

Title: General Relativity for Cosmology - Lecture 8

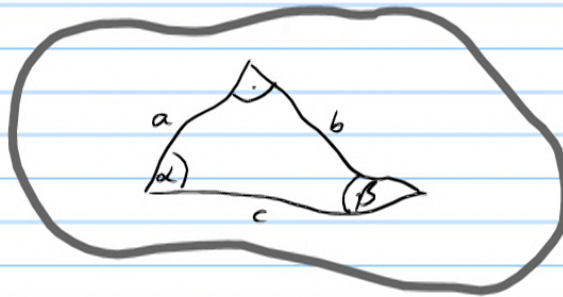
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URL: <http://pirsa.org/17100000>

Abstract:

How to describe the "shape" of a manifold?

Historically:



E.g., on a potato-shaped surface:

$$a^2 + b^2 \neq c^2$$

$$\alpha + \beta + 90^\circ \neq 180^\circ$$

Helmholtz & Gauss already considered checking for curvature of space this way.

Recall:

Defined $g_{\mu\nu}(x)$

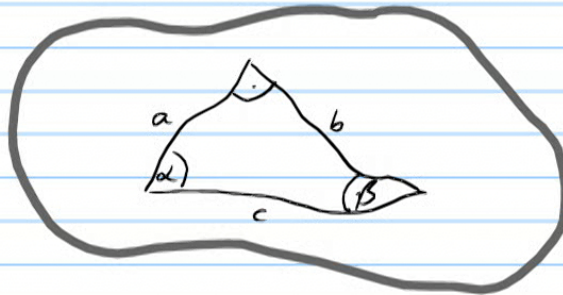
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GR for Cosmology, Achim Kempf, Fall 17, Lecture 8

Note Title

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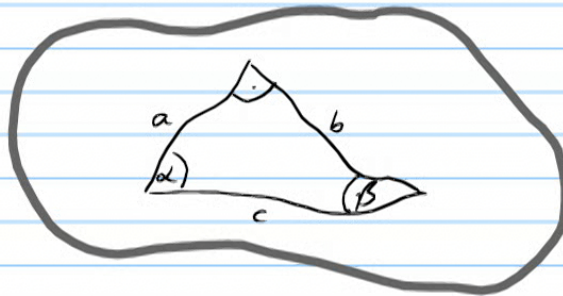
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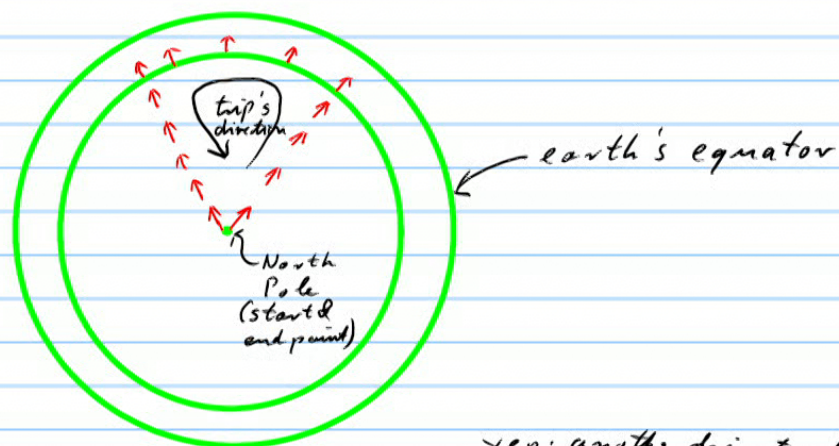
\Rightarrow infinitesimal distances \Rightarrow finite distances \Rightarrow shape

Alternative idea:

A manifold's shape, i.e., its curvature, also reveals itself in the nontriviality of the parallel transport of vectors on the manifold:

Example:

- start with a vector at North Pole.
- parallel transport down to some lower latitude, along that latitude and then back to pole.
- vector will arrive at pole rotated, in spite of having only been parallel transported!



