Title: S-Matrix Bootstrap with Linear Spectrum

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Abstract: <p>We work out constraints imposed by channel duality and analyticity on tree-level amplitudes of four identical real scalars, with the assumptions of a linear spectrum of exchanged particles and Regge asymptotic behaviour. We reduce the requirement of channel duality to a countably infinite set of equations in the general case. We show that channel duality uniquely fixes the soft Regge behaviour of the amplitudes to that found in String theory, (-s)^(2t). Specialising to the case of tachyonic external particles, we use channel duality to show that the amplitude can be any one in an infinite-dimensional parameter space, and present evidence that unitarity doesn't significantly reduce the dimension of the space of amplitudes.
This talk is based on 1707.08135 by Pranjal Nayak, Rohan R. Poojary and RMS.
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$a = -\alpha' s - \alpha(0)$ Poles at $a = -n$.

b t

c u

Brief History of S-Matrix Boostrap

- Large coupling constant and rich spectrum of particles made theory of strong interactions less amenable to Lagrangian approach (or so it looked before QCD was discovered!)
- S-Matrix bootstrap program: directly constrain the scattering matrices using symmetries and other consistency requirements.
- In search of amplitudes that had infinitely many resonances (= hadrons/mesons) and Regge asymptotic behaviour, Veneziano came up with Veneziano ampltiude, generalised by Virasoro etc.
- The rest is string theory.

Why should we revisit the program?

Our motivation was 2-fold:

1. CEMZ[1] argue that a theory of gravity is consistent only if graviton-graviton-spin-s 3-point coupling:
   a. is same as that in Einstein gravity (0 for $s > 2$) or
   b. non-zero for all $s > 2$ – theory has *infinitely many higher spin fields!*
2. Recent success in conformal bootstrap program and improved computational techniques might shed more light on this old program.

[1] Camanho, Edelstein, Maldacena, Zhiboedov, arXiv:1407.5597

Problem in Question

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Even Simpler Question: What is the most general 4-identical scalar scattering amplitude consistent with some minimal number of consistency conditions *and a linear spectrum*?

Short Answer: Regge fall-off in String theory, $(-s)^{2t}$ (with t rescaled in a particular way), is unique.

Side effect: pretty *Bootstrap Equations*.

Postulates: The Obvious Ones

1. Lorentz-Invariance: Amplitude is function of s, t, u.
2. Causality: Poles in the s, t, u complex planes only where particle is exchanged. $s = m_n^2$ etc.
3. Unitarity: Residues are sums of Gegenbauer polynomials ($\sim$ Legendre) with positive coefficients.
4. Crossing Symmetry: $A(s, t, u)$ can be analytically continued to $A(t, s, u)$, etc.
5. Tree-Level Amplitudes (Classicality): Only poles are simple poles, at masses of particles in the spectrum.
   (Loops would give branch cuts.)

Simplifying Postulate 1: Linear Spectrum

Poles are at

$$m_n^2 = \frac{n - \alpha(0)}{\alpha'} \quad /n \in \{0, 1, 2 \cdots\}. \tag{1}$$

We define new Mandelstam-like variables

$$\begin{aligned} a &= -\alpha' s - \alpha(0) \\ b &= -\alpha' t - \alpha(0) \\ c &= -\alpha' u - \alpha(0) \end{aligned} \tag{2}$$

so that poles are at $a = -n$ etc.
Mass-shell condition becomes

$$s + t + u = 4M_{ext^2} \to a + b + c \equiv P = -4\alpha' M_{ext}^2 - 3\alpha(0) \tag{3}$$

