Title: General Relativity for Cosmology - Lecture 13

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Abstract:

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GR for Cosmology, Achim Kempf, Fall 2017, Lecture 14

Note Title

On $T^{\mu\nu}$, continued:

Recall: We defined $T^{\mu\nu}$ as that tensor which obeys for all $\delta g_{\mu\nu}(\lambda, x)$:

$$\left.\frac{dS'}{d\lambda}\right|_{\lambda=0} = \frac{1}{2}\int_B T^{\mu\nu} \underbrace{\delta g_{\mu\nu}}_{= \left.\frac{dg_{\mu\nu}(\lambda,x)}{d\lambda}\right|_{\lambda=0}} \sqrt{g}\, d^4x$$

(we choose $T^{\mu\nu}$ symmetric because $g_{\mu\nu}$ is symmetric)

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(we choose $T^{\mu\nu}$ symmetric because $g_{\mu\nu}$ is symmetric)

The above is meant when writing:

$$T^{\mu\nu} = \frac{2}{\sqrt{g}} \frac{\delta}{\delta g_{\mu\nu}} S$$

We found that ${T^{\mu\nu}}_{;\nu} = 0$ always holds.

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We found that $T^{\mu\nu}{}_{;\nu} = 0$ always holds.

(Since is consequence of diffeomorphism invariance)

However: $T^{\mu\nu}{}_{;\nu} = 0$ is not conservation law!

Why? $T^{\mu\nu}{}_{;\nu}\sqrt{g}$ is not a divergence, unlike $K^{\mu}{}_{;\mu}\sqrt{g} = \mathrm{div}_{\Omega} K$

Except: if space-time possesses isometries, i.e., covariant so-called Killing fields, ξ, i.e., fields obeying:

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Except: if space-time possesses isometries, i.e., covariant so-called Killing fields, ξ, i.e., fields obeying:

$$L_\xi g = 0, \quad \text{i.e.,} \quad \xi_{\mu;\nu} = -\xi_{\nu;\mu}$$

Because then: $P^\mu := T^{\mu\nu}\xi_\nu$ obeys $P^\mu{}_{;\mu} = 0$

Thus: $\int P^\mu{}_{;\mu}\sqrt{g}\, d^4x \overset{\text{Stokes}}{=} \int i_P \Omega = 0$

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Thus: $$\int_B \underbrace{P^\mu{}_{;\mu}\sqrt{g}}_{\mathrm{div}_\Omega P \,=\, d\, i_P \Omega} d^4x \overset{\text{Stokes}}{\underset{(\text{Gauss})}{=}} \underbrace{\int_{\partial B} i_P \Omega = 0}_{\text{a conservation law}}$$

Proposition: maximal number of indep. Killing vector fields on spacetime: 10

Actual spacetime has no Killing vector fields, but realistic simplified models of parts or all of spacetime often do:

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Definition: A space-time (M, g) is called "stationary" if it possesses

Definition: A space-time (M, g) is called "stationary" if it possesses energy conservation, i.e., if it possesses a time-like Killing vector field, i.e., if it possesses a field ξ which obeys:

$$L_\xi g = 0 \quad \text{and} \quad \xi^\mu \xi_\mu = g(\xi, \xi) < 0$$

(Recall: if
$=0$ would be called "null" or light-like
>0 would be called space-like)

$\rightsquigarrow$ Since ξ is timelike, observers can travel along the integral curves of ξ and set up a coordinate system with their eigen-time as the time coordinate.

time as the time coordinate.

In such a "Comoving coordinate system": $\xi = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

$\Rightarrow$ $0 = L_\xi g = \xi^\alpha g_{\mu\nu,\alpha} + g_{\alpha\nu}\xi^\alpha{}_{,\mu} + g_{\mu\alpha}\xi^\alpha{}_{,\nu}$ becomes:

$0 = g_{\mu\nu,0} = \partial_t g_{\mu\nu}$, i.e.: $g_{\mu\nu}(x) =$ constant in time.

