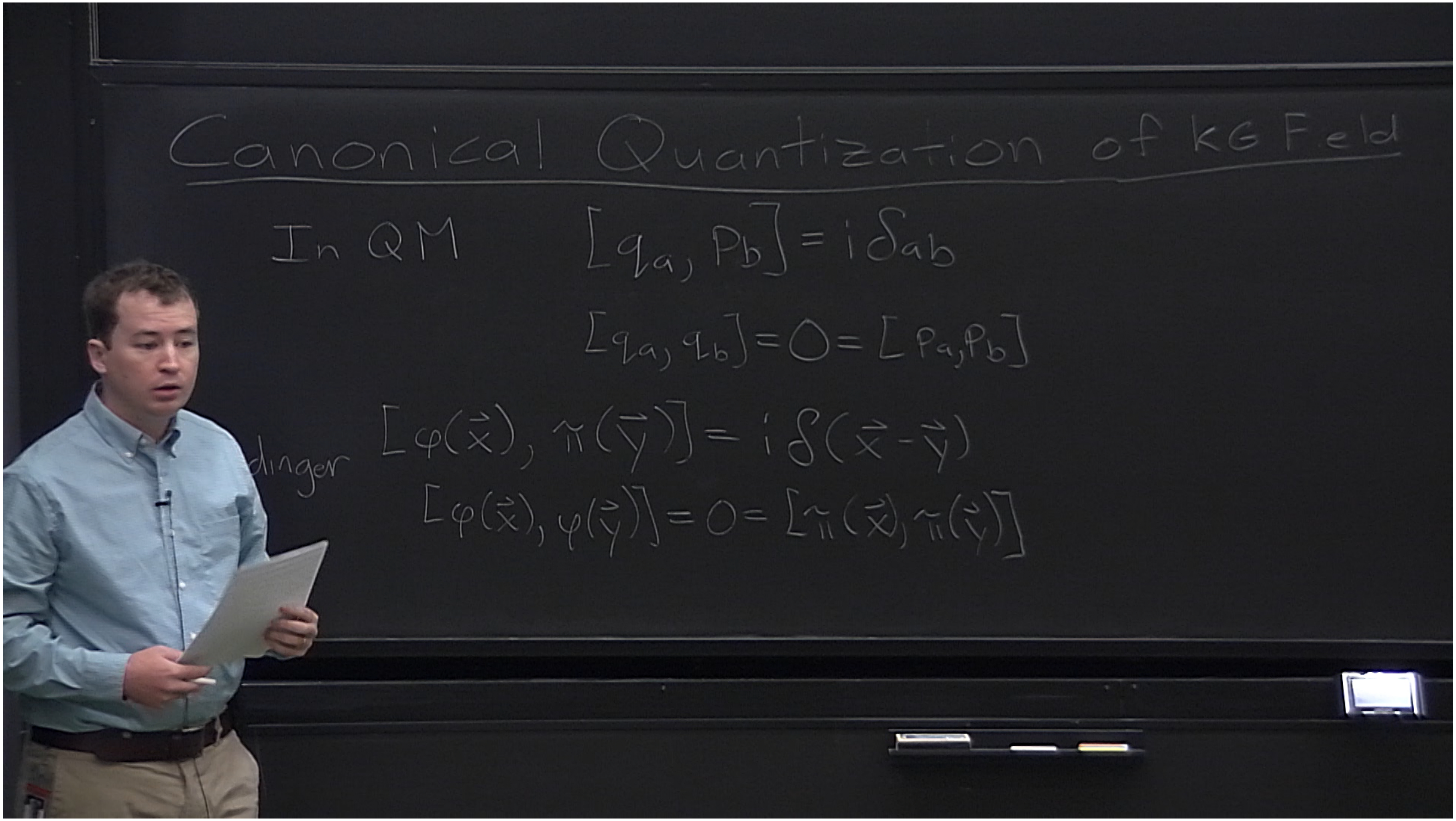
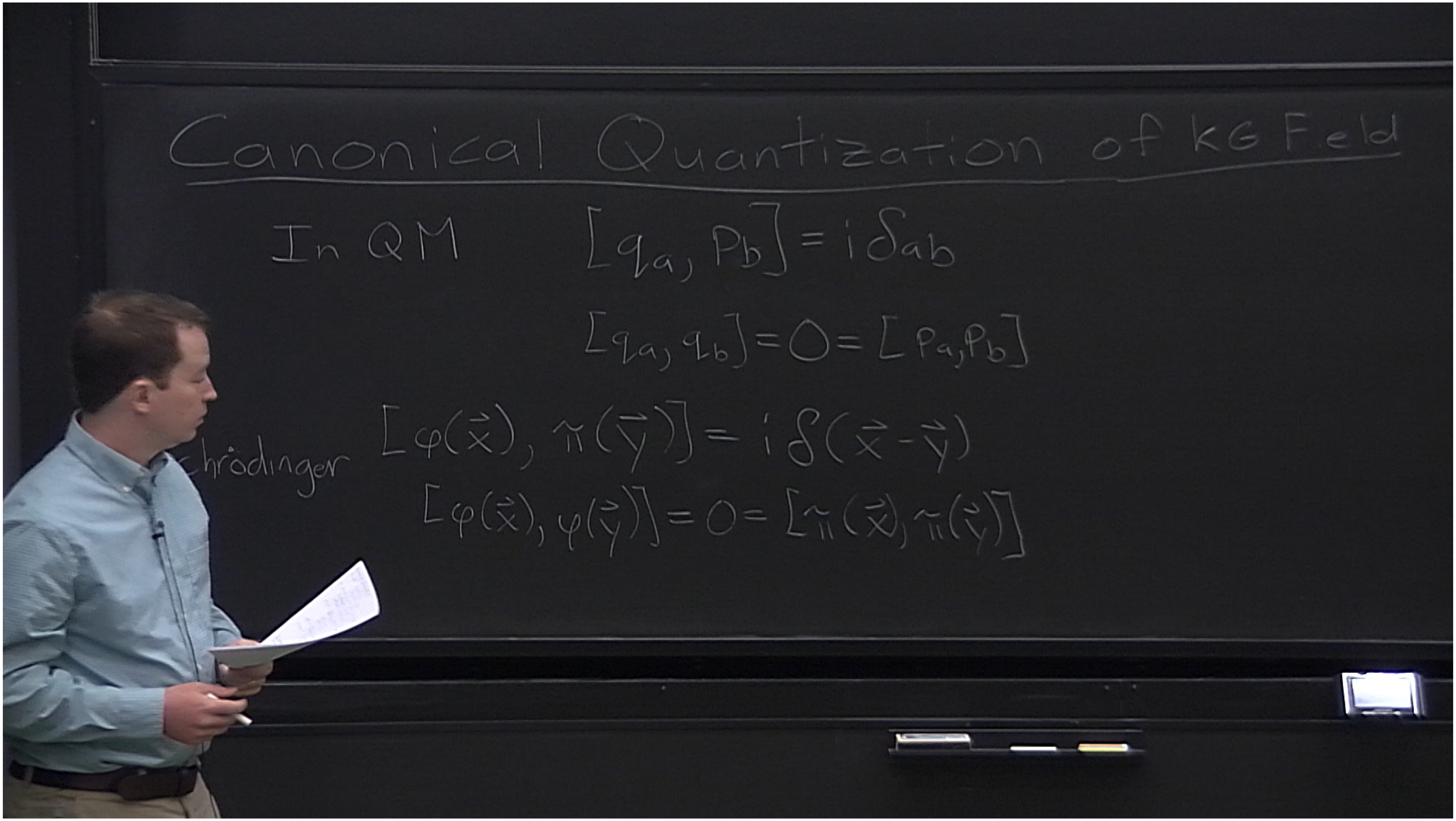
Title: PSI 17/18 - Quantum Field Theory I - Lecture 2

Date: Oct 11, 2017 09:00 AM

URL: http://pirsa.org/17100021

Abstract:


Canonical Quantization of KG Field
In QM $[q_a, p_b] = i\delta_{ab}$
$[q_a, q_b] = 0 = [p_a, p_b]$
dinger
$[\varphi(\vec{x}), \pi(\vec{y})] = i\delta(\vec{x}-\vec{y})$
$[\varphi(\vec{x}), \varphi(\vec{y})] = 0 = [\pi(\vec{x}), \pi(\vec{y})]$


Canonical Quantization of KG Field
In QM
$[q_a, p_b] = i\delta_{ab}$
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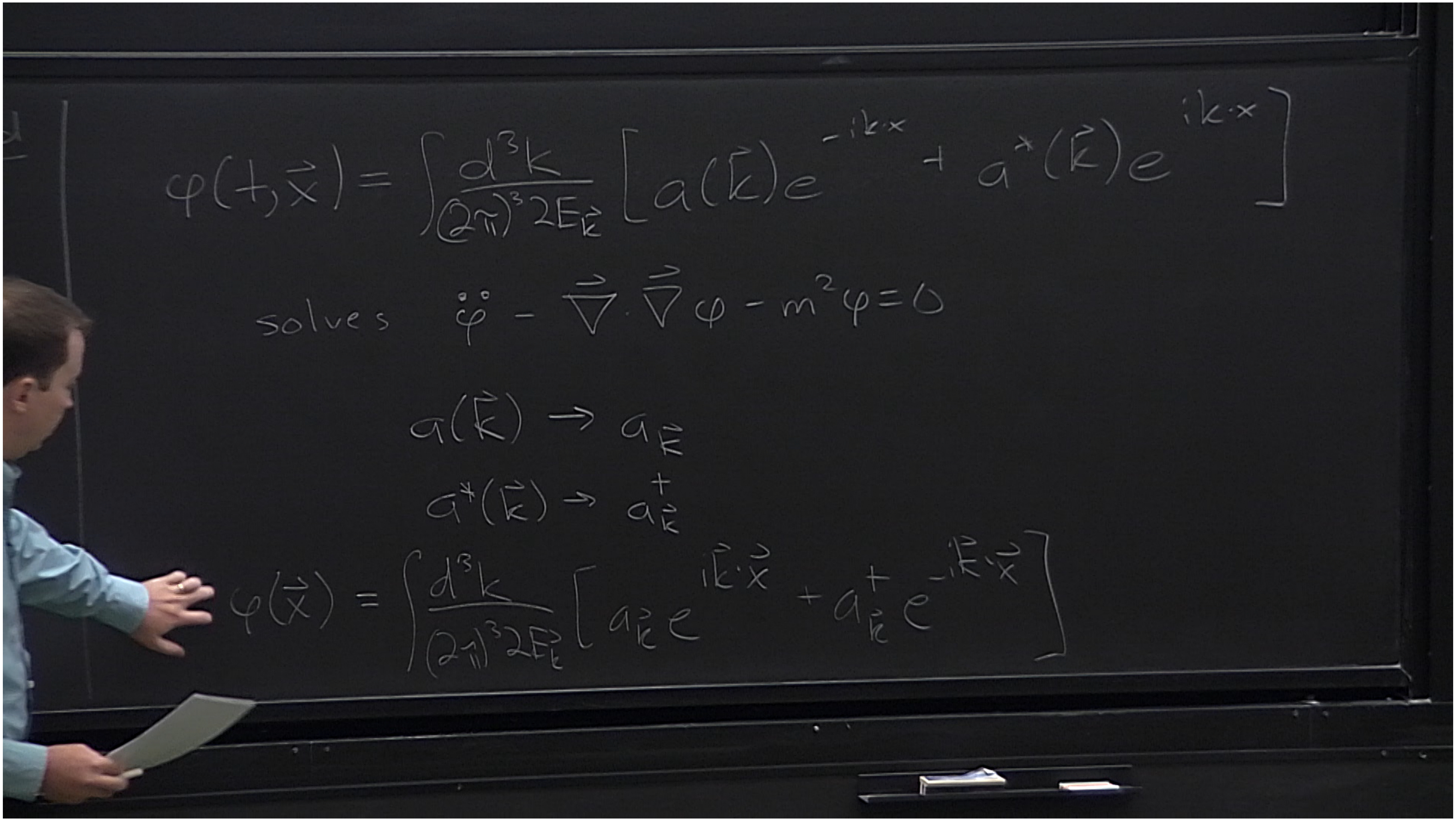
chrödinger
$[\varphi(\vec{x}), \pi(\vec{y})] = i\delta(\vec{x}-\vec{y})$
$[\varphi(\vec{x}), \varphi(\vec{y})] = 0 = [\pi(\vec{x}), \pi(\vec{y})]$

