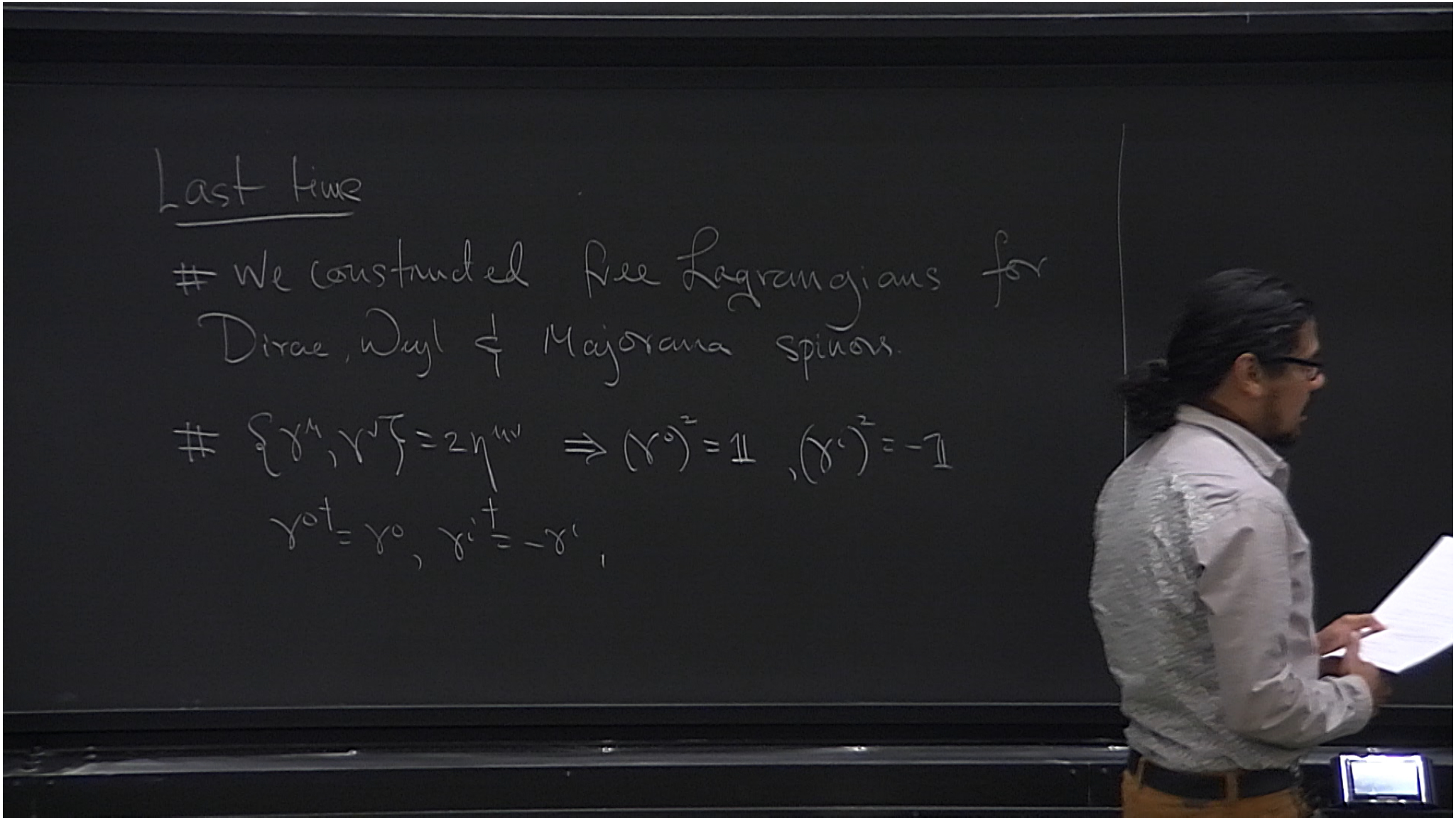
Title: PSI 17/18 - Quantum Field Theory I - Lecture 10

Date: Oct 23, 2017 09:00 AM

URL: http://pirsa.org/17100029

Abstract:


Last time
We constructed free Lagrangians for Dirac, Weyl & Majorana spinors.
$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \Rightarrow (\gamma^0)^2 = \mathbb{1}$, $(\gamma^i)^2 = -\mathbb{1}$
$\gamma^{0\dagger} = \gamma^0$, $\gamma^{i\dagger} = -\gamma^i$,

Plane Wave solutions to free D.E.

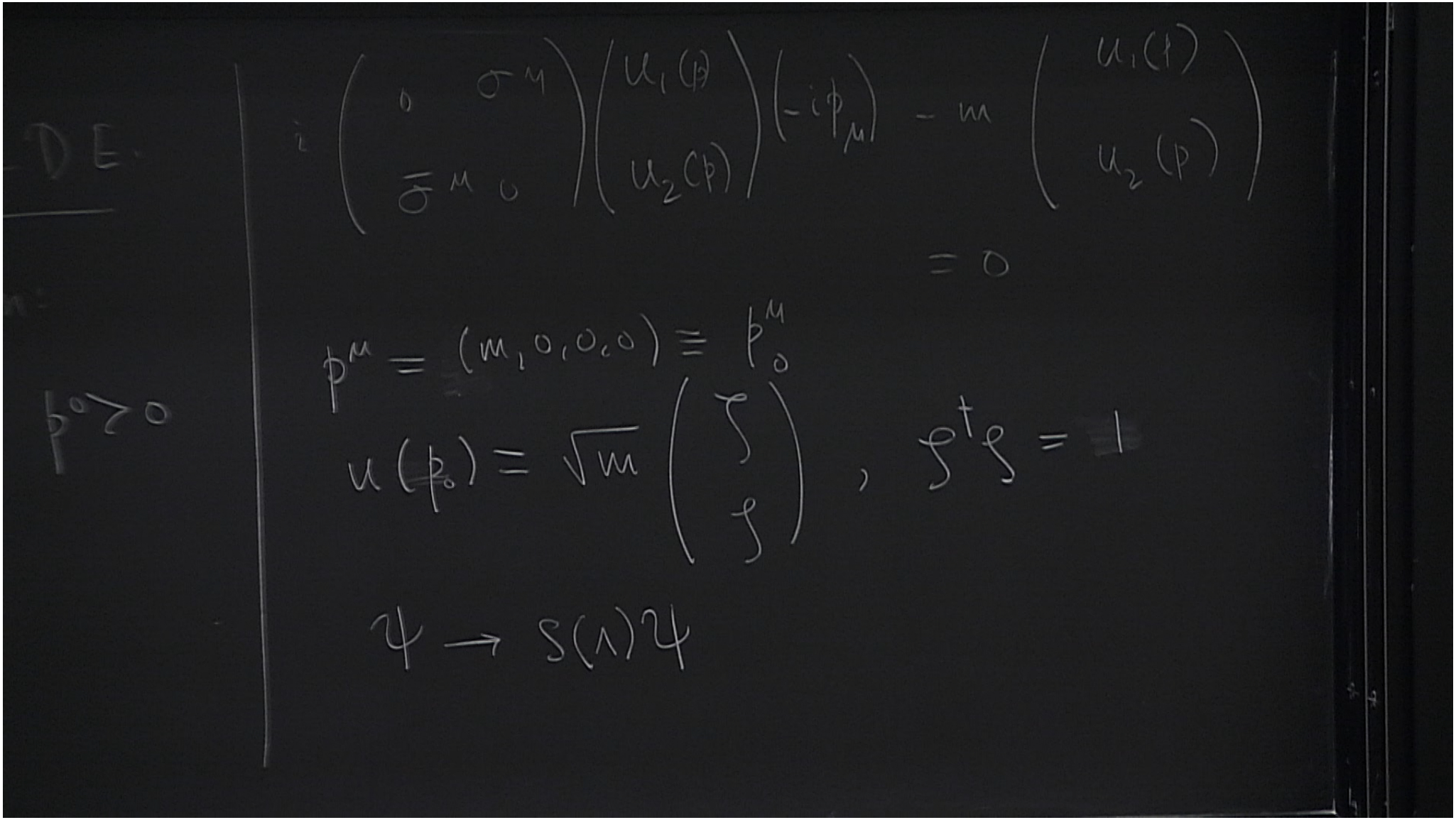
Positive frequency solution:

