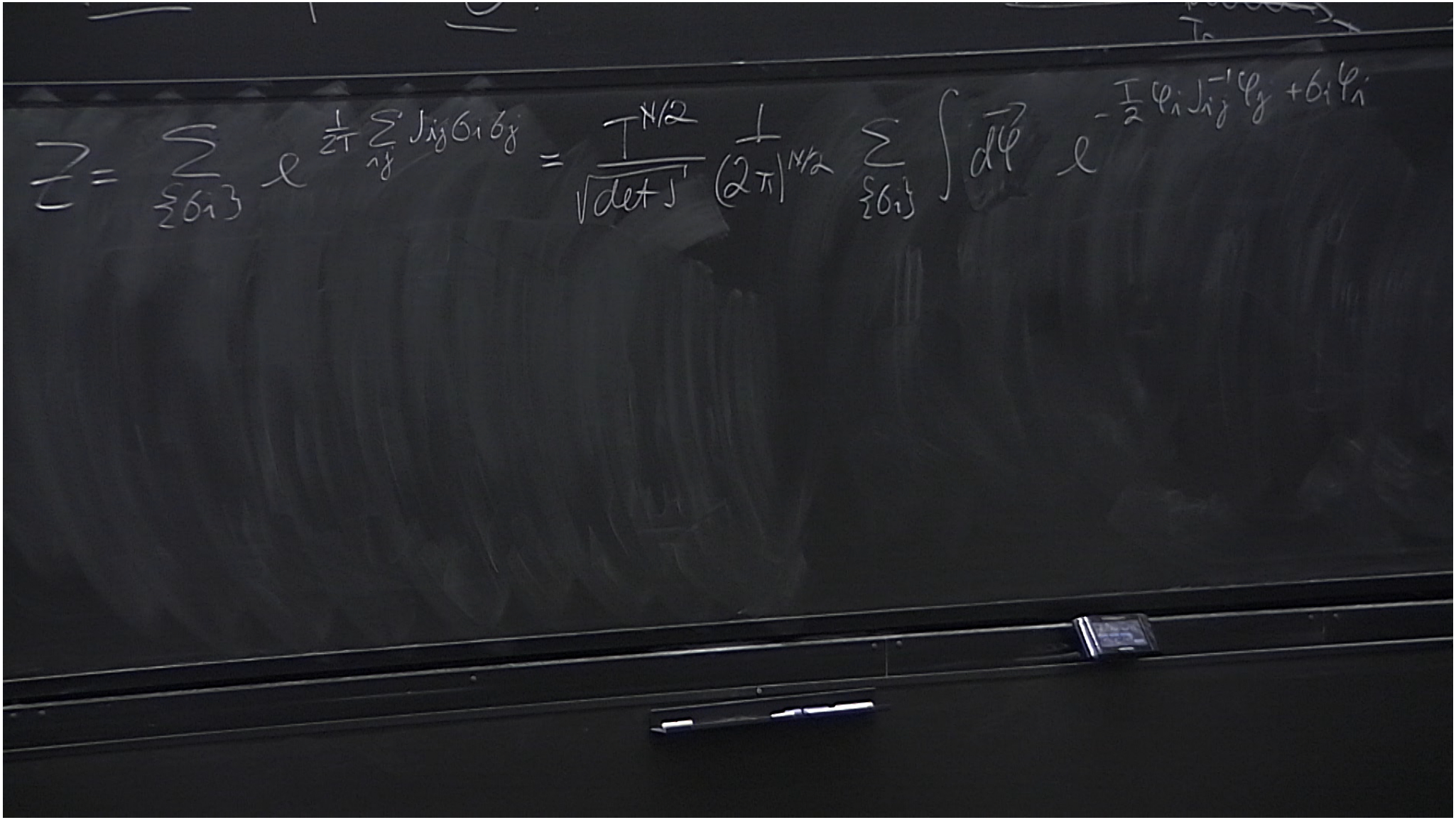
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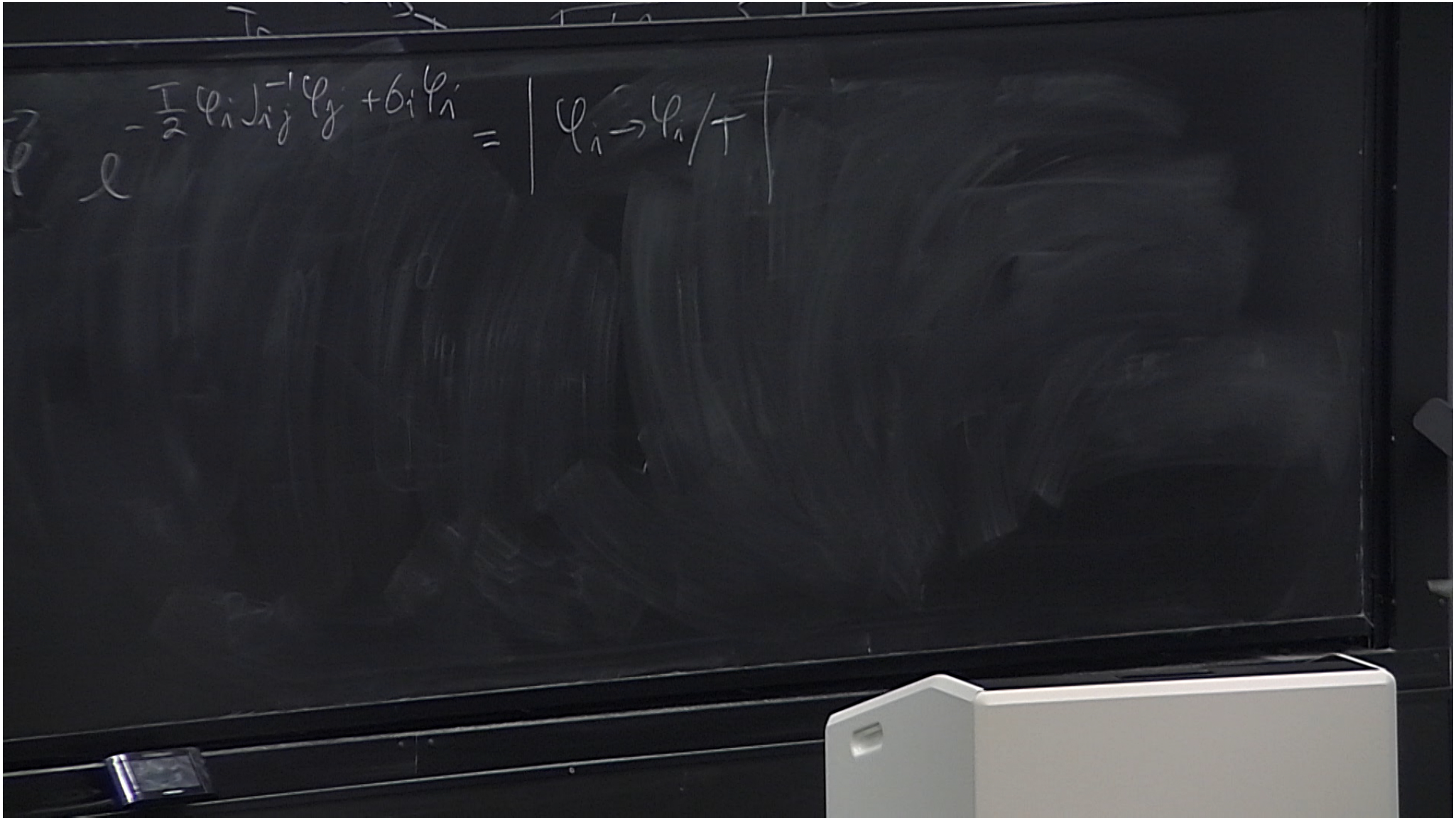
$$A_{ij} = J_{ij}/T$$