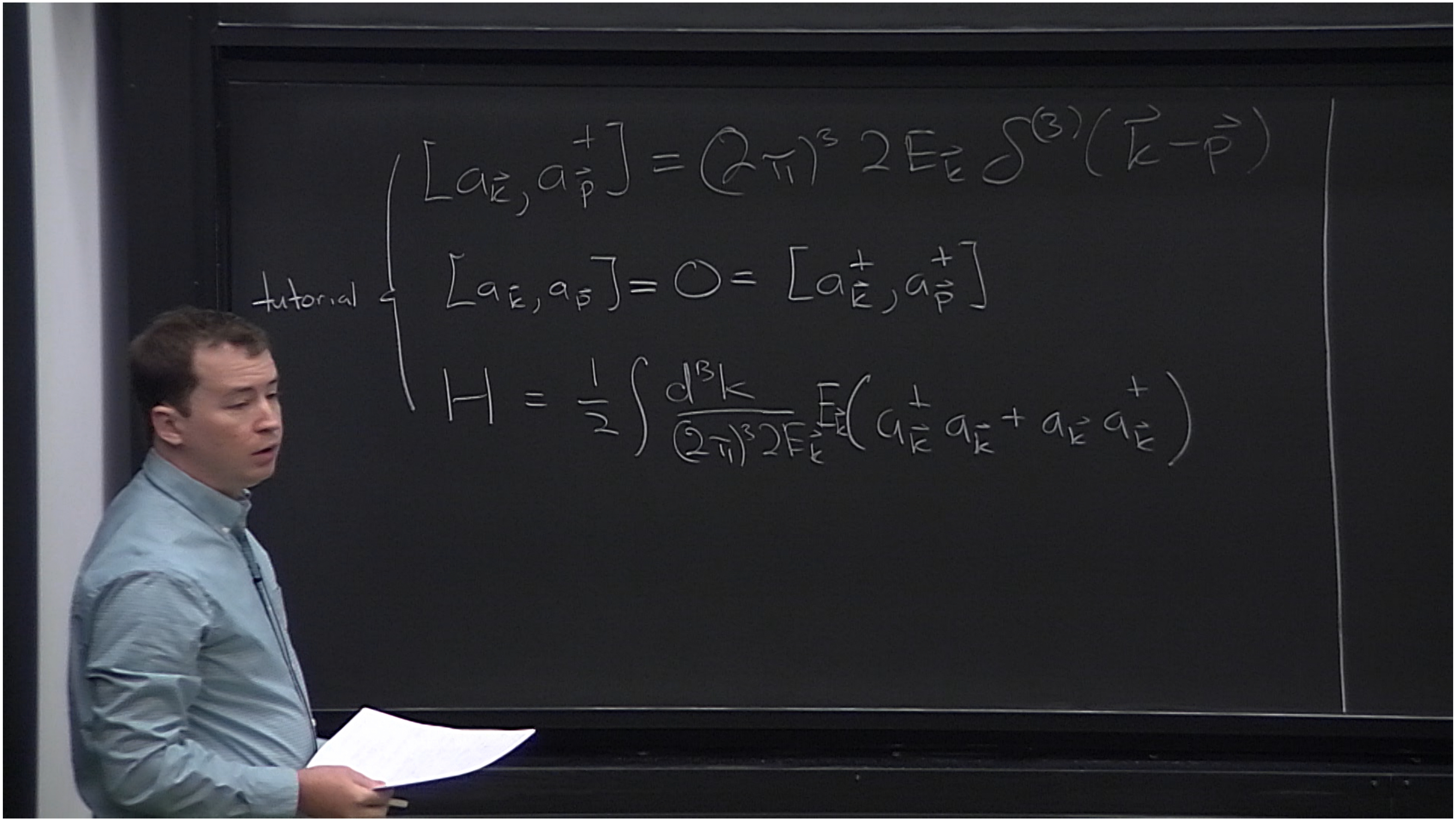
Field

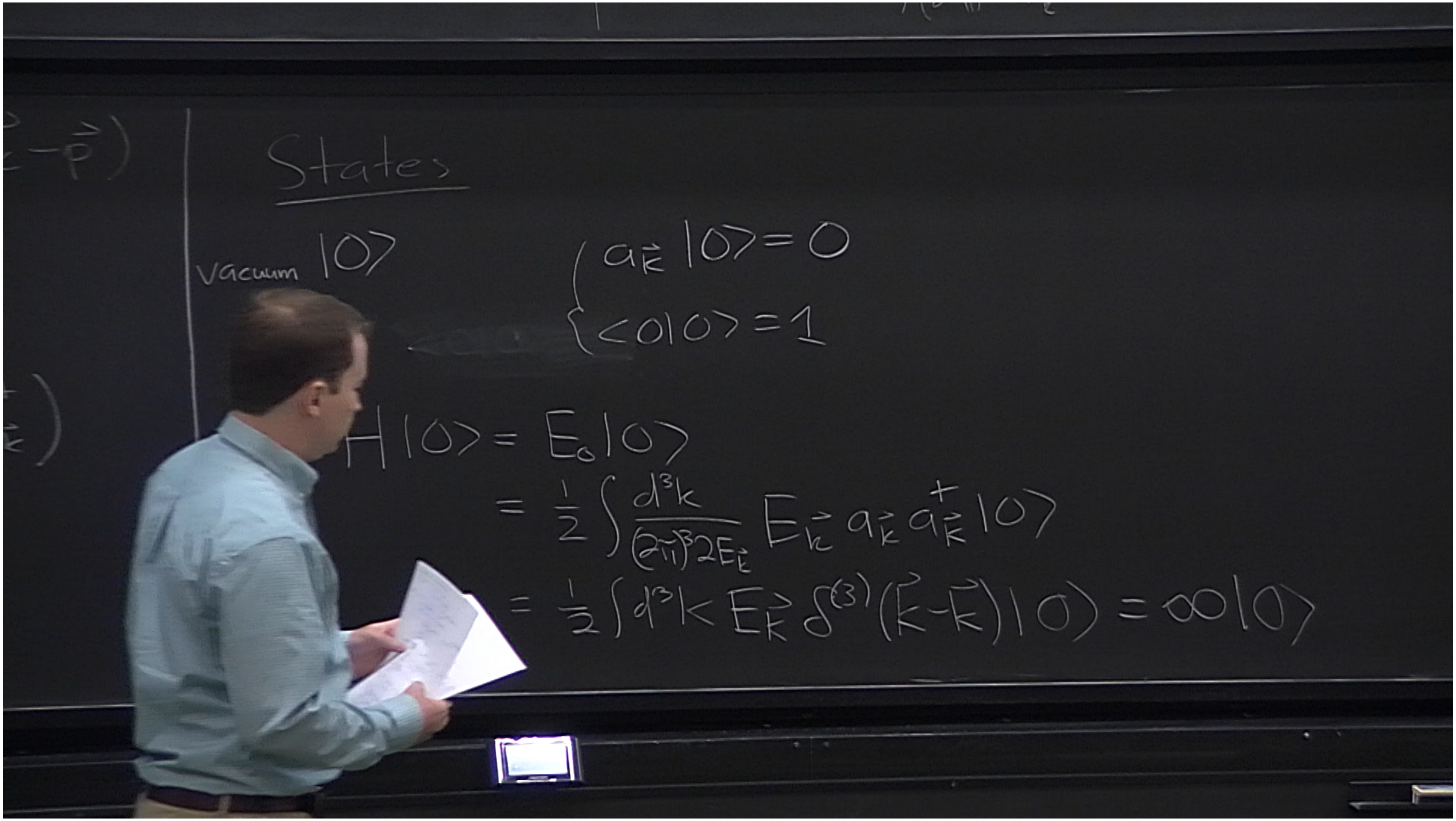
$$\varphi(t,\vec{x}) = \int \frac{d^3k}{(2\pi)^3 2E_{\vec{k}}} \left[a(\vec{k}) e^{-ik\cdot x} + a^*(\vec{k}) e^{ik\cdot x} \right]$$

solves $\ddot{\varphi} - \vec{\nabla}\cdot\vec{\nabla}\varphi - m^2\varphi = 0$



$\varphi(t,\vec{x}) = \int \frac{d^3k}{(2\pi)^3 2E_{\vec{k}}} \left[a(\vec{k}) e^{-ik\cdot x} + a^*(\vec{k}) e^{ik\cdot x} \right]$
solves $\ddot{\varphi} - \vec{\nabla}\cdot\vec{\nabla}\varphi - m^2\varphi = 0$
$a(\vec{k}) \to a_{\vec{k}}$
$a^*(\vec{k}) \to a^\dagger_{\vec{k}}$
$\varphi(\vec{x}) = \int \frac{d^3k}{(2\pi)^3 2E_{\vec{k}}} \left[a_{\vec{k}} e^{i\vec{k}\cdot\vec{x}} + a^\dagger_{\vec{k}} e^{-i\vec{k}\cdot\vec{x}} \right]$


$[a_{\vec{k}}, a^{+}_{\vec{p}}] = (2\pi)^3\, 2E_{\vec{k}}\, \delta^{(3)}(\vec{k}-\vec{p})$
tutorial
$[a_{\vec{k}}, a_{\vec{p}}] = 0 = [a^{+}_{\vec{k}}, a^{+}_{\vec{p}}]$
$H = \frac{1}{2}\int \frac{d^3k}{(2\pi)^3 2E_{\vec{k}}} E_{\vec{k}} (a^{+}_{\vec{k}} a_{\vec{k}} + a_{\vec{k}} a^{+}_{\vec{k}})$


States
vacuum |0>
$a_{\vec{k}}|0\rangle = 0$
$\langle 0|0\rangle = 1$
$H|0\rangle = E_0|0\rangle$
$= \frac{1}{2}\int \frac{d^3k}{(2\pi)^3 2E_{\vec{k}}} E_{\vec{k}}\, a_{\vec{k}} a^{\dagger}_{\vec{k}} |0\rangle$
$= \frac{1}{2}\int d^3k\, E_{\vec{k}}\, \delta^{(3)}(\vec{k}-\vec{k})|0\rangle = \infty|0\rangle$