$$\psi_p(x) = u(p)\, e^{-ip\cdot x} \quad ; \; p^0 > 0$$

$u(p)$: 4-component spinor

$\Rightarrow$ D.E. in Weyl basis

$$i\begin{pmatrix} 0 & \\ \bar{\sigma} & \end{pmatrix}$$


DE.
$p^0 > 0$
$i \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \begin{pmatrix} u_1(p) \\ u_2(p) \end{pmatrix} (-ip_\mu) - m \begin{pmatrix} u_1(p) \\ u_2(p) \end{pmatrix}$
$= 0$
$p^\mu = (m, 0, 0, 0) \equiv p_0^\mu$
$u(p_0) = \sqrt{m} \begin{pmatrix} \xi \\ \xi \end{pmatrix}$, $\xi^\dagger \xi = 1$
$\psi \to S(\Lambda)\psi$

$$u(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi \\ \sqrt{p\cdot\bar{\sigma}}\,\xi \end{pmatrix}$$

$$\bar{u}(p) = u^{\dagger}(p)\gamma^{0}$$

$$\bar{u}(p)u(p) = 2m$$

$$\xi_s^{\dagger}\xi_r = \delta_{sr}$$

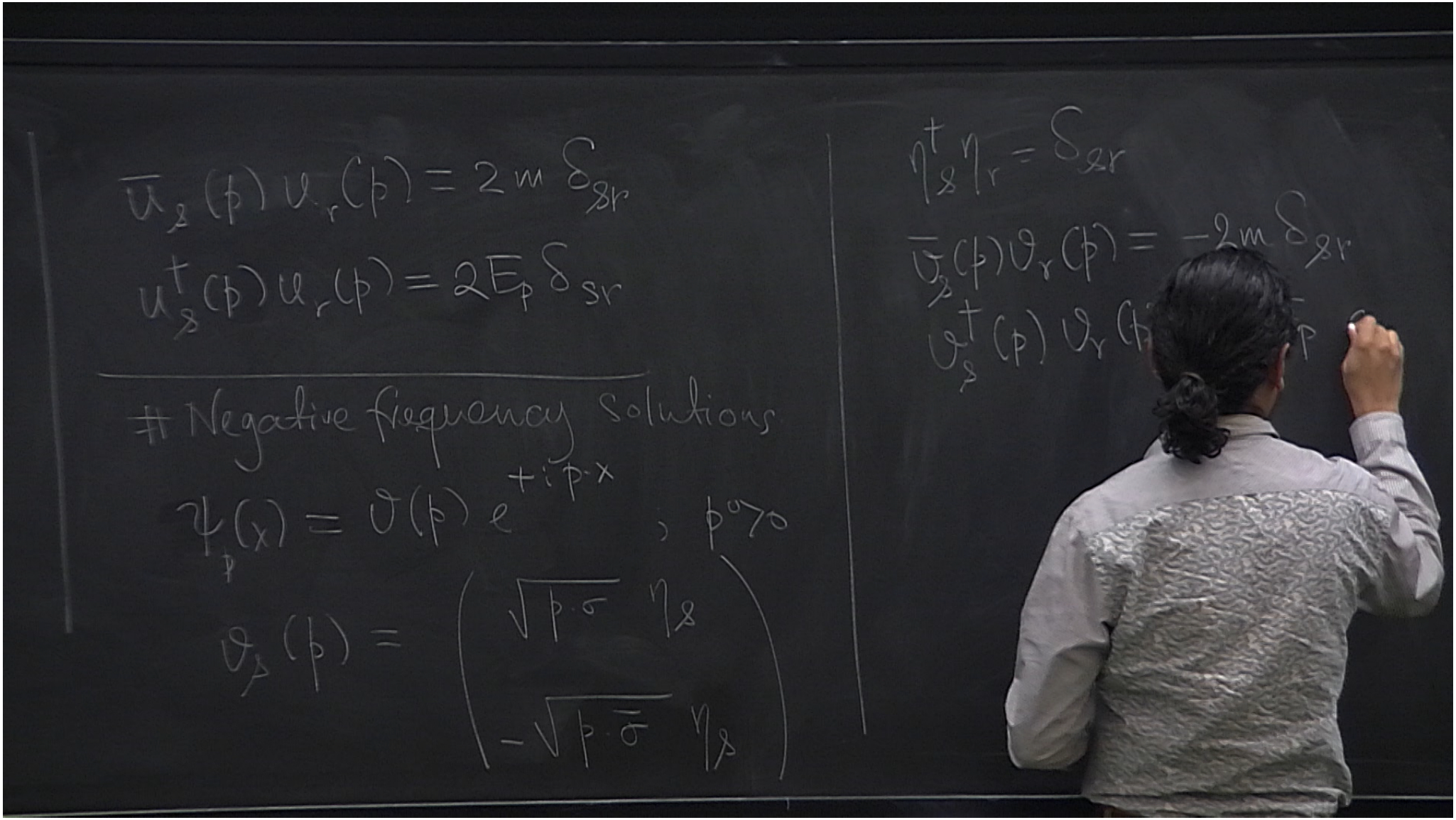
$\bar{u}_s(p)\,u_r($

$u_s^{\dagger}(p)\,u_r($

\# Negative f

$\psi_p(x) =$

$v_s(p) =$


$\bar{u}_s(p)\, u_r(p) = 2m\,\delta_{sr}$
$u_s^\dagger(p)\, u_r(p) = 2E_p\,\delta_{sr}$
Negative frequency solutions
$\psi_p(x) = v(p)\, e^{+ip\cdot x}$; $p^0 > 0$
$v_s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\;\eta_s \\ -\sqrt{p\cdot\bar{\sigma}}\;\eta_s \end{pmatrix}$
$\eta_s^\dagger \eta_r = \delta_{sr}$
$\bar{v}_s(p) v_r(p) = -2m\,\delta_{sr}$
$v_s^\dagger(p)\, v_r(p)$

$$\eta^{\dagger}_{s}\eta_{r} = \delta_{sr}$$

$$\bar{v}_{s}(p)v_{r}(p) = -2m\,\delta_{sr}$$

$$v^{\dagger}_{s}(p)v_{r}(p) = 2E_{p}\,\delta_{sr}$$

$\eta_s^\dagger \eta_r = \delta_{sr}$

$\bar{v}_s(p) v_r(p) = -2m \delta_{sr}$

$v_s^\dagger(p) v_r(p) = 2E_p \delta_{sr}$

Orthogonality

$\bar{u}_s(p) v_r(p) = \bar{v}_s(p) u_r(p) = 0$

$u_s^\dagger(-p) v_r(p) = 0$

$v_s^\dagger(-p) u_r(p) = 0$

Spin Sums:

$u_r(p) = 0$

Spin Sums:

$$\sum_{s=1}^{2} u_s(p)\,\bar{u}_s(p) = (\not{p} - m\mathbb{1})$$

$$\sum_{s=1}^{2} v_s(p)\,\bar{v}_s(p) = (\not{p} + m)$$

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3 2E_p} \sum_{s=1}^{2} \left\{ u_s(p)\, b_s(p)\, e^{-ipx} + v_s(p)\, c_s^*(p)\, e^{ipx} \right\}$$

$$\bar{\psi}(x) = \int \frac{d^3p}{(2\pi)^3 2E_p} \sum_{s=1}^{2} \left\{ \bar{u}_s(p)\, b_s^*(p)\, e^{+ipx} + \bar{v}_s(p)\, c_s(p)\, e^{-ipx} \right\}$$