This motivates:

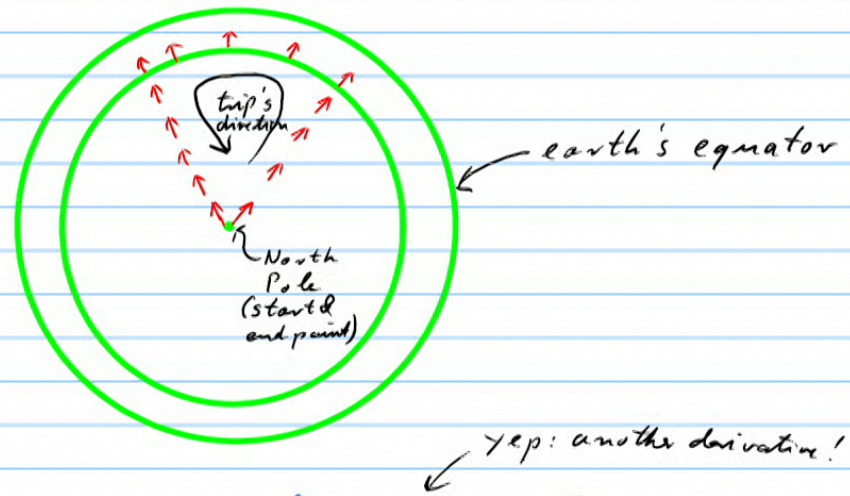
Try to define local shape through "derivative" of vectors with respect to parallel transport!

Recall: Lie derivative insensitive to g . Indeed, for $\xi_1 = \frac{\partial}{\partial x^1}$, $\xi_2 = \frac{\partial}{\partial x^2}$.

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Try to define local shape through "derivative" of vectors with respect to parallel transport!

Recall: Lie derivative insensitive to g . Indeed, for $\xi_1 = \frac{\partial}{\partial x^1}$, $\xi_2 = \frac{\partial}{\partial x^2}$, we have $[\xi_1, \xi_2] = L_{\xi_1} \xi_2 = 0 \Rightarrow$ No shape info from L_ξ !

The Covariant Differentiation, ∇ :

Definition: Any linear map of tangent vector fields

$$\nabla : T'(M) \times T'(M) \rightarrow T'(M)$$

$$\nabla : \quad \eta, \xi \rightarrow \nabla_{\xi} \eta$$

obeying

$$(I) \quad \nabla_{f\xi} \eta = f \nabla_{\xi} \eta, \quad \forall f \in \mathcal{F}(M)$$

$$(II) \quad \nabla_{\xi}(f\eta) = \underbrace{\nabla_{\xi} f}_{\xi(f)} \eta + f \nabla_{\xi} \eta \quad (\text{Leibniz rule})$$

is called a covariant derivative or affine connection.

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For now, let us assume a metric has not (yet) been specified, so we are free to choose ∇ , and this choice defines the shape of M !

∇ in a chart: \square Choose as bases for $T_x(M)$, e.g.: $\left\{ \frac{\partial}{\partial x^i} \right\}$

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\square Given a covariant derivative ∇ , consider its action on basis vectors, such as, e.g.: $\xi = \frac{\partial}{\partial x^i}, \eta = \frac{\partial}{\partial x^j}$:

Recall: $L_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = 0$

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} := \Gamma^k_{ij}(x) \frac{\partial}{\partial x^k}$$

The Γ^k_{ij} are called "Christoffel symbol" or "connection coefficients".

Thus, via the axioms:

$$\begin{aligned} \nabla_{\xi} \eta &= \nabla_{\xi} \left(\eta^j \frac{\partial}{\partial x^j} \right) \stackrel{(I)}{=} \xi^i \nabla_{\frac{\partial}{\partial x^i}} \left(\eta^j \frac{\partial}{\partial x^j} \right) \\ &\stackrel{(II)}{=} \xi^i \left(\eta^j_{,i} \frac{\partial}{\partial x^j} + \eta^j \Gamma^k_{ij}(x) \frac{\partial}{\partial x^k} \right) \\ &= (\xi^i \eta^k_{,i} + \xi^i \eta^j \Gamma^k_{ij}) \frac{\partial}{\partial x^k} \end{aligned}$$

function
vector

← recall: $\nabla_{\alpha} f = \alpha(f) \quad \forall \alpha \in T'_p(M), f \in F_p(M)$

Notation:

$$\eta^k_{;i} := \eta^k_{,i} + \eta^j \Gamma^k_{ij}$$

↑ semi-colon for covariant derivatives

Thus: $\nabla_{\xi} \eta = \xi^i \eta^k_{;i} \frac{\partial}{\partial x^k} \quad (*)$

