Title: PSI 17/18 - Quantum Field Theory I - Lecture 9

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Abstract:

Last time $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$

$\{\Gamma^{a+}, \Gamma^{a-}\} = \delta^{ab}$

$\Rightarrow \gamma^\mu \to 4\times 4$, $\psi \to 4$ dim

$\psi \to \psi(x)$

$$x^\mu \longrightarrow \Lambda^\mu{}_\nu x^\nu$$

$$\psi \longrightarrow \exp\left[-\frac{i}{2}\omega_{\mu\nu}\Sigma^{\mu\nu}\right]\psi$$

$$\Sigma^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$$

$\Sigma \rightarrow$ Lie alg of $SO^+(1,3)$

$\psi \to 4$ dim

$S(\Lambda)^{\dagger} \neq S^{-1}(\Lambda)$

Is Dirac reducible?

Schur's Lemma:

$\psi \to 4$ dim

$S(\Lambda)^\dagger \neq S^{-1}(\Lambda)$

Is Dirac reducible?

Schur's lemma: If a matrix commutes with all the group matrices of an irreducible rep then this matrix

is proportional to $\mathbb{1}$.

$$\gamma_5 = -\frac{i}{4!}\,\epsilon_{\mu\nu\rho\sigma}\,\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma$$

$$= -i\,\gamma^0\gamma^1\gamma^2\gamma^3$$

$$\{\gamma_5, \gamma^\mu\} = 0 \quad \Rightarrow \gamma_5 \neq \mathbb{1}$$

$$\gamma_5^\dagger = \gamma_5$$

$$(\gamma_5)^2 = \mathbb{1} \Rightarrow \text{evals}(\gamma_5) = \pm 1$$

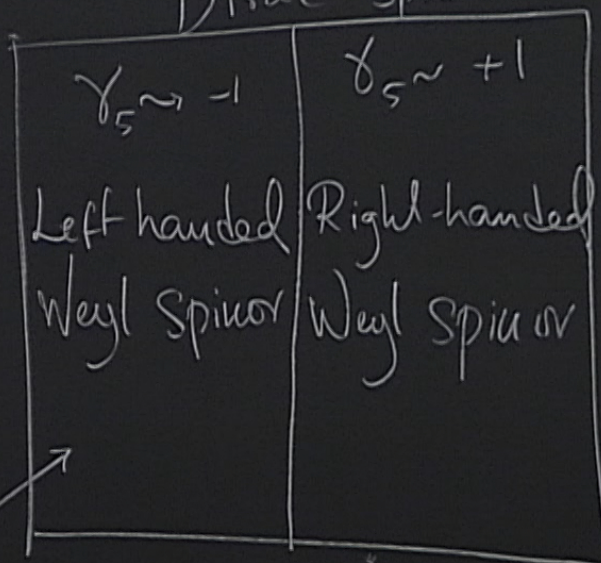
$[\gamma_5, \Sigma^{\mu\nu}] = 0$

$[\gamma_5, S(\Lambda)] = 0$

Dirac is reducible!

$$S(\Lambda) = \begin{pmatrix} \text{////} & 0 \\ 0 & \text{////} \end{pmatrix}$$

Dirac Spinors



$$P_L = \frac{1}{2}(1 - \gamma_5) \qquad P_R = \frac{1}{2}(1 + \gamma_5)$$

$$P_L \psi_D = \psi_L \qquad \psi_R = P_R \psi_D$$

Weyl / Chiral Basis

$$\gamma_5 = \begin{pmatrix} -\mathbb{1}_2 & 0 \\ 0 & \mathbb{1}_2 \end{pmatrix}$$

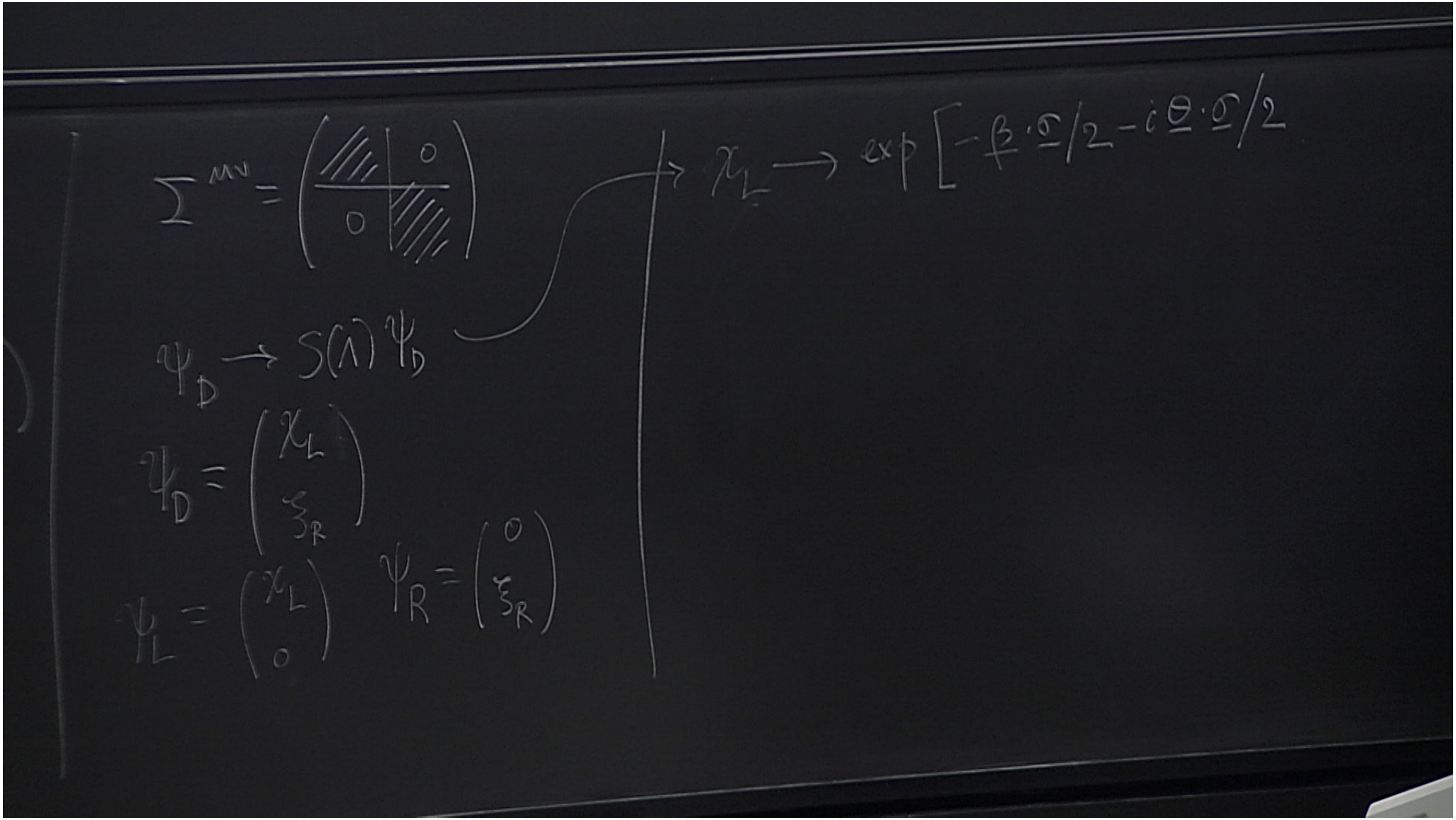
$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$$