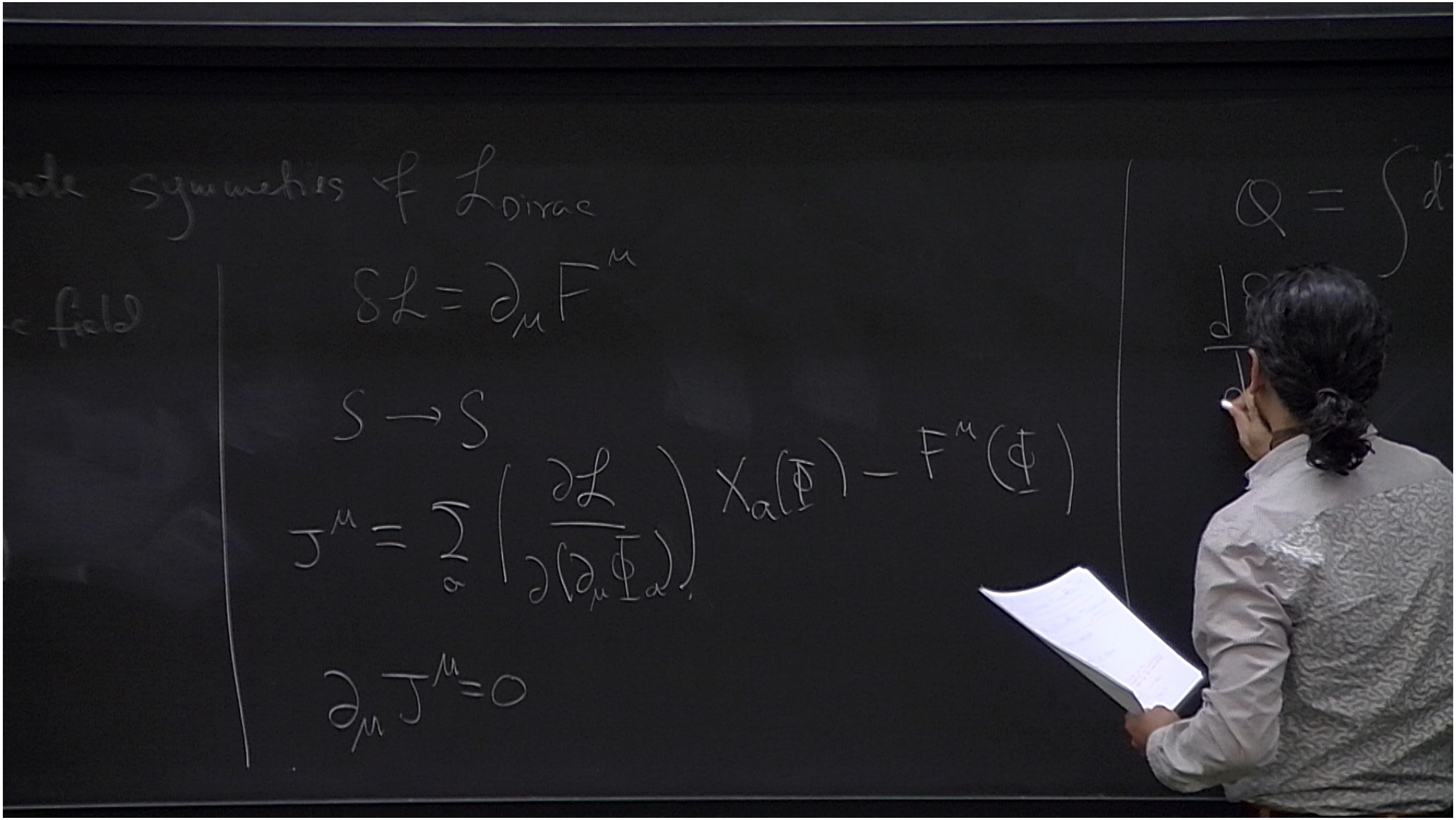
Continuous and discrete symmetries of $\mathcal{L}_{Dirac}$

Quantize the Dirac field

$\delta\mathcal{L} = \partial_\mu F^\mu$

Noether's Theorem

Global, infinitesimal $\delta\bar{\Phi}_a = X_a(\Phi)$


symmetries of $\mathcal{L}_{Dirac}$
field
$\delta\mathcal{L} = \partial_\mu F^\mu$
$S \to S$
$J^\mu = \sum_a \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi_a)} \right) X_a(\Phi) - F^\mu(\Phi)$
$\partial_\mu J^\mu = 0$
$Q = \int d$

$$Q = \int d^3x \, j^0$$

$$\frac{dQ}{dt} = 0$$

Global Phase Symmetry

$$\mathcal{L} = i\bar{\psi}\partial\!\!\!/\psi - m\bar{\psi}\psi$$

$$\left.\begin{aligned} \psi &\to e^{i\alpha}\psi \\ \bar{\psi} &\to e^{-i\alpha}\bar{\psi} \end{aligned}\right\} \to J^{\mu} = ?$$

Global Phase Symmetry

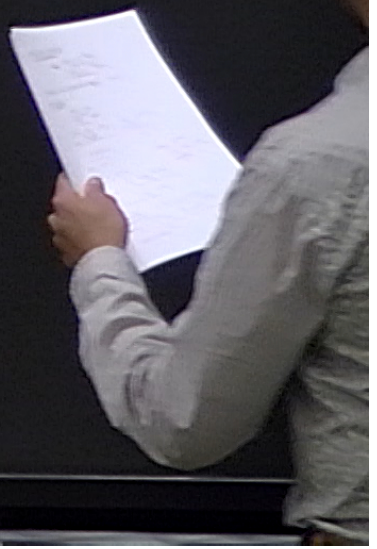
$$\mathcal{L} = i\bar{\psi}\partial\!\!\!/\psi - m\bar{\psi}\psi$$

$$\psi \to e^{i\alpha}\psi$$
$$\bar{\psi} \to e^{-i\alpha}\bar{\psi}$$

$$\to \quad J^\mu = \bar{\psi}\gamma^\mu\psi$$

$$\delta\psi = i\alpha\psi$$
$$\delta\bar{\psi} = -i\alpha\bar{\psi}$$

$J^\mu =$



$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}(i\alpha\psi) - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi})}(i\alpha\bar{\psi})$$

$$= -\alpha\,\bar{\psi}\gamma^\mu\psi$$

Axial Symmetry: $m \to 0$

$$\psi \to e^{i\alpha\gamma_5}\psi$$

$$\bar{\psi} \to \bar{\psi} e^{i\alpha\gamma_5}$$

Translational Symmetry

$$x^\mu \to x^\mu - a^\mu$$

$$T^\mu{}_\nu = i\bar{\psi}\gamma^\mu\partial_\nu\psi$$

$$P_\nu = \int d^3x\, T^0{}_\nu$$

Translational Symmetry

$$x^\mu \to x^\mu - a^\mu$$

$$T^\mu{}_\nu = i\bar{\psi}\gamma^\mu \partial_\nu \psi \sim \text{Energy-momentum stress tensor}$$

$$P_\nu = \int d^3x\, T^0{}_\nu \to \text{" " " } 4\text{-vector}$$

P_0

$$P_0 = H = \int d^3x \; T^{0}{}_{0}$$

Lorentz Symmetry

$$J^{\mu\nu} = \int d^3x \; i\bar{\psi}\gamma^0 \left[\frac{i}{2}\Sigma^{\mu\nu} - \partial^\mu (x^\nu \mathcal{L}) \right] \psi$$

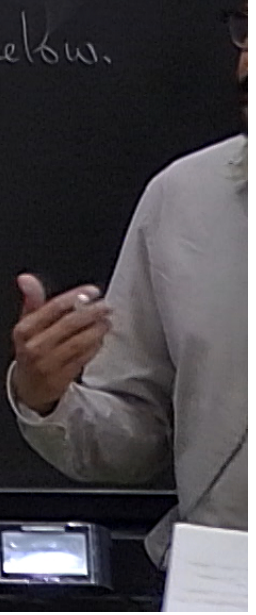
$\partial_\mu J^\mu = 0$

Quantization of the Dirac Field

Spin-Statistics Theorem $\Rightarrow$ anticommutators to quantize ψ

If you to quantize using commutators $\Rightarrow$ H is unbounded from below.

In the non-rel setting: Stability of $H \Rightarrow$ spin-statistics



$\psi, \psi^\dagger \quad t=0$

$\psi_a \rightarrow \pi_a = \frac{\partial \mathcal{L}}{\partial \dot{\psi}_a} = (i\bar{\psi}\gamma^0)_a = i\psi^\dagger_a$

ψ_a ↑ Spinor index

$\{\psi_a(\underline{x}), \pi_b(\underline{y})\} = i\hbar\,\delta_{ab}\,\delta^3(\underline{x}-\underline{y})$

$\boxed{\{\psi_a(\underline{x}), \psi^\dagger_b(\underline{y})\} = \delta_{ab}\,\delta^3(\underline{x}-\underline{y})}$

$\{\psi_a(\underline{x})$

$\{\psi^\dagger_a($

$$\{\psi_a(x), \psi_b(y)\} = 0$$

$$\{\psi_a^\dagger(x), \psi_b^\dagger(y)\} = 0$$

$$\{b_s(\vec{p}), b_r^\dagger(\vec{p}')\} = (2\pi)^3 2E_p \, \delta_{sr} \, \delta^3(\vec{p} - \vec{p}')$$

$$\{c_s(\vec{p}), c_r^\dagger(\vec{p}')\} = (2\pi)^3 2E_p \, \delta_{sr} \, \delta^3(\vec{p} - \vec{p}')$$