GAUSSIAN INTEGRATION
$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\varphi\, e^{-\frac{1}{2}a\varphi^2 + b\varphi} = \frac{1}{\sqrt{a}} e^{\frac{b^2}{2a}}$ $a>0$
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$A_{ij} = J_{ij}/T$ $\gamma_i = \sigma_i$


$Z = \sum_{\{\sigma_i\}} e^{\frac{1}{2T}\sum_{ij} J_{ij}\sigma_i\sigma_j} = \frac{T^{N/2}}{\sqrt{\det J}} \frac{1}{(2\pi)^{N/2}} \sum_{\{\sigma_i\}} \int D\varphi$


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$e^{-\frac{I}{2}\varphi_i J^{-1}_{ij}\varphi_j + G_i\varphi_i} = \left| \varphi_i \to \varphi_i/T \right|$

$$Z = \sum_{\{\sigma_i\}} e^{\frac{1}{2T}\sum_{ij} J_{ij}\sigma_i\sigma_j} = \frac{T^{N/2}}{\sqrt{\det J}} \frac{1}{(2\pi)^{N/2}} \sum_{\{\sigma_i\}} \int d\vec{\varphi}\; e^{-\frac{1}{2}\varphi_i J^{-1}_{ij}\varphi_j + \sigma_i\varphi_i} =$$