$$\sigma^\mu = (\mathbb{1}_2, \sigma^i)$$

$$\bar{\sigma}^\mu = (\mathbb{1}_2, -\sigma^i)$$

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$$

$$\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$


$\Sigma^{\mu\nu} = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}$
$\psi_D \to S(\Lambda)\psi_D$
$\psi_D = \begin{pmatrix} \chi_L \\ \xi_R \end{pmatrix}$
$\psi_L = \begin{pmatrix} \chi_L \\ 0 \end{pmatrix}$ $\psi_R = \begin{pmatrix} 0 \\ \xi_R \end{pmatrix}$
$\to \chi_L \to \exp\left[-\beta\cdot\sigma/2 - i\theta\cdot\sigma/2\right.$

$\chi_L \rightarrow \exp\left[-\underline{\beta}\cdot\underline{\sigma}/2 - i\underline{\theta}\cdot\underline{\sigma}/2\right]\chi_L$ (Left-handed) =

$\xi_R \rightarrow \exp\left[+\underline{\beta}\cdot\underline{\sigma}/2 - i\underline{\theta}\cdot\underline{\sigma}/2\right]\xi_R$ (Right-handed) =

$\sigma^2 \chi_L^* \rightarrow$ Right-handed spinor.

Dirac = $(1/2, 0) \oplus (0, 1/2)$

Vector = $(1/2, 1/2)$

Take-home discussion

In 5D spacetime: What is the dim of the Dirac rep?

Is Dirac rep in 5D reducible to Weyl reps?

Lesson objectives

- Construct $\mathcal{L}$ for Dirac & Weyl spinors
- Majorana Spinors

Q: Is $\psi^\dagger\psi$ Lorentz invariant?

$$\psi \to \exp\left[-\frac{i}{2}\omega_{\mu\nu}\Sigma^{\mu\nu}\right]\psi$$

$$\psi^\dagger \to \psi^\dagger \exp\left[\frac{i}{2}\omega_{\mu\nu}\Sigma^{\mu\nu\dagger}\right]$$

Q: Is $\psi^\dagger\psi$ Lorentz invariant?

$$\psi \to \exp\left[\underbrace{-\frac{i}{2}\omega_{\mu\nu}\Sigma^{\mu\nu}}_{S}\right]\psi$$

$$\psi^\dagger \to \psi^\dagger \exp\left[\underbrace{\frac{i}{2}\omega_{\mu\nu}\Sigma^{\mu\nu\dagger}}_{S^\dagger}\right]$$

$$\psi^\dagger\psi \to \psi^\dagger S^\dagger S \psi$$

$S^\dagger = S^{-1}$

$$\Sigma^{\mu\nu\dagger} = \left(\frac{i}{4}[\gamma^\mu, \gamma^\nu]\right)^\dagger$$

$$= -\frac{i}{4}[\gamma^{\nu\dagger}, \gamma^{\mu\dagger}]$$

$$\gamma^{0\dagger} = \gamma^0$$
$$\gamma^{i\dagger} = -\gamma^i$$

$$\Sigma^{0i\dagger} = -\frac{i}{2}[\gamma^{i\dagger}, \gamma^{0\dagger}]$$

$$= -\frac{i}{2}[\gamma^0, \gamma^i]$$

$$= -\Sigma^{0i}$$

$$\Sigma^{ij\dagger} = \Sigma^{ij}$$

$$\Sigma^{ij\,\dagger} = \Sigma^{ij}$$

$$\gamma^0 \gamma^{\mu\,\dagger} \gamma^0 = \gamma^\mu$$

$$\Sigma^{\mu\nu\,\dagger} = \gamma^0 \Sigma^{\mu\nu} \gamma^0$$

$$\psi^\dagger \longrightarrow \psi^\dagger \gamma^0 \exp\left[\frac{i}{2} \omega_{\mu\nu} \Sigma^{\mu\nu} \right] \gamma^0$$

Dirac conjugate of ψ

$$\bar{\psi} = \psi^\dagger \gamma^0$$

$$\bar{\psi} \to \underbrace{\psi^\dagger \gamma^0}_{\bar{\psi}} \exp\left[\frac{i}{2}\omega_{\mu\nu}\Sigma^{\mu\nu}\right](\gamma^0)^2$$

$$= \bar{\psi} S^{-1}(\Lambda)$$

$$\bar{\psi}\psi \longrightarrow \bar{\psi}\psi \sim \text{Lorentz scalar}$$

Discrete Symmetries – Parity

C, P, T

CPT theorem: Any relativistic QFT invariant under CPT.

How does ψ transform under P?

$$\begin{pmatrix} -m & (\sigma^\mu)\partial_\mu \\ (\bar{\sigma}^\mu)\partial_\mu & m \end{pmatrix} \begin{pmatrix} \chi_L \\ \xi_R \end{pmatrix} = 0$$