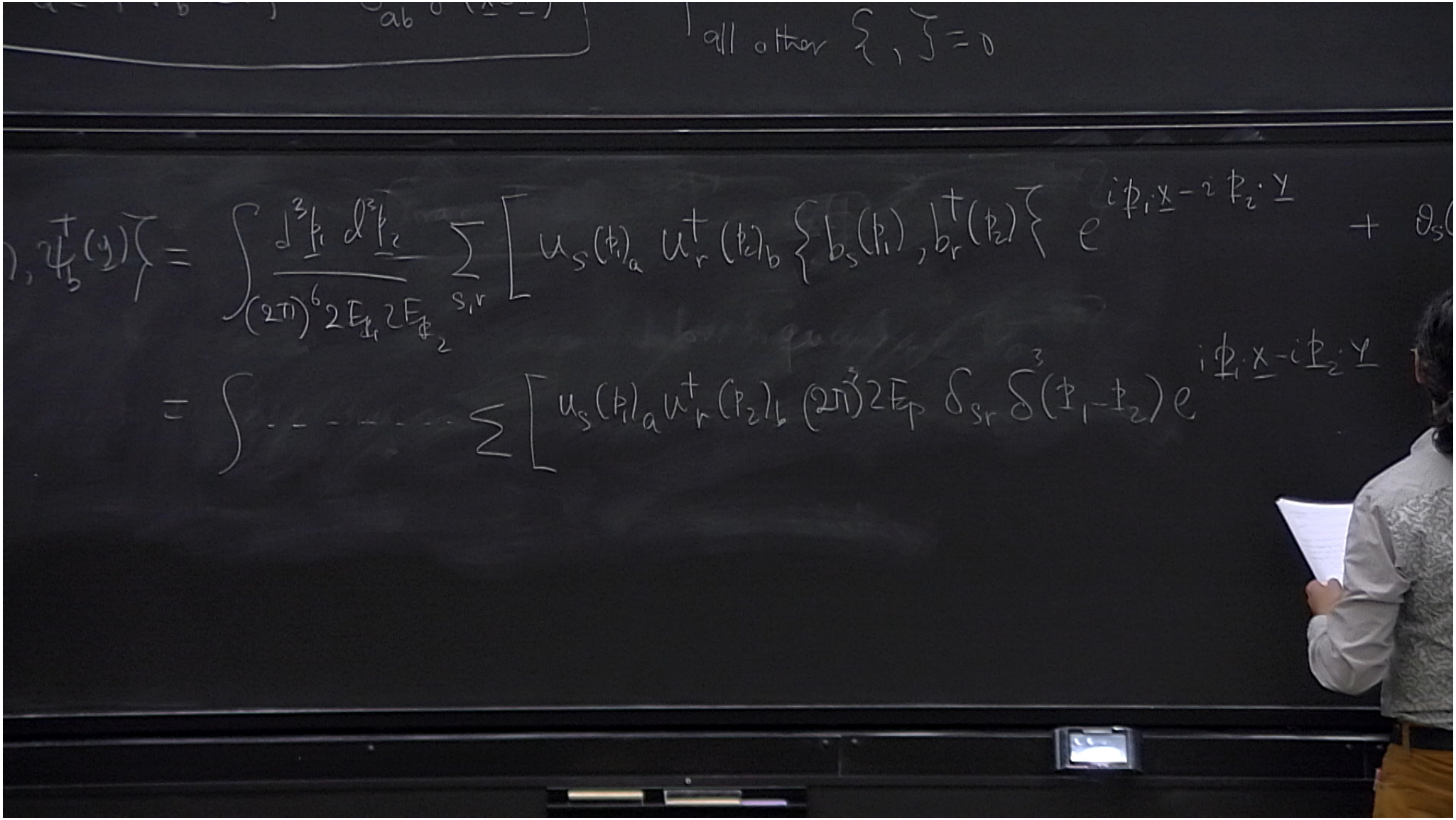
all other $\{\,,\,\} = 0$

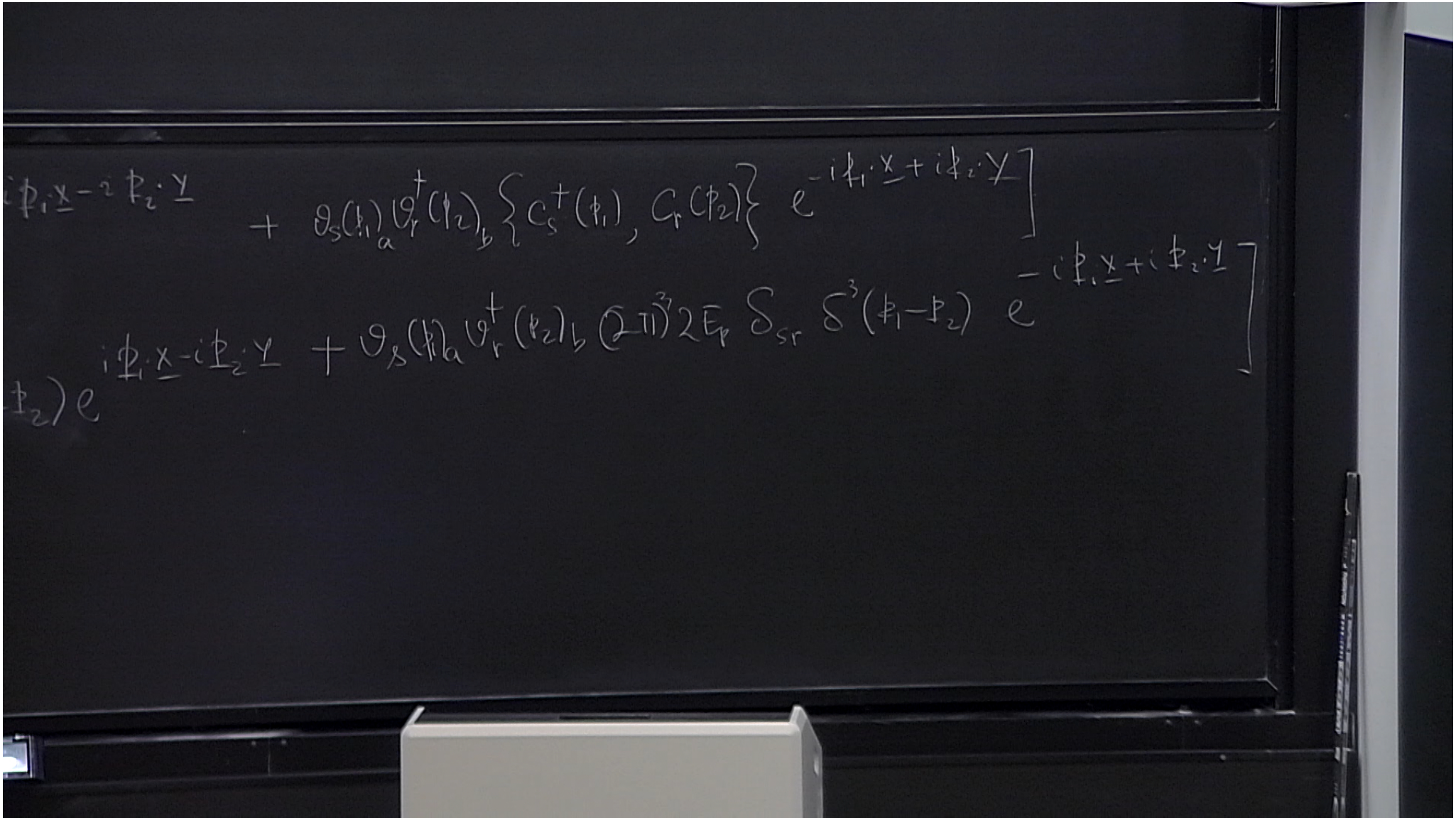
$\delta_{ab}\,\delta(\underline{x}-\underline{y})$