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ANALYTIC FACTOR

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ANALYTIC FACTOR

$$\| \quad 1$$

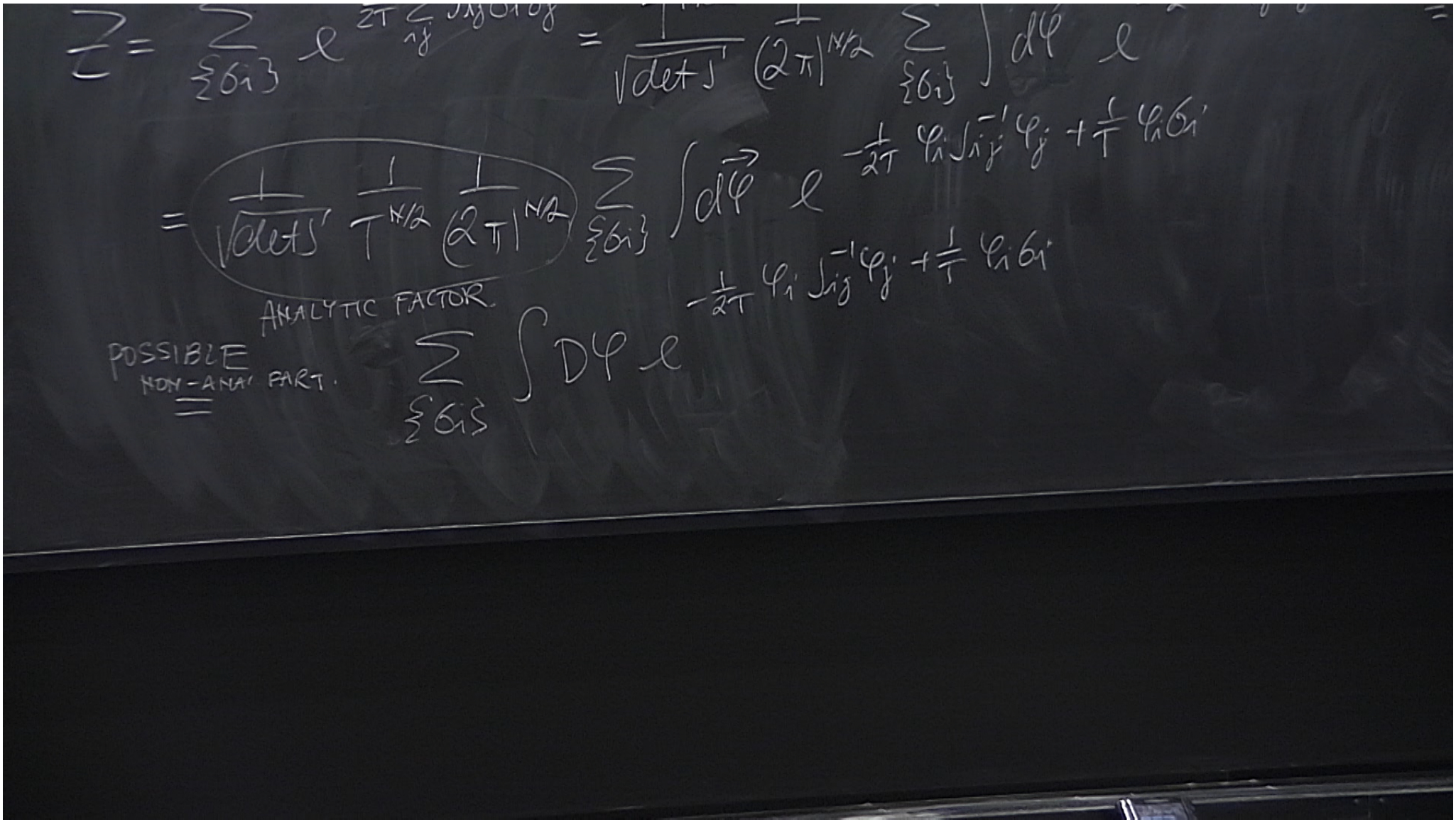


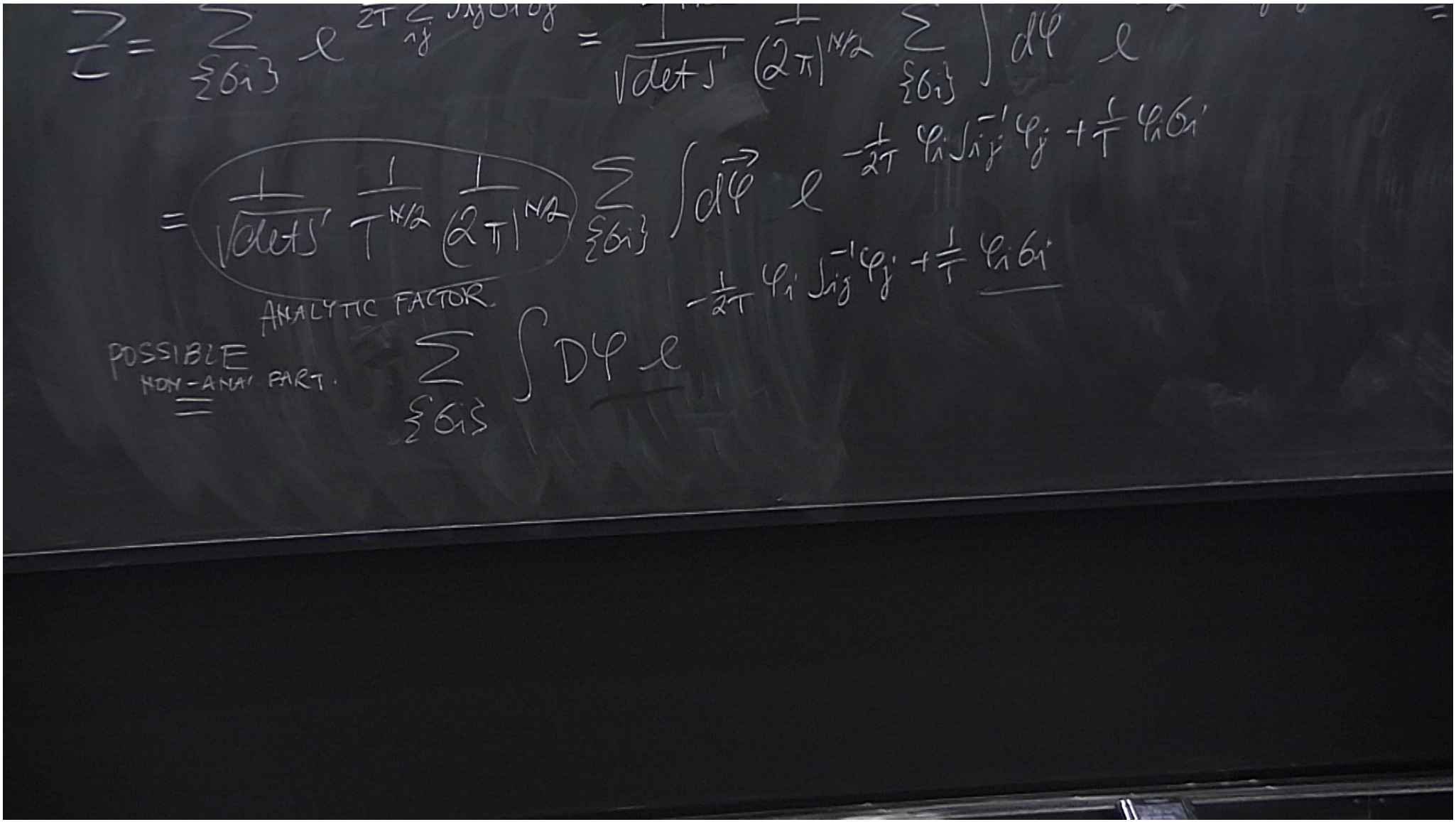
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