$\Rightarrow$ In static spacetimes, one can find a (so-called comoving) coordinate system, in which:

$$\frac{\partial}{\partial x^0} g_{\mu\nu}(x^0, x^1, x^2, x^3) = 0$$

□ **However:** Stationarity does <u>not</u> imply that there is a cds in which

$$g = \begin{pmatrix} g_{00}, 0, 0, 0 \\ 0, & \\ 0, & \times \\ 0, & \end{pmatrix} \qquad (\times)$$

Example: The g of a stationary black hole that is rotating, given by the "Kerr metric".

Definition: A space-time is called "<u>static</u>", if the time-like

Definition: A space-time is called "static", if the time-lik[e] Killing field ξ, viewed as a 1-form,

$$\xi = \xi_\mu dx^\mu$$

also obeys the "Frobenius condition":

$$\xi \wedge d\xi = 0 \qquad (F)$$

Exercise: write it out in coordinates

Significance? (F) holds $\Leftrightarrow$ $g = \begin{pmatrix} g_{00}, 0, 0, 0 \\ 0 \\ 0 & (*) \\ 0 \end{pmatrix}$ in suitable cds.

- If, in a suitable coordinate system, g is of the form $(*)$ and the time-like ξ

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Generic properties of $T^{\mu\nu}$:

- $T^{\mu\nu}$ has contributions from known and also from as yet unknown matter fields (e.g., from dark matter).
- Thus, in order to draw generic conclusions about, e.g.,
  - **a.)** the occurrence of singularities, or
  - **b.)** the overall positivity of the energy (despite universal attraction!)

  (Note: black hole formation stops energy dropping)

  one needs plausible conjectures about the full $T_{\mu\nu}$:

(assumption)

The "weak energy condition":

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The weak energy condition:

$$T_{\mu\nu} v^\mu v^\nu \geq 0 \quad \text{for all timelike } v: \; g(v,v) < 0$$

(by continuity it then also holds for lightlike v)

Why assume it?

(Note: Negative energy would be anti-gravitating i.e. repulsive.) $\rightsquigarrow$ All observers travel with a time-like tangent v. They then see a positive local energy density: $T_{\mu\nu} v^\mu v^\nu \geq 0$

The "dominant energy condition":

$T_{\mu\nu} v^\mu v^\nu \geq 0$ for all timelike v (i.e. ...

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The "dominant energy condition":

$T_{\mu\nu} v^\mu v^\nu \geq 0$ for all timelike v (i.e., weak energy condition)

and

$K_\mu := T_{\mu\nu} v^\nu$ obeys $K_\mu K^\mu \leq 0$ (i.e., $T_{\mu\nu} v^\nu$ is non-space-like)

Why assume it?

- The local energy-momentum flow vector, K, may not be conserved but should be non-space-like
  "All flow should be into the future."

- In an orthonormal basis, the dominant energy

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Why assume it?

- The local energy-momentum flow vector, K, may not be conserved but should be non-space-like

"All flow should be into the future."

- In an orthonormal basis, the dominant energy condition takes the form:

$$T^{00} \geq |T^{ab}|$$

i.e. "energy dominates over momentum."

(Note: This is all intuition from fluid mechanics analogy. Quantum fields may or may not behave this way.)

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The dynamics of space-time!

- Consider the full matter action:

$$S[g, \psi] = \int_M L(g, \psi) \sqrt{g}\, d^4x$$

all matter fields: e^-, photons, quarks, gluons etc.

