Title: General Relativity for Cosmology - Lecture 9

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Abstract:

GR for Cosmology, Achim Kempf, Fall 2017, Lecture 10

Note Title

Recall:

- The curvature map, R, is defined through:

$$R:\ \xi_1,\xi_2,\xi_3 \rightarrow R(\xi_1,\xi_2)\xi_3 = \underbrace{\left(\nabla_{\xi_1}\nabla_{\xi_2} - \nabla_{\xi_2}\nabla_{\xi_1} - \nabla_{[\xi_1,\xi_2]}\right)}_{\overset{\shortparallel}{R(\xi_1,\xi_2)}}\xi_3$$

So, "R" can stand for the tensor, the map and this R!

- 1st Bianchi Identity:

$$\sum_{\text{cyclic}} R(\xi,\eta)\nu = \sum_{\text{cyclic}}\Big(T(T(\xi,\eta),\nu) + (\nabla_\xi T)(\eta,\nu)\Big)$$

- 2nd Bianchi Identity:

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☐ **2nd Bianchi Identity:**

$$\sum \Big((\nabla_\xi R)(\eta,\nu)+R(T(\xi,\eta),\nu)\Big)=0$$

In a chart? (Assuming no torsion, and using $\frac{\partial}{\partial x^i}$, dx^i bases)

1st Bianchi: $$\sum_{(jkl)} R^i{}_{jkl} = 0$$

↑ cyclic sum

2nd Bianchi: $$\sum_{(klm)} R^i{}_{jkl;m} = 0$$

↑ cyclic sum

Other useful properties:

(Note: This antisymmetry will be useful because it allows one to view R as a 2 form, which is (1,1) tensor-valued)

☐ $R^i{}_{jkl} = - R^i{}_{jlk}$

$$R:\ \xi_1,\xi_2,\xi_3 \rightarrow R(\xi_1,\xi_2)\xi_3 = \underbrace{\left(\nabla_{\xi_1}\nabla_{\xi_2} - \nabla_{\xi_2}\nabla_{\xi_1} - \nabla_{[\xi_1,\xi_2]}\right)}_{=\,R(\xi_1,\xi_2)}\xi_3$$

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$\uparrow$ cyclic sum

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Other useful properties:

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- ☐ $R^{i}{}_{jk\ell} = -R^{i}{}_{j\ell k} \longleftarrow$
- ☐ $R_{ijk\ell} = -R_{jik\ell}$
- ☐ $R_{ijk\ell} = R_{k\ell ij}$

$$\left(\begin{array}{l} \langle R(\xi,\eta)v,\zeta\rangle = \langle R(\xi,\eta)\zeta, v\rangle \\ \langle R(\xi,\eta)v,\zeta\rangle = -\langle R(v,\zeta)\xi,\eta\rangle \end{array} \right)$$

Contractions of R:

The Ricci Tensor: $R_{jl} := R^{i}{}_{jil}$

⇒ clearly: $R_{jl}\, dx^{j} \otimes dx^{l} \in T_p(M)_2$

The Curvature Scalar: $R := g^{jl} R_{jl}$

Then, 2nd Bianchi identity implies:

$$\left(R_{i}{}^{k} - \frac{1}{2}\delta_{i}^{k} R\right)_{;k} = 0$$

⇒ The so-called "Einstein tensor" $G_{i}{}^{k} := R_{i}{}^{k} - \frac{1}{2}\delta_{i}^{k} R$ obeys:

$G^{K} \qquad = 0$ (this property was crucial)

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$$G_i^{\ k}{}_{;k} = 0$$

(this property was crucial guidance for Einstein, as we will see)

Recall strategy:

☐ Specified g $\Rightarrow$ specified distances in $\mathcal{M}$

$\Rightarrow$ implicitly specified "shape" of $\mathcal{M}$

Then, alternatively:

☐ Specified ∇ $\Rightarrow$ specified parallel transport in $\mathcal{M}$

$\Rightarrow$ specified "shape" of $\mathcal{M}$, namely:

∇ specifies Torsion T and Curvature R.

Now assume a manifold is specified by giving a metric g.

There ought to exist a ∇ which describes the same manifold.

