Title: General Relativity for Cosmology - Lecture 11

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Abstract:

Note Title

GR for Cosmology, Achim Kempf, Fall 2017, Lecture 12

Plan: I The dynamics of matter & radiation in curved spacetime

II Energy-momentum tensor

III The dynamics of spacetime itself.

1. **Recall:** On a (pseudo)-Riemannian mfld, equations are well-defined only if defined independently of any chart.

$\Rightarrow$ Any eqn, including the eqns of motions for matter fields must be eqns among tensors and their covariant derivatives.

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Plan: I The dynamics of matter & radiation in curved spacetime

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III The dynamics of spacetime itself.

1. Recall: On a (pseudo)-Riemannian mfld, equations are well-defined only if defined independently of any chart.

$\Rightarrow$ Any eqn, including the eqns of motions for matter fields must be eqns among tensors and their covariant derivatives.

$\Rightarrow$ Need a tensor field, Ψ, for each species of particle:

e^-, q, gluon, $\pi^{\pm}$, photon, $W^{\pm}$, etc...

Notation: $\Psi_{(i)}{}^{a \ldots b}{}_{c \ldots d}$

contravariant

covariant

species label

Note: any spinor equation can also be expressed as a (complicated) tensor equation (see e.g. Hawking & Ellis, p 59)

Question:

Could we have also an additional connection field $\tilde{\Gamma}^k_{ij}$?

Yes, we could: But, the difference field $Q^k{}_{ij} := \Gamma^k{}_{ij} - \tilde{\Gamma}^k{}_{ij}$ is actually a tensor field!

$$\Gamma^r{}_{ab} \longrightarrow \frac{\partial \bar{x}^r}{\partial x^k}\frac{\partial^2 x^j}{\partial \bar{x}^a \partial \bar{x}^b} + \frac{\partial \bar{x}^r}{\partial x^k}\frac{\partial x^i}{\partial \bar{x}^a}\frac{\partial x^j}{\partial \bar{x}^b}\Gamma^k{}_{ij}$$

$$\tilde{\Gamma}^r{}_{ab} \longrightarrow \frac{\partial \bar{x}^r}{\partial x^k}\frac{\partial^2 x^j}{\partial \bar{x}^a \partial \bar{x}^b} + \frac{\partial \bar{x}^r}{\partial x^k}\frac{\partial x^i}{\partial \bar{x}^a}\frac{\partial x^j}{\partial \bar{x}^b}\tilde{\Gamma}^k{}_{ij}$$

$$\Rightarrow \quad \left(\Gamma^r{}_{ab} - \tilde{\Gamma}^r{}_{ab}\right) \longrightarrow \frac{\partial \bar{x}^r}{\partial x^k}\frac{\partial x^i}{\partial \bar{x}^a}\frac{\partial x^j}{\partial \bar{x}^b}\left(\Gamma^k{}_{ij} - \tilde{\Gamma}^k{}_{ij}\right)$$

$$\Rightarrow \quad \boxed{Q^r{}_{ab} \longrightarrow \frac{\partial \bar{x}^r}{\partial x^k}\frac{\partial x^i}{\partial \bar{x}^a}\frac{\partial x^j}{\partial \bar{x}^b}\, Q^k{}_{ij}}$$

i.e. $Q^r{}_{ab}$ is a tensor due to having the correct transformation property according to the physicist's definition of a tensor.

$\Rightarrow$ Introducing an additional connection $\tilde{\Gamma}$ is same as introducing simply a new tensor field Q.

Remark: $\Longrightarrow$ "variations" $\delta\Gamma^r{}_{ab}$ will behave tensorially!

Eqns of motion of matter fields?

Action principle: (As in special relativity)

Any theory of matter fields can be defined by specifying the so-called Lagrangian function, L, namely a scalar function of the matter fields $\Psi_{(i)}{}^{a\ldots b}{}_{c\ldots d}$ and their first
← we'll sometimes omit the indices
covariant derivatives, and now also of the metric g:

$$\boxed{L(\Psi) = L^{(matter)}\left(\left\{\Psi_{(i)}{}^{a\ldots b}{}_{c\ldots d}\right\}, \left\{\Psi_{(i)}{}^{a..b}{}_{c..d;e}\right\}, g\right)}$$

Define the action functional:

$$S[\psi] := \int_B \underbrace{\overbrace{L(\psi)}^{\text{scalar}}\,\overbrace{\sqrt{g}\,d^4x}^{\Omega = \text{volume form}}}_{n\text{-form}} \in \mathbb{R}$$

B ← some bounded and closed 4-dim region in M.