$$\chi_L \sim e^{ip\cdot x}$$

$$(E - \vec{\sigma}\cdot\vec{p})\,\xi_R = m\chi_L$$

$$(E + \vec{\sigma}\cdot\vec{p})\,\chi_L = m\xi_R$$

$$m \to 0$$

Helicity op: $\dfrac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|}$

$\gamma_5 \sim$ chirality operator

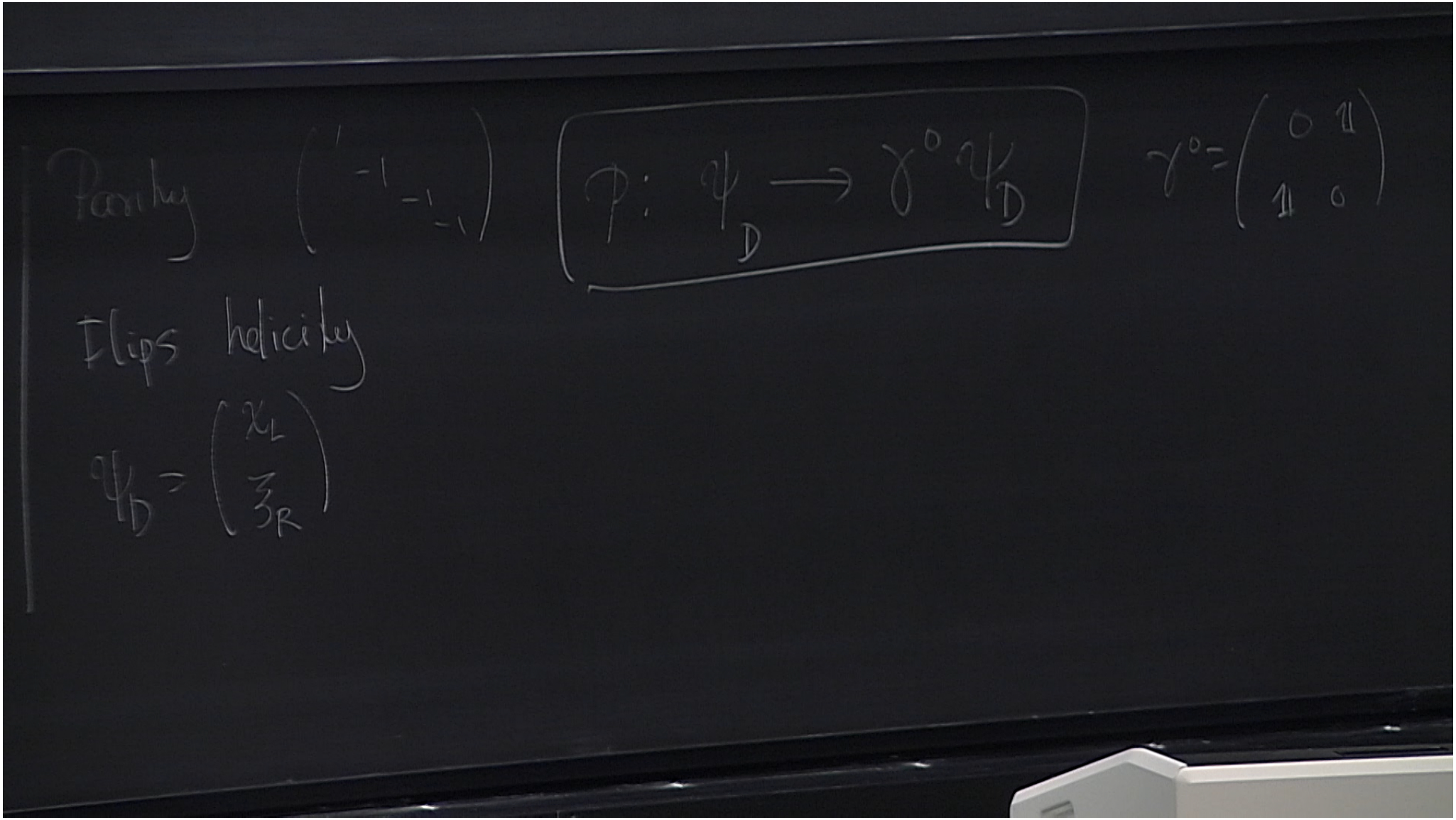
Helicity op: $\frac{\sigma \cdot p}{|p|}$

$\gamma_5 \rightsquigarrow$ chirality operator

When $m \to 0$

helicity ~ chirality

Parity


Parity
$\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$
$P: \psi_D \longrightarrow \gamma^0 \psi_D$
$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$
Flips helicity
$\psi_D = \begin{pmatrix} \chi_L \\ \xi_R \end{pmatrix}$

$P: \quad \psi(\underline{x},t) \longrightarrow \gamma^0 \psi(-\underline{x},t)$

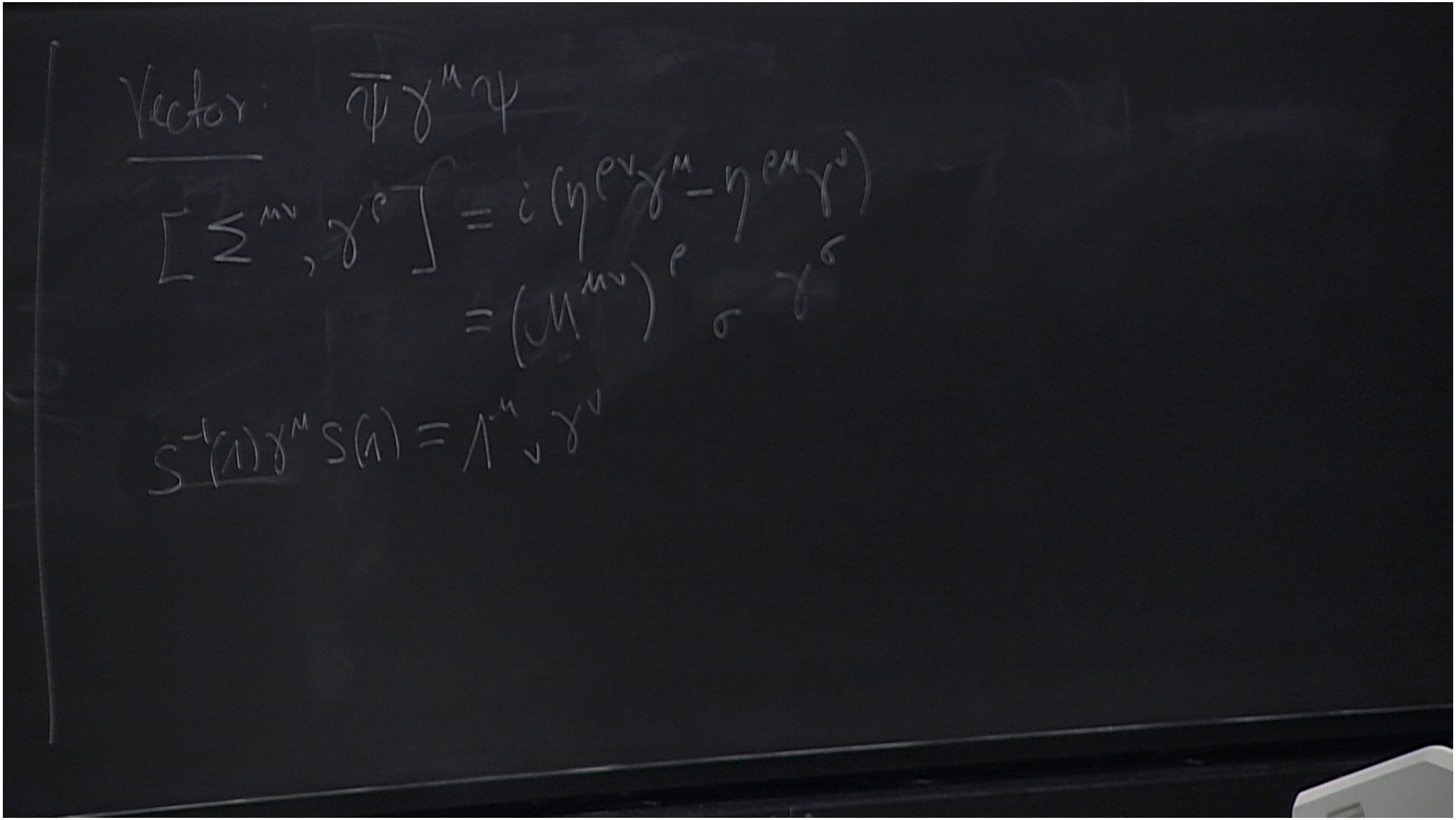
$\psi^\dagger \longrightarrow \psi^\dagger \gamma^0$