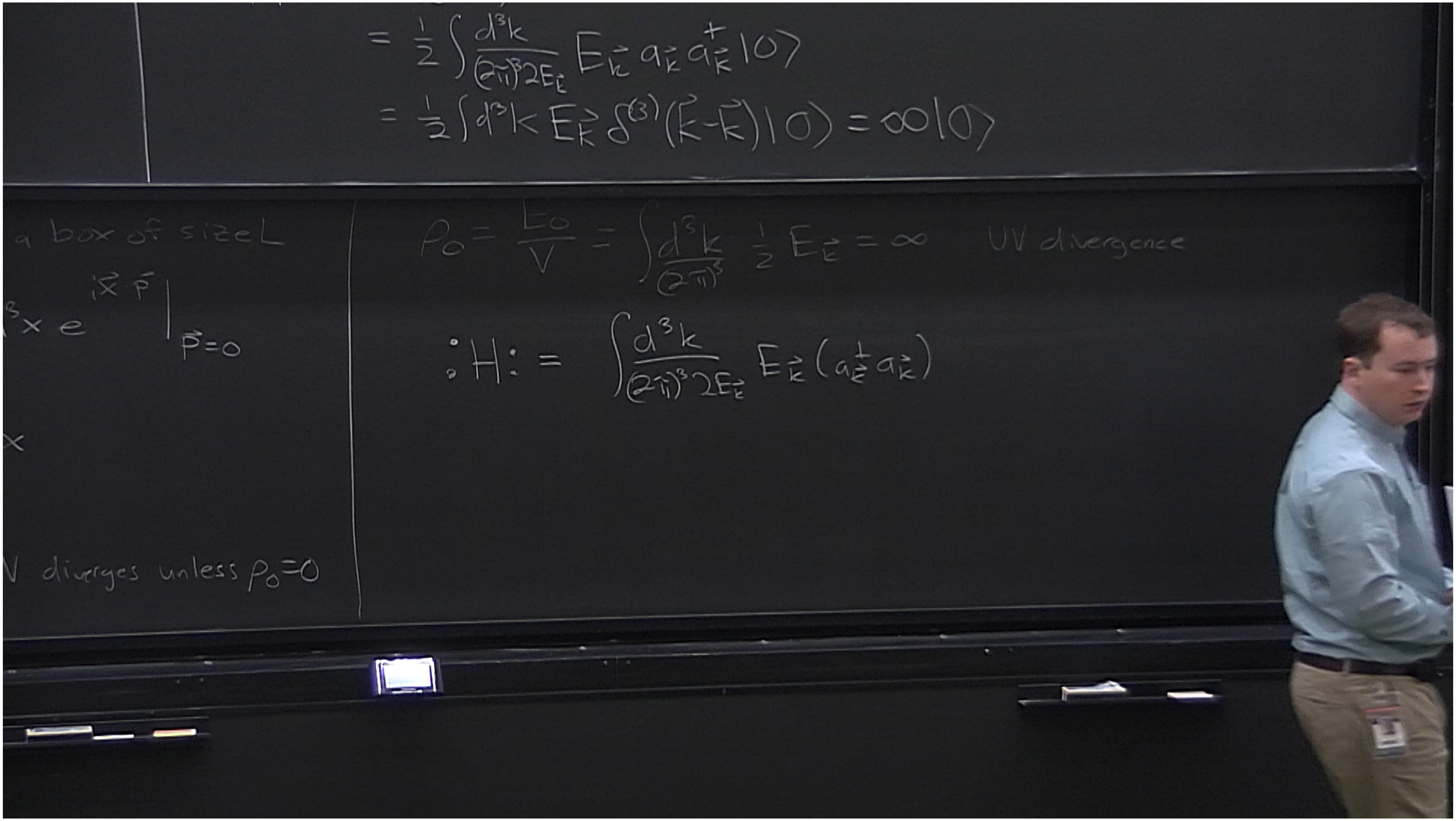
IR-regulate by putting theory in a box of size L

$$(2\pi)^3 \delta^{(3)}(\vec{0}) = \lim_{L\to\infty} \int_{-\frac{L}{2}}^{\frac{L}{2}} d^3x \, e^{i\vec{x}\cdot\vec{p}} \Big|_{\vec{p}=0}$$

$$= \lim_{L\to\infty} \int_{-\frac{L}{2}}^{\frac{L}{2}} d^3x$$

$$= V$$

total energy diverges if V diverges unless $\rho_0 = 0$


$= \frac{1}{2}\int \frac{d^3k}{(2\pi)^3 2E_{\vec{k}}} E_{\vec{k}} a_{\vec{k}} a^{\dagger}_{\vec{k}} |0\rangle$
$= \frac{1}{2}\int d^3k \, E_{\vec{k}} \, \delta^{(3)}(\vec{k}-\vec{k}) |0\rangle = \infty |0\rangle$
a box of size L
$\rho_0 = \frac{E_0}{V} = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2} E_{\vec{k}} = \infty$
UV divergence
$:H: = \int \frac{d^3k}{(2\pi)^3 2E_{\vec{k}}} E_{\vec{k}} (a^{\dagger}_{\vec{k}} a_{\vec{k}})$
V diverges unless $\rho_0 = 0$

IR-regulate by putting theory in a box of size L

$$(2\pi)^3\,\delta^{(3)}(\vec{0}) = \lim_{L\to\infty}\int_{-\frac{L}{2}}^{\frac{L}{2}} d^3x\, e^{i\vec{x}\cdot\vec{p}}\Big|_{\vec{p}=0}$$

$$= \lim_{L\to\infty}\int_{-\frac{L}{2}}^{\frac{L}{2}} d^3x$$

$$= \lim_{L\to\infty} V$$

total energy diverges if V diverges unless $\rho_0 = 0$

$$\rho_0 = \frac{E_0}{V} = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2} E_{\vec{k}} = \infty \quad \text{UV divergence}$$

$$:H: = \int \frac{d^3k}{(2\pi)^3 2E_{\vec{k}}} E_{\vec{k}} \left(a^\dagger_{\vec{k}} a_{\vec{k}} \right)$$

One particle states

$$|\vec{k}\rangle = a^\dagger_{\vec{k}} |0\rangle \quad \text{definite 3-momentum + energy}$$

$$\| \quad |k\rangle$$

$$\langle \vec{p} | \vec{k} \rangle = \langle 0 | a_{\vec{p}} a^\dagger_{\vec{k}} | 0 \rangle$$

$$= (2\pi)^3 \, 2E_{\vec{k}} \, \delta(\vec{p} - \vec{k})$$

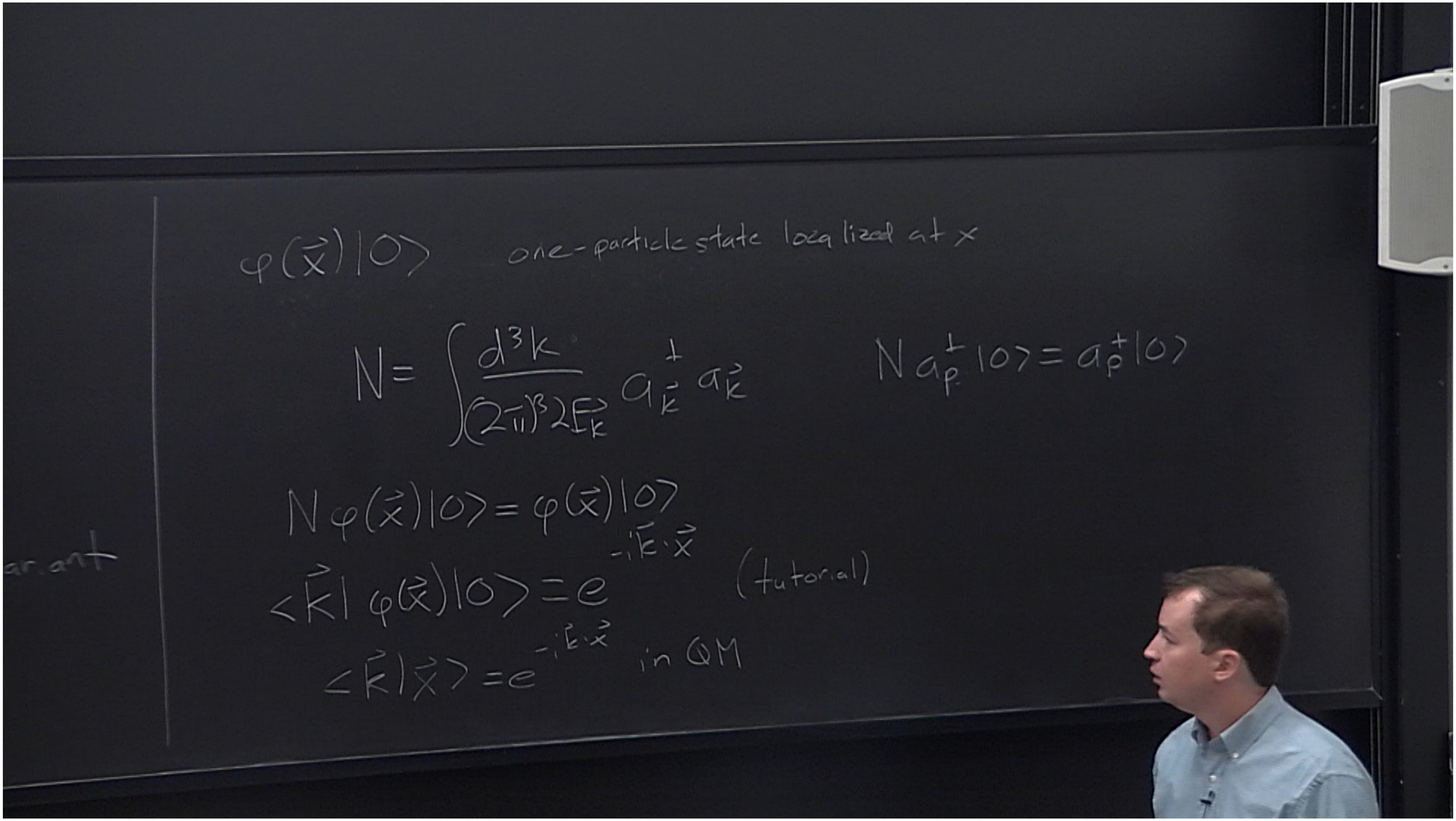
one particle states

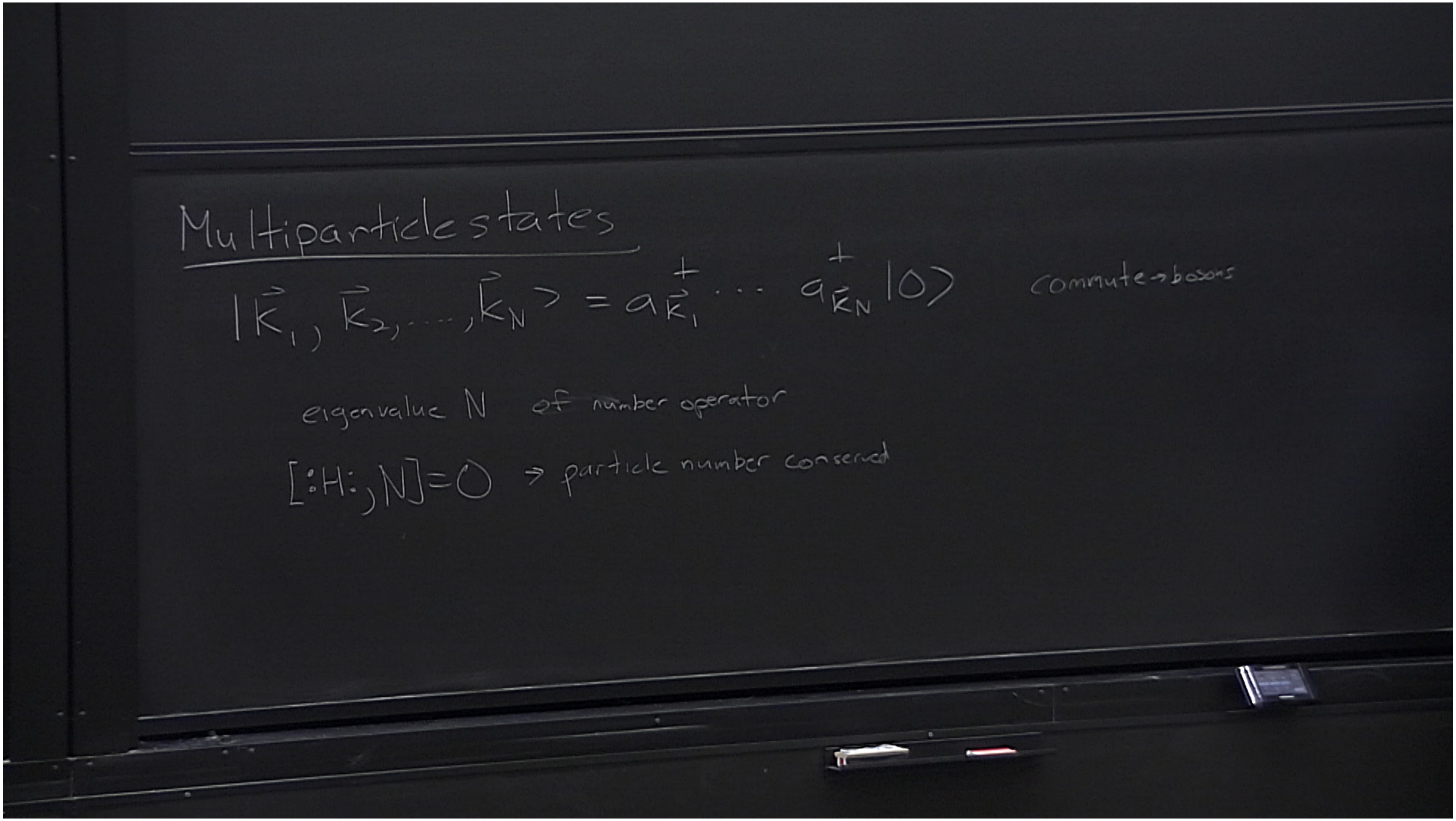
$$|\vec{k}\rangle = a^{\dagger}_{\vec{k}}|0\rangle$$ definite 3-momentum + energy