Regge Asymptotic Behaviour, Our Lord and Saviour

At large a,

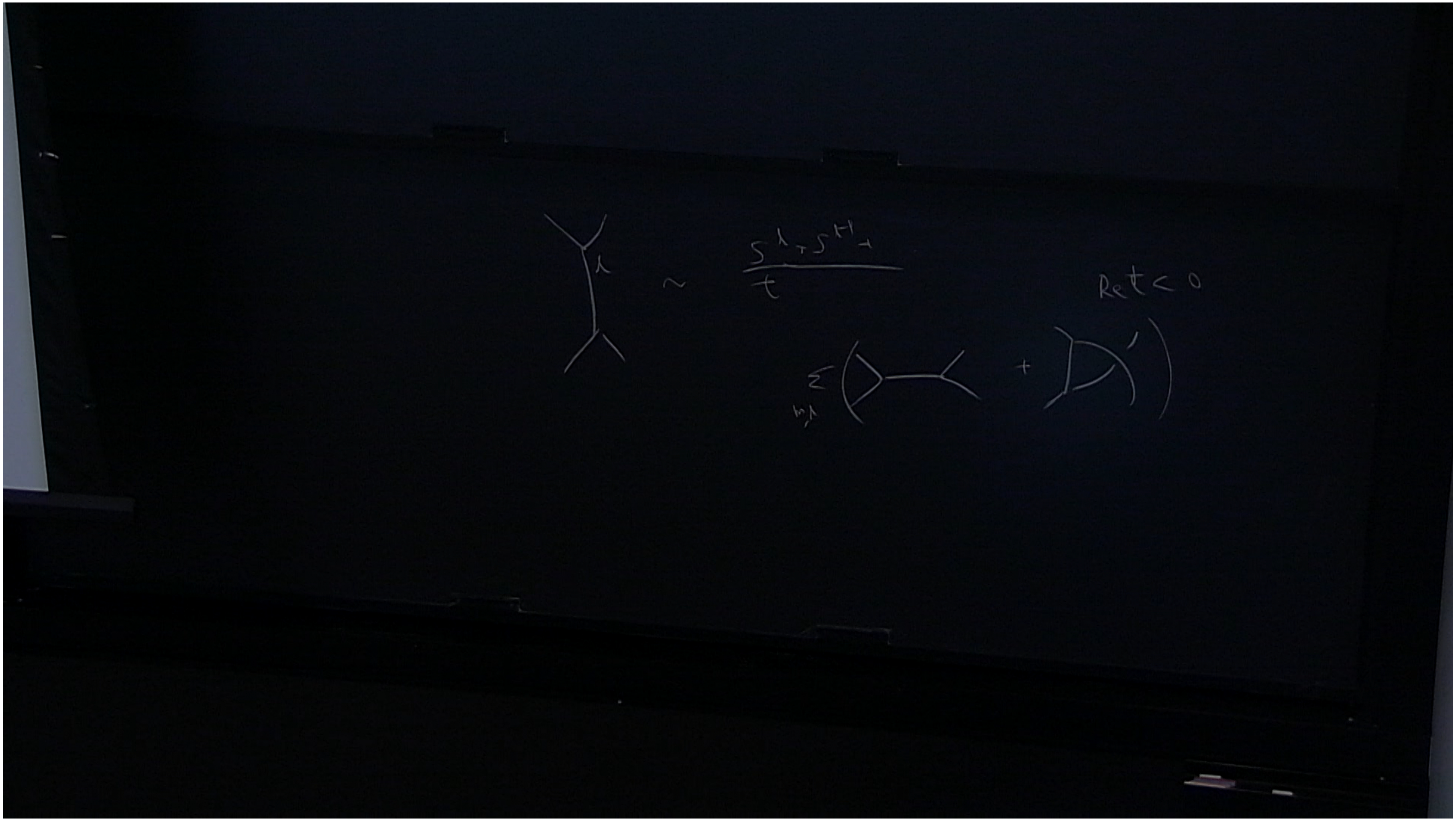
$$A(a,b,c) \xrightarrow{|a|\to\infty} a^{-k(b)} \sim (-s)^{-k(-t)}, \tag{4}$$

where $k(b) > 0$ in s-channel scattering region.
In particular, we take the case $k(b) = kb$. $kb + l$ won't change much.