Scalar: $P: \bar{\psi}\psi \longrightarrow \psi^\dagger \gamma^0 \gamma^0 \gamma^0 \psi$

$= \bar{\psi}\psi$

pseudo-scalars: $P: \bar{\psi}\gamma_5\psi \longrightarrow \psi^\dagger \gamma^0 \gamma^0 \gamma_5 \gamma^0 \psi$

$= -\bar{\psi}\gamma_5\psi$


Vector: $\bar{\psi}\gamma^\mu\psi$
$[\Sigma^{\mu\nu}, \gamma^\rho] = i(\eta^{\rho\nu}\gamma^\mu - \eta^{\rho\mu}\gamma^\nu)$
$= (\mathcal{M}^{\mu\nu})^\rho{}_\sigma \gamma^\sigma$
$S^{-1}(\Lambda)\gamma^\mu S(\Lambda) = \Lambda^\mu{}_\nu \gamma^\nu$

$$\bar{\psi}\gamma^\mu\psi \longrightarrow \bar{\psi}\underbrace{S^{-1}(\Lambda)\gamma^\mu S(\Lambda)}_{\Lambda^\mu{}_\nu\gamma^\nu}\psi$$

$$= \Lambda^\mu{}_\nu\,\bar{\psi}\gamma^\nu\psi$$

$$P:\ \bar{\psi}\gamma^\mu\psi \longrightarrow \begin{cases} \bar{\psi}\gamma^0\psi \\ -\bar{\psi}\gamma^i\psi \end{cases}$$

Axial- or pseudo-vector:

P: $\bar{\psi}\gamma^{\mu}\gamma_5\psi \longrightarrow \begin{cases} -\bar{\psi}\gamma^0\gamma_5\psi \\ +\bar{\psi}\gamma^i\gamma_5\psi \end{cases}$

Lagrangian for $\mathcal{L}_D$

Kinetic term: $i\bar{\psi}\gamma^\mu\partial_\mu\psi \longrightarrow \Lambda_\mu{}^\nu\,\Lambda^\mu{}_\rho = \delta^\nu_\rho$

$\partial_\mu \rightarrow$ Covector

$$i\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi \longrightarrow i\bar{\psi}S^{-1}\gamma^{\mu}\Lambda_{\mu}{}^{\rho}\partial_{\rho}S\psi$$

$$= i\bar{\psi}(\Lambda^{\mu}{}_{\sigma}\gamma^{\sigma})\Lambda_{\mu}{}^{\rho}\partial_{\rho}\psi$$

$$= i\bar{\psi}\gamma^{\sigma}\partial_{\sigma}\psi$$

$$S_{Kin} = i\int d^{4}x\,\bar{\psi}\gamma^{\sigma}\partial_{\sigma}\psi$$

$$S^{*}_{Kin} = S_{Kin}$$

Mass dimensions

$[S] = 0$ $\quad [d^4x] = -4$ $\quad [\gamma^\mu] = 0$

$[\partial_\mu] = +1$ $\quad [\psi] = [\bar{\psi}] = \frac{3}{2}$

$\mathcal{L} = -m\bar{\psi}\psi \rightsquigarrow [m] = 1$

$\mathcal{L}_{Dirac} = i\bar{\psi}\not{\partial}\psi - m\bar{\psi}\psi$

$\not{\partial} = \gamma^\mu\partial_\mu$

$\mathcal{L}$ for Weyl Spinors

$$S(\Lambda) = \begin{pmatrix} S_L & 0 \\ 0 & S_R \end{pmatrix}$$

$$S_L = \exp\left[-\underline{\beta}\cdot\underline{\sigma} + i\underline{\theta}\cdot\frac{\underline{\sigma}}{2}\right]$$