Idea: The parallel transport of vectors η, ν must be such that their inner product (i.e. their lengths and relative angles) stays constant:

Consider any path γ and any two vector fields η, ν that are $\underbrace{\text{parallel transported}}$ along γ, i.e., for which:

(i.e., autoparallel to γ)

$$\nabla_{\dot\gamma}\, \eta(\gamma(t)) = 0, \quad \nabla_{\dot\gamma}\, \nu(\gamma(t)) = 0 \quad \text{for all } t.$$

Then, require: $$\frac{d}{dt}\left(g(\gamma(t))_{bc}\, \eta^b(\gamma(t))\, \nu^c(\gamma(t))\right) = 0$$

$$= \nabla_{\dot\gamma}\langle g, \eta\otimes\nu\rangle =$$

by ∇ obeying Leibniz rule

because $\nabla_{\dot\gamma}\eta = 0$

because $\nabla_{\dot\gamma}\nu = 0$

their inner product (i.e. their lengths and relative angles) stays constant:

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(i.e., autoparallel to γ) →

$$\nabla_{\dot\gamma}\,\eta(\gamma(t)) = 0, \quad \nabla_{\dot\gamma}\,v(\gamma(t)) = 0 \quad \text{for all } t.$$

Then, require: $\frac{d}{dt}\left(g(\gamma(t))_{bc}\ \eta^b(\gamma(t))\ v^c(\gamma(t))\right) = 0$

$$\text{i.e.:}\quad 0 = \overbrace{\dot\gamma^a\left(g_{bc}\,\eta^b v^c\right)_{;a}}^{\nabla_{\dot\gamma}\langle g,\,\eta\otimes v\rangle} \overset{\text{by } \nabla \text{ obeying Leibniz rule}}{=} \dot\gamma^a\left(g_{bc;a}\,\eta^b v^c + g_{bc}\,\overbrace{\eta^b{}_{;a}}^{0 \text{ because } \nabla_{\dot\gamma}\eta = 0} v^c + g_{bc}\,\eta^b\,\overbrace{v^c{}_{;a}}^{0 \text{ because } \nabla_{\dot\gamma} v = 0}\right)$$

$\Rightarrow$ $0 = g_{bc;a}\,\dot{x}^a \eta^b v^c$ for all arbitrary $\dot{x}, \eta, v$!

$\Rightarrow$ Compatibility of ∇ with g means:

$$\boxed{\nabla_\xi g = 0 \quad \text{for all } \xi}$$

Is there a ∇ for each choice of g? Indeed:

Fund. theorem of (pseudo) Riemannian geometry:

For each (pseudo) Riemannian manifold (M, g) there exists a unique ∇ that is torsionless and compatible with g, i.e., which obeys $\nabla g = 0$, the Levi-Civita connection.

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More generally: $\forall (M, g)$ and a tensor field T with $T_{ij}^{k} = -T_{ji}^{k}$: there is a metric-preserving ∇ whose torsion is T.

In a chart: How to obtain the Levi-Civita ∇ from g?

$\nabla g = 0$ means $$g_{\mu\nu,\alpha} - g_{\mu\beta}\Gamma^{\beta}{}_{\nu\alpha} - g_{\beta\nu}\Gamma^{\beta}{}_{\mu\alpha} = 0 \qquad \underline{I}$$

i.e. $$g_{\alpha\mu,\nu} - g_{\alpha\beta}\Gamma^{\beta}{}_{\mu\nu} - g_{\beta\mu}\Gamma^{\beta}{}_{\alpha\nu} = 0 \qquad \underline{II}$$

and $$g_{\nu\alpha,\mu} - g_{\nu\beta}\Gamma^{\beta}{}_{\alpha\mu} - g_{\beta\alpha}\Gamma^{\beta}{}_{\nu\mu} = 0 \qquad \underline{III}$$

take: $\frac{1}{2}\left(-\underline{I} + \underline{II} + \underline{III}\right)$

$$\Rightarrow \frac{1}{2}\left(g_{\alpha\mu,\nu} + g_{\nu\alpha,\mu} - g_{\mu\nu,\alpha}\right) = g_{\alpha\beta}\Gamma^{\beta}{}_{\nu\mu}$$

Thus: $$\Gamma^{\beta}{}_{\nu\mu} = \frac{1}{2} g^{\alpha\beta}\left(g_{\alpha\mu,\nu} + g_{\nu\alpha,\mu} - g_{\mu\nu,\alpha}\right)$$

i.e. $g_{\alpha\mu,\nu} - g_{\alpha\beta}\Gamma^{\beta}{}_{\mu\nu} - g_{\beta\mu}\Gamma^{\beta}{}_{\alpha\nu} = 0$ $\underline{II}$

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Thus: $$\Gamma^{\beta}{}_{\nu\mu} = \frac{1}{2} g^{\alpha\beta}\left(g_{\alpha\mu,\nu} + g_{\nu\alpha,\mu} - g_{\mu\nu,\alpha}\right)$$

↑ "Levi-Civita" connection or also called "Riemannian" connection.