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Thus:

$$\nabla_{\xi} \eta = \xi^i \eta^k{}_{,i} \frac{\partial}{\partial x^k} \quad (*)$$

Important: the Γ^k_{ij} ; transform non-tensorially when $x \rightarrow \bar{x}$:

On one hand:

$$\nabla_{\partial/\partial \bar{x}^a} \frac{\partial}{\partial \bar{x}^b} = \Gamma^c_{ab} \frac{\partial}{\partial \bar{x}^c} = \Gamma^c_{ab} \frac{\partial x^k}{\partial \bar{x}^c} \frac{\partial}{\partial x^k} \quad (\text{I})$$

because $\frac{\partial}{\partial x}$ is tangent vector

On the other hand:

$$\begin{aligned} \nabla_{\partial/\partial \bar{x}^a} \frac{\partial}{\partial \bar{x}^b} &= \nabla_{\frac{\partial x^i}{\partial \bar{x}^a} \frac{\partial}{\partial x^i}} \left(\frac{\partial x^j}{\partial \bar{x}^b} \frac{\partial}{\partial x^j} \right) && \text{use axiom (b)} \Rightarrow \\ &= \frac{\partial x^i}{\partial \bar{x}^a} \nabla_{\frac{\partial}{\partial x^i}} \left(\frac{\partial x^j}{\partial \bar{x}^b} \frac{\partial}{\partial x^j} \right) && \text{use Leibniz rule (c)} \Rightarrow \\ &= \frac{\partial x^i}{\partial \bar{x}^a} \left[\left(\frac{\partial}{\partial x^i} \frac{\partial x^j}{\partial \bar{x}^b} \right) \frac{\partial}{\partial x^j} + \frac{\partial x^j}{\partial \bar{x}^b} \Gamma^k_{ij} \frac{\partial}{\partial x^k} \right] && (\text{II}) \end{aligned}$$

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$$\begin{aligned} \nabla_{\partial/\partial x^a} \frac{\partial}{\partial x^b} &= \nabla_{\frac{\partial x^i}{\partial x^a} \frac{\partial}{\partial x^i}} \left(\frac{\partial x^j}{\partial x^b} \frac{\partial}{\partial x^j} \right) && \text{use axiom (b)} \Rightarrow \\ &= \frac{\partial x^i}{\partial x^a} \nabla_{\frac{\partial}{\partial x^i}} \left(\frac{\partial x^j}{\partial x^b} \frac{\partial}{\partial x^j} \right) && \text{use Leibniz rule (c)} \Rightarrow \\ &= \frac{\partial x^i}{\partial x^a} \left[\left(\frac{\partial}{\partial x^i} \frac{\partial x^j}{\partial x^b} \right) \frac{\partial}{\partial x^j} + \frac{\partial x^j}{\partial x^b} \Gamma^k_{ij} \frac{\partial}{\partial x^k} \right] \\ &= \left(\frac{\partial}{\partial x^a} \frac{\partial x^j}{\partial x^b} \right) \frac{\partial}{\partial x^j} + \frac{\partial x^i}{\partial x^a} \frac{\partial x^j}{\partial x^b} \Gamma^k_{ij} \frac{\partial}{\partial x^k} \quad (\text{II}) \end{aligned}$$

Compare I, II \Rightarrow

$$\bar{\Gamma}_{ab}^c \frac{\partial x^k}{\partial \bar{x}^c} = \frac{\partial^2 x^k}{\partial \bar{x}^a \partial \bar{x}^b} + \frac{\partial x^i}{\partial \bar{x}^a} \frac{\partial x^j}{\partial \bar{x}^b} \Gamma^k_{ij} \quad \left(\frac{\partial \bar{x}^r}{\partial x^k} \Rightarrow \right)$$

\Rightarrow

$$\bar{\Gamma}^r_{ab} = \underbrace{\frac{\partial \bar{x}^r}{\partial x^k} \frac{\partial^2 x^k}{\partial \bar{x}^a \partial \bar{x}^b}}_{\text{This term is indep. of } \Gamma} + \underbrace{\frac{\partial \bar{x}^r}{\partial x^k} \frac{\partial x^i}{\partial \bar{x}^a} \frac{\partial x^j}{\partial \bar{x}^b} \Gamma^k_{ij}}_{\text{Only this term would be there, if the } \Gamma^k_{ij} \text{ were tensor coefficients in the } \frac{\partial}{\partial x^i}, dx^i \text{ bases.}}$$

This term is indep. of Γ
 $\Rightarrow \Gamma$ can be zero in one coordinate system and nonzero in another!