$$S^{-1}(\Lambda)\gamma^\mu S(\Lambda) = \Lambda^\mu{}_\nu \gamma^\nu$$

$$S_L^{-1}\sigma^\mu S_R = \Lambda^\mu{}_\nu \sigma^\nu$$

$$S_R^{-1}\bar{\sigma}^\mu S_L = \Lambda^\mu{}_\nu \bar{\sigma}^\nu$$

$$\psi_D \to S(\Lambda)\psi_D$$

$$\chi_L \to S_L \chi_L$$

$$\xi_R \to S_R \xi_R$$

$$\chi_L^\dagger \to \chi_L^\dagger S_R^{-1}$$

$$\xi_R^\dagger \to \xi_R^\dagger S_L^{-1}$$

$\psi_D \to S(\Lambda)\psi_D$

$\chi_L \to S_L \chi_L$

$\xi_R \to S_R \xi_R$

$\chi_L^\dagger \to \chi_L^\dagger S_R^{-1}$

$\xi_R^\dagger \to \xi_R^\dagger S_L^{-1}$

$\bar{\psi} \to \bar{\psi} S^{-1}$

$= \psi^\dagger \gamma^0 S^{-1}$

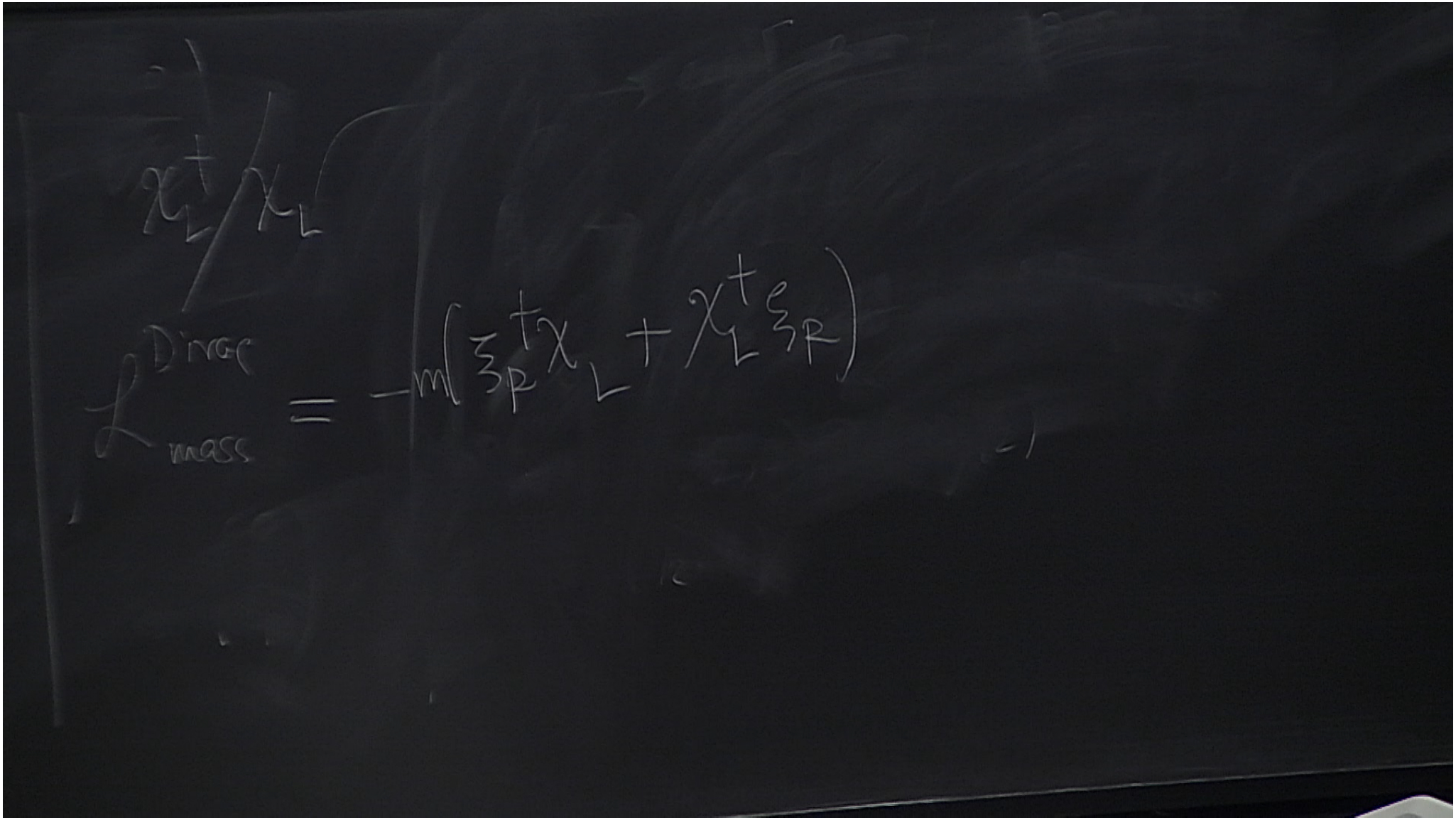
$= (\chi_L^\dagger, \xi_R^\dagger)\gamma^0 S^{-1}$

$= (\xi_R^\dagger, \chi_L^\dagger) S^{-1}$

Basics

$$\mathcal{L}^{\text{Dirac}}_{\text{kin}} = i\,\bar{\psi}\,\gamma^\mu \partial_\mu \psi$$

$$= i\,\underbrace{\chi_L^\dagger\,\bar{\sigma}^\mu \partial_\mu \chi_L}_{\text{Lorentz inv.}} + i\,\underbrace{\xi_R^\dagger\,\sigma^\mu \partial_\mu \xi_R}_{\text{Lorentz inv.}}$$


$\chi_L^\dagger / \chi_L$
$\mathcal{L}^{Dirac}_{mass} = -m(\xi_R^\dagger \chi_L + \chi_L^\dagger \xi_R)$