- The equations of motion of matter fields are

$$\frac{\delta S}{\delta \psi_{(i)}{}^{a..b}{}_{c..d}} = 0$$

▫ The equations of motion of matter fields are

$$\frac{\delta S'}{\delta \psi_{(i)}{}^{a..b}{}_{c..d}} = 0$$

i.e.:

$$\frac{\partial L}{\partial \psi_{(i)}{}^{a..b}{}_{c..d}} = \left(\frac{\partial L}{\partial \psi_{(i)}{}^{a..b}{}_{c..d;e}} \right)_{;e}$$

▫ Do we obtain suitable equations of motion for $g_{\mu\nu}$ by setting

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☐ Do we obtain suitable equations of motion for $g_{\mu\nu}$ by setting

$$\frac{\delta S'}{\delta g_{\mu\nu}} = 0 \quad ?$$

Apparently not, because it would mean:

$$\frac{\delta S'}{\delta g_{\mu\nu}} = \frac{1}{2} T^{\mu\nu}\sqrt{g} = 0 \; !$$

Thus, the universe would have to be empty of matter (assuming all matter has positive energy).

☐ Andrei Sakharov (1968):

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- Andrei Sakharov (1968):

The quantum effects of matter induce suitable extra terms in the action!

Sakharov's reasoning: (modernized version)

- Classical deterministic evolution of matter obeys:

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Sakharov's reasoning: (modernized version)

- Classical deterministic evolution of matter obeys:

$$\frac{\delta S}{\delta \psi_{(i)}} = 0$$

- But quantum theory allows every evolution $\psi_{(i)}(x,t)$ to happen "virtually", with probability amplitudes:

a normalization constant.

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- But quantum theory allows every evolution $\psi_{(i)}(x,t)$ to happen "virtually", with probability amplitudes:

a normalization constant,

$$\text{prob.ampl.}\left[\psi_{(i)}\right] = N\, e^{\frac{i S'[g,\psi]}{\hbar}}$$

- As usual in quantum theory, the actual or "effective" matter evolution $\langle\psi_{(i)}(x,t)\rangle$ is close

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□ As usual in quantum theory, the actual or "effective" matter evolution $\langle \psi_{(i)}(x,t) \rangle$ is close

to but not identical to the classical matter evolution $\psi_{(i)}(x,t)$.

Why? Path integral picture: The field evolutions with close-to-extremal actions have very similar values $e^{\frac{iS}{\hbar}}$ because for them $\frac{\delta S}{\delta \psi} \simeq 0$ i.e. their prob. amplitudes add up. Other matter evolutions $\psi_{(i)}(x,t)$

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- Thus, the effective quantum fields obey equations of motion that are somewhat modified!

⇒ Aim: Calculate the "effective action"

$$S_{eff}[g, \psi]$$

which yields the effective evolution of matter fields when matter quantum effects are taken into account.

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Problems:

- These calculations are very difficult.

⇒ Use perturbative methods.

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Problems:

- These calculations are very difficult.

  $\Rightarrow$ Use perturbative methods.

- There occur integrals that are divergent at short distances.

  $\Rightarrow$ Introduce a cutoff at some minimum length l_c (or maximum momentum $\frac{\hbar}{l_c}$).

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- There occur integrals that are divergent at short distances.

⇒ Introduce a cutoff at some minimum length ℓ_c (or maximum momentum $\frac{\hbar}{\ell_c}$).

- But as always in quantum theory, the effective action will contain terms of all possible forms that are consistent with the symmetries of the theory i.e. here with general covariance (i.e. that are scalars):

The only question is which prefactors these terms will have.

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But as always in quantum theory, the effective action will contain terms of all possible forms that are consistent with the symmetries of the theory i.e. here with general covariance (i.e. that are scalars):

The only question is which prefactors these terms will have.

quantum "vacuum energy" of matter

$$S_{eff}[g,\psi] = \int_M \left(L^{matter} + L^{matter}_{\substack{quantum\\effects}} + c_1 + c_2 R + c_3\, O(R^2) \right) \sqrt{g}\, d^4x$$

this is the local change of the vacuum energy due to curvature deforming the quantum harmonic oscillators of the field modes.