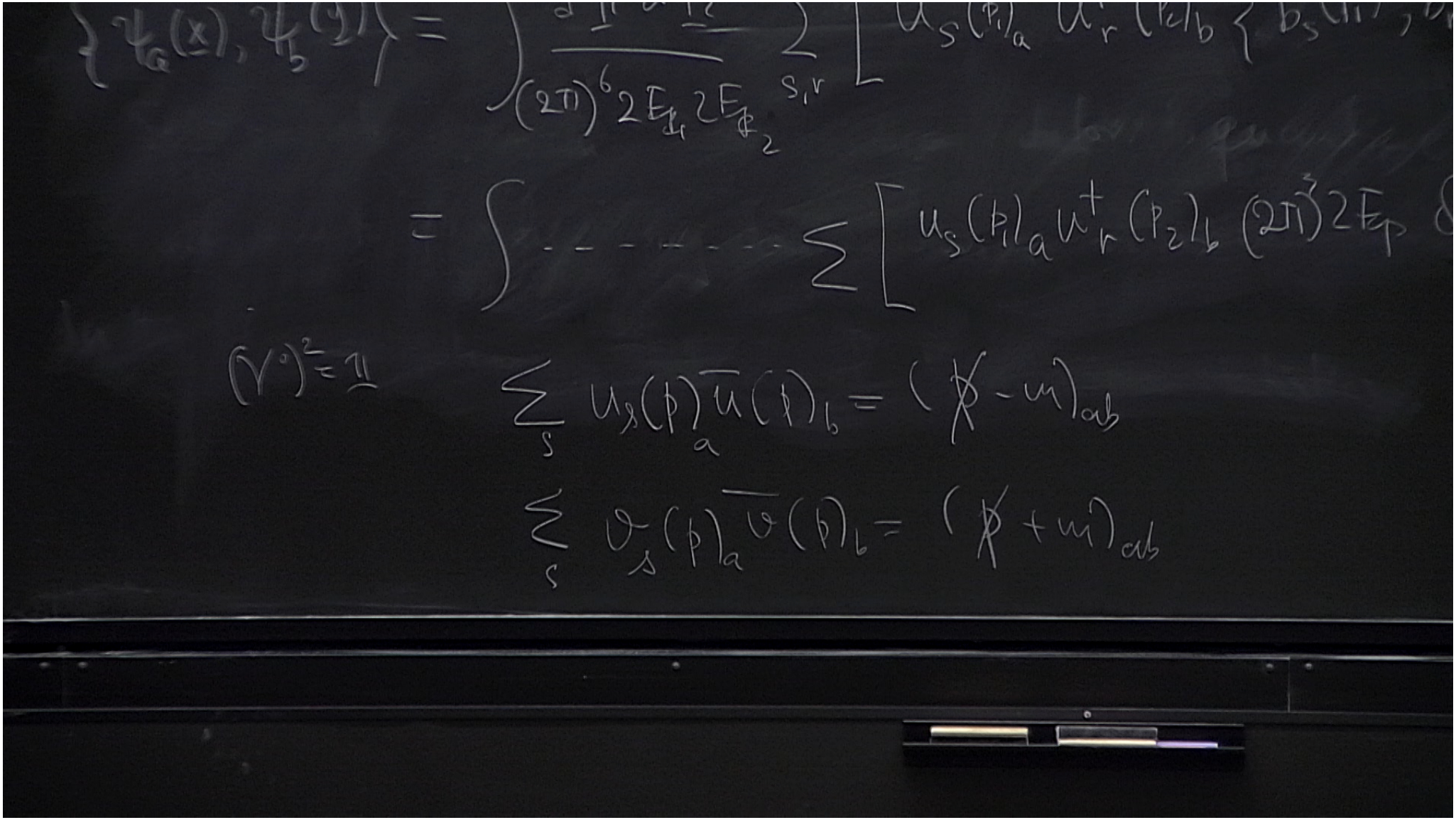
all other $\{\,,\,\}=0$

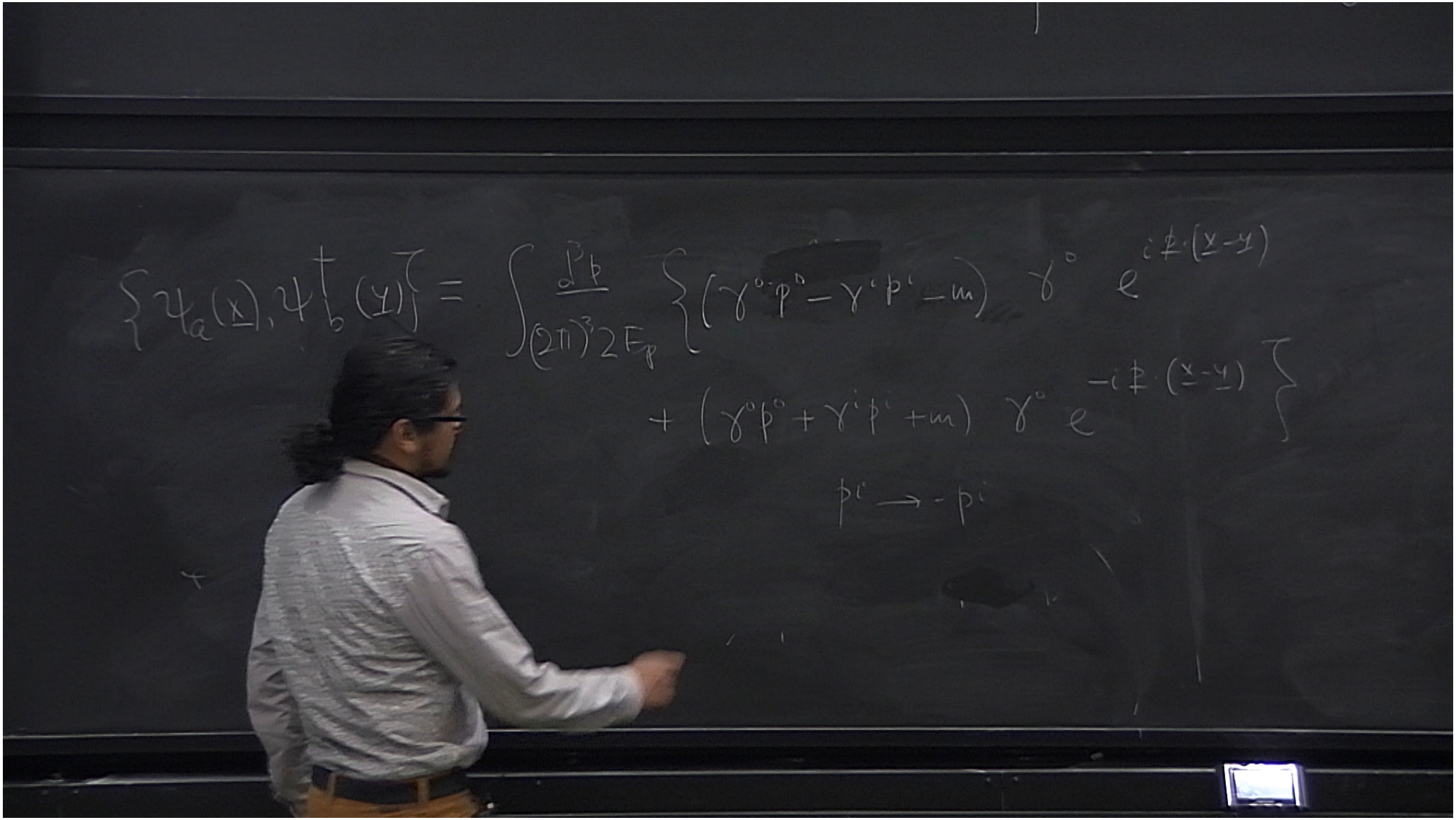
$$\{\psi_a(\underline{x}),\psi_b^\dagger(\underline{y})\}=\int\frac{d^3\underline{p}_1\,d^3\underline{p}_2}{(2\pi)^6\,2E_{\underline{p}_1}\,2E_{\underline{p}_2}}\sum_{s,r}\Big[u_s(\underline{p}_1)_a\,u_r^\dagger(\underline{p}_2)_b\,\{b_s(\underline{p}_1),b_r^\dagger(\underline{p}_2)\}\,e^{i\underline{p}_1\cdot\underline{x}-i\underline{p}_2\cdot\underline{y}}$$

$$\cdot \underline{Y} \quad + \theta_S(\mathbf{k}_1)\,\theta_r^{\dagger}(\mathbf{k}_2)\left\{c_S^{\dagger}(\mathbf{k}_1),\, c_r(\mathbf{k}_2)\right\} e^{-i\mathbf{k}_1\cdot\underline{X}+i\mathbf{k}_2\cdot\underline{Y}}\Big]$$


all other {,} = 0
$\psi_b^\dagger(y)\} = \int \frac{d^3p_1\, d^3p_2}{(2\pi)^6 2E_{p_1} 2E_{p_2}} \sum_{s,r} \left[u_s(p_1)_a\, u_r^\dagger(p_2)_b \{ b_s(p_1), b_r^\dagger(p_2) \} e^{i p_1 \cdot x - i p_2 \cdot y} + \dots \right.$
$= \int \dots\dots\dots \sum \left[u_s(p_1)_a\, u_r^\dagger(p_2)_b\, (2\pi)^3 2E_p\, \delta_{sr}\, \delta^3(p_1 - p_2)\, e^{i p_1 \cdot x - i p_2 \cdot y} \right.$