Majorana mass

$\chi_L \rightarrow$

ξ_R

$\mathcal{L}^{\text{Majorana}}_{\text{mass}} = -m\, \xi_R^T\, \sigma_2\, \xi_R$

$\sigma_2 \sigma_i^T \sigma_2 = -\sigma_i$

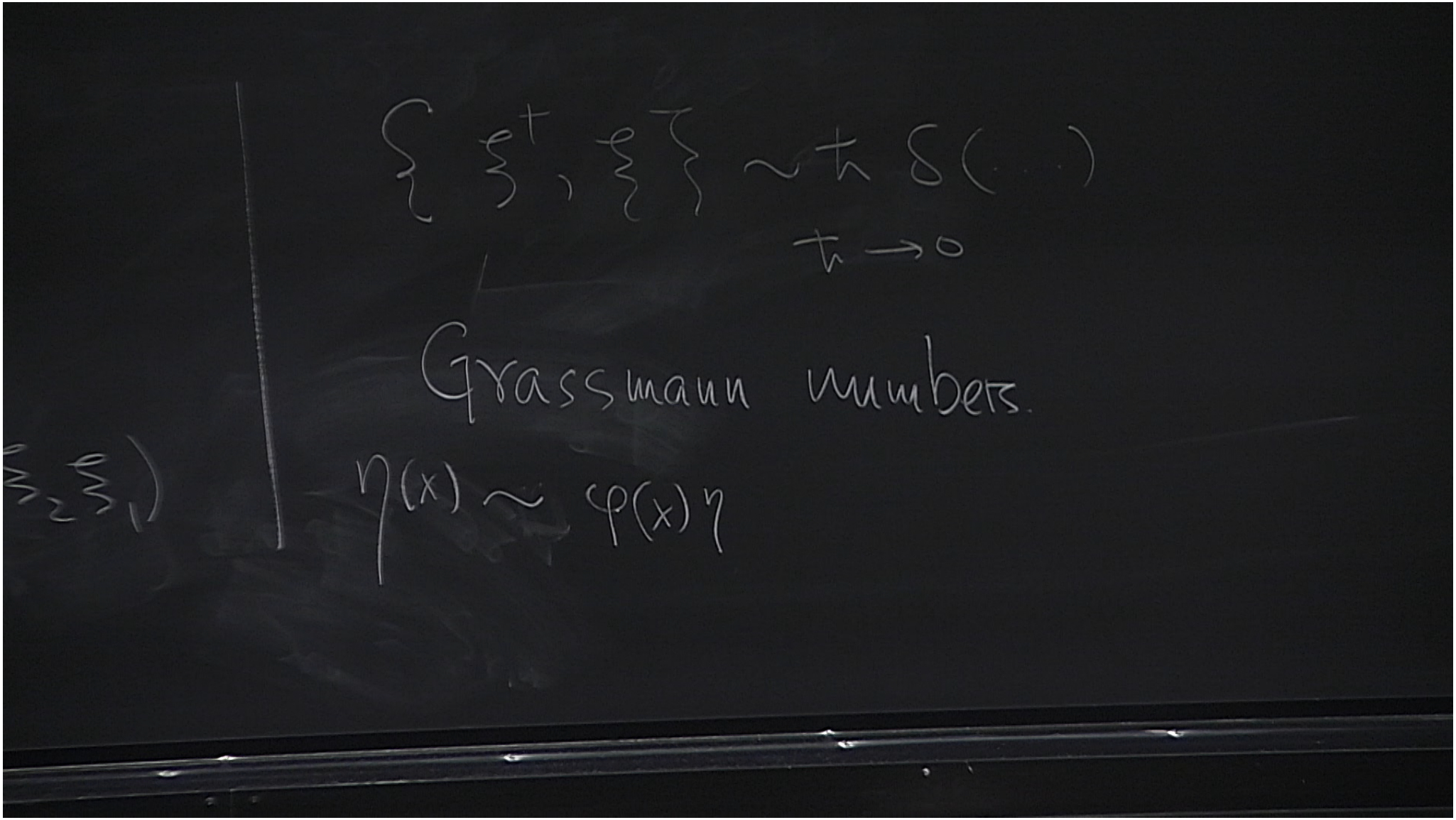
ξ_R

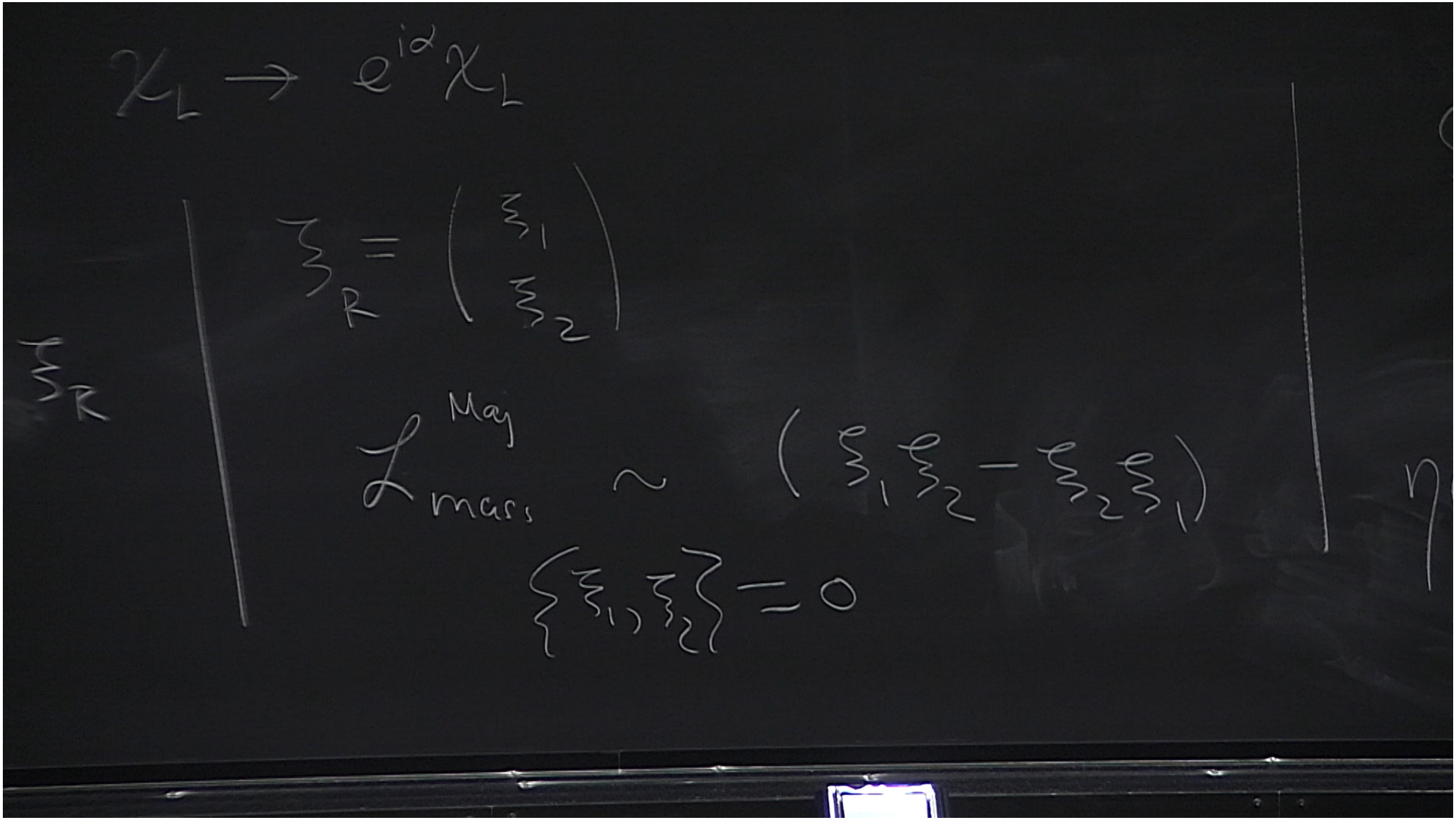