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- The constants c_1, c_2 etc depend on:
  - the details of the matter action L^{matter}. (Bosons and Fermions tend to contribute with opposite signs)
  - the order of perturbation
  - the value of the short-distance cutoff:

The λ_i are unitless numbers that are roughly of order one,

$c_1 = \lambda_1 \ell_c^{-4}$ (must make up for $[length]^4$ from d^4x)

$c_2 = \lambda_2 \ell_c^{-2}$ (because R has units $[length]^{-2}$)

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• the value of the short-distance cut off:

The λ_i are unitless numbers that are roughly of order one, depending on the precise matter Lagrangian

$c_1 = \lambda_1 \ell_c^{-4}$ (must make up for $[\text{length}]^4$ from d^4x)

$c_2 = \lambda_2 \ell_c^{-2}$ (because R has units $[\text{length}]^{-2}$)

$c_3 = \lambda_3 \ell_c^{0}$ (terms R^2 or $R^{\mu\nu} R_{\mu\nu}$ etc have units $[\text{length}]^{-4}$)

$c_4 = \lambda_4 \ell_c^{2}$ (higher powers in R have prefactors $\sim \ell_c^{\text{positive power}}$)

$\Rightarrow$ For small ℓ_c, we have:

$$c_1 \gg c_2 \gg c_3 \gg c_4 \gg \ldots$$

Consider the lowest order terms:

$$S'_{\psi}[g,\psi] = \int_M \Big(\overset{\swarrow \text{total effective matter Lagrangian}}{L} + \underbrace{c_1}_{\substack{\parallel \\ -\Lambda \\ \text{cosmological constant}}} + c_2 \underbrace{R}_{\substack{\uparrow \\ \text{Einstein action}}}\Big)\sqrt{g}\, d^4x$$

and postulate now that the equations of motion for the metric follow from the action principle:

$$\frac{\delta S_{\psi}[g,\psi]}{\delta g_{\mu\nu}} = 0$$

Einstein had postulated the same action principle!

Einstein had postulated the same action principle:

We note that:

Every (effective) quantum field theory with minimum length induces Einstein gravity.

See, e.g., review: gr-qc/0204062

The equations of motion for g:

The action principle, $\frac{\delta S_{eff}}{\delta \ldots} = 0$, yields:

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The action principle, $\frac{\delta S'_{eff}}{\delta g_{\mu\nu}} = 0$, yields:

$$\frac{\delta}{\delta g_{\mu\nu}} \int_B \left(c_1 + c_2 R_{\mu\nu} g^{\mu\nu}\right) \sqrt{g}\, d^4x = -\frac{1}{2}\sqrt{g}\, T^{\mu\nu}$$

in principle, it is the effective quantum expectation value

Evaluate the left hand side:

recall: $= \frac{1}{2} g^{\mu\nu}\sqrt{g}$

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a.) $$\delta \int_B c_1 \sqrt{g}\, d^4x = \int_B c_1 \underbrace{\frac{\delta\sqrt{g}}{\delta g_{\mu\nu}}} \delta g_{\mu\nu}\, d^4x$$

$\delta g_{\mu\nu}$

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$$= \int_B c_1 \frac{1}{2} g^{\mu\nu} \sqrt{g}\, \delta g_{\mu\nu}\, d^4x$$

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$\delta g_{\mu\nu}$

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Evaluate the left hand side:

a.)