$+ \vartheta_s(\vec{p}_1)_a \vartheta_r^\dagger(\vec{p}_2)_b \{c_s^\dagger(\vec{p}_1), c_r(\vec{p}_2)\} e^{-i\vec{p}_1\cdot\vec{x} + i\vec{p}_2\cdot\vec{y}}]$
$+ \vartheta_s(\vec{p}_1)_a \vartheta_r^\dagger(\vec{p}_2)_b (2\pi)^3 2E_p \delta_{sr} \delta^3(\vec{p}_1 - \vec{p}_2) e^{-i\vec{p}_1\cdot\vec{x} + i\vec{p}_2\cdot\vec{y}}]$


$\{\psi_a(x), \psi_b(y)\} = \int \frac{d^3p_1\, d^3p_2}{(2\pi)^6 2E_{p_1} 2E_{p_2}} \sum_{s,r} \left[u_s(p_1)_a u^*_r(p_2)_b \{ b_s(p_1), b^\dagger_r(p_2) \} \right.$
$= \int \cdots \sum \left[u_s(p_1)_a u^\dagger_r(p_2)_b (2\pi)^3 2E_p \delta \right.$
$(\gamma^0)^2 = \mathbb{1}$
$\sum_s u_s(p)_a \bar{u}(p)_b = (\not{p} - m)_{ab}$
$\sum_s v_s(p)_a \bar{v}(p)_b = (\not{p} + m)_{ab}$


$\{\psi_a(\underline{x}), \psi^\dagger_b(\underline{y})\} = \int \frac{d^3p}{(2\pi)^3 2E_p} \{(\gamma^0 p^0 - \gamma^i p^i - m)\gamma^0 e^{i\not{p}\cdot(\underline{x}-\underline{y})}$
$+ (\gamma^0 p^0 + \gamma^i p^i + m)\gamma^0 e^{-i\not{p}\cdot(\underline{x}-\underline{y})}\}$
$p^i \longrightarrow -p^i$

$$\{\psi_a(\underline{x}), \psi^\dagger_b(\underline{y})\} = \int \frac{d^3p}{(2\pi)^3 2E_p} \Big\{ (\gamma^0 p^0 - \gamma^i p^i - m)\, \gamma^0\, e^{i\underline{p}\cdot(\underline{x}-\underline{y})}$$

$$+ (\gamma^0 p^0 + \gamma^i p^i + m)\, \gamma^0\, e^{+i\underline{p}\cdot(\underline{x}-\underline{y})} \Big\}_{ab}$$

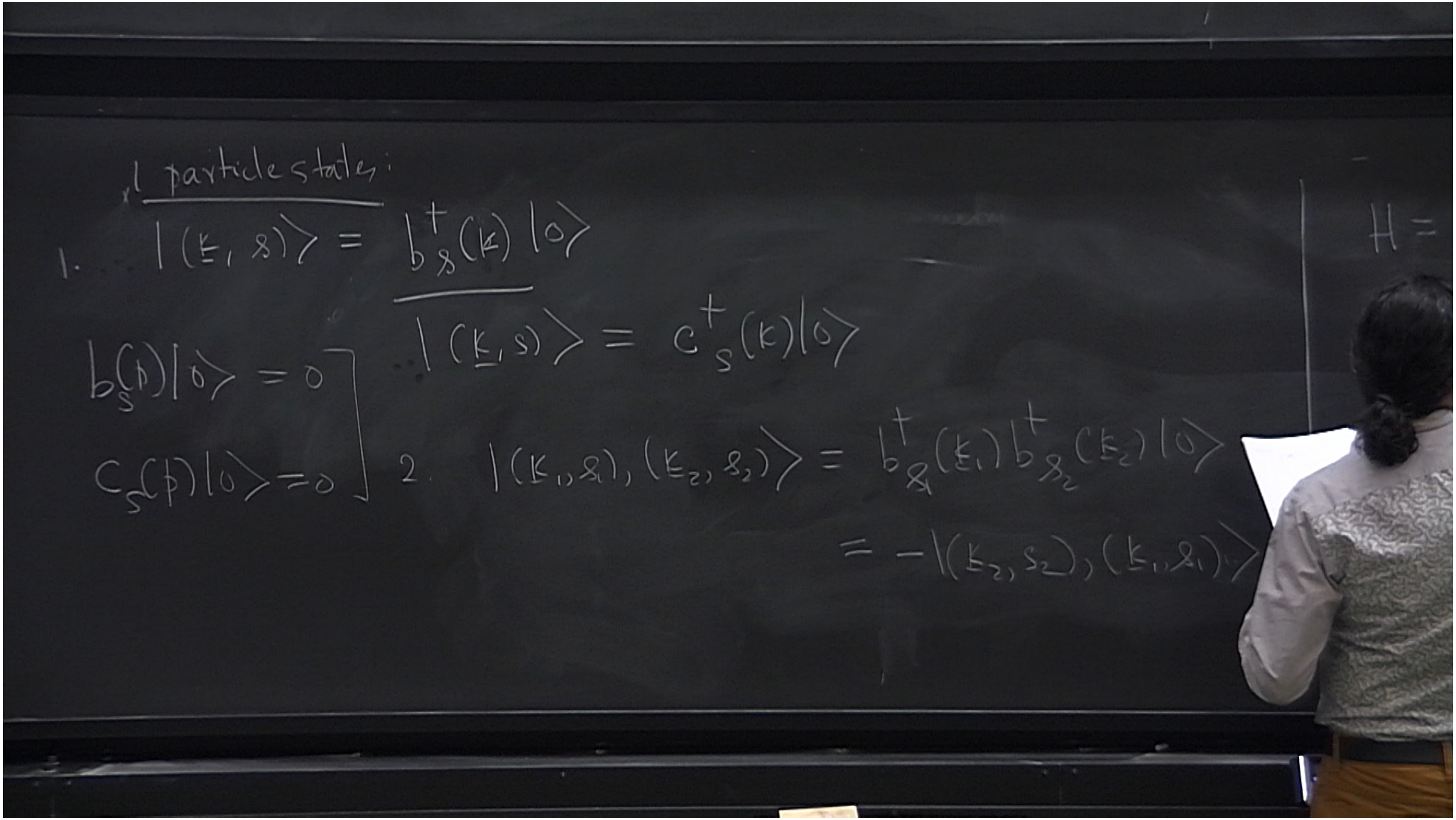
$$(p^i \to -p^i)$$

$$= \int \frac{d^3p}{(2\pi)^3 2E_p} 2p^0\, \delta_{ab}\, e^{i\underline{p}\cdot(\underline{x}-\underline{y})} = \delta_{ab}\, \delta^3(\underline{x}-\underline{y})$$

$\hbar \to 0$

$-\{\psi(\underline{x}), \psi^{\dagger}(\underline{y})\} = 0$

$\psi, \psi^{\dagger}$ → Grassmann variable


1 particle states:
1. $|(\underline{k}, s)\rangle = b^\dagger_s(\underline{k})|0\rangle$
$|(\underline{k}, s)\rangle = c^\dagger_s(\underline{k})|0\rangle$
$b_s(\underline{k})|0\rangle = 0$
$c_s(\underline{k})|0\rangle = 0$
2. $|(\underline{k}_1, s_1), (\underline{k}_2, s_2)\rangle = b^\dagger_{s_1}(\underline{k}_1) b^\dagger_{s_2}(\underline{k}_2)|0\rangle$
$= -|(\underline{k}_2, s_2), (\underline{k}_1, s_1)\rangle$
H =

$$H = \int d^3x \left[\pi_a \dot{\psi}_a - \mathcal{L} \right],$$

$$= \int d^3x \left[-i \bar{\psi} \gamma^i \partial_i \psi - m \bar{\psi} \psi \right]$$

$$H = \int d^3x \left[\pi_a \dot{\psi}_a - \mathcal{L} \right],$$

$$= \int d^3x \left[-i \bar{\psi} \gamma^i \partial_i \psi - m \bar{\psi} \psi \right] -$$

$$= \int \frac{d^3p}{(2\pi)^3 2E_p} \sum_r E_p \left\{ b_r^\dagger(p) b_r(p) + c_r^\dagger(p) c_r(p) - (2\pi)^3 2E_p \delta_{rr} \delta(0) \right\}$$

$$:H: = :\int d^3x \left[\pi_a \dot{\psi}_a - \mathcal{L} \right]:$$

$$= :\int d^3x \left[-i \bar{\psi} \gamma^i \partial_i \psi - m \bar{\psi} \psi \right]:$$

$$= \int \frac{d^3p}{(2\pi)^3 2E_p} \sum_r E_p \left\{ b^\dagger_r(p) b_r(p) + c^\dagger_r(p) c_r(p) - (2\pi)^3 2E_p \delta_{rr} \delta(0) \right\}$$