Thus, each physical field $\psi(x,t)$ (as a function of both space and time) is mapped into a number $S[\psi]$.

Action principle (or postulate) of classical physics:

In nature, physical fields ψ are such that $S[\psi]$ is extremal in the space of all fields ψ.

- **Thus:** The matter fields ψ obey:

$$\boxed{\frac{\delta S[\psi]}{\delta \psi} = 0} \qquad (*)$$

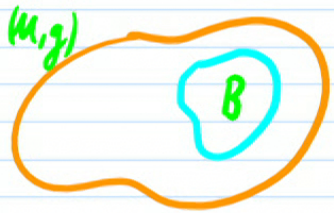
These will be the eqns of motion for the fields ψ.

- **Definition of $(*)$?**

Def: A "variation $\delta\psi$" of the fields $\psi_{(i)}(\overset{p \in B \subset M}{\overset{\downarrow}{p}})$ in a region B is a one-parameter deformation, $\psi_{(i)}(\lambda, p)$, with $\lambda \in \overbrace{(-\varepsilon, \varepsilon)}^{\text{some finite interval}}$,
$\uparrow$ deformation parameter

so that

i.e. $\lambda=0$ is non-deformation



1.) $\Psi_{(i)}(0,p) = \Psi_{(i)}(p) \quad \forall p \in M$

2.) $\Psi_{(i)}(\lambda,p) = \Psi_{(i)}(p) \quad \forall \lambda, \text{ if } p \in M - B$

i.e. no deformation at all outside region B.

Def: Then, we define:

$$\delta \Psi_{(i)}(p) := \left. \frac{\partial \Psi_{(i)}(\lambda,p)}{\partial \lambda} \right|_{\lambda=0}$$

Def: The action principle now reads:

$$\boxed{0 = \left. \frac{\partial S[\Psi]}{\partial \lambda} \right|_{\lambda=0} \quad \text{for all variations } \delta\Psi_{(i)}.}$$

Evaluate:

$$0 = \frac{\partial S}{\partial \lambda}\Big|_{\lambda=0} = \sum_i \int_B \Bigg[\overbrace{\frac{\partial L}{\partial \psi_{(i)}{}^{a\ldots b}{}_{c\ldots d}} \underbrace{\delta \psi_{(i)}{}^{a\ldots b}{}_{c\ldots d}}_{\text{recall: } = \frac{d\psi_{(i)}{}^{a\ldots b}{}_{c\ldots d}}{d\lambda}\big|_{\lambda=0}}}^{\text{Term I}}$$

$$+ \overbrace{\frac{\partial L}{\underbrace{\partial \psi_{(i)}{}^{a\ldots b}{}_{c\ldots d;e}}_{\substack{\text{by assumption,} \\ L \text{ depends also on} \\ \text{the 1st cov. derivatives.}}}} \delta\left(\psi_{(i)}{}^{a\ldots b}{}_{c\ldots d;e}\right)}^{\text{Term II}} \Bigg] \sqrt{g}\, d^4x$$

Evaluate terms I, II separately:

Term II:

We notice:

Recall: At origin of geodesic coordinate system, $\Gamma^k{}_{ij} = 0$, i.e. $\Psi_{;e} = \Psi_{,e}$. But then $\frac{\partial}{\partial x^i}$ and $\frac{\partial}{\partial \lambda}$ commute. True in any coordinate system.

$$\delta\left(\Psi_{(i)}{}^{a\ldots b}{}_{c\ldots d;e}\right) = \left(\delta\Psi_{(i)}{}^{a\ldots b}{}_{c\ldots d}\right)_{;e}$$

$$\Rightarrow \text{Term II} = \sum_i \int_B \frac{\partial L}{\partial \Psi_{(i)}{}^{a\ldots b}{}_{c\ldots d;e}} \left(\delta\Psi_{(i)}{}^{a\ldots b}{}_{c\ldots d}\right)_{;e} \sqrt{g}\, d^4x$$

$$= \sum_i \int_B \Bigg[\Bigg(\overbrace{\frac{\partial L}{\partial \Psi_{(i)}{}^{a\ldots b}{}_{c\ldots d;e}} \delta\Psi_{(i)}{}^{a\ldots b}{}_{c\ldots d}}^{=:K^e} \Bigg)_{;e}$$

(use Leibniz rule to verify)

$$- \left(\frac{\partial L}{\partial \Psi_{(i)}{}^{a\ldots b}{}_{c\ldots d;e}}\right)_{;e} \delta\Psi_{(i)}{}^{a\ldots b}{}_{c\ldots d} \Bigg] \sqrt{g}\, d^4x$$