Only this term would be there, if the Γ^k_{ij} were tensor coefficients in the $\frac{\partial}{\partial x^i}, dx^i$ bases.

(Can be shown to be equivalent)

Physicists' definition of ∇ : Any set of n^3 functions $\Gamma_{ab}^c(x)$ which transform this way are defining a cov. derivative ∇ .

The "absolute" covariant derivative ∇ :

Consider the covariant derivative but:
without choosing a direction vector ξ :

$$\nabla : T_x(M)' \rightarrow T_x(M)'$$

$$\nabla : \eta = \eta^i(x) \frac{\partial}{\partial x^i} \rightarrow \nabla \eta(x) = \eta^k{}_{;i}(x) dx^i \otimes \frac{\partial}{\partial x^k}$$

(I.e. feed the open covariant slot
of $\nabla \eta$ with contravariant ξ .)

Indeed: The contraction of $\nabla \eta$ with ξ yields:

$$\nabla \eta(\xi) = \eta^k{}_{;i} \underbrace{dx^i(\xi)}_{\xi^i} \frac{\partial}{\partial x^k} = \eta^k{}_{;i} \xi^i \frac{\partial}{\partial x^k} = \nabla_{\xi} \eta \quad \text{ok with } (*)$$

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We defined ∇ algebraically. Now, extract the

Geometric meaning of ∇ :

(∇ describes infinitesimal parallel transport
It should also describe finite parallel transport)

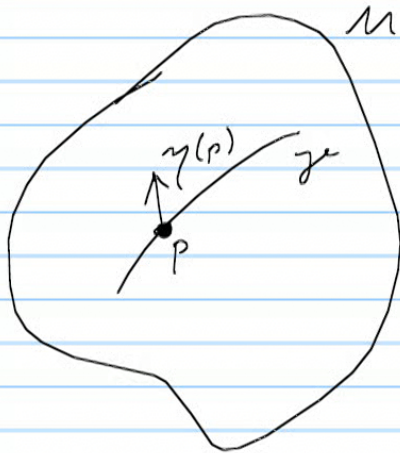
Definition: Assume ∇ is given. Choose a path $\gamma: \mathbb{R} \rightarrow M$.

Then, a tangent vector field η is called auto-parallel along γ , if

$$\nabla_{\dot{\gamma}} \eta = 0$$

i.e. if η doesn't change under parallel transport along the path γ .

Note: We'll see that they always exist, i.e., we



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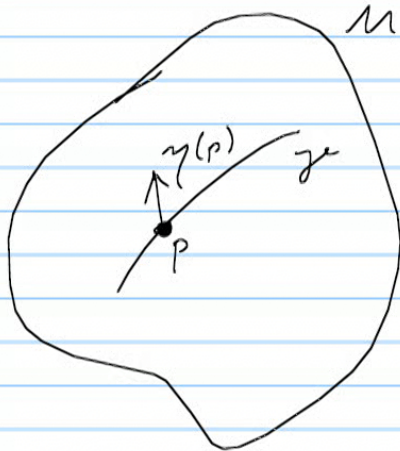
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Note: We'll see that they always exist, i.e., we can always parallel transport a vector η finite distances.

□ In a chart,

$$\eta = \eta^i(x) \frac{\partial}{\partial x^i}$$

and

$$x: [a, b] \rightarrow \mathcal{M}$$

$$x: t \rightarrow x^i(t)$$

and the tangent vector:

$$\dot{x}(x(t)) = \frac{dx^k}{dt} \frac{\partial}{\partial x^k}$$

Thus: $\nabla_{\dot{x}} \eta = \nabla_{\frac{dx^k}{dt} \frac{\partial}{\partial x^k}} \left(\eta^i \frac{\partial}{\partial x^i} \right) = \frac{dx^k}{dt} \nabla_{\frac{\partial}{\partial x^k}} \left(\eta^i \frac{\partial}{\partial x^i} \right)$

$$= \frac{dx^k}{dt} \left(\frac{\partial \eta^i}{\partial x^k} \frac{\partial}{\partial x^i} + \eta^i \Gamma^i_{ki} \frac{\partial}{\partial x^i} \right)$$

$$= \left(\frac{d\eta^i}{dt} + \frac{dx^k}{dt} \eta^j \Gamma^i_{ki} \right) \frac{\partial}{\partial x^i} \stackrel{!}{=} 0$$