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$$= \int_B c_1 \frac{1}{2} g^{\mu\nu} \sqrt{g}\, \delta g_{\mu\nu}\, d^4x$$

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The equations of motion for g:

The action principle, $\frac{\delta S'_{eff}}{\delta g_{\mu\nu}} = 0$, yields:

$$\frac{\delta}{\delta g_{\mu\nu}} \int_B \left(c_1 + c_2 R_{\mu\nu} g^{\mu\nu}\right)\sqrt{g}\, d^4x = -\frac{1}{2}\sqrt{g}\, T^{\mu\nu}$$

in principle, it is the effective quantum expectation value

Evaluate the left hand side:

recall: $= \frac{1}{2} g^{\mu\nu}\sqrt{g}$

a) $\delta \int c_1 \sqrt{g}\, d^4x = \int c_1 \delta\sqrt{g}\, d^4x$

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Evaluate the left hand side:

recall: $= \frac{1}{2} g^{\mu\nu}\sqrt{g}$

a.) $$\delta \int_B c_1 \sqrt{g}\, d^4x = \int_B c_1 \underbrace{\frac{\delta\sqrt{g}}{\delta g_{\mu\nu}}} \delta g_{\mu\nu}\, d^4x$$

$$= \int_B c_1 \frac{1}{2} g^{\mu\nu} \sqrt{g}\, \delta g_{\mu\nu}\, d^4x$$

b.) $$\delta \int_B c_2 R_{\mu\nu} g^{\mu\nu} \sqrt{g}\, d^4x$$

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b.)

$$\delta \int_B c_2 R_{\mu\nu} g^{\mu\nu} \sqrt{g}\, d^4x$$

$$= \underbrace{\int_B c_2 (\delta R_{\mu\nu}) g^{\mu\nu} \sqrt{g}\, d^4x}_{\text{Term I}} + \underbrace{\int_B c_2 R_{\mu\nu} \delta(g^{\mu\nu} \sqrt{g})\, d^4x}_{\text{Term II}}$$

Proposition: Term I $= 0$

Proof: Choose origin of geodesic coordinate system vanish at origin because $\Gamma = 0$

Thus:

$$\delta R_{\mu\nu} = (\delta \Gamma^\alpha{}_{\mu\nu})_{,\alpha} - (\delta \Gamma^\alpha{}_{\mu\alpha})_{,\nu}$$

$$\Rightarrow \quad g^{\mu\nu} \delta R_{\mu\nu} = g^{\mu\nu} (\delta \Gamma^\alpha{}_{\mu\nu})_{,\alpha} - g^{\mu\nu} (\delta \Gamma^\alpha{}_{\mu\alpha})_{,\nu}$$

$$= w^\alpha{}_{,\alpha} \text{ for } w^\alpha = g^{\mu\nu} \delta \Gamma^\alpha{}_{\mu\nu} - g^{\mu\alpha} \delta \Gamma^\nu{}_{\mu\nu}$$

recall: $g^{\mu\nu}{}_{,\alpha} = 0$ here.

Thus, in arbitrary coordinate system:

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b.) $$\delta \int_B c_2 R_{\mu\nu} g^{\mu\nu} \sqrt{g}\, d^4x$$

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Proposition: Term I $= 0$

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$$\Rightarrow \quad g^{\mu\nu}\delta R_{\mu\nu} = g^{\mu\nu}(\delta\Gamma^{\alpha}{}_{\mu\nu})_{,\alpha} - g^{\mu\nu}(\delta\Gamma^{\alpha}{}_{\mu\alpha})_{,\nu}$$

$$= w^{\alpha}{}_{,\alpha} \text{ for } w^{\alpha} = g^{\mu\nu}\delta\Gamma^{\alpha}{}_{\mu\nu} - g^{\mu\alpha}\delta\Gamma^{\nu}{}_{\mu\nu}$$

recall: $g^{\mu\nu}{}_{,\alpha} = 0$ here.