$:H:$

normal ordering

$$:c b^\dagger: = -:b^\dagger c: = b^\dagger c$$

$$:H: = :\int d^3x \left[\pi_a \dot{\psi}_a - \mathcal{L} \right]:$$

$$= :\int d^3x \left[-i \bar{\psi} \gamma^i \partial_i \psi - m \bar{\psi} \psi \right]:$$

$$= \int \frac{d^3p}{(2\pi)^3 2E_p} \sum_r E_p \left\{ b^\dagger_r(p) b_r(p) + c^\dagger_r(p) c_r(p) \right\}$$

$:H:$

normal ordering

$$:c b^\dagger: = -:b^\dagger c: = b^\dagger c$$

all other $\xi, \zeta = 0$

$$:Q: = \int \frac{d^3p}{(2\pi)^3 2E_p} \sum_r \left\{ b_r^\dagger(p) b_r(p) - c_r^\dagger(p) c_r(p) \right\}$$

$$\{\psi_a(\underline{x}), \psi_b^\dagger(\underline{y})\} = \delta_{ab}\,\delta^3(\underline{x}-\underline{y})$$

$$\{C_s(\underline{p}), C_r^\dagger(\underline{p}')\} = (2\pi)^3 2E_{\underline{p}}\,\delta_{sr}\,\delta(\underline{p}-\underline{p}')$$

all other $\{\,,\,\} = 0$

$$:Q: = \int \frac{d^3\underline{p}}{(2\pi)^3 2E_{\underline{p}}} \sum_r \left\{ b_r^\dagger(\underline{p}) b_r(\underline{p}) - C_r^\dagger(\underline{p}) C_r(\underline{p}) \right\}$$

anti-particle interpretation of $C^\dagger|0\rangle$

$$\psi(x) = e^{iHt}\,\psi(\underline{x})\,e^{-iHt} = \int \frac{d^3\underline{p}}{(2\pi)^3 2E_{\underline{p}}} \sum_r \left[u_s(\underline{p}) b_s(\underline{p}) e^{-iE_{\underline{p}}t + i\underline{p}\cdot\underline{x}} + v_s(\underline{p}) C_s^\dagger(\underline{p})\, e^{+iE_{\underline{p}}t - i\underline{p}\cdot\underline{x}} \right]$$

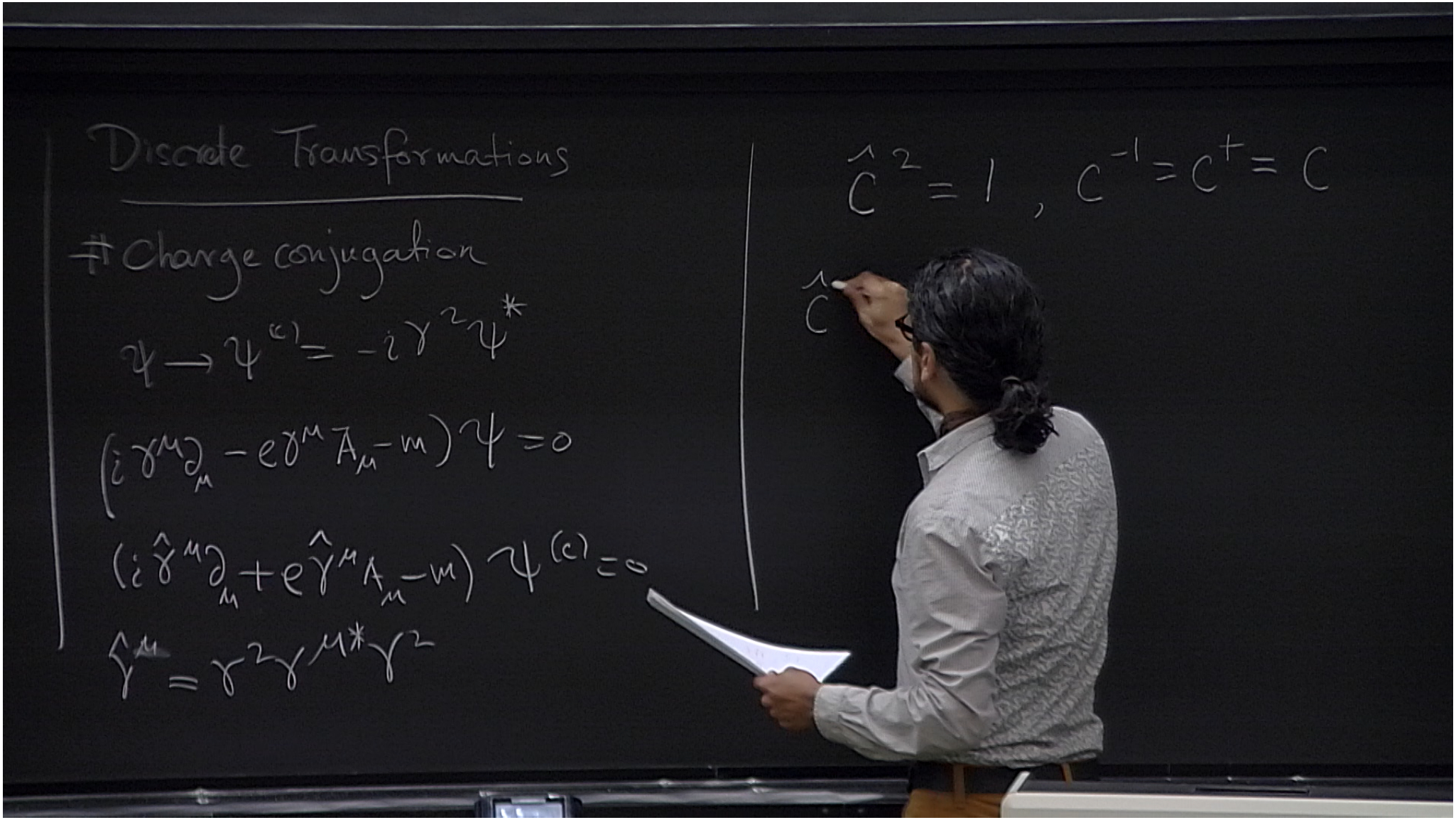
$b^\dagger|0\rangle$, $b^\dagger b^\dagger|\ldots b^\dagger c^\dagger|0\rangle$

$:H: b_r^\dagger(p)|0\rangle = E(p)|\ldots\rangle$

$\langle 0|0\rangle = 1$

$\langle(\underline{k}', s')|(\underline{k}, s)\rangle = (2\pi)^3 2E_p \delta_{ss'} \delta^3(\underline{k}' - \underline{k})$