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Thus:
$$\nabla_{\dot{\gamma}} \eta = \nabla_{\frac{dx^k}{dt} \frac{\partial}{\partial x^k}} \left(\eta^i \frac{\partial}{\partial x^i} \right) = \frac{dx^k}{dt} \nabla_{\frac{\partial}{\partial x^k}} \left(\eta^i \frac{\partial}{\partial x^i} \right)$$

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\Rightarrow η autoparallel to γ means:

$$\frac{d\eta^i}{dt} + \eta^j \frac{dx^k}{dt} \Gamma^i_{kj} = 0$$

i.e. this is the condition for the vectors of η being parallel translates of each other, along γ .

□ Conclusion:

This is 1st order ODEs for η . Thus:

Initial condition $\eta(\gamma(0)) \Rightarrow$ solution $\eta(\gamma(t))$ exists at least locally

\Rightarrow □ Proposition:

Given a path $\gamma: [t, s] \rightarrow M$, the autoparallel transport of a tangent vector η

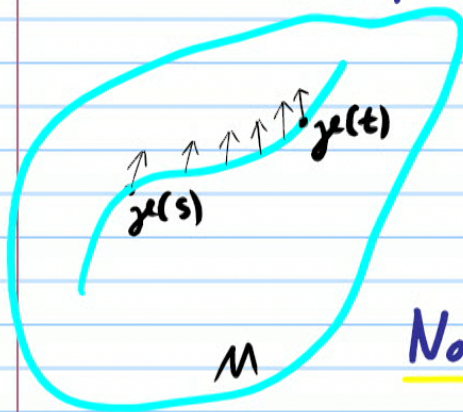
I.e., the path γ defines a parallel transport map τ :

$$\tau(t,s): T_{\gamma(t)} \rightarrow T_{\gamma(s)}$$

$$\tau(t,s): \eta(\gamma(t)) \rightarrow \eta(\gamma(s))$$

□ Q: Can one use τ to obtain ∇ as a Newton-Leibniz limit?

□ Proposition: (for the proof, see e.g. the text by Strömmermann)



$$\nabla_{j_i} \eta(\gamma(t)) = \left. \frac{d}{ds} \right|_{s=t} \tau(s,t) (\eta(\gamma(s)))$$

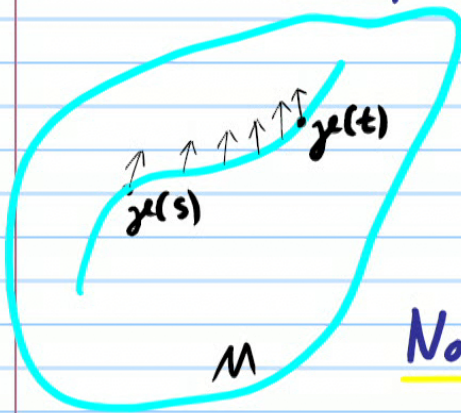
Note: Since we can choose paths with arbitrary j_i this equation can be used as a

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□ Q: Can one use τ to obtain ∇ as a Newton-Leibniz limit?

□ Proposition: (for the proof, see e.g. the text by Straumann)



$$\nabla_{\dot{\gamma}^i} \eta(\gamma(t)) = \frac{d}{ds} \Big|_{s=t} \tau(s,t) (\eta(\gamma(s)))$$

Note: Since we can choose paths with arbitrary $\dot{\gamma}$ this equation can be used as a **geometric definition of ∇** .

▽ for arbitrary tensors:

□ The parallel transport map $\tau(s,t)$ transports tangent vectors η from $\gamma(s)$ to $\gamma(t)$.

□ Definition: $\tau(s,t)$ also parallel transports the dual vectors ω , namely so that contraction is conserved:

$$\underbrace{\tau(\omega)}_{\text{parallel transported } \omega} \left(\underbrace{\tau(\eta)}_{\text{parallel transported } \eta} \right) = \omega(\eta) \quad (G)$$

□ Extension of τ to tensor products:

$$\tau(S_1 \otimes S_2) := \tau(S_1) \otimes \tau(S_2) \quad (T)$$

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↑ ↑
S₁ and S₂ are tensors of arbitrary rank.