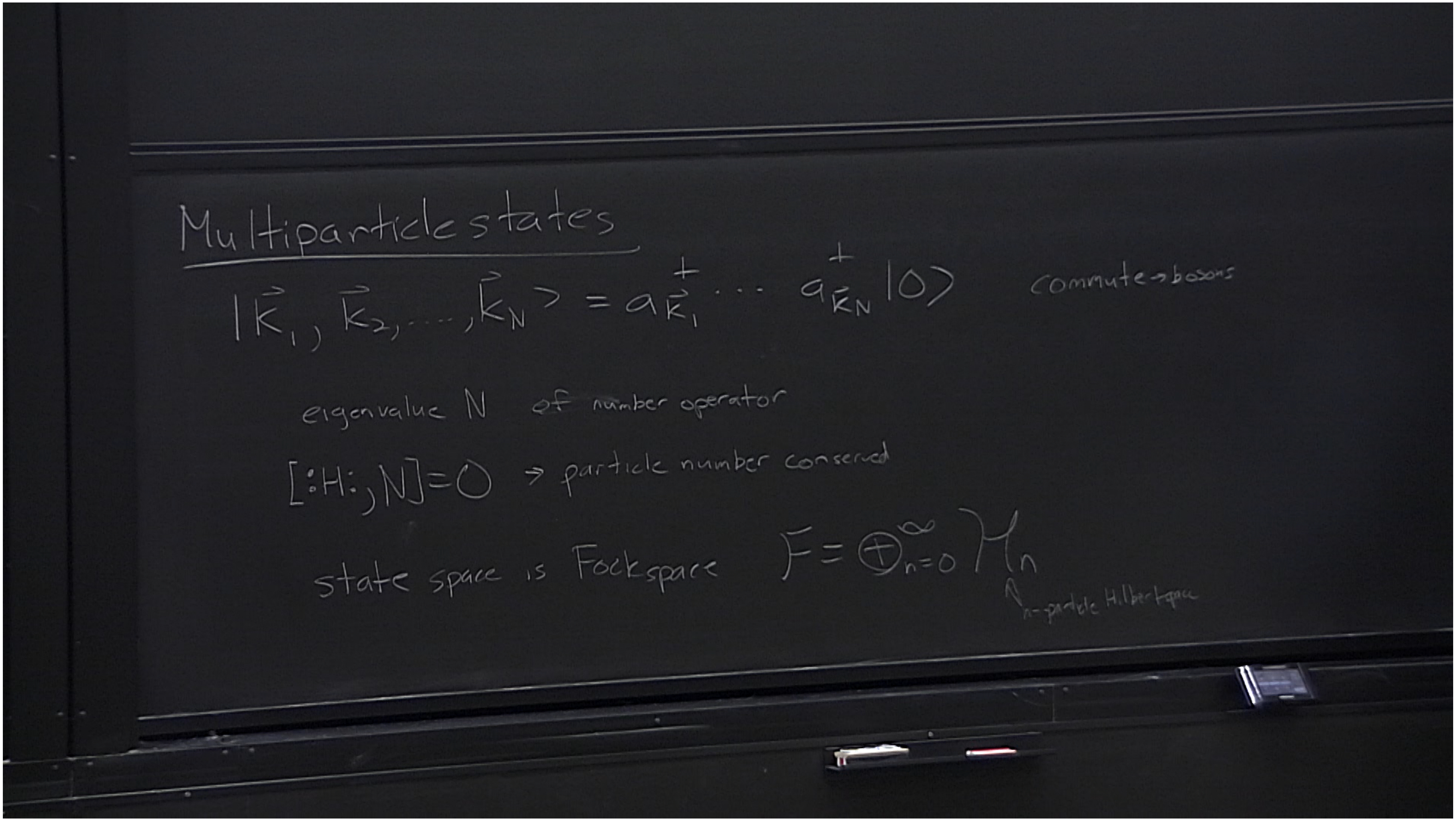
$$|\vec{k}\rangle = |k\rangle$$

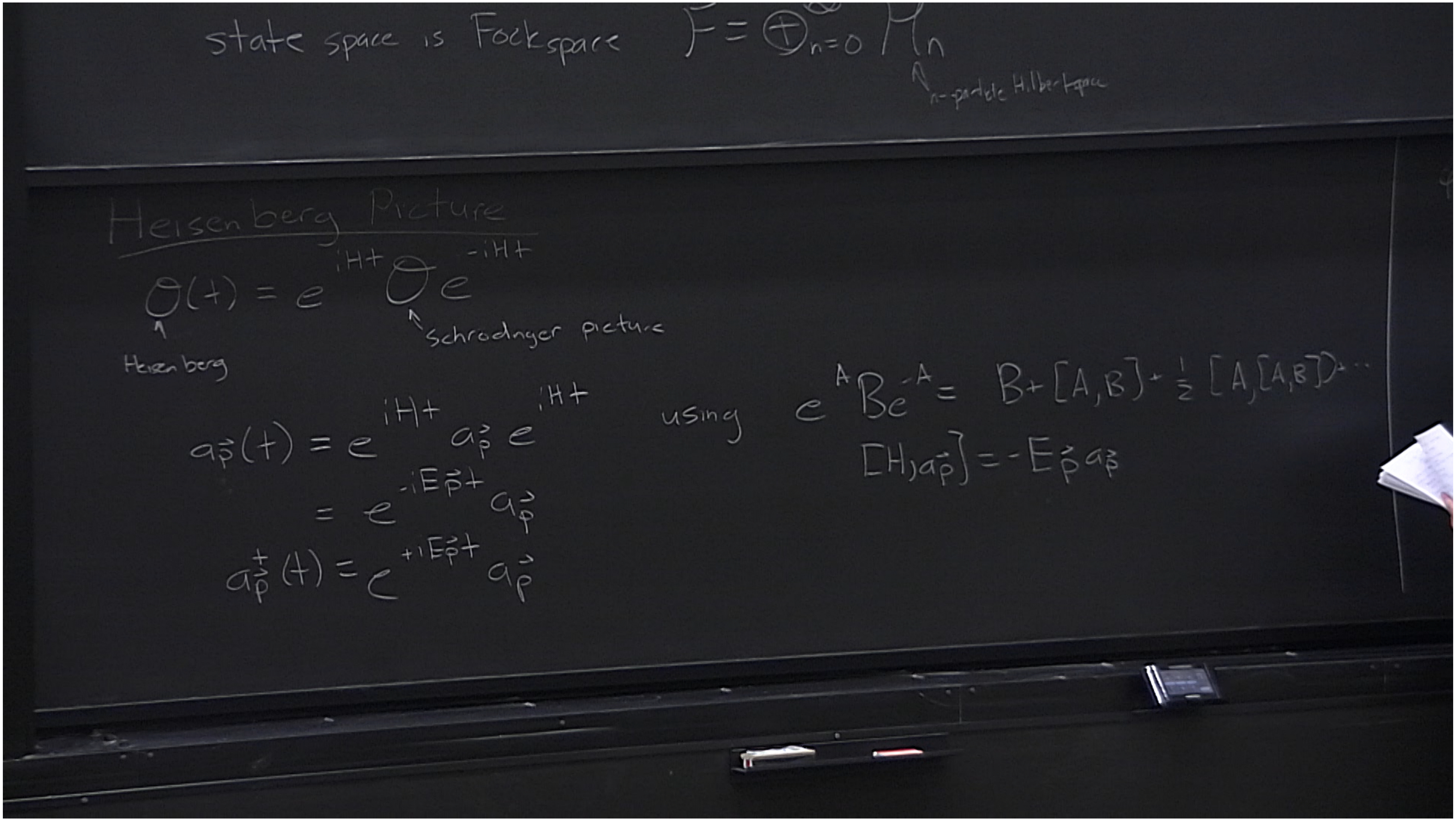
$$\langle\vec{p}|\vec{k}\rangle = \langle 0|a_{\vec{p}}\,a^{\dagger}_{\vec{k}}|0\rangle$$

$$= (2\pi)^3\, 2E_{\vec{k}}\,\delta(\vec{p}-\vec{k})$$ Lorentz invariant


$\varphi(\vec{x})|0\rangle$ one-particle state localized at x
$N = \int \frac{d^3k}{(2\pi)^3 2E_{\vec{k}}} a^\dagger_{\vec{k}} a_{\vec{k}}$
$N a^\dagger_{\vec{p}} |0\rangle = a^\dagger_{\vec{p}} |0\rangle$
$N \varphi(\vec{x})|0\rangle = \varphi(\vec{x})|0\rangle$
$\langle \vec{k} | \varphi(\vec{x}) | 0 \rangle = e^{-i\vec{k}\cdot\vec{x}}$ (tutorial)
$\langle \vec{k} | \vec{x} \rangle = e^{-i\vec{k}\cdot\vec{x}}$ in QM