Upgrade the math:

- Make use of arbitrary bases e_i, θ^i in (co-) tangent spaces: frames

- Allow forms to be tensor-valued: obtain, e.g., torsion and curvature forms. Also: connection forms.

$\Rightarrow$ We will obtain powerful, simple equations that relate ∇, g, R, T. (Even the Bianchi identities will look simple)

Now: Assume again that ∇ and g are still unrelated and $T \neq 0$.
(possible)

"Moving frames":

Def: A "moving frame" is a set, $\{e_i\}_{i=1}^{n}$, of contravariant vector fields e_i which, together, at each point $p \in M$ form a basis of $T_p(M)$'.

Def: We denote the dual basis $\{\theta^i\}_{i=1}^{n}$.

It obeys: $\theta^i(e_j) = \delta^i_j$.

german: 4 legs.

Def: For $n=4$ it may be called vierbein or tetrad.

Def: A **moving frame** is a set, $\{e_i\}_{i=1}$, of contravariant vector fields e_i which, together, at each point $p \in M$ form a basis of $T_p(M)'$.

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Def: For $n=4$ it may be called $\overbrace{\textbf{vierbein}}^{\text{german: 4 legs.}}$ or **tetrad.**
(in arb. dimensions: "vielbein" = many legs)

Coefficients:

- Torsion: $T^i{}_{kl} := \langle \theta^i, T(e_k, e_l) \rangle$
- Curvature: $R^i{}_{jkl} := \langle \theta^i, R(e_k, e_l) e_j \rangle$
- Metric: $g_{ik} := g(e_i, e_k) = \langle e_i, e_k \rangle$
- Christoffel: $\Gamma^i{}_{kj} e_i := \nabla_{e_k} e_j$

Consider arbitrary change of frame:

(has nothing to do with a change of chart!)

- assume $\bar{\theta}^i(x) = A^i{}_j(x)\, \theta^j(x)$
- then: $\bar{e}_i(x) = (A^{-1})_i{}^j(x)\, e_j(x)$

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(because we chose bases that are dual: $\bar{\theta}^i(\bar{e}_j) = \delta^i{}_j$)

Another step towards more abstract formulation:

Tensor-valued p-forms:

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$$\phi : \underbrace{T_q(M)' \times \ldots \times T_q'(M)}_{p \text{ factors}} \longrightarrow T_q(M)^r_s$$

Def: The p-forms $\phi^{i_1 \ldots i_r}_{j_1 \ldots j_s} := \phi(\theta^{i_1}, \ldots \theta^{i_r}; e_{i_1}, \ldots, e_{i_s})$ are called the component p-forms relative to the basis $\{e_i\}_{i=1}^n$.

Special cases:

☐ (r,s) tensors are (r,s) tensor-valued 0-forms.

☐ p-forms are $(0,0)$ tensor-valued forms.

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Torsion 2-form:

- We recall that $T(\xi,\eta) = -T(\eta,\xi) \Rightarrow$ can define the torsion's (1,0) tensor-valued 2-form through its action on 2 vector fields ξ, η:

"torsion 2-form" $\rightarrow$

$$\underbrace{\Theta^i(\xi,\eta)\,e_i}_{\substack{\text{the 2 form } \Theta^i \\ \text{fed 2 vectors to} \\ \text{yield a vector}}} := T(\xi,\eta)$$

- Given a frame:

using their antisymmetry

$$\Theta^i = \frac{1}{2} T^i{}_{kl}\, \theta^k \wedge \theta^l$$

- We know that $J(\xi,\eta) = -J(\eta,\xi) \Rightarrow$ can define the torsion's (1,0) tensor-valued 2-form through its action on 2 vector fields ξ,η:

"torsion 2-form" $\longrightarrow$

$$\underbrace{\Theta^i(\xi,\eta)\,e_i}_{\text{the 2 form } \Theta^i \text{ fed 2 vectors to yield a vector}} := J(\xi,\eta)$$

- Given a frame:

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$$\Theta^i = \frac{1}{2} J^i{}_{kl}\,\theta^k \wedge \theta^l$$

Curvature 2-form:

(recall that in canonical basis: $R^i{}_{jkl} = -R^i{}_{jlk}$)

- We recall that also $R(\xi,\eta) = -R(\eta,\xi)$

$\Rightarrow$ can define curvature's (1,1) tensor-valued 2-form:

$$\text{"curvature 2-form"} \longrightarrow \underbrace{\overbrace{\Omega^i{}_j(\xi,\eta)}^{\text{numbers}}e_i}_{\text{tangent vector}} := \underbrace{R(\xi,\eta)e_j}_{\text{tangent vector}}$$

Recall: $R: \xi\,\eta\,e_j \rightarrow \nabla_\xi \nabla_\eta e_j - \nabla_\eta \nabla_\xi e_j - \nabla_{[\eta,\xi]} e_j$

- Given a frame $\{\Theta^i\}_{i=1}^{n}$:

$$\left(R^i{}_{jkl} = - R^i{}_{jlk} \right) \swarrow$$

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Recall: $R: \xi\, \eta\, e_j \to \nabla_\xi \nabla_\eta e_j - \nabla_\eta \nabla_\xi e_j - \nabla_{[\eta,\xi]} e_j$

☐ Given a frame $\{\theta^i\}_{i=1}^n$:

$$\Omega^i{}_j = \frac{1}{2} R^i{}_{jkl}\, \theta^k \wedge \theta^l$$

The connection as a form?

- Nontrivial because:

  1. Christoffels $\Gamma^i{}_{kj} e_i := \nabla_{e_k} e_j$ are not tensors to start with!

  2. $\Gamma^i{}_{kj}$ is not anti-sym. in any indices, so can't be a 2-form (but can be 1-form):

- Define the connection 1-forms $\omega^i{}_j$: $\boxed{\omega^i{}_j := \Gamma^i{}_{kj}\,\theta^k}$

Thus: $\nabla e_j = \overbrace{\omega^i{}_j(e)}^{\text{scalars}} e_i$

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Thus:

$$\boxed{\underbrace{\nabla_\xi e_j}_{\text{vector}} = \underbrace{\overbrace{\omega^i{}_j(\xi)}^{\text{scalars}}\, e_i}_{\text{vector}}}$$

$\left(\text{because } \nabla_{\xi^k e_k} e_j = \xi^k \nabla_{e_k} e_j\right)$

Connection 1-forms are non-tensorial:

Proposition: Under change of frame $\bar{\theta}^i(x) = A^i{}_j(x)\,\theta^j(x)$ the transformation is:

functions   function   matrix inverse.

$$\underbrace{\bar{\omega}^a{}_b}_{\text{1-form}} = \underbrace{A^a{}_i \overbrace{\omega^i{}_j}^{\text{1 form}} A^{-1j}{}_b}_{\text{1-form}} - \underbrace{\underbrace{(dA)^a{}_i}_{\text{1 form}} \underbrace{\overbrace{(A^{-1})}^{}{}^i{}_b}_{\text{functions}}}_{\text{1-form}}$$

Proof:

$$-\bar{\omega}(\xi)^a{}_b\,\bar{\theta}^b = \nabla_\xi \bar{\theta}^a = \nabla_\xi (A^a_b \theta^b) \overset{\text{Leibniz rule}}{=} \Big(\overbrace{dA^a_b(\xi)}^{= \xi(A^a{}_b)}\Big)\theta^b + A^a_b \nabla_\xi \theta^b$$

$$= dA^a{}_b(\xi)\,\theta^b - A^a{}_b\,\omega(\xi)^b{}_c\,\theta^c$$

$$= dA^b{}_b(\xi)\,A^{-1b}{}_c\,\bar{\theta}^c - A^a{}_b\,\omega(\xi)^b{}_c\,A^{-1c}{}_d\,\bar{\theta}^d$$

the transformation is:

$$\underbrace{\bar{\omega}^a{}_b}_{\text{1-form}} = \underbrace{\overset{\text{functions}}{A^a{}_i}\,\overset{\text{1 form}}{\omega^i{}_j}\,\overset{\text{functions}}{A^{-1j}{}_b}}_{\text{1-form}} - \underbrace{\underbrace{\overset{\text{function}}{(dA)^a{}_i}}_{\text{1 form}}\,\underbrace{\overset{\text{matrix inverse}}{(A^{-1})^i{}_b}}_{\text{functions}}}_{\text{1-form}}$$