□ Definition:

arbitrary tensor \downarrow arb. point $\in M$

$$\nabla_{\xi} S'(p) := \nabla_{\xi^i} S'(x(t)) \Big|_{t=0}$$

arb. tangent vector \uparrow

$$:= \frac{d}{dt} \Big|_{t=0} \tau(t,0)(S'(x(t)))$$

here, γ is any path through p obeying:

$$\gamma(0) = \xi(p), \quad \dot{\gamma}(0) = p$$

Exercise:

Show that when S' is a scalar function $S' \in F(M)$, then:

$$\nabla_{\xi} S' = \xi(S') = \xi^i \frac{\partial}{\partial x^i} S'$$

□ Absolute covariant derivative:

(for abs. derivative one is not specifying)

led to ∇S which is (r,p) tensor

$$(\nabla S')(\eta_1, \dots, \eta_r, \omega_1, \dots, \omega_{r,p}) := \nabla_{\xi} S'(\eta_1, \dots, \eta_r, \omega_1, \dots, \omega_{r,p})$$

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Properties of ∇ :

* ∇ is a derivation: (because ∇ inherits the Leibniz rule from $\frac{d}{ds}$)

$$\nabla_{\xi} (S_1 \otimes S_2) = \frac{d}{ds} \Big|_{s=t} \tau(S_1 \otimes S_2) = \frac{d}{ds} \Big|_{s=t} \tau(S_1) \otimes \tau(S_2)$$

$\tau(S_1)$ ← $S_1(\gamma(s))$ $\tau(S_2)$ ← $S_2(\gamma(s))$

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$\tau(s,t)$ $S_1(\gamma(s))$ $S_2(\gamma(s))$

$$= \left[\frac{d}{ds} \Big|_{s=t} \tau(S_1) \right] \otimes \tau(S_2) \Big|_{s=t} + \tau(S_1) \Big|_{s=t} \otimes \frac{d}{ds} \Big|_{s=t} \tau(S_2)$$

$$= (\nabla_{\xi} S_1) \otimes S_2 + S_1 \otimes \nabla_{\xi} S_2 \quad (A)$$

* Eq. (A) implies that ∇ and contractions do commute.

Action of ∇ on tensors in a chart?

▢ Recall: $\nabla_{\xi} \frac{\partial}{\partial x^i} = \xi^k \Gamma^k_{li} \frac{\partial}{\partial x^k}$

▢ Problem: Find $\nabla_{\xi} dx^i = ?$

→ • Consider $\eta \otimes \omega$.

\downarrow tangent vector field \downarrow cotangent vector field

• Differentiate:

$$\nabla_{\xi} (\eta \otimes \omega) \stackrel{(A)}{=} (\nabla_{\xi} \eta) \otimes \omega + \eta \otimes \nabla_{\xi} \omega$$

• Contract: (use that ∇_{ξ} and contraction commute)

(by exercise above) $\nabla_{\xi} (\omega(\eta)) = \omega(\nabla_{\xi} \eta) + (\nabla_{\xi} \omega)(\eta)$

Same strategy will be used below for general tensors.

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$$(i.e. \quad \xi(\omega(\eta)) = \omega(\nabla_{\xi} \eta) + (\nabla_{\xi} \omega)(\eta))$$

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- Differentiate:

$$\nabla_{\xi}(\eta \otimes \omega) \stackrel{(A)}{=} (\nabla_{\xi} \eta) \otimes \omega + \eta \otimes \nabla_{\xi} \omega$$

- Contract: (use that ∇_{ξ} and contraction commute)

(by exercise above)

$$\xi(\omega(\eta)) \stackrel{=}{=} \nabla_{\xi}(\underbrace{\omega(\eta)}_{\text{scalar function}}) = \omega(\nabla_{\xi} \eta) + (\nabla_{\xi} \omega)(\eta)$$

$$\text{(i.e. } \xi(\omega(\eta)) = \omega(\nabla_{\xi} \eta) + (\nabla_{\xi} \omega)(\eta)\text{)}$$

\Rightarrow An expression for $\nabla_{\xi}(\omega)(\eta)$:

$$(\nabla_{\xi} \omega)(\eta) = \xi(\omega(\eta)) - \omega(\nabla_{\xi} \eta) \quad (*)$$