Not true for a negative and real, because poles.
Nevertheless, true in other directions, with poles giving oscillatory envelope but not modifiying the power law.
In fact, not even a true non-analyticity at ∞ (magic!).

A t-channel exchange of a spin l particle is $\frac{s^l}{t}$,
so this is what rules out QFT-type crossing (s-channel + t-channel + u-channel).

We'll actually use a slightly stronger assumption; we'll come to it later.


Re t < 0

Plan of Attack

1. Write amplitude in pole-sum form,

$$A(a,b) = \sum_{n=0}^{\infty} \frac{f_n(b)}{a+n} + \frac{f_n(b)}{c+n}, \quad Re(b) > 0, \tag{5}$$

   Condition on b because no explicit singularities in b, so we aren't in t-channel.

2. Analytically continue to t-channel and regularise infinite sum to find isolated t-channel poles.

A Couple of Facts about Analytic Continuation

1. Analytic continuation off an open set unique.
2. Carlson's theorem: Given a function f_n defined on the postive integers, there's a unique analytic continuation that satisfies
   2.1 $|f(z)| < Ce^{\tau|z|}, \quad \mathrm{Re}\, z > 0$ (diverges at most exponentially at ∞; also a condition on f_n) and
   2.2 $|f(iy)| < C' e^{c|y|}, \quad c < \pi$ and $y \in \mathbb{R}$ (diverges slower than $\sin(\pi z)$ on the imaginary axis).

Asymptotics of the Residue, and an Extra Assumption

Regge behaviour $\Rightarrow$ for large n, $f_n(b) \sim n^{-k(b)}$.

Assumption: the residue can be analytically continued to complex n plane and it admits a Laurent expansion about $n = \infty$ with finite radius of convergence,

$$f_n(b) = \sum_{j=0}^{\infty} g_j(b) n^{-k(b)-j}. \quad (6)$$

Almost justified by Carlson's theorem.

Unimportant assumption: Laurent expansion converges for $|n| > 1$. Changing this will cause mainly notational complication.

The Fun Part

Plugging in Laurent expansion of residue and also expanding $(a+n)^{-1}$ as $\sum \frac{(-a)^r}{n^{r+1}}$, we get

$$A(a,b,c) = \sum_{n=1}^{\infty} \left\{ \left(\sum_{j=0}^{\infty} g_j(b) n^{-k(b)-j} \right) \left(\sum_{r=0}^{\infty} \frac{(-a)^r + (-c)^r}{n^{r+1}} \right) \right\} + \text{finite sum},$$

where all subtleties validity of expansion is thrown into the finite sum. Infinite sums all converge, so we can rearrange and get

$$A(a,b,c) = \sum_{j,r=0}^{\infty} g_j(b) \left\{ (-a)^r + (-c)^r \right\} \sum_{n=1}^{\infty} n^{-k(b)-j-r-1} + reg.$$

Analytic continuation: replace with ζ function, giving

$$A(a,b,c) = \sum_{N=0}^{\infty} \left(\sum_{j=0}^{N} g_j(b) \left\{ (-a)^{N-j} + (-c)^{N-j} \right\} \right) \zeta(k(b)+N+1)$$

ζ function has isolated poles at $k(b) = -N$ or $b = -k^{-1}(-N)$.

What do we do with those equations?

Put in some conditions, of course!

1. Poles appear at $b = -k^{-1}(-N)$.

   Since we want poles at $b = -n$, $k(-n)$ better be a negative integer. Simplest choice: $k(b) = kb$.