One term is a "boundary term":

$$\sum_i \int_B K^e{}_{;e} \sqrt{g}\, d^4x$$

Exercise:

show that for all ξ^μ:

$$\xi^\mu{}_{;\mu} \Omega = \mathrm{div}_\Omega \xi$$

if $\Omega = \sqrt{g}\, dx^1 \wedge \ldots \wedge dx^n$

$$= \sum_i \int_B \mathrm{div}_\Omega K$$

Gauß' theorem $\Rightarrow$

$$= \sum_i \int_{\partial B} i_K \Omega$$

inner derivation

$$\left(\begin{aligned} \text{Recall: } \mathrm{div}_\Omega K &= L_K \Omega \\ &= (i_K \circ d + d \circ i_K)\, \Omega \\ &= d \circ i_K \Omega \end{aligned} \right)$$

but: $K \propto \delta\psi$ and $\delta\psi(p) = 0$ if $p \in \partial B$ by property 2) of variations.

$\Rightarrow \quad = 0 \quad !$

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inner derivation

(Recall: $\mathrm{div}_\Omega K = L_K \Omega$
$= (i_K \circ d + d \circ i_K)\, \Omega$
$= d \circ i_K \Omega$)

but: $K \propto \delta\psi$ and $\delta\psi(p) = 0$ if $p \in \partial B$
by property 2) of variations.

$\Rightarrow \quad = 0$!

Thus, term II simplifies and we obtain:

$$0 = \frac{\partial S}{\partial \lambda}\Big|_{\lambda=0} = \sum_i \int_B \Big[\overbrace{\frac{\partial L}{\partial \psi_{(i)}{}^{a\ldots b}{}_{c\ldots d}} \delta \psi_{(i)}{}^{a\ldots b}{}_{c\ldots d}}^{\text{Term I}} - \overbrace{\left(\frac{\partial L}{\partial \psi_{(i)}{}^{a\ldots b}{}_{c\ldots d;e}}\right)_{;e} \delta \psi_{(i)}{}^{a\ldots b}{}_{c\ldots d}}^{\text{Term II}} \Big] \sqrt{g}\, d^4x$$

Since must hold for all variations $\delta\psi$

$\Rightarrow$

$$\boxed{\frac{\partial L}{\partial \psi_{(i)}{}^{a\ldots b}{}_{c\ldots d}} - \left(\frac{\partial L}{\partial \psi_{(i)}{}^{a\ldots b}{}_{c\ldots d;e}}\right)_{;e} = 0}$$

"Euler-Lagrange equations"

Given $L(\psi)$, these eqns yield the eqns. of motion for ψ.

Example: A real-valued scalar field ψ ← real-valued

- Such ψ describe e.g.:
  - π^0 meson (quark + antiquark)
  - inflaton
- Lagrangian ?
  - Choose geodesic cds at arb. point and appeal to equiv. principle.
  - Obtain from spec. relat. Lagrangian:

$$L = -\frac{1}{2}\left(\psi_{;a}\,\psi_{;b}\,g^{ab} + \frac{m^2}{\hbar^2}\psi^2\right)$$

- Euler-Lagrange equation: Klein-Gordon equation

(Exercise: verify)

$$\boxed{\psi_{;ab}\,g^{ab} - \frac{m^2}{\hbar^2}\psi = 0}$$

▫ The Lagrangian (from equiv. principle):

$$L = -\frac{1}{16\pi} F_{ab} F_{cd}\, g^{ac} g^{bd}$$ (Exercise: write in terms of forms)

▫ Varying w. resp. to A, the E.L. equations read:

$$\boxed{F_{ab;c}\, g^{bc} = 0}$$ recall: this is $\delta F = 0$

"Maxwell eqns".

▫ It is also true that

$$F_{ab;c} + F_{ca;b} + F_{bc;a} = 0$$

but this is not an Euler Lagrange eqn. It is simply: $\boxed{dF = 0}$ (which holds because $F = dA$ and $d^2 = 0$)

<u>Example:</u> A charged scalar field ψ, ← complex-valued

(such ψ describe, e.g., $\pi^{\pm}$ mesons)

together with electromagnetism.

Equiv. principle yields from spec. relativity:

Why ψ complex?