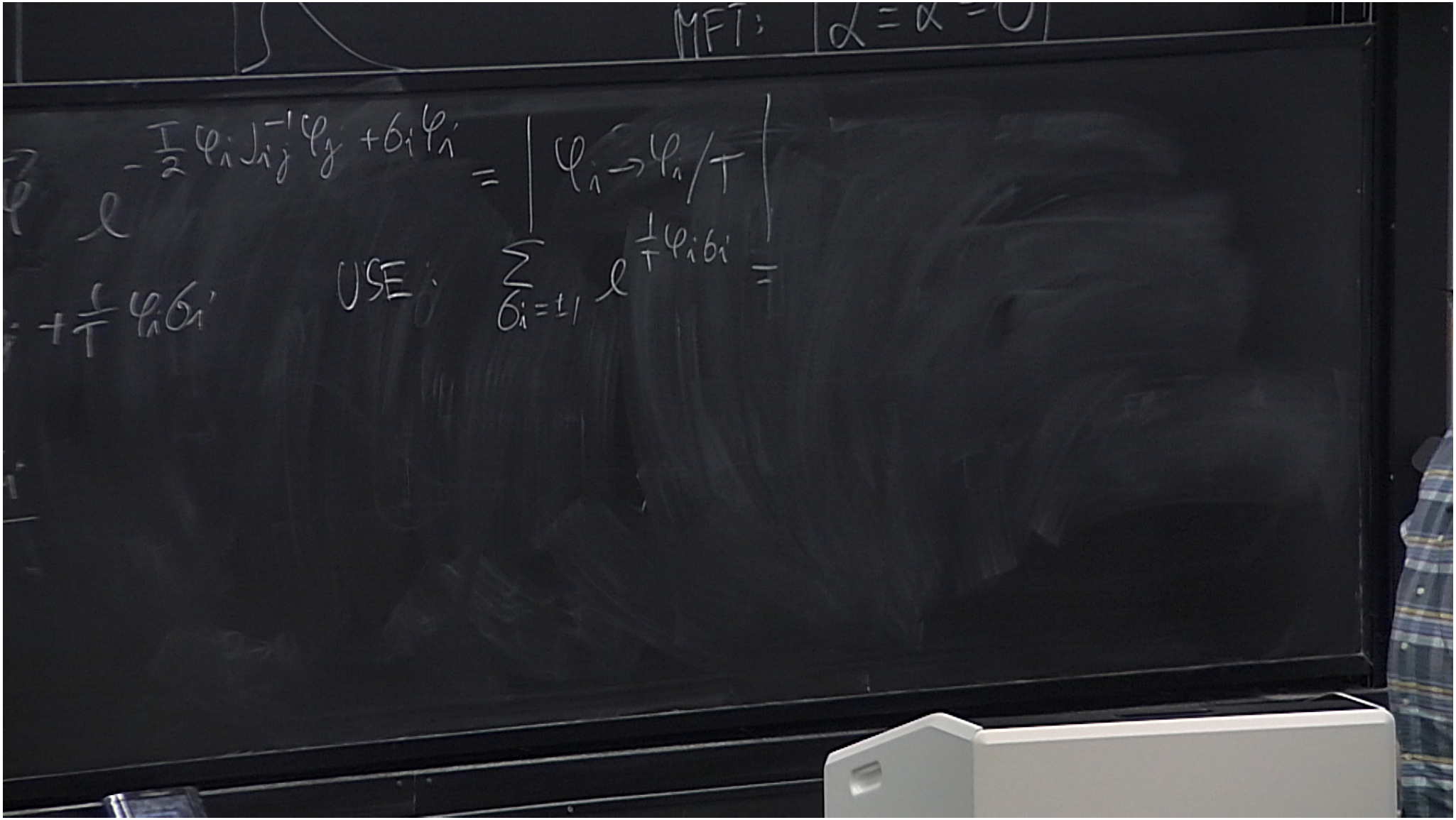
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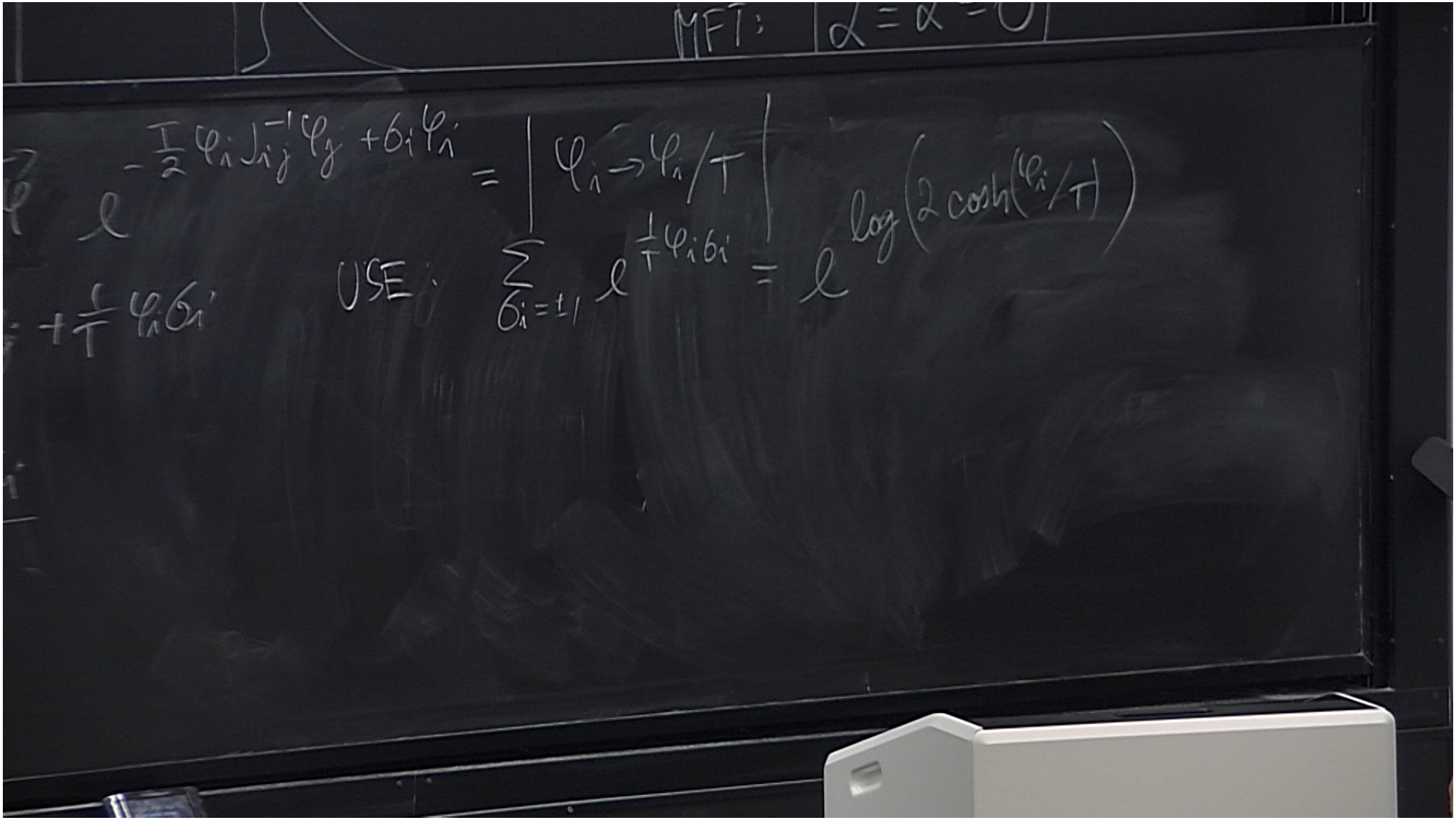
ANALYTIC FACTOR.