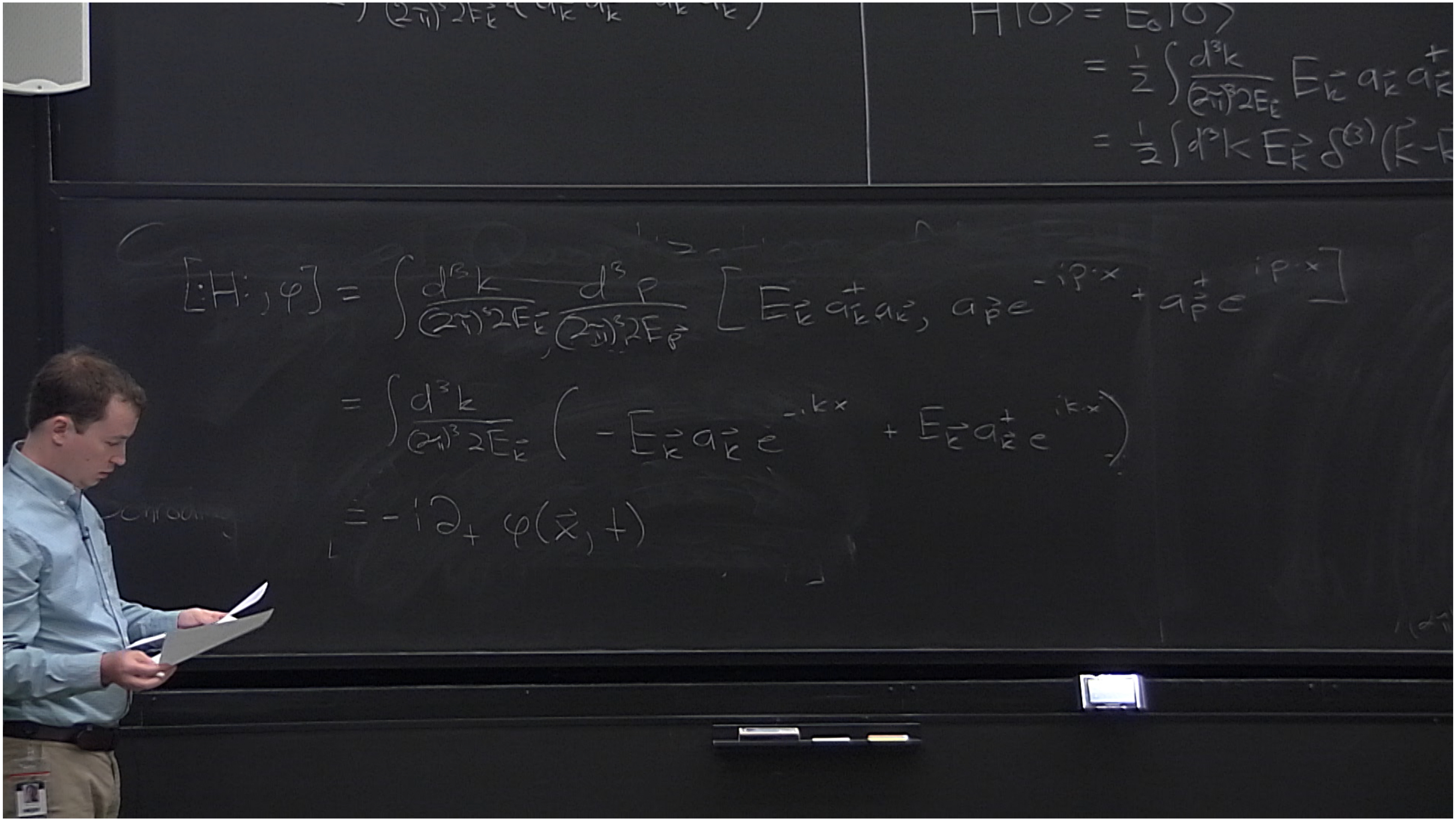

Multiparticle states
$|\vec{k}_1, \vec{k}_2, \dots, \vec{k}_N\rangle = a^\dagger_{\vec{k}_1} \cdots a^\dagger_{\vec{k}_N}|0\rangle$
commute → bosons
eigenvalue N of number operator
$[:H:, N] = 0$ → particle number conserved

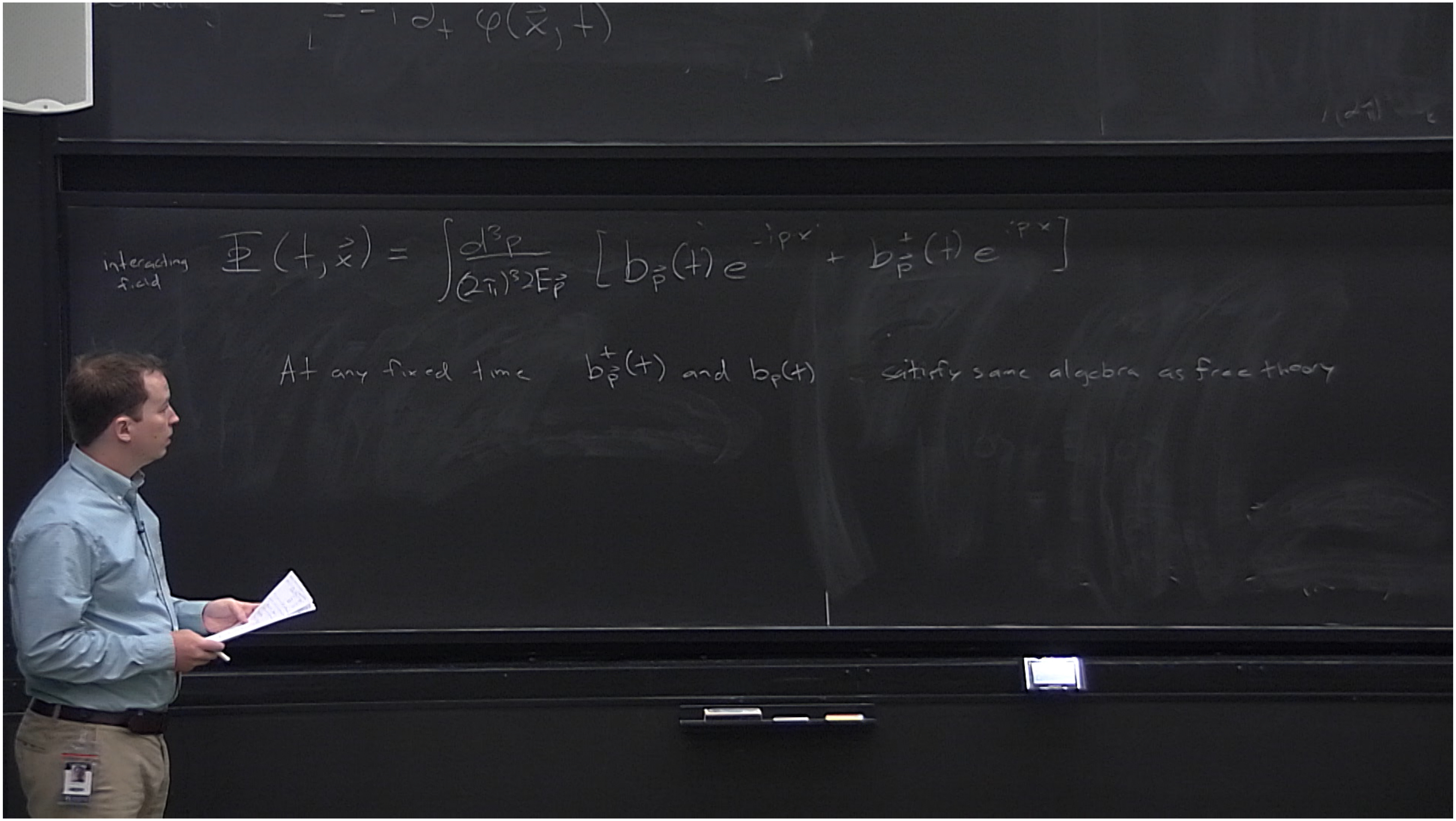

Multiparticle states
$|\vec{k}_1, \vec{k}_2, \ldots, \vec{k}_N\rangle = a^\dagger_{\vec{k}_1} \cdots a^\dagger_{\vec{k}_N}|0\rangle$
commute → bosons
eigenvalue N of number operator
$[:H:, N] = 0$ → particle number conserved
state space is Fock space $F = \oplus_{n=0}^{\infty} \mathcal{H}_n$
n-particle Hilbert space

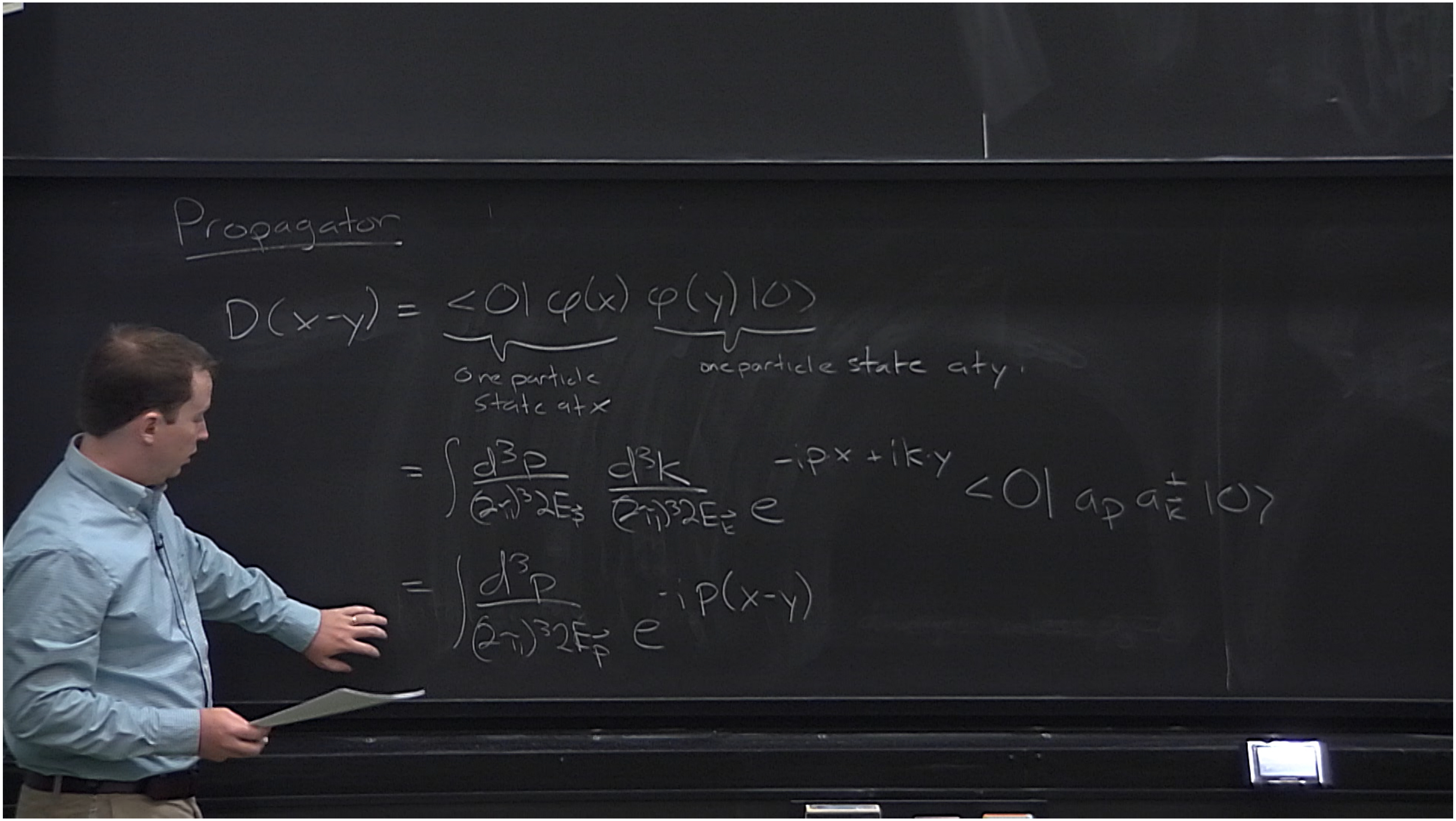

state space is Fock space $\mathcal{F} = \bigoplus_{n=0}^{\infty} \mathcal{H}_n$
n-particle Hilbert space
Heisenberg Picture
$\mathcal{O}(t) = e^{iHt} \mathcal{O} e^{-iHt}$
Heisenberg
Schrodinger picture
$a_{\vec{p}}(t) = e^{iHt} a_{\vec{p}} e^{iHt}$
$= e^{-iE_{\vec{p}}t} a_{\vec{p}}$
$a^{+}_{\vec{p}}(t) = e^{+iE_{\vec{p}}t} a_{\vec{p}}$
using $e^{A} B e^{-A} = B + [A,B] + \frac{1}{2}[A,[A,B]] + \cdots$
$[H, a_{\vec{p}}] = -E_{\vec{p}} a_{\vec{p}}$

$$\varphi(t,\vec{x}) = \int \frac{d^3k}{(2\pi)^3 2E_{\vec{k}}} \left[a_{\vec{k}} e^{-ik\cdot x} + a_{\vec{k}}^\dagger e^{ik\cdot x} \right]$$