Mixed term is Lorentz force

If ψ was real, it would be absent:

$-ieA_a\psi^{*}\psi_{;b}g^{ab}$

$+ieA_b\psi^{*}_{,a}\psi g^{ab}$

$= ieA_a g^{ab}(\psi^{*}_{,a}\psi - \psi_{,a}\psi^{*})$

$= 0$ if $\psi^{*} = \psi$

electric charge constant

$$L = -\frac{1}{2}\left(\psi^{*}_{;a} - ieA_a\psi^{*}\right)\left(\psi_{;b} + ieA_b\psi\right)g^{ab} - \frac{1}{2}\frac{m^2}{\hbar^2}\psi^{*}\psi - \frac{1}{16\pi}F_{ab}F_{cd}g^{ac}g^{bd}$$

- Vary w. resp. to ψ^* $\Rightarrow$ E.L. eqn:

$$\underbrace{\psi_{;ab}\, g^{ab} - \frac{m^2}{\hbar^2}\psi}_{\text{Klein Gordon part}} + \underbrace{ie A_a g^{ab}\left(\psi_{;b} + ie A_b \psi\right) + ie A_{a;b}\, g^{ab}\psi}_{\psi \text{ is affected by } A} = 0$$

and varying w. resp. to ψ yields the compl. conj. equation.

- Vary w. resp. to A_a $\Rightarrow$ E.L. eqn:

$$\underbrace{\frac{1}{4\pi} F_{ab;c}\, g^{bc}}_{\text{plain Maxwell part}} \underbrace{- ie\psi\left(\psi^*_{;a} - ie A_a \psi^*\right) + ie\psi^*\left(\psi_{;a} + ie A_a \psi\right)}_{A \text{ is affected by } \psi, \psi^*.} = 0$$

Dirac equation: (Brief treatment of basics only of Dirac spinors)

In special relativity: (with units such that $\hbar = 1$)

$$\boxed{\left(i\,\gamma^\mu \frac{\partial}{\partial x^\mu} - m\right)\psi(x) = 0}$$

"Dirac equation" (D)

where $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$ is a "Spinor"

describes spin ½ particles such as electrons and quarks

and the four 4×4 matrices γ^μ obey:

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu} \quad (*)$$

$\eta^{\mu\nu} = \begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}^{\mu\nu}$

□ Why (*)? Equation (*) is specifically chosen so that each component of Ψ obeys the Klein Gordon equation. Indeed:

$$(D) \Rightarrow \left(-i\gamma^\mu \partial_\mu - m\right)\left(i\gamma^\nu \partial_\nu - m\right)\Psi = 0$$

$$\Rightarrow \left(+\gamma^\mu \gamma^\nu \partial_\mu \partial_\nu + i\gamma^\mu \partial_\mu m - im\gamma^\nu \partial_\nu + m^2\right)\Psi = 0$$

symmetric under $\mu \leftrightarrow \nu$

$$\Rightarrow \left(\gamma^\mu \gamma^\nu \partial_\mu \partial_\nu + m^2\right)\Psi = 0$$

anti-symmetric part not needed, it would drop out.

$$\Rightarrow \left(\frac{1}{2}\left(\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu\right)\partial_\mu \partial_\nu + m^2\right)\Psi = 0$$

$$\overset{(*)}{\Rightarrow} \quad 1\left(\eta^{\mu\nu}\partial_\mu \partial_\nu + m^2\right)\Psi = 0$$

which is the Klein Gordon equation in flat space.

In general relativity:

- By choosing an orthonormal tetrad, $\{\theta^i\}$, we achieve

$$g^{\mu\nu} = \eta^{\mu\nu} \quad \forall p \in M$$

i.e. one set of matrices γ^μ obeying $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}$ suffices.

- This motivates:

$$(i\gamma^\mu \nabla_{e_\mu} - m)\psi = 0$$

- **But what is the covariant derivative of a spinor?**

$$\nabla_{e_\mu}\psi = ?$$

Recall: The covariant derivative of a vector yields the infinitesimal Lorentz transformation by which the vector rotates under infinitesimal parallel transport.

Idea: The covariant derivative of a spinor should yield the rotation of the spinor by the same infinitesimal Lorentz transformation.

Recall: Infinitesimal parallel transport of a vector e_ρ in direction e_μ:

$$e_\rho \to e_\rho + \nabla_{e_\mu} e_\rho = e_\rho + \omega^\sigma_\rho(e_\mu)\, e_\sigma$$

Recall: the curvature 1-form takes values that are infinitesimal Lorentz transformations.

This is an infinitesimal Lorentz transformation $\Lambda_\rho{}^\sigma$:

$$e_\rho \to \Lambda_\rho^\sigma e_\sigma \quad \text{with} \quad \Lambda^\sigma_\rho = \delta^\sigma_\rho + \omega^\sigma_\rho(e_\mu)$$

because ω^σ_ρ obeys: $\omega_{\sigma\rho} = -\omega_{\rho\sigma}$. (Which is the defining equation for infinitesimal Lorentz transformations)

(Recall intuition why parallel transport yields Lorentz transformation: Parallel transport preserves the lengths of vectors, i.e. they can at most "rotate" and in 3+1 dim. this is Lorentz transformations.)