Now: Choose $\omega := dx^i$ and $\eta := \underline{\underline{e}}$.

same strategy will be used below for general tensors.

$$\nabla_{\xi}(\eta \otimes \omega) \stackrel{(A)}{=} (\nabla_{\xi} \eta) \otimes \omega + \eta \otimes \nabla_{\xi} \omega$$

• **Contract:** (use that ∇_{ξ} and contraction commute)

(by exercise above) $\xi(\omega(\eta)) \stackrel{=}{=} \nabla_{\xi}(\omega(\eta)) = \omega(\nabla_{\xi} \eta) + \underbrace{(\nabla_{\xi} \omega)}_{\text{scalar function}}(\eta)$

(i.e. $\xi(\omega(\eta)) = \omega(\nabla_{\xi} \eta) + (\nabla_{\xi} \omega)(\eta)$)

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$$(\nabla_{\xi} \omega)(\eta) = \xi(\omega(\eta)) - \omega(\nabla_{\xi} \eta) \quad (*)$$

Now: Choose $\omega := dx^i$ and $\eta := \frac{\partial}{\partial x^i}$

$\Rightarrow (\nabla_{\rho} dx^i) \left(\frac{\partial}{\partial x^i} \right) = \xi \left(\underbrace{\langle dx^i, \frac{\partial}{\partial x^i} \rangle}_{\delta^i_i = \text{const.}} \right) - \langle dx^i, \nabla_{\rho} \frac{\partial}{\partial x^i} \rangle$

\Rightarrow An expression for $\nabla_{\xi}(\omega)(\eta)$:

$$(\nabla_{\xi}\omega)(\eta) = \xi(\omega(\eta)) - \omega(\nabla_{\xi}\eta) \quad (*)$$

Now: Choose $\omega := dx^j$ and $\eta := \frac{\partial}{\partial x^i}$

$$\Rightarrow (\nabla_{\xi} dx^j)\left(\frac{\partial}{\partial x^i}\right) = \xi\left(\underbrace{\langle dx^j, \frac{\partial}{\partial x^i} \rangle}_{\delta_{ij} = \text{const.}}\right) - \underbrace{\langle dx^j, \nabla_{\xi} \frac{\partial}{\partial x^i} \rangle}_{=0}$$

Notation:
 $\langle \omega, \xi \rangle = \omega(\xi)$
(inner product,
contraction)

$$= - \langle dx^j, \xi^l \Gamma_{li}^k \frac{\partial}{\partial x^k} \rangle$$

$$= - \xi^l \Gamma_{li}^j$$

\Rightarrow

$$\nabla_{\xi} dx^j = - \xi^l \Gamma_{li}^j dx^i$$

For general tensors: (by exactly same strategy as above but applied to multiple tensor products, we obtain:

$$\nabla_{\xi} S(\eta_1, \dots, \eta_r, \omega_1, \dots, \omega_s) \quad (\text{as in Eq. (*) above})$$

$$= S(\eta_1, \dots, \eta_r, \omega_1, \dots, \omega_s)$$

$$- S(\nabla_{\xi} \eta_1, \eta_2, \dots, \eta_r, \omega_1, \dots, \omega_s) - \dots$$

$$- S(\eta_1, \dots, \nabla_{\xi} \eta_r, \omega_1, \dots, \omega_s)$$

$$- S(\eta_1, \dots, \eta_r, \nabla_{\xi} \omega_1, \omega_2, \dots, \omega_s) + \dots$$

$$- S(\eta_1, \dots, \eta_r, \omega_1, \dots, \nabla_{\xi} \omega_s)$$

\rightsquigarrow • Consider $\eta \otimes \omega$.