Thus, in arbitrary coordinate system:

$$g^{\mu\nu}\delta R_{\mu\nu} = w^{\alpha}{}_{;\alpha}$$

$$\Rightarrow \int_B c_2\, g^{\mu\nu}\delta R_{\mu\nu}\sqrt{g}\, d^4x = \int_B w^{\alpha}{}_{;\alpha}\sqrt{g}\, d^4x$$

$$= \int \operatorname{div} \Omega$$

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$$g^{\mu\nu}\delta R_{\mu\nu} = w^{\alpha}{}_{;\alpha}$$

$$\Rightarrow \int_B c_2\, g^{\mu\nu}\delta R_{\mu\nu}\sqrt{g}\, d^4x = \int_B w^{\alpha}{}_{;\alpha}\sqrt{g}\, d^4x$$

$$= \int_B \mathrm{div}_w \Omega$$

$$\overset{\text{Gauß}}{=} \int_{\partial B} i_w \Omega$$

$= 0$ on ∂B, assuming $\delta g_{\mu\nu}$ and $\delta g_{\mu\nu,\rho} = 0$ on ∂B.

$$= 0$$

$$\int_B c_2 R_{\mu\nu}\,\delta\left(g^{\mu\nu}\sqrt{g}\right)d^4x = ?$$

We have :

$$\delta\left(g^{\mu\nu}\sqrt{g}\right) = \left(\delta g^{\mu\nu}\right)\sqrt{g} + g^{\mu\nu}\frac{\partial\sqrt{g}}{\partial g_{ab}}\,\delta g_{ab}$$

$$= -g^{\mu a}g^{\nu b}\,\delta g_{ab}\sqrt{g} + g^{\mu\nu}\,\frac{1}{2}\,g^{ab}\sqrt{g}\,\delta g_{ab}$$

$\Rightarrow$

$$\int_B c_2 R_{\mu\nu}\,\delta\left(g^{\mu\nu}\sqrt{g}\right)d^4x = -\int_B c_2\underbrace{\left(+R^{ab}-\frac{1}{2}g^{ab}R\right)}\sqrt{g}\,\delta g_{ab}\,d^4x$$

$$\delta(g^{\mu\nu}\sqrt{g}) = (\delta g^{\mu\nu})\sqrt{g} + g^{\mu\nu}\frac{\partial\sqrt{g}}{\partial g_{ab}}\delta g_{ab}$$

$$= -g^{\mu a}g^{\nu b}\delta g_{ab}\sqrt{g} + g^{\mu\nu}\frac{1}{2}g^{ab}\sqrt{g}\,\delta g_{ab}$$

$$\Longrightarrow$$

$$\int_B c_2 R_{\mu\nu}\,\delta(g^{\mu\nu}\sqrt{g})\,d^4x = -\int_B c_2 \underbrace{\left(+R^{ab} - \frac{1}{2}g^{ab}R\right)}_{\text{recall: } = G^{ab}} \sqrt{g}\,\delta g_{ab}\,d^4x$$

"Einstein tensor"

Bringing together a) + b) + c) $\Longrightarrow$

$$\delta \int_B (c_1 + c_2 R_{\mu\nu} g^{\mu\nu}) \sqrt{g}\, d^4x$$

$$= \int_B \underbrace{\left(c_1 \frac{1}{2} g^{\mu\nu} - c_2 G^{\mu\nu}\right)}_{} \sqrt{g}\, \delta g_{ab}\, d^4x$$

symmetric

as in the case of the $T^{\mu\nu}$ calculation, one could add an antisymmetric part here and it would drop from the integrand.

$$\Rightarrow \frac{\delta}{\delta g_{\mu\nu}} \int_B \left(c_1 + c_2 R_{\mu\nu} g^{\mu\nu}\right) \sqrt{g}\, d^4x$$

(up to anti-sym. components, which we set to zero)

$$= \left(c_1 \frac{1}{2} g^{\mu\nu} - c_2 G^{\mu\nu}\right) \sqrt{g}$$

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Finally, we conclude:

$$\frac{\delta S'_{g}}{\delta g_{\mu\nu}} = 0$$

leads to this equation of motion for g:

$$\left(\frac{1}{2} c_1 g^{\mu\nu} - c_2 G^{\mu\nu}\right)\sqrt{g} = -\frac{1}{2}\sqrt{g}\, T^{\mu\nu}$$

i.e.:

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R - \frac{c_1}{2c_2} g^{\mu\nu} = \frac{1}{2c_2} T^{\mu\nu}$$

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leads to this equation of motion for g:

$$\left(\frac{1}{2} c_1 g^{\mu\nu} - c_2 G^{\mu\nu}\right)\sqrt{g} = -\frac{1}{2}\sqrt{g}\, T^{\mu\nu}$$

i.e.: $$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R - \frac{c_1}{2c_2} g^{\mu\nu} = \frac{1}{2c_2} T^{\mu\nu}$$

As is well-known, comparison with experiment requires:

$$c_2 = \frac{1}{16\pi G}$$

Newton's constant.

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⇒ Einstein equation:

(Notice: the "reduced Bianchi identity" $(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R)_{;\nu} = 0$ now is equivalent to the statement $T^{\mu\nu}{}_{;\nu} = 0$)

Notice: The symmetry of $R^{\mu\nu}$ (due to $g_{\mu\nu;\rho} = 0$) enforces the symmetry of $T^{\mu\nu}$.

$$\boxed{R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \underbrace{- 8\pi G c_1}_{= \Lambda \text{ "cosmol. constant"}} g^{\mu\nu} = 8\pi G T^{\mu\nu}}$$

What is Λ ?

Recall: The λ_i are constants

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What is Λ ?

Recall: The λ_i are constants prefactor of order one which depends on the matter Lagrangian.

□ First find ℓ_c:

(Recall: $c_i \sim \ell^2$)

Given that $\frac{1}{\quad} = c_i - \lambda_i \ell^{-2}$

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What is Λ ?

Recall: The λ_i are constants prefactor of order one which depends on the matter Lagrangian.

□ **First find ℓ_c:**

Given that $\frac{1}{16\pi G} = c_2 = \lambda_2 \ell_c^{-2}$

(I.e. also: $G \approx \ell_c^2$, which we'll need on the next slide)

undoing $\hbar = 1$ and $c = 1$ so that get kg, m, s:

we obtain: $\ell_c = \lambda_2 \sqrt{16\pi G} = \sqrt{16\pi \frac{\hbar G}{c}}$

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undoing $\hbar = 1$ and $c = 1$ so that get kg, m, s:

$= \lambda_2 \cdot 4\sqrt{\pi} \cdot \underbrace{1.616 \times 10^{-35} m}_{\text{the "Planck length"}}$

Recall: The λ_i are some numbers of order $O(1)$, depending on details which sort of particles are in the matter action, and their quantum effects

We note therefore: When quantum field theories are cut off at

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We note therefore: When quantum field theories are cut off at with the correct eqns of motion and coupling strength!

Now find c_1:

Given that $\ell_c \approx 10^{-35}$ m, quantum field theories generate a value of c_1, i.e., a cosmological constant of about:

$$c_1 = \underset{\text{of } \mathcal{O}(1)}{\lambda_1} \ell_c^{-4}$$

Finally, find Λ:

$$\Lambda_{theory} = -8\pi G c_1 = -8\pi G \lambda_1 \ell_c^{-4}$$

$$\approx G \ell_c^{-4} \approx \ell_c^{-2} \quad (\text{using } \lambda_1 \approx 1 \text{ and } G \approx \ell_c^2)$$

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$$\approx 10^{70}\, m^{-2}$$

This is the worst physical prediction ever:

Experiment:

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This is the worst physical prediction ever:

- **Experiment:**

  Based on cosmic microwave background data and on supernova brightness versus redshift data:

$$\Lambda_{experiment} \approx 10^{-52}\, m^{-2} \qquad \text{i.e. } \frac{\Lambda_{th}}{\Lambda_{exp}} \approx 10^{122}$$

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For some unknown reason, the constant part, c_1, of the vacuum energy of quantum field theories does essentially not gravitate – while its disturbance through curvature, $c_2 R$, is real: it induces regular gravity.