Now that we know the inf. Lorentz transf. for any inf. parallel transport:

⇝ <u>Strategy:</u> Apply the same inf. Lorentz transformation on spinors for their parallel transport.

<u>To this end:</u> Recall from Special Relativity how an infinitesimal Lorentz transformation acts on a spinor:

- Assume $\{s_i\}_{i=1}^{4}$ are ON basis in Spinor space, i.e.

$$\psi = \psi^i(x)\, s_i$$

these are Spinor indices: $i = 1,2,3,4$

- How do the s_i transform under Lorentz transformations? I.e., what is $\nabla_{e_i} s_j = ?$ (In analogy to $\nabla_{e_a} e_\mu = \omega^\nu{}_\mu(e_a)\, e_\nu$)

From special relativity it is known that under infinitesimal Lorentz transformations,

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu$$

vectors transform as

$$e_\mu \to e_\mu + \omega_\mu{}^\nu e_\nu$$

and the Dirac spinors transform as:

$$\psi_i \to \psi_i \underbrace{- \frac{1}{4}\omega_\mu{}^\nu [\gamma^\mu, \gamma_\nu] \psi_i}$$

$\Rightarrow$ under infinitesimal Lorentz transf. the spinor "rotates" by this amount.

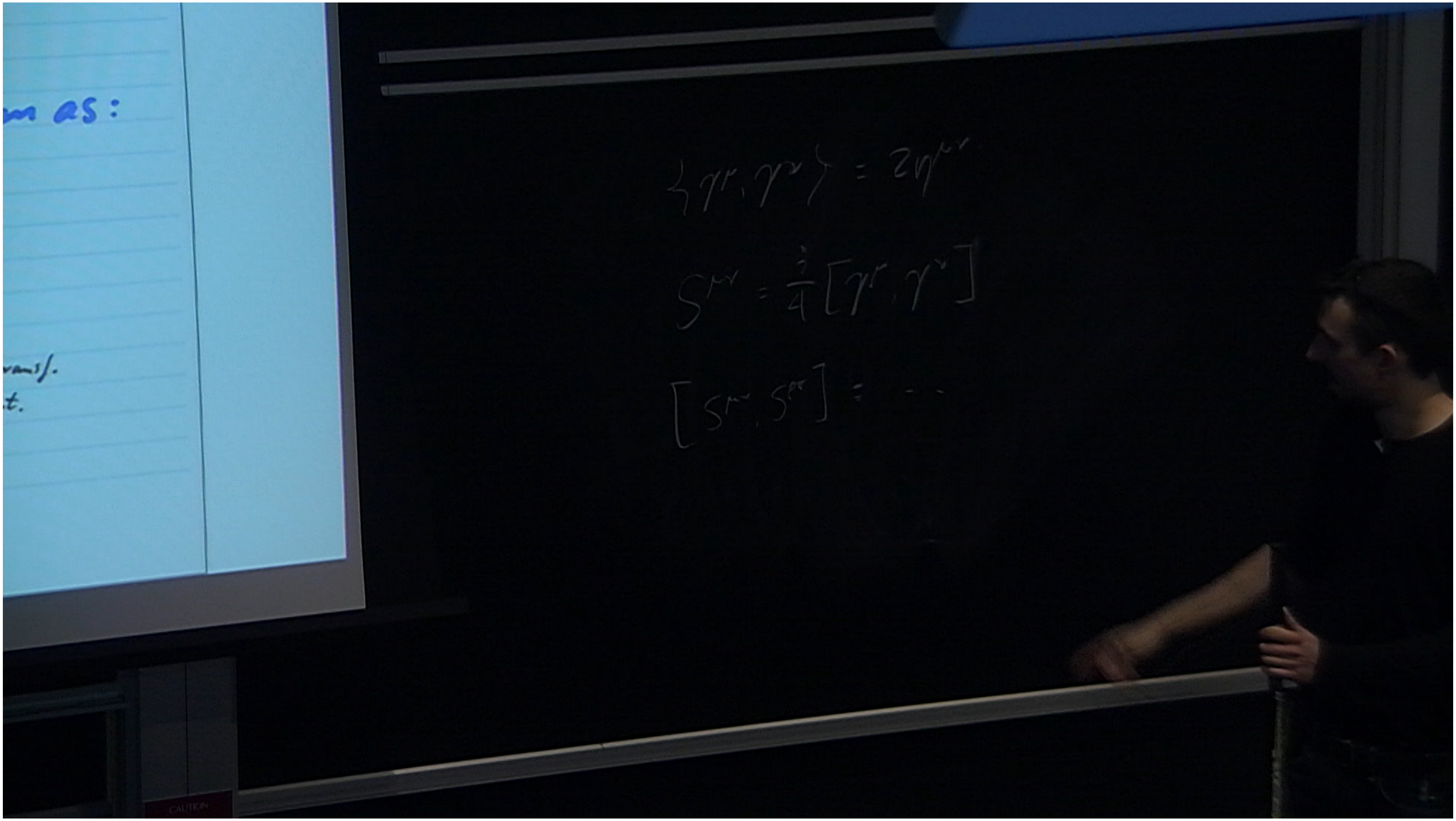
Where does $[\gamma^\mu, \gamma^\nu]$ come from?