Proof:

$$-\bar{\omega}(\xi)^a{}_b\,\bar{\theta}^b = \nabla_\xi\bar{\theta}^a = \nabla_\xi(A^a{}_b\theta^b) \overset{\text{Leibniz rule}}{=} \Big(\overset{=\xi(A^a{}_b)}{dA^a{}_b(\xi)}\Big)\theta^b + A^a{}_b\nabla_\xi\theta^b$$

$$= dA^a{}_b(\xi)\,\theta^b - A^a{}_b\,\omega(\xi)^b{}_c\,\theta^c$$

$$= dA^b{}_b(\xi)\,A^{-1b}{}_c\,\bar{\theta}^c - A^a{}_b\,\omega(\xi)^b{}_c\,A^{-1c}{}_d\,\bar{\theta}^d$$

✓

true for all $\bar{\theta}$ ⇒ proposition above.

The "absolute exterior differential" D:

(It generalizes both ∇ and d)

☐ **Proposition:** (proof, see e.g. Straumann: check tensorial behavior under frame change)

For every (r,s) tensor-valued p-form ϕ there exists a unique (r,s) tensor-valued $(p+1)$ form $D\phi$ whose components relative to $\{\theta^i\}$ are:

$$(D\phi)^{i_1\dots i_r}_{j_1\dots j_s} = d\underbrace{\phi^{i_1\dots i_r}_{j_1\dots j_s}}_{p\text{-form}} + \overbrace{\overbrace{\omega^{i_1}{}_l}^{1\text{-form}} \wedge \overbrace{\phi^{l i_2\dots i_r}_{j_1\dots j_s}}^{p\text{-form}}}^{(p+1)\text{-form}} + \dots \qquad (*)$$

(It generalizes both ∇ and d)

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$$- \omega^{\ell}{}_{j_1}\wedge\phi^{i_1,\dots,i_r}_{\ell,\dots,j_s} - \dots \qquad (*)$$

Proposition: D is an anti-derivation:

degree of ϕ

$$D(\phi\wedge\psi) = D\phi\wedge\psi + (-1)^{p}\,\phi\wedge D\psi$$

Special cases:

- An ordinary p-form is $(0,0)$ tensor-valued. In this case, clearly:

$$D = d$$

- An ordinary tensor field is a tensor-valued 0-form. In this case:

$$D = \nabla$$

Exercise: Verify

Hint: Choose frame $\theta^i = dx^i$, use $\omega^i{}_j = \Gamma^i{}_{kj}\theta^k$,

then show (✱) implies indeed:

(It generalizes both ∇ and d)

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$$- \omega^{\ell}{}_{j_1} \wedge \phi^{i_1, \dots, i_r}_{\ell, \dots, j_s} - \dots \qquad (*)$$

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Exercise: Verify

Hint: Choose frame $\theta^i = dx^i$, use $\omega^i{}_j = \Gamma^i{}_{kj}\theta^k$, then show (✗) implies indeed:

$$\phi^{i_1 \ldots i_r}{}_{j_1 \ldots j_s ; k} = \phi^{i_1 \ldots i_r}{}_{j_1 \ldots j_s , k} + \Gamma^{i_1}{}_{kl}\,\phi^{l i_2 \ldots i_r}{}_{j_1 \ldots j_s} + \ldots - \Gamma^{l}{}_{k j_1}\,\phi^{i_1 \ldots i_r}{}_{l j_2 \ldots j_s} - \ldots$$

How are ω, g, Θ, Ω related now?

Proposition: (Exercise: check)

An affine connection ∇ is metric, if and only if $Dg = 0$, i.e., iff:

$$\boxed{\underbrace{dg_{ik}}_{(0,2)\text{ tensor-valued 1-form}} - \omega_{ik} - \omega_{ki} = 0}$$

They express torsion and curvature in terms of the connection

Theorem: "The Cartan structure equations"

In special case of frame $\theta^i = dx^i$:

$$J^i_{\ kj} = \Gamma^i_{\ kj} - \Gamma^i_{\ jk}$$

= 0 for metric connection

Torsion $\Theta = \Theta^i e_i$ is (1,0) tensor-valued 2-form

1.) $\Theta^i = d\theta^i + \omega^i_{\ j} \wedge \theta^j$ i.e. $\Theta^i = D\theta^i$

(The frame, $\theta = \theta^i e_i$, is a $\underbrace{(1,0)\text{ tensor}}_{\text{notice the upper index}}$-valued 1-form.)