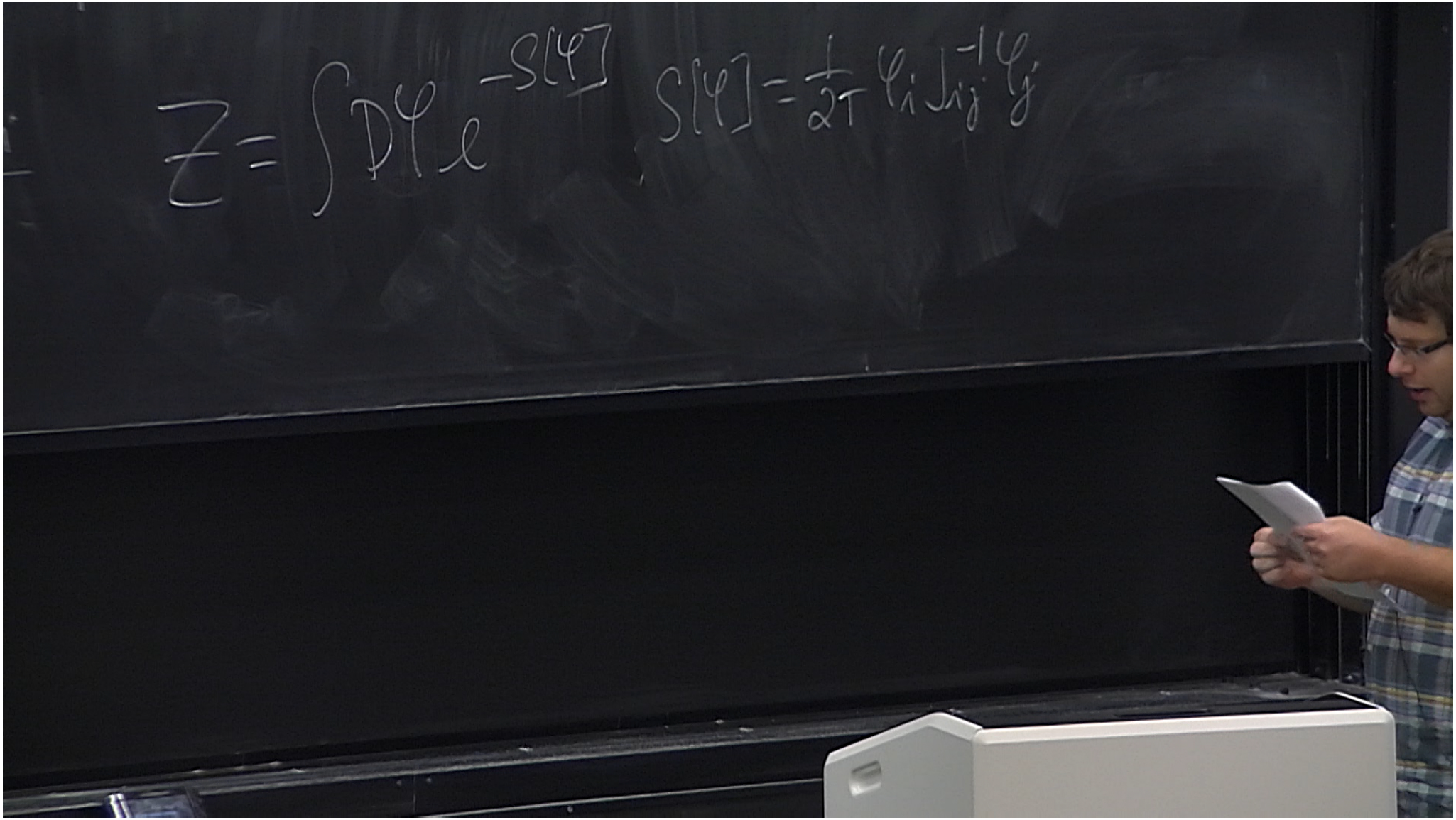
POSSIBLE NON-ANAL. PART.