$\begin{matrix} \text{tangent vector field} \\ \downarrow \\ \eta \\ \text{cotangent vector field} \\ \downarrow \\ \omega \end{matrix}$

• Differentiate:

$$\nabla_{\xi}(\eta \otimes \omega) \stackrel{(A)}{=} (\nabla_{\xi} \eta) \otimes \omega + \eta \otimes \nabla_{\xi} \omega$$

• Contract: (use that ∇_{ξ} and contraction commute)

(by exercise above) $\xi(\omega(\eta)) \stackrel{=}{=} \nabla_{\xi}(\underbrace{\omega(\eta)}_{\text{scalar function}}) = \omega(\nabla_{\xi} \eta) + (\nabla_{\xi} \omega)(\eta)$

Same strategy will be used below for general tensors.

$$(i.e. \quad \xi(\omega(\eta)) = \omega(\nabla_{\xi} \eta) + (\nabla_{\xi} \omega)(\eta))$$

\Rightarrow An expression for $\nabla_{\xi}(\omega)(\eta)$:

$$(\nabla \omega)(\cdot) = \rho(\dots) - \omega(\nabla \dots) \quad (\star)$$

$$\nabla_{\xi} S(\eta_1, \dots, \eta_r, \omega_1, \dots, \omega_s) \quad (\text{as in Eq. } (*) \text{ above})$$

$$= \xi(S(\eta_1, \dots, \eta_r, \omega_1, \dots, \omega_s))$$

$$- S(\nabla_{\xi} \eta_1, \eta_2, \dots, \eta_r, \omega_1, \dots, \omega_s) - \dots$$

$$- S(\eta_1, \dots, \nabla_{\xi} \eta_r, \omega_1, \dots, \omega_s)$$

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$$- S(\eta_1, \dots, \eta_r, \omega_1, \dots, \nabla_{\xi} \omega_s)$$

Choosing the basis vectors dx^i and $\frac{\partial}{\partial x^i}$, we obtain

for

$$S = \sum_{j_1, \dots, j_q}^{i_1, \dots, i_p} \frac{\partial}{\partial x^{j_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{j_q}} \otimes dx^{i_1} \otimes \dots \otimes dx^{i_p}$$

that $\nabla_{\xi} S$ reads

$$\nabla_{\xi} S = \xi^k \sum_{j_1, \dots, j_q}^{i_1, \dots, i_p} \Gamma_{jk}^{i_1} \frac{\partial}{\partial x^{j_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{j_q}} \otimes dx^{i_1} \otimes \dots \otimes dx^{i_p}$$

with:

$$\begin{aligned} \sum_{j_1, \dots, j_q}^{i_1, \dots, i_p} \Gamma_{jk}^{i_1} &:= \sum_{j_1, \dots, j_{p-1}, k}^{i_1, \dots, i_p} + \Gamma_{ke}^{i_1} \sum_{j_1, \dots, j_q}^{e, i_2, \dots, i_p} \\ &\quad + \dots + \Gamma_{ke}^{i_p} \sum_{j_1, \dots, j_q}^{i_1, \dots, e} \\ &\quad - \Gamma_{kj}^{i_1} \sum_{e, \dots, j_q}^{i_1, \dots, i_p} \end{aligned}$$

for

$$S = \sum_{j_1, \dots, j_q}^{i_1, \dots, i_p} \frac{\partial}{\partial x^{i_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{i_p}} \otimes dx^{j_1} \otimes \dots \otimes dx^{j_q}$$

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$$\nabla_{\xi} S = \xi^k \sum_{j_1, \dots, j_q, k}^{i_1, \dots, i_p} \frac{\partial}{\partial x^{i_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{i_p}} \otimes dx^{j_1} \otimes \dots \otimes dx^{j_q}$$

with:

$$\begin{aligned} \sum_{j_1, \dots, j_q, k}^{i_1, \dots, i_p} &:= \sum_{j_1, \dots, j_q, k}^{i_1, \dots, i_p} + \Gamma_{k\ell}^{i_1} \sum_{j_1, \dots, j_q}^{\ell i_2, \dots, i_p} \\ &\quad + \dots + \Gamma_{k\ell}^{i_p} \sum_{j_1, \dots, j_q}^{i_1, \dots, \ell} \\ &\quad - \Gamma_{kj_1}^{\ell} \sum_{\ell, \dots, j_q}^{i_1, \dots, i_p} \\ &\quad - \dots - \Gamma_{kj_q}^{\ell} \sum_{j_1, \dots, \ell}^{i_1, \dots, i_p} \end{aligned}$$

Special cases:

□ Tangent vector fields:

$$\xi^i_{jk} = \xi^i_{,k} + \xi^j \Gamma^i_{kj}$$

□ Cotangent vector fields:

$$\omega_{ijk} = \omega_{i,k} - \omega_j \Gamma^l_{ki}$$

Recall: Specifying ∇ specifies parallel transport of vectors and this should specify the manifold's shape, but how?

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