Proposition: (Exercise: check)

An affine connection ∇ is metric,
if and only if $Dg = 0$, i.e., iff:

$$\boxed{\underbrace{dg_{ik}}_{(0,2)\text{ tensor-valued 1-form}} - \omega_{ik} - \omega_{ki} = 0}$$

They express torsion and curvature in terms of the connection

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(The frame, $\theta = \theta^i e_i$, is a $\underbrace{\text{(1,0) tensor-valued}}_{\text{notice the upper index}}$ $\underbrace{\text{1-form}}_{\text{clear}}$.)

$$R^i{}_{jkl} = \Gamma^i_{lj,k} - \Gamma^i_{kj,l} + \Gamma^s_{lj}\Gamma^i_{ks} - \Gamma^s_{kj}\Gamma^i_{ls}$$

2.) $$\Omega^i{}_j = d\omega^i{}_j + \omega^i{}_k \wedge \omega^k{}_j$$

Proposition: D is an anti-derivation:

degree of ϕ

$$D(\phi \wedge \psi) = D\phi \wedge \psi + (-1)^{p}\, \phi \wedge D\psi$$

Special cases:

- An ordinary p-form is $(0,0)$ tensor-valued. In this case, clearly:

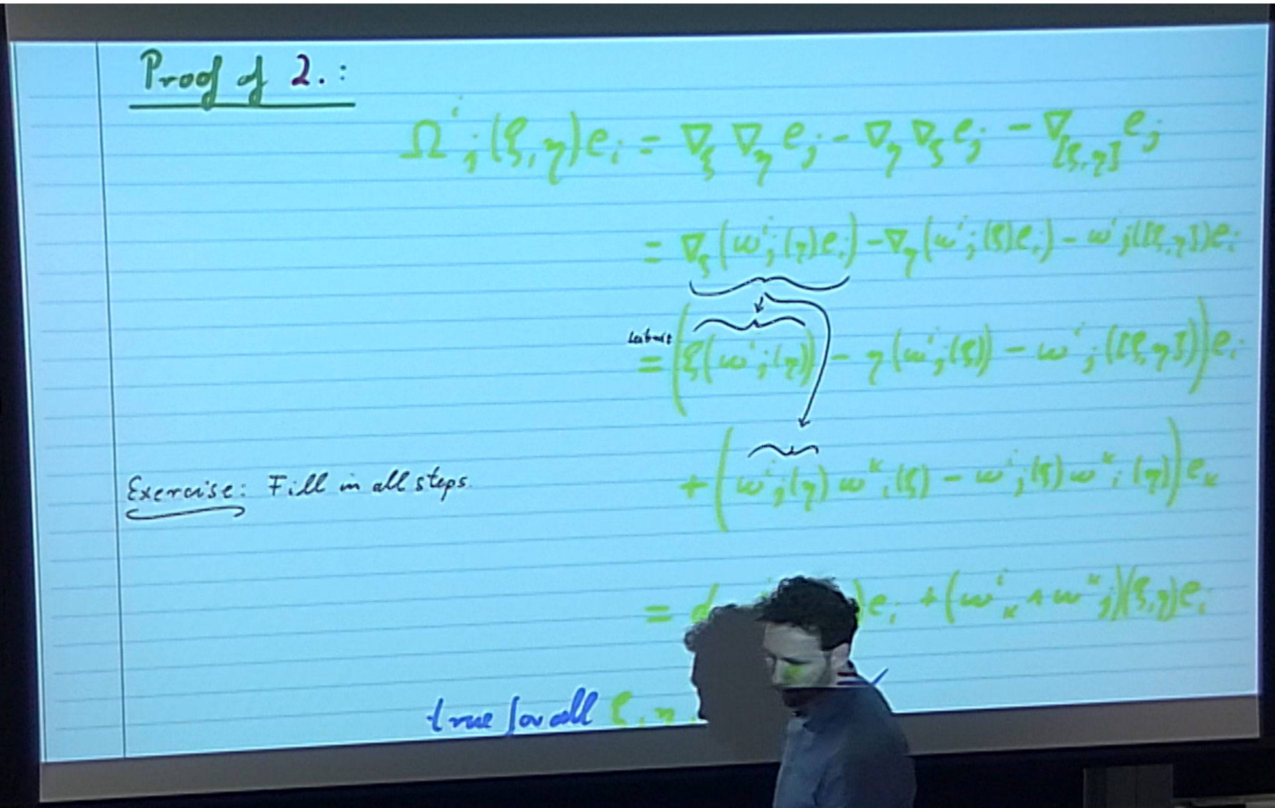
$$D = d$$

- An ordinary tensor field is a tensor-valued 0-form. In this case:

$$D = \nabla$$

Exercise: Verify

Hint: Choose frame $\theta^i = dx^i$, use $\omega^i = \Gamma^i\,\theta^k$



Proof of 2.:

$$\Omega^i{}_j(\xi,\eta)e_i = \nabla_\xi \nabla_\eta e_j - \nabla_\eta \nabla_\xi e_j - \nabla_{[\xi,\eta]} e_j$$

$$= \nabla_\xi(\omega^i{}_j(\eta)e_i) - \nabla_\eta(\omega^i{}_j(\xi)e_i) - \omega^i{}_j([\xi,\eta])e_i$$

Leibniz

$$= \Big(\xi(\omega^i{}_j(\eta)) - \eta(\omega^i{}_j(\xi)) - \omega^i{}_j([\xi,\eta])\Big)e_i$$

$$+ \Big(\omega^i{}_j(\eta)\,\omega^k{}_i(\xi) - \omega^i{}_j(\xi)\,\omega^k{}_i(\eta)\Big)e_k$$

Exercise: Fill in all steps.

$= d$ [illegible] $e_i + (\omega^i{}_k \wedge \omega^k{}_j)(\xi,\eta)e_i$ ✓

true for all ξ, η

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$$= \Big(\overbrace{\xi(\omega^i{}_j(\eta))} - \eta(\omega^i{}_j(\xi)) - \omega^i{}_j([\xi,\eta])\Big)e_i$$

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$$+ \Big(\overbrace{\omega^i{}_j(\eta)}\,\omega^k{}_i(\xi) - \omega^i{}_j(\xi)\,\omega^k{}_i(\eta)\Big)e_k$$

$$= d\omega^i{}_j(\xi,\eta)e_i + (\omega^i{}_k \wedge \omega^k{}_j)(\xi,\eta)e_i$$

true for all $\xi, \eta, e_i \Rightarrow$ ✓

Use of the Cartan Structure equations?

- ☐ Allow proof of simple formulation of the Bianchi identities:

1st Bianchi: $$D\Theta^i = \Omega^i{}_j \wedge \theta^j$$

2nd Bianchi: $$D\Omega^i{}_j = 0$$

← i.e. "Riemannian", i.e. "Levi Civita", without torsion.

- ☐ Thus, for metric connection, i.e. when $dg_{ik} = \omega_{ik} + \omega_{ki}$ and $\Theta^i = 0$ (same as $\nabla g = 0$, and $\Gamma_{ij} = \Gamma_{ji}$)

then: $$\boxed{\Omega^i{}_j \wedge \theta^j = 0}$$

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then:

$$\boxed{\begin{array}{l} \Omega^i{}_j \wedge \theta^j = 0 \\ D\Omega^i{}_j = 0 \end{array}}$$

Proposition:

- In the case of metric connection, the Cartan equations yield for arbitrary bases:

$C^l_{ki} = 0$ in canonical frame $\{dx^i\}$

$$\Gamma^l_{ki} = \frac{1}{2}\left(\overbrace{C^l_{ki} - g_{is}\, g^{lj} C^s_{kj} - g_{ks}\, g^{lj} C^s_{ij}}\right)$$

$$+\frac{1}{2}\, g^{lj}\left(g_{ij,k} + g_{jk,i} - g_{ki,j}\right)$$

Recall:

$$d\theta^i = \overbrace{-\frac{1}{2}}^{\text{convention}}\, C^i_{jk}\; \underbrace{\theta^j \wedge \theta^k}_{\text{basis for space of all 2 forms}}$$

coefficient functions depend on choice of frame

absent in canonical frame

- In this case, also:

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- In this case, also:

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$$R^i_{jab} = \Gamma^i_{bj,a} - \Gamma^i_{aj,b} + \Gamma^i_{ae}\Gamma^e_{bj} - \Gamma^i_{be}\Gamma^e_{aj} - \Gamma^i_{ej}\,C^e_{ab}$$