$[A,[A,B]]+\ldots$


$H|0\rangle = E_0|0\rangle$
$= \frac{1}{2}\int \frac{d^3k}{(2\pi)^3 2E_{\vec{k}}} E_{\vec{k}} a_{\vec{k}} a^\dagger_{\vec{k}}$
$= \frac{1}{2}\int d^3k\, E_{\vec{k}}\, \delta^{(3)}(\vec{k}-\vec{k}$
$[:H:, \varphi] = \int \frac{d^3k}{(2\pi)^3 2E_{\vec{k}}} \frac{d^3p}{(2\pi)^3 2E_{\vec{p}}} \left[E_{\vec{k}} a^\dagger_{\vec{k}} a_{\vec{k}}, a_{\vec{p}} e^{-ip\cdot x} + a^\dagger_{\vec{p}} e^{ip\cdot x} \right]$
$= \int \frac{d^3k}{(2\pi)^3 2E_{\vec{k}}} \left(-E_{\vec{k}} a_{\vec{k}} e^{-ik\cdot x} + E_{\vec{k}} a^\dagger_{\vec{k}} e^{ik\cdot x} \right)$
$= -i\partial_t \varphi(\vec{x}, t)$


interacting field
Φ(t, x⃗) = ∫ d³p / (2π)³ 2E_p⃗ [b_p⃗(t) e^{-ip·x} + b†_p⃗(t) e^{ip·x}]
At any fixed time b†_p⃗(t) and b_p(t) satisfy same algebra as free theory

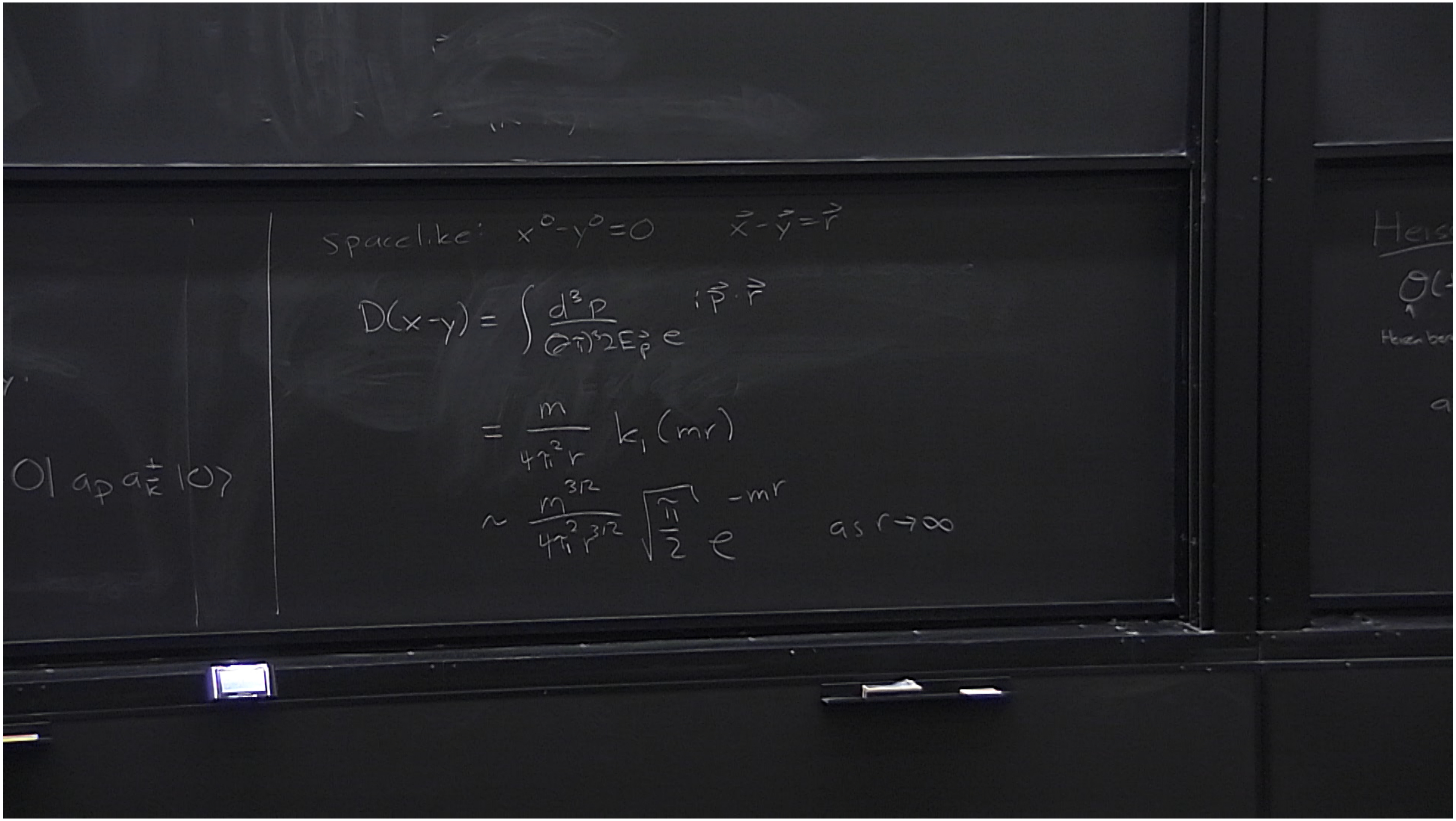

Propagator
$D(x-y) = \langle 0|\varphi(x)\,\varphi(y)|0\rangle$
one particle state at x
one particle state at y
$= \int \frac{d^3p}{(2\pi)^3 2E_{\vec{p}}} \frac{d^3k}{(2\pi)^3 2E_{\vec{k}}} e^{-ip\cdot x + ik\cdot y} \langle 0|\, a_p\, a^{\dagger}_{k} |0\rangle$
$= \int \frac{d^3p}{(2\pi)^3 2E_{\vec{p}}} e^{-ip(x-y)}$

Propagator

$$D(x-y) = \underbrace{\langle 0|\varphi(x)}_{\text{one particle state at } x}\ \underbrace{\varphi(y)|0\rangle}_{\text{one particle state at } y}$$

$$= \int \frac{d^3p}{(2\pi)^3 2E_{\vec{p}}} \frac{d^3k}{(2\pi)^3 2E_{\vec{k}}} e^{-ip\cdot x + ik\cdot y} \langle 0|a_p a^\dagger_k|0\rangle$$

$$= \int \frac{d^3p}{(2\pi)^3 2E_{\vec{p}}} e^{-ip(x-y)}$$


spacelike: $x^0 - y^0 = 0$ $\vec{x} - \vec{y} = \vec{r}$
$D(x-y) = \int \frac{d^3p}{(2\pi)^3 2E_{\vec{p}}} e^{i\vec{p}\cdot\vec{r}}$
$= \frac{m}{4\pi^2 r} K_1(mr)$
$\sim \frac{m^{3/2}}{4\pi^2 r^{3/2}} \sqrt{\frac{\pi}{2}}\, e^{-mr}$ as $r \to \infty$
$\langle 0| a_p a_k^\dagger |0\rangle$

$|0\rangle$

Spacelike: $x^0 - y^0 = 0 \qquad \vec{x} - \vec{y} = \vec{r}$

$$D(x-y) = \int \frac{d^3p}{(2\pi)^3 2E_{\vec{p}}} e^{i\vec{p}\cdot\vec{r}}$$

$$= \frac{m}{4\pi^2 r} K_1(mr)$$

$$\sim \frac{m^{3/2}}{4\pi^2 r^{3/2}} \sqrt{\frac{\pi}{2}}\, e^{-mr} \qquad \text{as } r \to \infty$$