Recall that e.g. translations in space are generated by momentum operators, $e^{-i\vec{a}\vec{p}} f(x) e^{i\vec{a}\vec{p}} = f(x+\vec{a})$, if they obey the commutation relations $[x_i, p_j] = i\delta_{ij}$.

Similarly, Lorentz transformations are generated by operators $M^{\mu\nu}$: antisym. antisym.

$$e^{-i\omega_{\mu\nu}M^{\mu\nu}} f \; e^{i\omega_{\mu\nu}M^{\mu\nu}} = \Lambda(f)$$

if these $M^{\mu\nu}$ obey certain commutation relations. In spinor space, the unique objects that obey these commutation relations are the $M^{\mu\nu} = [\gamma^\mu, \gamma^\nu]$.


m as:
$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$
$S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$
$[S^{\mu\nu}, S^{\rho\sigma}] = \ldots$

Apply to GR:

If a vector e_μ is infinitesimally parallel transported in the direction of e_a then it obtains an infinitesimal "rotation", namely the infinitesimal Lorentz transformation

$$\omega^\mu{}_\nu(e_a)$$

which is the value of the connection 1-form, i.e.:

$$e_\mu \to e_\mu + \overbrace{\omega_\mu^\nu(e_a)}^{\text{local value of the connection form}} e_\nu$$

→ From this one can immediately read off again the covariant derivative for vectors:

$$\nabla_{e_a} e_\mu = \omega^\nu{}_\mu(e_a) e_\nu$$

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□ Now, when a spinor s_i is infinitesimally parallel transportet in the direction of e_a then it too experiences the infinitesimal rotation, i.e., the infinitesimal Lorentz transformation

$$\omega^\mu{}_\nu(e_a)$$

which is the value of the connection 1 form. Thus:

$$s_i \to s_i - \frac{1}{4}\overbrace{\omega(e_a)_\mu{}^\nu}^{\substack{\text{local infinitesimal Lorentz transformation,}\\ \text{i.e., local value of the connection 1-form.}}}[\gamma^\mu, \gamma_\nu]\, s_i$$

□ Since, under infinitesimal parallel transport:

$$s_i \to s_i + \underbrace{\nabla_{e_a}}_{\text{to be determined}} s_i$$

⇒ The covariant derivative of the basis vectors $\{s_i\}$ of Dirac spinors is:

$$\nabla_{e_a} s_i = -\frac{1}{4}\,\omega_\mu{}^\nu(e_a)\,[\gamma^\mu,\gamma_\nu]\,s_i$$

⇒ For general Dirac spinors $\Psi(x) = \Psi^i(x)\,s_i$

the Leibniz rule for ∇ yields:

scalar coefficient functions

$$\nabla_{e_a}\Psi = \nabla_{e_a}\left(\Psi^i(x)\,s_i\right) = \left(\nabla_{e_a}\Psi^i(x)\right)s_i + \Psi^i(x)\,\nabla_{e_a}s_i$$

i.e.: $$\boxed{\nabla_{e_a}\Psi = e_a(\Psi) - \frac{1}{4}\,\omega(e_a)_\mu{}^\nu\,[\gamma^\mu,\gamma_\nu]\,\Psi}$$

function

$e_a(\Psi) = s_i\, e_a(\Psi^i)$

vector field

Dirac equation:

The general relativistic Dirac equation

$$\boxed{\left(i\gamma^\mu \nabla_{e_\mu} - m\right)\psi = 0}$$

now takes this explicit form:

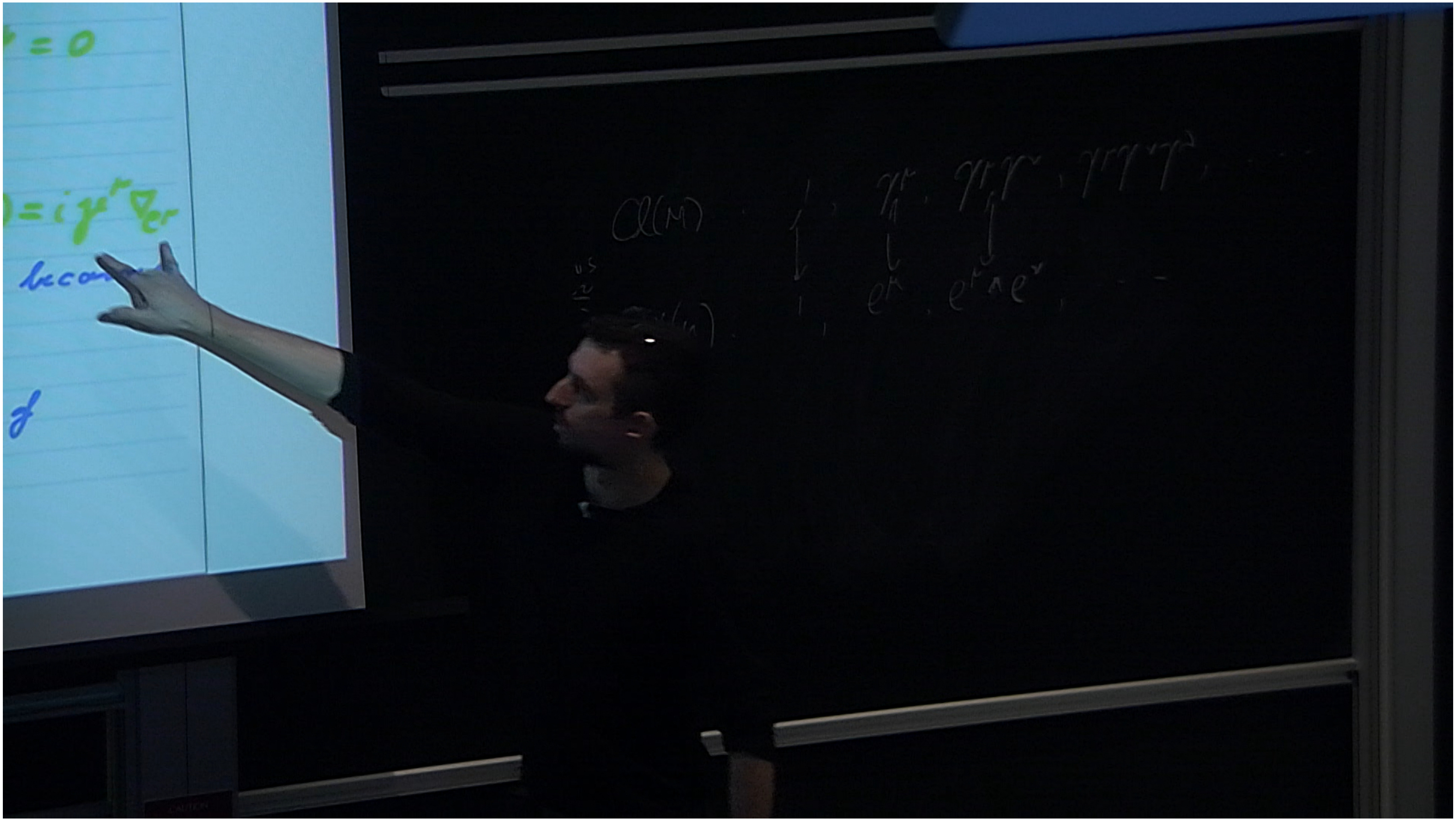
$$i\gamma^\mu e_\mu(\psi) - i\frac{1}{4}\omega(e_\mu)^\nu{}_\rho\, \gamma^\mu [\gamma^\rho, \gamma_\nu]\psi - m\psi = 0$$