2. At these physical poles, the residues should be $f_n(a)$.

$$\operatorname*{Res}_{b\,=\,k^{-1}(-N)} A(a,b) = f_n(a), \qquad \text{when } k^{-1}(-N) = -n \tag{7}$$

3. $\forall b \notin \{-n\}$, s.t., $k(b) = -N$, the residues should vanish.

$$\operatorname*{Res}_{b\,=\,k^{-1}(-N)} A(a,b) = 0, \qquad \text{when } k^{-1}(-N) \neq -n \tag{8}$$

Note: haven't used linearity of $k(b)$ or spectrum yet.
We'll use them now.

$k(b) = kb$ and $c = P - a - b$

Plugging these two in, we get

$$N \neq kn \Rightarrow \sum_{J=0}^{N} g_J\left(-\frac{N}{k}\right)\left\{(-a)^{N-J} + \left(a - \frac{N}{k} - P\right)^{N-J}\right\} = 0$$
$$N = kn \Rightarrow \frac{1}{k}\sum_{J=0}^{kn} g_J(-n)\left\{(-a)^{kn-J} + (a-n-P)^{kn-J}\right\} = f_n(a) \tag{9}$$

Organize these equations in powers of a, $f_n(a) = \sum_{J=0}^{kn} h_J(-n)(-a)^{kn-J}$

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Spurious Pole Equations (SPE)

$$0 = [1 + (-1)^{N-J}] g_J\left(-\frac{N}{k}\right) + (-1)^{N-J}\sum_{j=1}^{J}(-1)^j \binom{N-J+j}{j}\left(\frac{N}{k}+P\right)^j g_{J-j}\left(-\frac{N}{k}\right) \quad (10)$$

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Residue Matching Equations (RME)

$$kh_J(-n) = [1 + (-1)^{kn-J}]g_J(-n) + (-1)^{kn-J}\sum_{j=1}^{J}(-1)^j\binom{kn-J+j}{j}(n+P)^j g_{J-j}(-n) \quad (11)$$

Constraining k

k odd

Consider $k = 1$ for definiteness,

There are no SPEs!
RMEs are

$$f_n(a) = \sum_{J=0}^{n} g_J(-n)(-a)^{n-J} + (-1)^n \sum_{J=0}^{n} \sum_{j=0}^{J} (-1)^{j+J} \binom{n-J+j}{j} (n+P)^j g_{J-j}(-n)(-a)^{n-J} \tag{12}$$

Essential singularity at $n = \infty$! Inconsistent with the initial assumption!

Constraining k

k even, $k > 2$

Write SPEs and RMEs together as

$$kh_J\left(-\frac{N}{k}\right) = \left[1 + (-1)^{N-J}\right] g_J\left(-\frac{N}{k}\right) + (-1)^{N-J} \sum_{j-1}^{J} c_{N,k,J,j}\, g_{J-j}\left(-\frac{N}{k}\right) \tag{13}$$

with $h_J(\text{non-integer}) = 0$.

RHS defines analytic continuation of LHS.

To avoid essential singularity, we use only even values of N to do analytic continuation; this also ensures that RHS satisfies conditions of Carlson's theorem.

Now, notice: analytic continuation has property that $h_J\left(-n - \frac{2}{k}\right) = 0$. Since RHS satisfies Carlson's theorem conditions, this means that the analytic continuation is 0!

Constraining k

$k = 2$

For $k = 2$,
odd N gives SPEs,

$$0 = \left[1 - (-1)^J\right] g_J \left(-n - \frac{1}{2}\right) - (-1)^J \sum_{j=1}^{J} c_{2n+1,2,J,j}\, g_{J-j} \left(-n - \frac{1}{2}\right), \tag{14}$$

which we can analytically continue to arbitrary values of n.
Even N gives RMEs,

$$2h_J(-n) = \left[1 + (-1)^J\right] g_J(-n) + (-1)^J \sum_{j=1}^{J} c_{2n,2,J,j}\, g_{J-j}(-n). \tag{15}$$

Plugging in SPEs, we find

$$h_J(-n) = g_J(-n). \tag{16}$$

All string theory amplitudes have $k = 2$.