Propagator

$$D(x-y) = \langle 0|\, \varphi(x)\, \varphi(y)\, |0\rangle$$

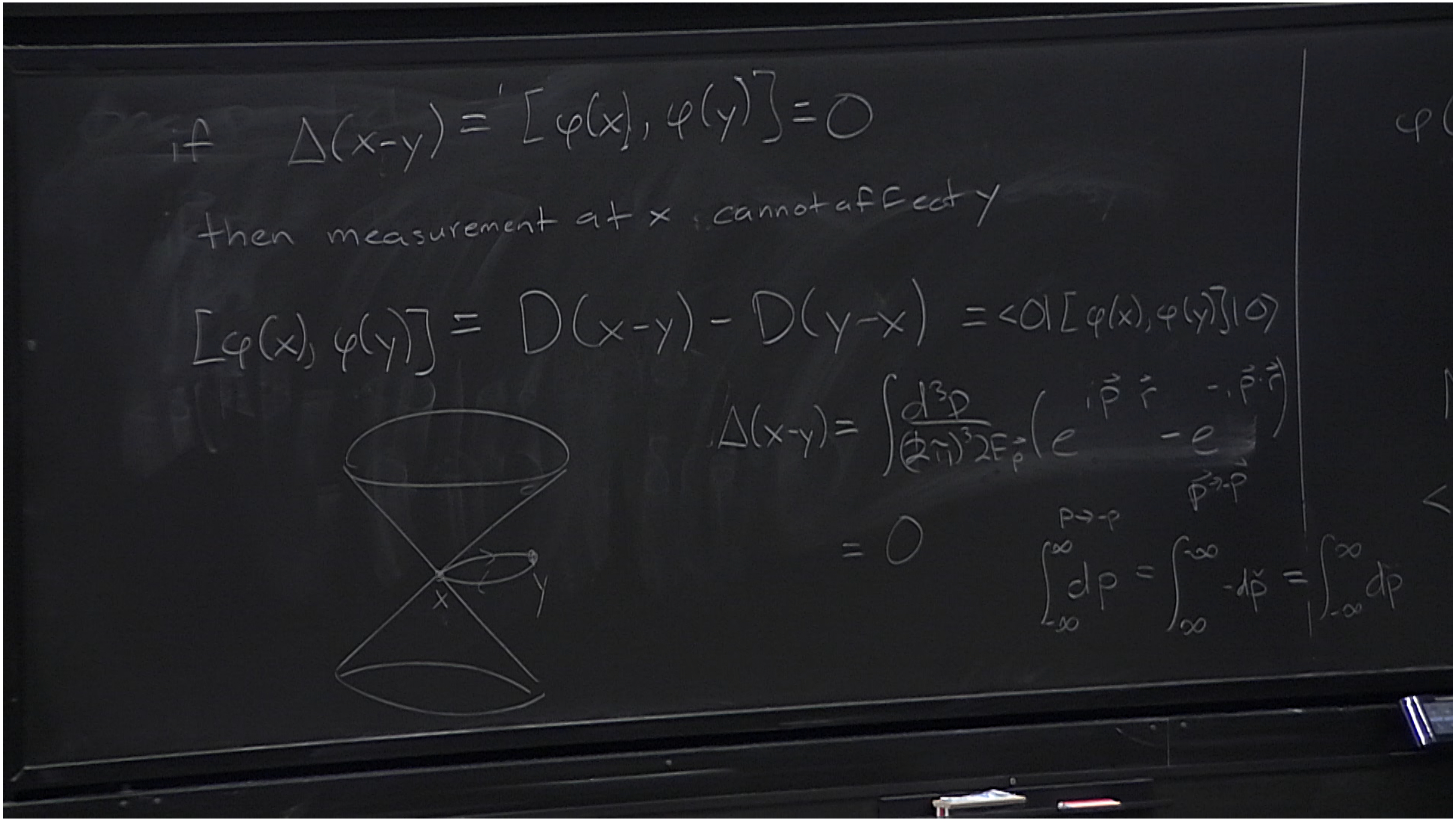
one particle state at x

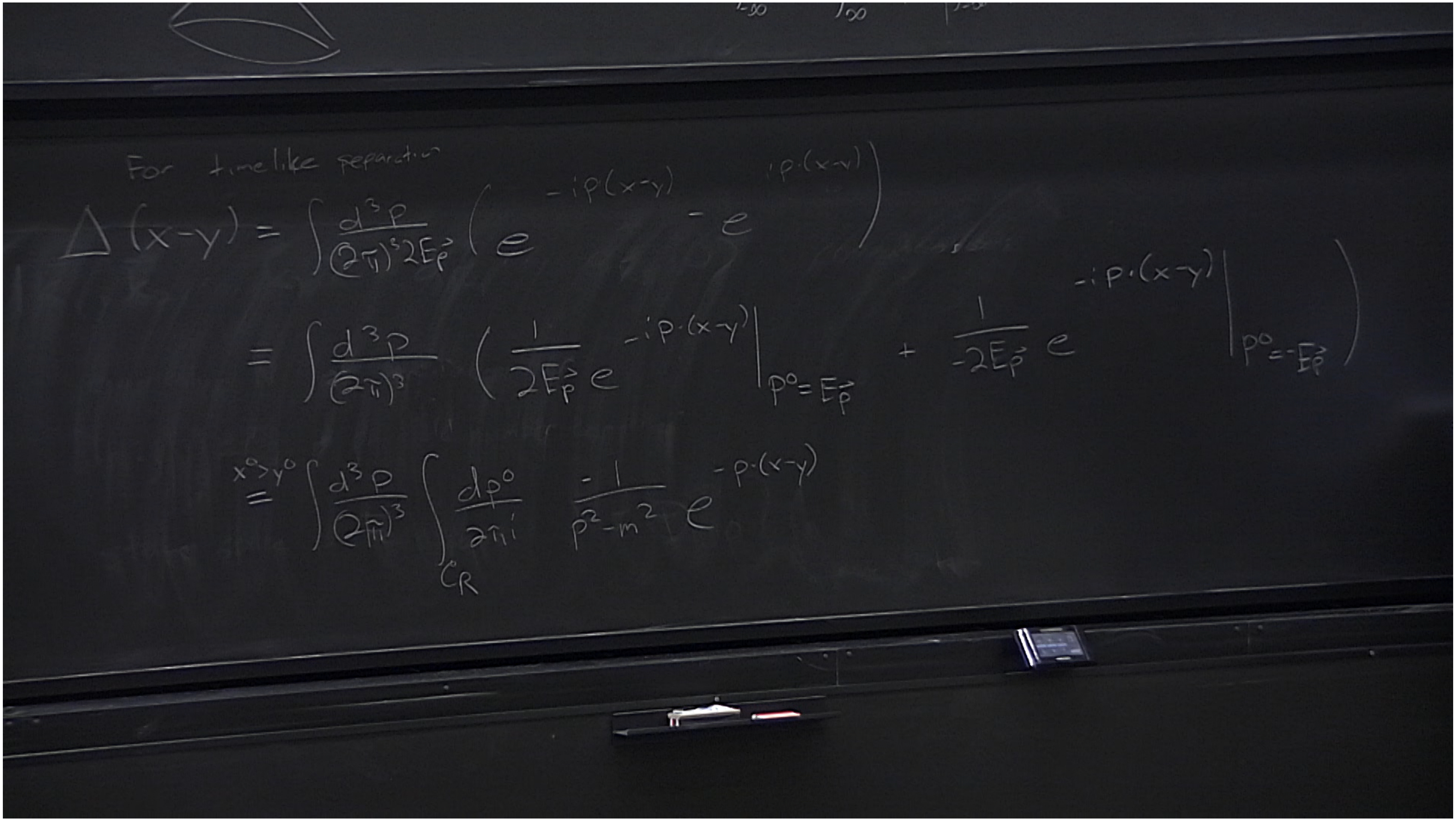
one particle state at y.

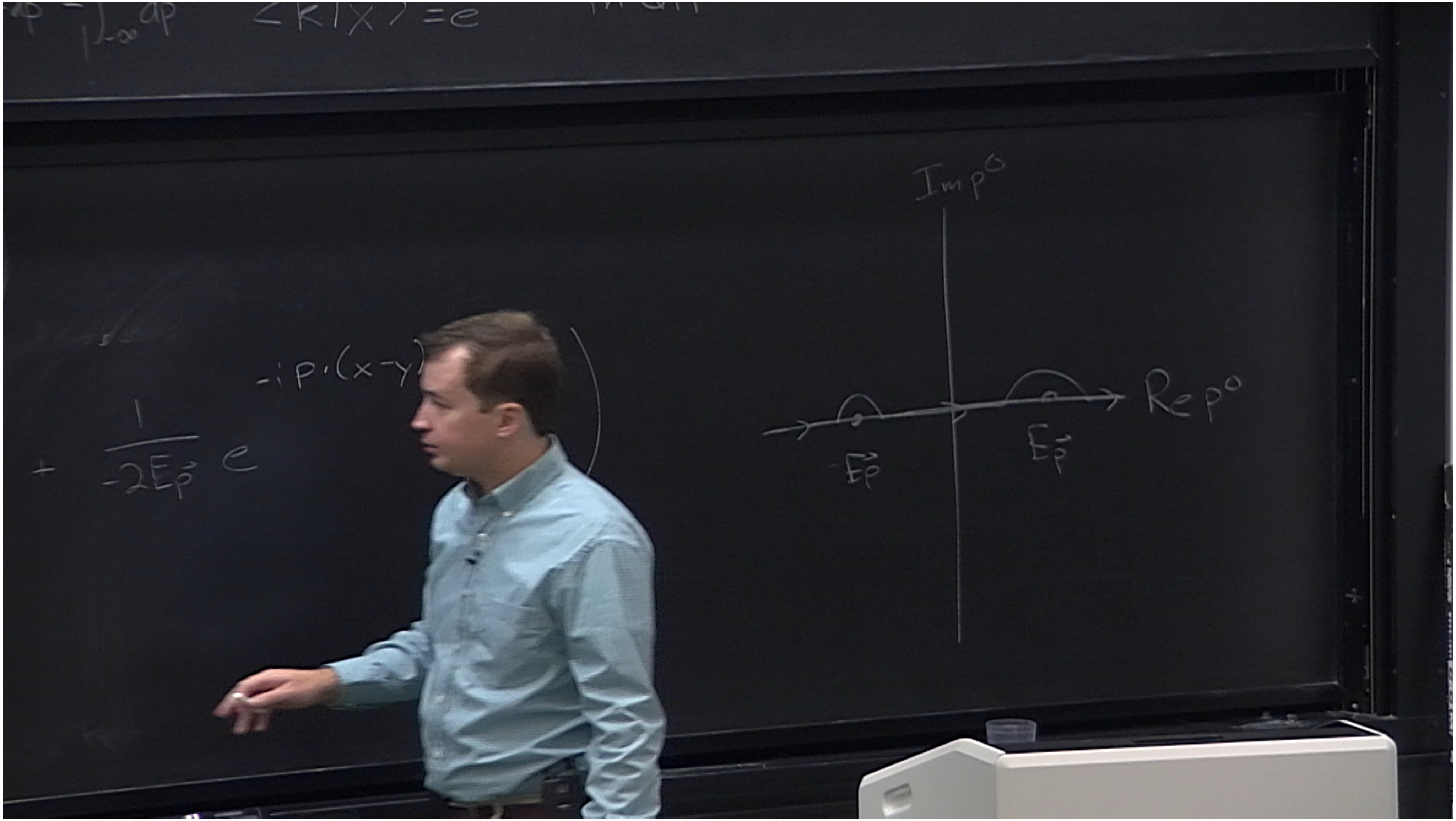
$$= \int \frac{d^3p}{(2\pi)^3 2E_{\vec p}} \frac{d^3k}{(2\pi)^3 2E_{\vec k}}\, e^{-ip\cdot x + ik\cdot y} \langle 0|\, a_{\vec p}\, a^{\dagger}_{\vec k}\, |0\rangle$$