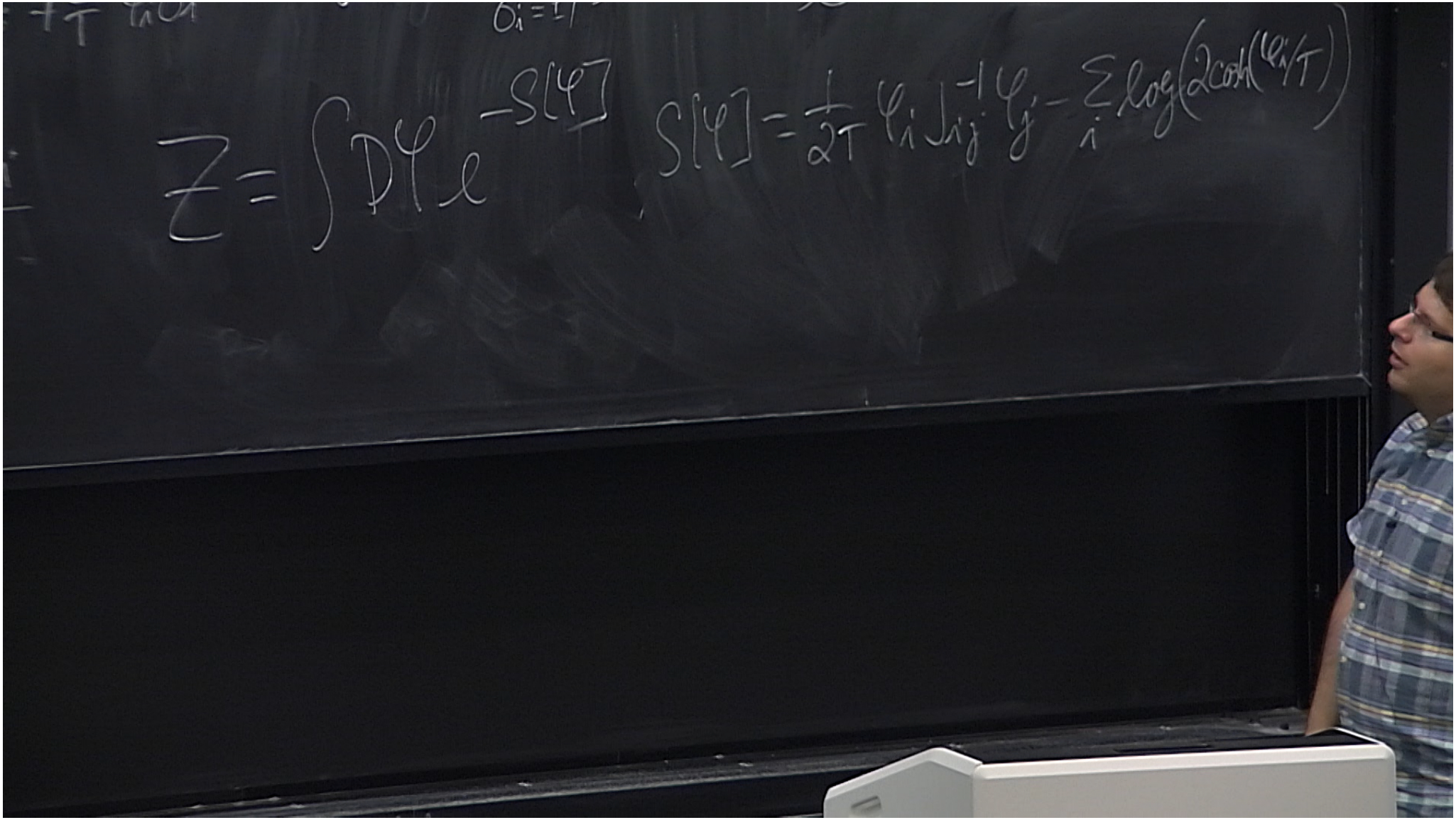

ANALYTIC FACTOR
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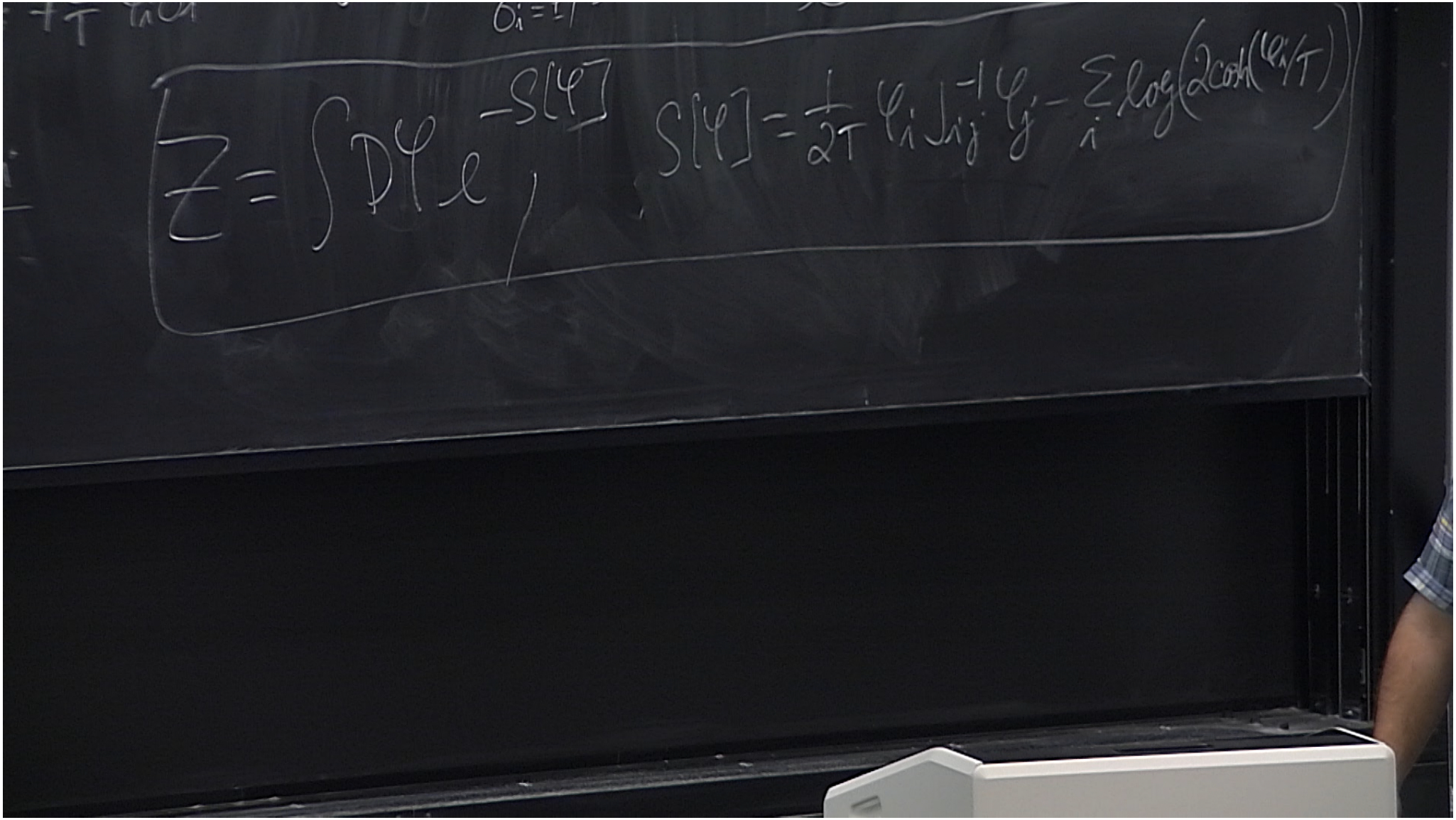

ANALYTIC FACTOR
POSSIBLE NON-ANA' PART.


MFT:
$\alpha = \alpha_c = 0$
$-\frac{I}{2}\varphi_i J_{ij}^{-1}\varphi_j + \sigma_i\varphi_i$
$\varphi_i \to \varphi_i/T$
$+\frac{1}{T}\varphi_i\sigma_i$
USE:
$\sum_{\sigma_i=\pm 1} e^{\frac{1}{T}\varphi_i\sigma_i} =$


MFT:
$\alpha = \alpha' = 0$
$e^{-\frac{I}{2}\varphi_i J_{ij}^{-1}\varphi_j + G_i\varphi_i} = \left| \varphi_i \to \varphi_i/T \right|$
USE: $\sum_{G_i=\pm 1} e^{\frac{1}{T}\varphi_i G_i} = e^{\log\left(2\cosh(\varphi_i/T)\right)}$
$+\frac{1}{T}\varphi_i G_i$


$Z = \int D\varphi\, e^{-S[\varphi]}$
$S[\varphi] = \frac{1}{2T}\varphi_i J^{-1}_{ij} \varphi_j$


$Z = \int D\varphi\, e^{-S[\varphi]}$
$S[\varphi] = \frac{1}{2T}\varphi_i J_{ij}^{-1}\varphi_j - \sum_i \log\left(2\cosh(\varphi_i/T)\right)$


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