$$\chi_L \to e^{i\alpha} \chi_L$$

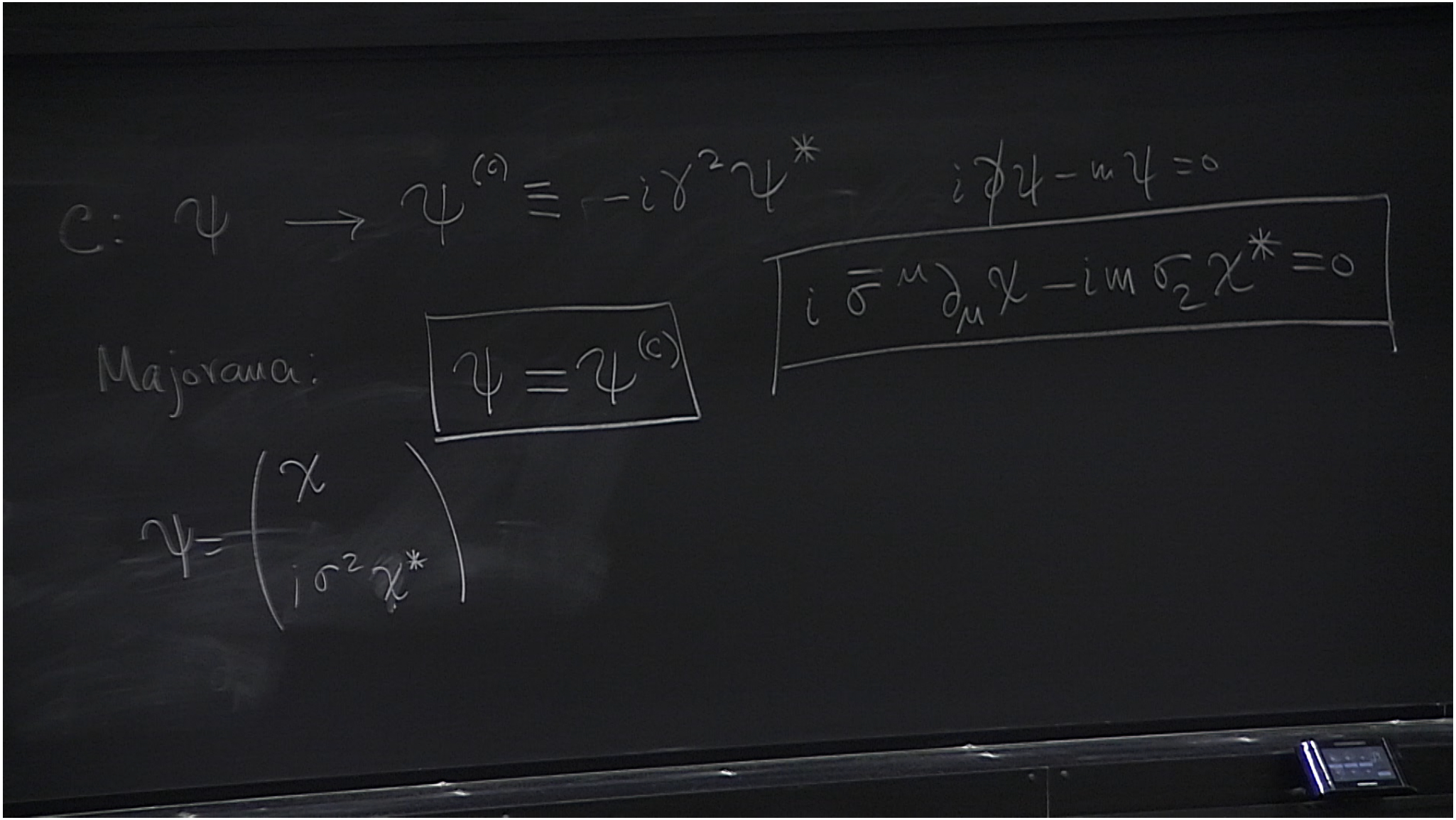
$$\xi_R = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$$

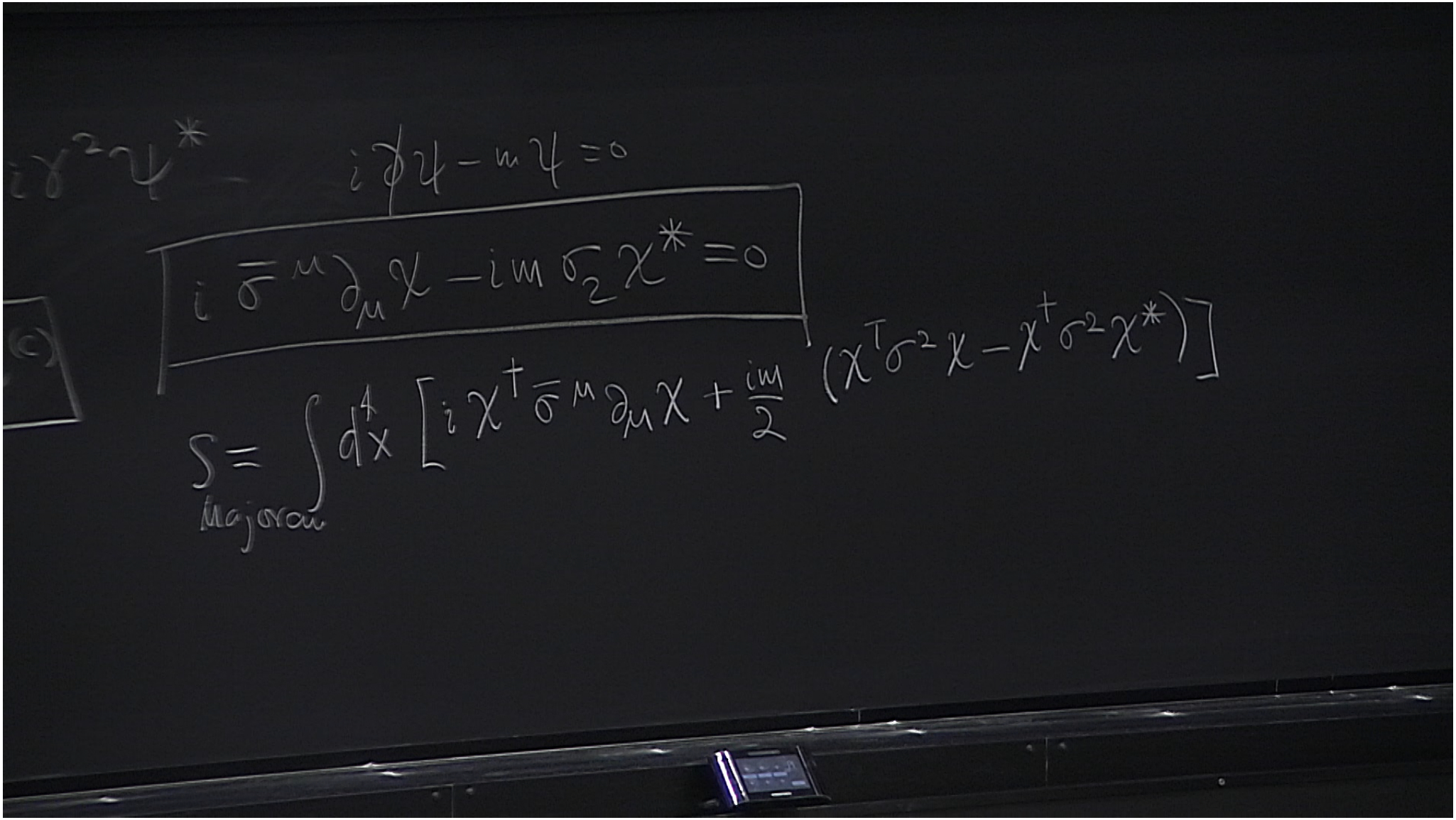
$$\mathcal{L}^{\text{Maj}}_{\text{mass}} \sim (\xi_1 \xi_2 - \xi_2 \xi_1)$$