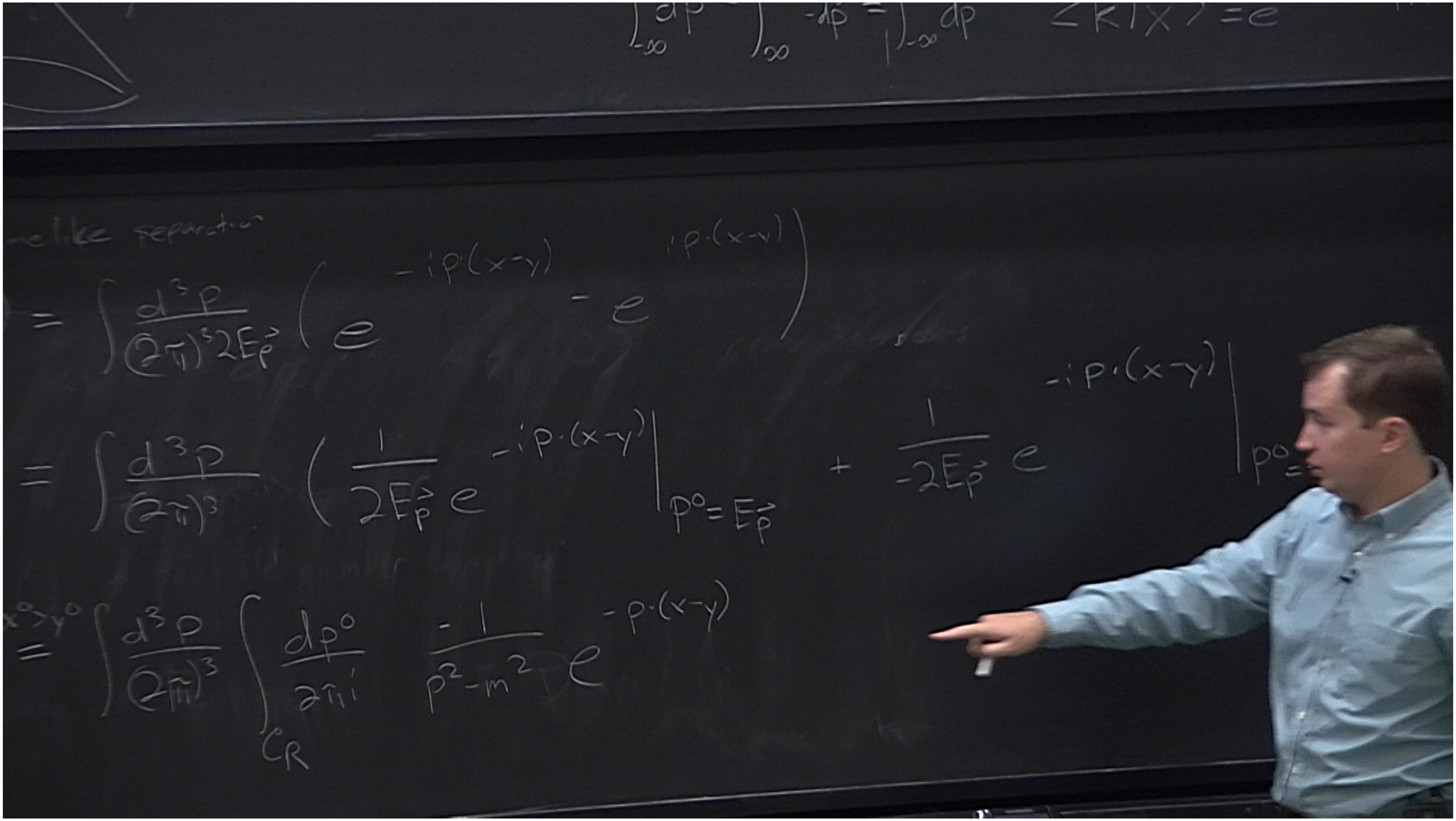
$a_{\vec p}\, a_{\vec k}$

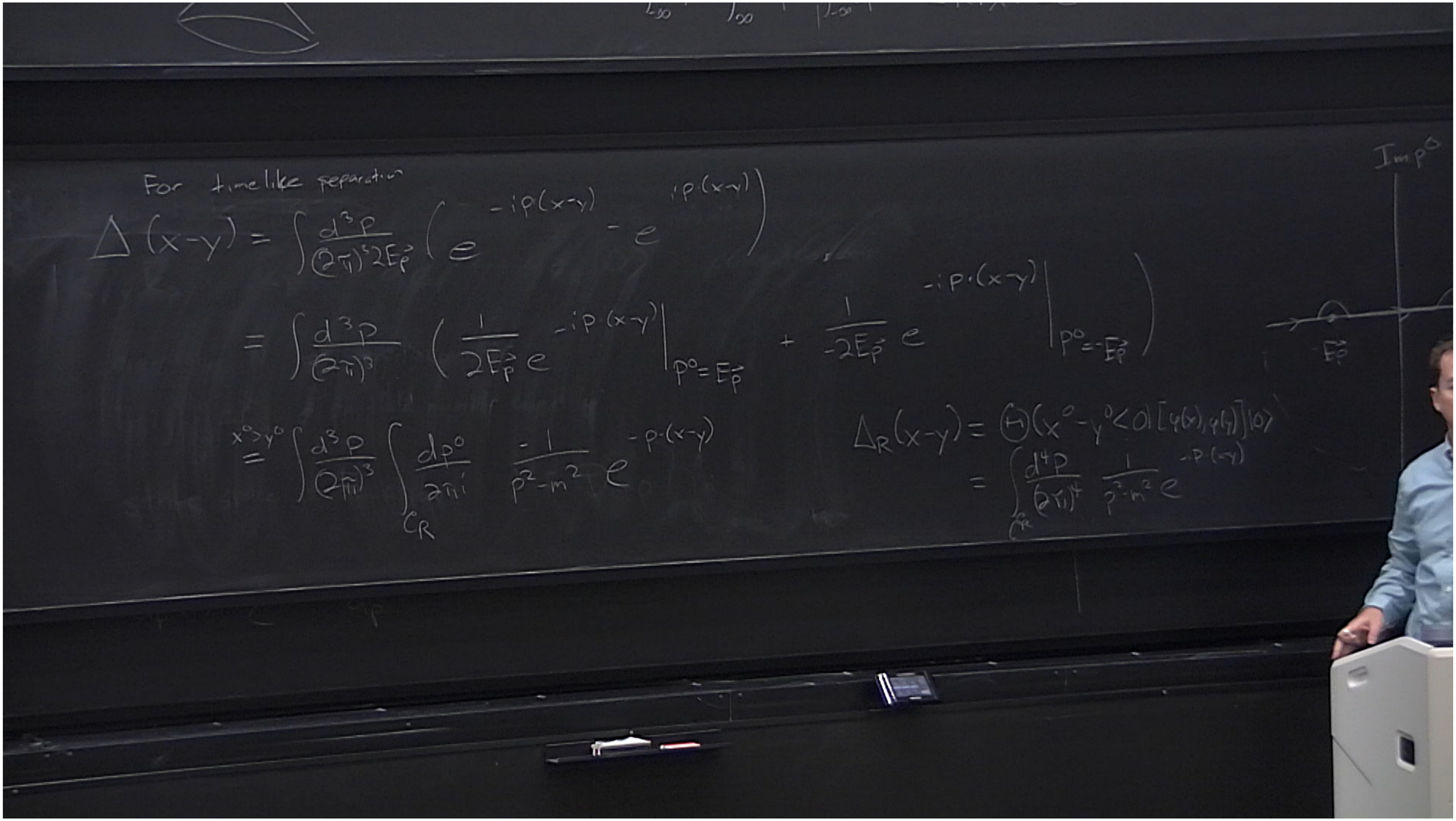
$$= \int \frac{d^3p}{(2\pi)^3 2E_{\vec p}}\, e^{-ip(x-y)}$$

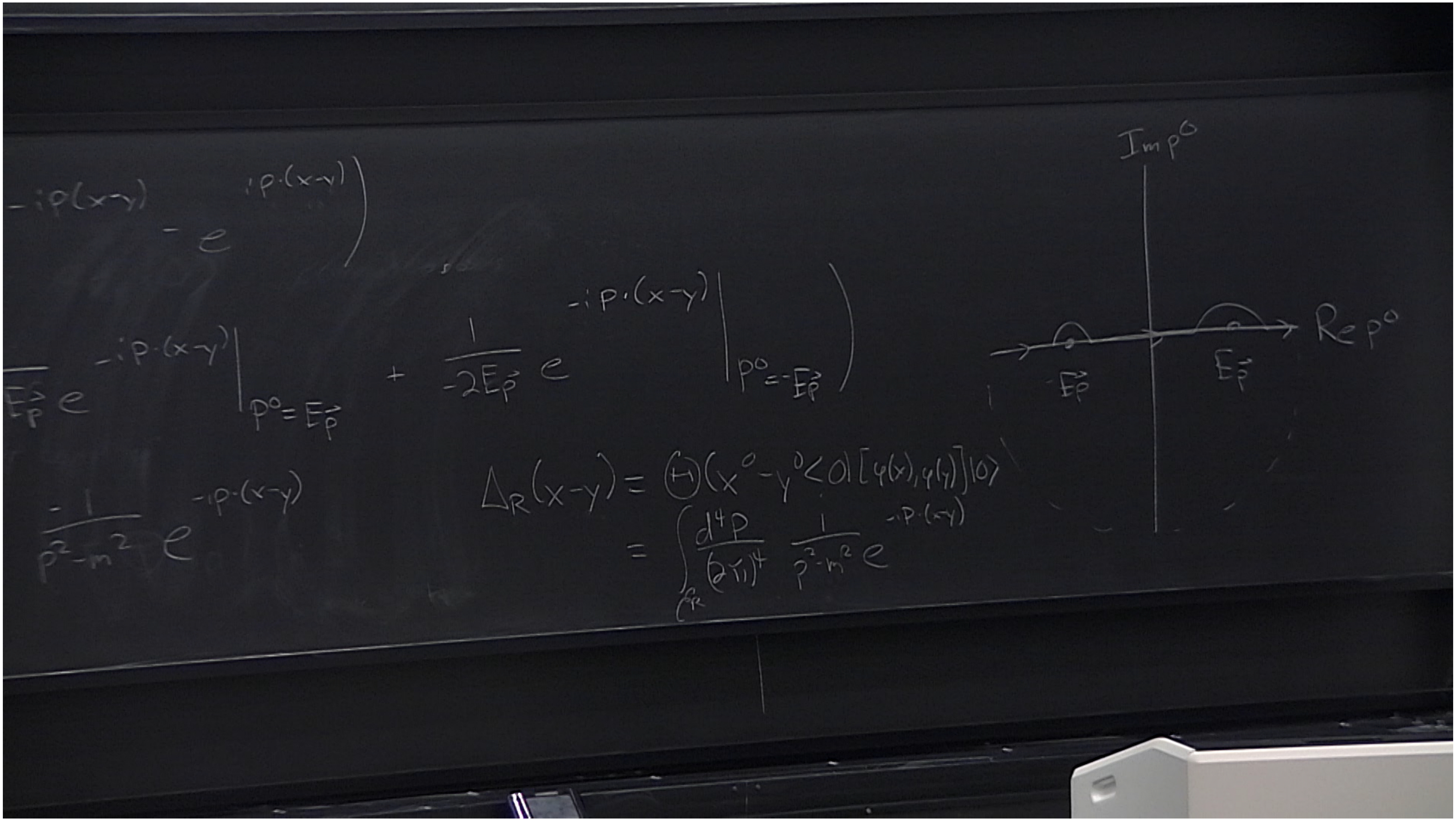

if Δ(x-y) = [φ(x), φ(y)] = 0
then measurement at x cannot affect y
[φ(x), φ(y)] = D(x-y) - D(y-x) = <0|[φ(x), φ(y)]|0>
Δ(x-y) = ∫ d³p / (2π)³ 2E_p (e^{i p·r} - e^{-i p·r})
p → -p
= 0
x
y
∫_{-∞}^{∞} dp = ∫_{∞}^{-∞} -dp = ∫_{-∞}^{∞} dp


For timelike separation
$\Delta(x-y) = \int \frac{d^3p}{(2\pi)^3 2E_{\vec{p}}} \left(e^{-ip(x-y)} - e^{ip\cdot(x-y)} \right)$
$= \int \frac{d^3p}{(2\pi)^3} \left(\frac{1}{2E_{\vec{p}}} e^{-ip\cdot(x-y)}\Big|_{p^0=E_{\vec{p}}} + \frac{1}{-2E_{\vec{p}}} e^{-ip\cdot(x-y)}\Big|_{p^0=-E_{\vec{p}}} \right)$
$\overset{x^0>y^0}{=} \int \frac{d^3p}{(2\pi)^3} \int_{C_R} \frac{dp^0}{2\pi i} \frac{-1}{p^2-m^2} e^{-p\cdot(x-y)}$


Im p°
Re p°
-Ep
Ep
-ip·(x-y)
1
-2Ep
e




For timelike separation
$\Delta(x-y) = \int \frac{d^3p}{(2\pi)^3 2E_{\vec p}} \left(e^{-ip(x-y)} - e^{ip(x-y)} \right)$
$= \int \frac{d^3p}{(2\pi)^3} \left(\frac{1}{2E_{\vec p}} e^{-ip(x-y)} \Big|_{p^0 = E_{\vec p}} + \frac{1}{-2E_{\vec p}} e^{-ip\cdot(x-y)} \Big|_{p^0 = -E_{\vec p}} \right)$
$\overset{x^0>y^0}{=} \int \frac{d^3p}{(2\pi)^3} \int_{C_R} \frac{dp^0}{2\pi i} \frac{-1}{p^2-m^2} e^{-p\cdot(x-y)}$
$\Delta_R(x-y) = \Theta(x^0-y^0) \langle 0| [\varphi(x), \varphi(y)] |0\rangle$
$= \int_{C_R} \frac{d^4p}{(2\pi)^4} \frac{1}{p^2-m^2} e^{-p\cdot(x-y)}$
Im p^0
$-E_{\vec p}$


Im p^0
Re p^0
-E_p
E_p
1/(-2E_p) e^{-ip·(x-y)} |_{p^0=-E_p}
e^{-ip·(x-y)} |_{p^0=E_p}
Δ_R(x-y) = Θ(x^0-y^0)<0|[φ(x),φ(y)]|0>
= ∫_{C_R} d^4p/(2π)^4 1/(p^2-m^2) e^{-ip·(x-y)}
1/(p^2-m^2) e^{-ip·(x-y)}