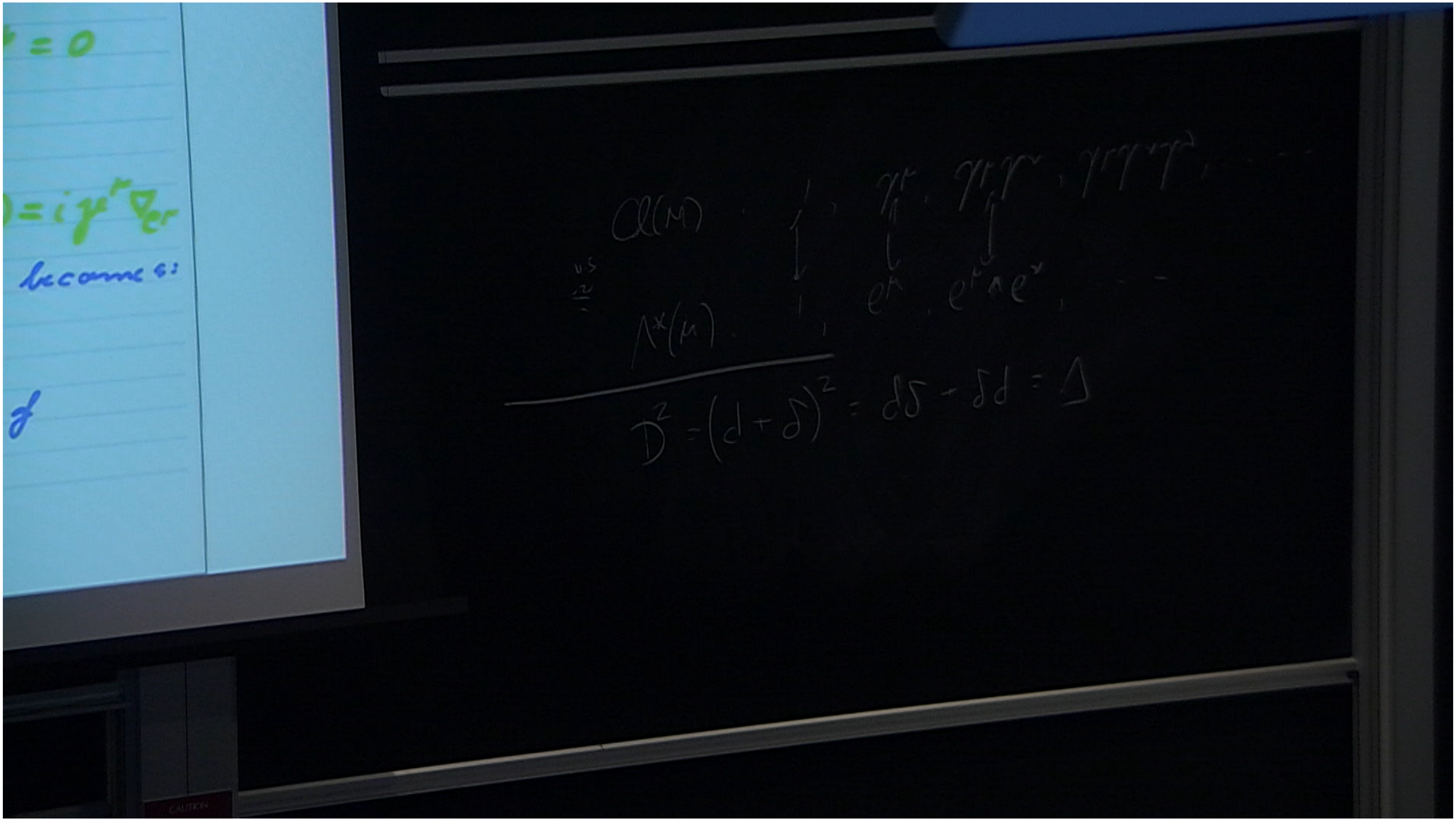
in a chart, this becomes a directional derivative of ψ.

Remark: The relationship between the Dirac operator $D = i\gamma^\mu \nabla_{e_\mu}$ and the Laplace or d'Alembert operator $\Box$ also becomes:

$$D = d + \delta\,.$$

To this end, one re-interprets the Grassmann algebra of differential forms as a so-called Clifford algebra.




$= 0$
$= i\gamma^\mu \nabla_{e_\mu}$
becomes:
$Cl(M)$: $1, \gamma^\mu, \gamma^\mu\gamma^\nu, \gamma^\mu\gamma^\nu\gamma^\rho, \ldots$
$\Lambda^*(M)$: $1, e^\mu, e^\mu \wedge e^\nu, \ldots$
$D^2 = (d+\delta)^2 = d\delta + \delta d = \Delta$
CAUTION

Equiv. ppl.

("heuristic")

$$\eta_{\mu\nu} \longrightarrow g_{\mu\nu}$$

$$\partial_\mu \longrightarrow \nabla_\mu$$

"minimal coupling"

$$S_{KG}(\phi, \phi_{;\mu}, g) + \int \sqrt{|g|}\, \frac{1}{2} \xi R \phi^2$$

"non-minimal coupling"