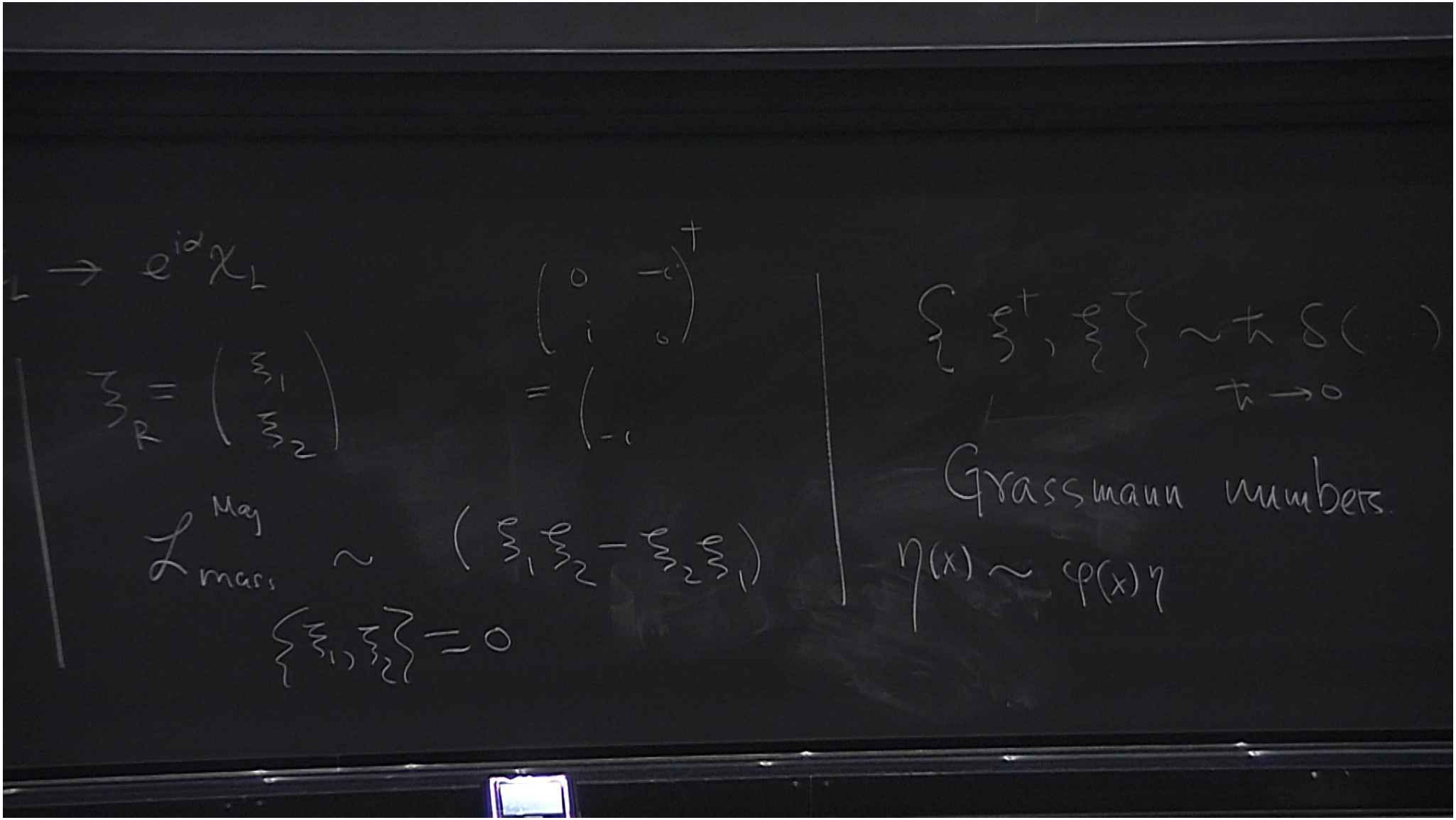
$\xi_2 \; \xi_R$


$\{\xi^\dagger, \xi\} \sim \hbar\,\delta(\ldots)$
$\hbar \to 0$
Grassmann numbers.
$\eta(x) \sim \varphi(x)\,\eta$


$\chi_L \to e^{i\alpha} \chi_L$
$\xi_R = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$
ξ_R
$\mathcal{L}^{Maj}_{mass} \sim (\xi_1 \xi_2 - \xi_2 \xi_1)$
$\{\xi_1, \xi_2\} = 0$


C: $\psi \rightarrow \psi^{(c)} \equiv -i\gamma^2\psi^*$
$i\not{\partial}\psi - m\psi = 0$
Majorana: $\psi = \psi^{(c)}$
$i\bar{\sigma}^\mu\partial_\mu\chi - im\sigma_2\chi^* = 0$
$\psi = \begin{pmatrix} \chi \\ i\sigma^2\chi^* \end{pmatrix}$


$i\gamma^2\psi^*$
$i\partial\!\!\!/\psi - m\psi = 0$
$i\bar{\sigma}^\mu\partial_\mu\chi - im\sigma_2\chi^* = 0$
$S = \int d^4x \left[i\chi^\dagger\bar{\sigma}^\mu\partial_\mu\chi + \frac{im}{2}(\chi^T\sigma^2\chi - \chi^\dagger\sigma^2\chi^*) \right]$
Majorana


$\chi_L \to e^{i\alpha}\chi_L$
$\xi_R = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$
$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}^\dagger$
$= \begin{pmatrix} \cdot & \cdot \\ -i & \cdot \end{pmatrix}$
$\mathcal{L}^{May}_{mass} \sim (\xi_1\xi_2 - \xi_2\xi_1)$
$\{\xi_1, \xi_2\} = 0$
$\{\xi^\dagger, \xi\} \sim \hbar\,\delta(\;)$
$\hbar \to 0$
Grassmann numbers
$\eta(x) \sim \varphi(x)\eta$