The Final Bootstrap Equations

Definitions: $$f_n(b) = \sum_{J=0}^{2n} h_J(-n)(-b)^{2n-J} = \sum_{J=0}^{\infty} g_J(b) n^{-2b-J}, \qquad (17)$$

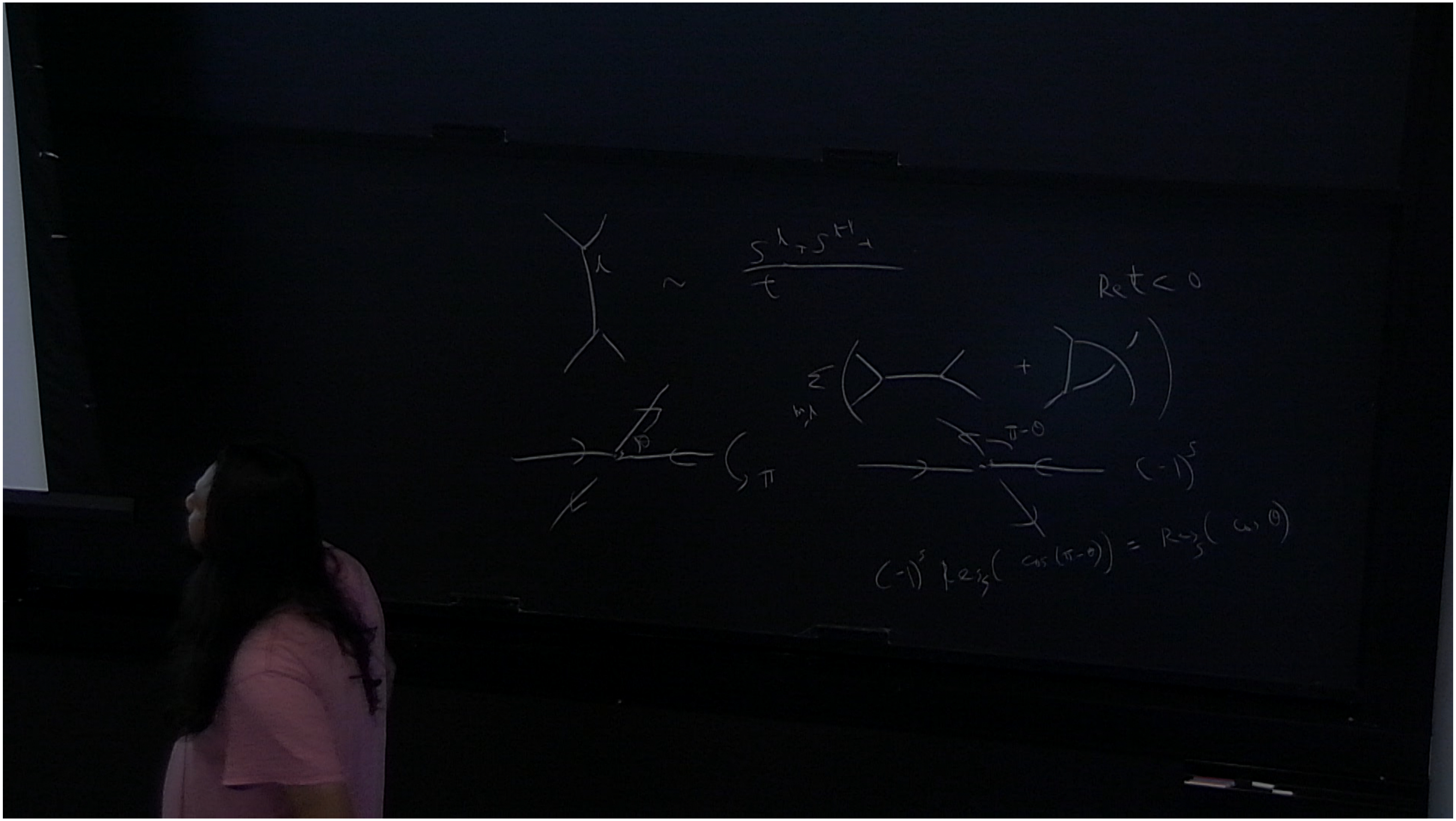
RMEs: $$g_j(-n) = h_j(-n), \quad j \leq 2n, \qquad (18)$$