$U(\Lambda) = e^{-\frac{i}{2}\omega_{\mu\nu}:J^{\mu\nu}:}$


Discrete Transformations
Charge conjugation
$\psi \rightarrow \psi^{(c)} = -i\gamma^2\psi^*$
$(i\gamma^\mu\partial_\mu - e\gamma^\mu A_\mu - m)\psi = 0$
$(i\hat{\gamma}^\mu\partial_\mu + e\hat{\gamma}^\mu A_\mu - m)\psi^{(c)} = 0$
$\hat{\gamma}^\mu = \gamma^2\gamma^{\mu *}\gamma^2$
$\hat{C}^2 = 1, \; C^{-1} = C^\dagger = C$
$\hat{C}$

$$\hat{C}^2 = 1 \quad , \quad C^{-1} = C^\dagger = C$$

$$\hat{C}\, b_s^\dagger(p)\, \hat{C} = c_s^\dagger(p)$$

$$\hat{C}\, c_s^\dagger(p)\, \hat{C} = b_s^\dagger(p)$$

Parity

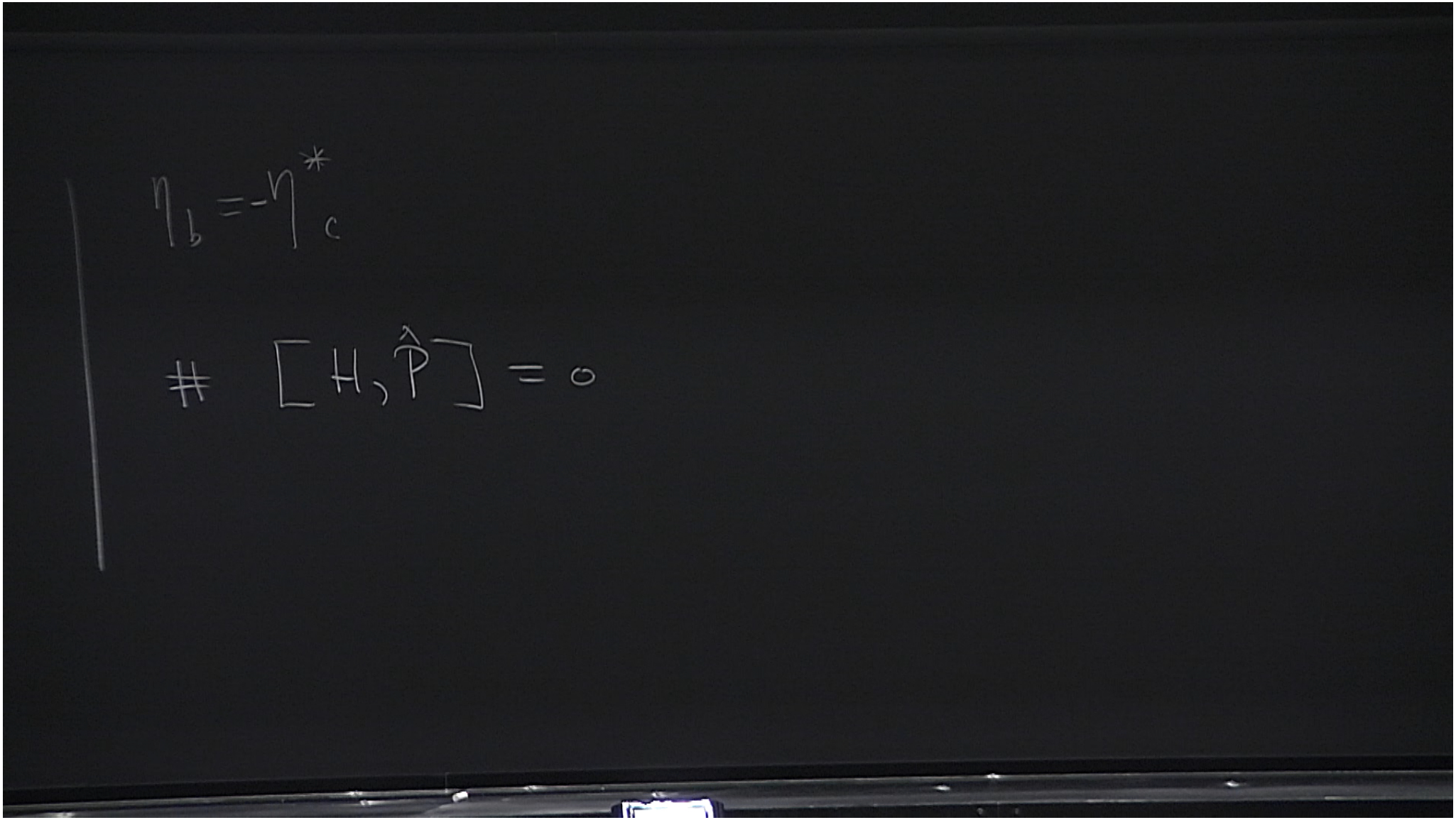
$$\hat{P}\, b_s(\underline{p})\, \hat{P} = \eta_b\, b_s(-\underline{p})$$

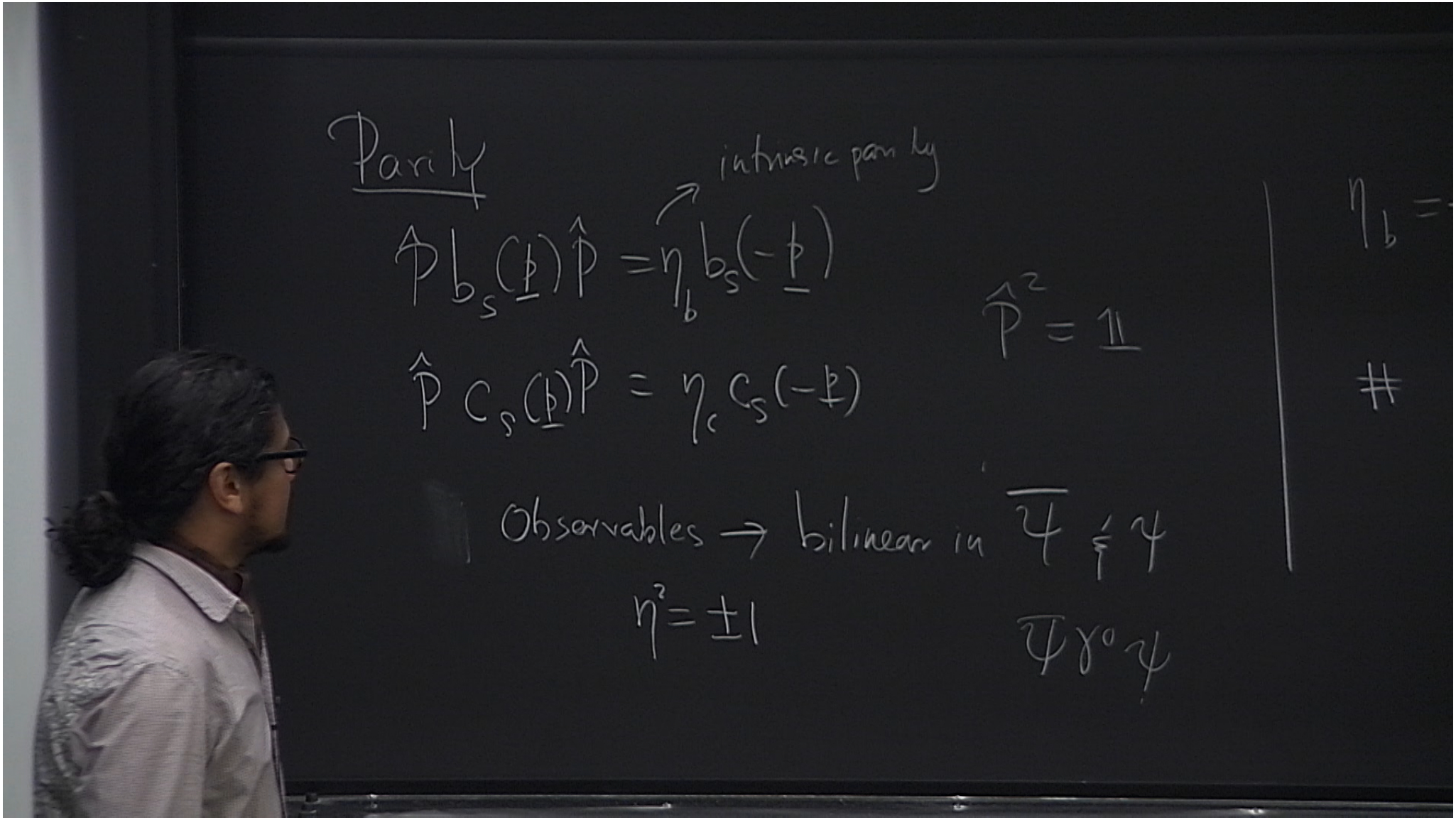
(η_b: intrinsic parity)

$$\hat{P}\, c_s(\underline{p})\, \hat{P} = \eta_c\, c_s(-\underline{p})$$

Observables → bilinear in $\overline{\psi}$ & ψ

$$\eta^2 = \pm 1$$


$\eta_b = -\eta_c^*$
$\# \quad [H, \hat{P}] = 0$


Parity
intrinsic parity
$\hat{P} b_s(\underline{p}) \hat{P} = \eta_b b_s(-\underline{p})$
$\hat{P} c_s(\underline{p}) \hat{P} = \eta_c c_s(-\underline{p})$
$\hat{P}^2 = \mathbb{1}$
Observables → bilinear in $\overline{\psi}$ & ψ
$\eta^2 = \pm 1$
$\overline{\psi}\gamma^0\psi$
$\eta_b =$
#