SPEs: $$2g_J(b) = \sum_{j=1}^{J} (-1)^{j+1} \frac{\Gamma(-2b-J+j+1)}{\Gamma(j+1)\Gamma(-2b-J+1)} (P-b)^j g_{J-j}(b),$$

J odd,

$$0 = \sum_{j=1}^{J} (-1)^{j} \frac{\Gamma(-2b-J+j+1)}{\Gamma(j+1)\Gamma(-2b-J+1)} (P-b)^j g_{J-j}(b), \quad J \text{ even}$$

(19)


Re t < 0

Why is $k = 2$ Special?

SPEs fix coefficients of a^{2n-1} in terms of coefficients of a^{2n}.
This reflects the fact that only even spins are allowed in four identical-scalar scattering.

For odd k, since highest power of a is kn, alternate levels have leading spin odd.
It's to cancel this that that $(-1)^n$ was there; impossible to solve the equations.

For higher even k, leading spin jumps by 4, which means that half the levels with leading spin even are set to 0.
Overconstrains the system, only solution is 0.

Constraining k

k even, $k > 2$

Write SPEs and RMEs together as

$$kh_J\left(-\frac{N}{k}\right) = \left[1 + (-1)^{N-J}\right] g_J\left(-\frac{N}{k}\right) + (-1)^{N-J} \sum_{j-1}^{J} c_{N,k,J,j}\, g_{J-j}\left(-\frac{N}{k}\right) \quad (13)$$

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Solutions to the Boostrap Equations

Consider an amplitude $A_0(a,b,c)$ that satisfies all the assumptions that are discussed above.
Then an arbitrary amplitude constructed as follows also obeys all the assumptions of dual amplitudes [N. Khuri '69, E. Weimar '74]:

$$A(a,b,c) = \sum_{m=0}^{\infty} a_m A_0(a+m, b+m, c+m) \equiv \sum_{m=0}^{\infty} a_m A_m(a,b,c) \quad (20)$$

Most general?
Small amount of freedom that we don't know how to deal with.

Unitarity Bounds

If at some mass m, the amplitude constitutes of exchange particles of spin $0 \ldots L$, then one can decompose the reside at pole $s = m^2$ as:

$$\operatorname*{Res}_{s=m^2} A(s,t) = \sum_{l=0}^{L} \lambda_{m,l}^2 C_l^{\left(\frac{D-3}{2}\right)}(\cos(\theta)) \tag{21}$$

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Weak Unitarity $\lambda_{m,l}^2 \geq 0$.

Stronger Unitarity If there are multiple particles of same spin and mass with coupling constant $\lambda_{m,l,i}^2 \geq 0$.

Checked 'weak unitarity' type for perturbations of Virasoro-Shapiro, and doesn't seem to be particularly constrained.

Summary and Conclusions

Main results of this work are:

1. $k = 2$: Regge asymptotic behaviour necessarily same as string theory.
2. The bootstrap equations RMEs and SPEs, and infinite class of solutions (20).
3. Unitarity doesn't seem to be very constraining.

What next?

1. Can we derive the Virasoro-Shapiro Γ functions from the bootstrap equations?
   They seem very sigguestive.
2. Higher-point Amplitudes/Non-identical Particles.
3. Finding constraints on *graviton* scatterings.
4. Non-linear Spectra.
5. Can we get some mileage by making assumptions on the density of states?
6. Can we generalize the method to loop amplitudes?
7. Develop numerics to constrain the space of amplitudes.