Title: The Markov property of the CFT vacuum and the entropic a-theorem

Date: Sep 19, 2017 02:30 PM

URL: http://pirsa.org/17090059

Abstract: <p>A state is called a Markov state if it fulfil the important condition of saturating the Strong Subadditivity inequality. I will show how the vacuum state of any relativistic QFT is a Markov state when reduced to certain geometric regions of spacetime. A characterisation of this regions will be presented as well as two independent proofs of the Markov condition in QFT. </p>

<p>For the CFT vacuum, the Markov property is the key ingredient to prove the a-theorem (irreversibility of the RG flow in QFT in d=4 spacetime dimensions) using vacuum entanglement entropy. This extends the entropic proofs of the c and F theorems in dimensions d=2 and d=3 and gives a unified picture of all the known irreversibility theorems in QFT. </p>

<p>I will also comment on the relation of this Markov property with the unitarity bound and other information theory inequalities.</p>

The Markov property of the vacuum and the entropic a-theorem

Eduardo Testé
Instituto Balseiro
Centro Atómico Bariloche

Perimeter Institute, Sep 19th, 2017

Based on: 1703.10656,1704.01870
with **Horacio Casini** and **Gonzalo Torroba**

Plan of the talk

- Statement of the Markov property in QFT
- Sketch of a general proof (modular Hamiltonians)
- Some intuitions about being Markovian
- Entropic proof of the a-theorem
- concluding remarks

Eduardo Testé
Instituto Balseiro
Centro Atómico Bariloche

Statement of the Markov property

A state is a Markov state if it saturates the SSA

$$S(\rho_A) + S(\rho_B) \geqslant S(\rho_{A\cap B}) + S(\rho_{A\cup B})$$ ρ a generic state

$$S(\sigma_A) + S(\sigma_B) = S(\sigma_{A\cap B}) + S(\sigma_{A\cup B})$$ σ a Markov state

Eduardo Testé
Instituto Balseiro
Centro Atómico Bariloche

Statement of the Markov property

A state is a Markov state if it saturates the SSA

$$S(\rho_A) + S(\rho_B) \geqslant S(\rho_{A\cap B}) + S(\rho_{A\cup B}) \qquad \rho \text{ a generic state}$$

$$S(\sigma_A) + S(\sigma_B) = S(\sigma_{A\cap B}) + S(\sigma_{A\cup B}) \qquad \sigma \text{ a Markov state}$$

Why the name?

Classically, a distribution $p(x, y, z)$ is Markovian if the marginals fulfil the Markov condition

$$p(x|y, z) = p(x|y)$$

Eduardo Testé
Instituto Balseiro
Centro Atómico Bariloche

Statement of the Markov property

A state is a Markov state if it saturates the SSA

$$S(\rho_A) + S(\rho_B) \geqslant S(\rho_{A\cap B}) + S(\rho_{A\cup B})$$ ρ a generic state

$$S(\sigma_A) + S(\sigma_B) = S(\sigma_{A\cap B}) + S(\sigma_{A\cup B})$$ σ a Markov state

Why the name?

Classically, a distribution $p(x,y,z)$ is Markovian if the marginals fulfil the Markov condition

$$p(x|y,z) = p(x|y)$$

From this condition the full distribution can be reconstructed from its marginal in the form

$$p(x,y,z) \overset{\text{def}}{=} p(z|y,z)p(y,z) \overset{\text{Markov}}{=} p(z|y)p(y,z) \overset{\text{def}}{=} \frac{p(z,y)}{p(y)}p(y,z)$$

Eduardo Testé
Instituto Balseiro
Centro Atómico Bariloche

Statement of the Markov property

A state is a Markov state if it saturates the SSA

$$S(\rho_A) + S(\rho_B) \geqslant S(\rho_{A\cap B}) + S(\rho_{A\cup B})$$ ρ a generic state

$$S(\sigma_A) + S(\sigma_B) = S(\sigma_{A\cap B}) + S(\sigma_{A\cup B})$$ σ a Markov state

Why the name?

Classically, a distribution $p(x,y,z)$ is Markovian if the marginals fulfil the Markov condition

$$p(x|y,z) = p(x|y)$$

From this condition the full distribution can be reconstructed from its marginal in the form

$$p(x,y,z) \overset{\text{def}}{=} p(z|y,z)p(y,z) \overset{\text{Markov}}{=} p(z|y)p(y,z) \overset{\text{def}}{=} \frac{p(z,y)}{p(y)}p(y,z)$$

$$\log p(x,y) + \log p(y,z) = \log p(x,y,z) + \log p(y)$$

$$S(p(x,y)) + S(p(y,z)) = S(p(x,y,z)) + S(p(y))$$

Eduardo Testé
Instituto Balseiro
Centro Atómico Bariloche

Statement of the Markov property

Classically

$$p(x|y,z) = p(x|y)$$

$$p(x,y,z) = \frac{p(x,y)p(y,z)}{p(y)}$$

$$\log p(x,y) + \log p(y,z) = \log p(x,y,z) + \log p(y)$$

$$S(p(x,y)) + S(p(y,z)) = S(p(x,y,z)) + S(p(y))$$

Eduardo Testé
Instituto Balseiro
Centro Atómico Bariloche

Statement of the Markov property

Classically

$$p(x|y,z) = p(x|y)$$

$$p(x,y,z) = \frac{p(x,y)p(y,z)}{p(y)}$$

$$\log p(x,y) + \log p(y,z) = \log p(x,y,z) + \log p(y)$$

$$S(p(x,y)) + S(p(y,z)) = S(p(x,y,z)) + S(p(y))$$

QM (we call σ Markovian if)

$$\longrightarrow \quad H_A + H_B = H_{A\vee B} + H_{A\wedge B} \qquad (H_A = -\log \rho_A \otimes 1_{\bar{A}})$$

$$\longrightarrow \quad S(\sigma_A) + S(\sigma_B) = S(\sigma_{A\vee B}) + S(\sigma_{A\wedge B})$$

Eduardo Testé
Instituto Balseiro
Centro Atómico Bariloche

Statement of the Markov property

Classically

$$p(x|y,z) = p(x|y)$$

$$p(x,y,z) = \frac{p(x,y)p(y,z)}{p(y)}$$

$$\log p(x,y) + \log p(y,z) = \log p(x,y,z) + \log p(y)$$

$$S(p(x,y)) + S(p(y,z)) = S(p(x,y,z)) + S(p(y))$$

QM (we call σ Markovian if)

$\longrightarrow$ $$\sigma_{A\vee B} = e^{\log\sigma_A + \log\sigma_B - \log\sigma_{A\wedge B}}$$

$\longrightarrow$ $$H_A + H_B = H_{A\vee B} + H_{A\wedge B} \qquad (H_A = -\log\rho_A \otimes 1_{\bar{A}})$$

$\longrightarrow$ $$S(\sigma_A) + S(\sigma_B) = S(\sigma_{A\vee B}) + S(\sigma_{A\wedge B})$$

Eduardo Testé
Instituto Balseiro
Centro Atómico Bariloche

Statement of the Markov property

Classically

$p(x|y,z) = p(x|y)$

$p(x,y,z) = \frac{p(x,y)p(y,z)}{p(y)}$

$\log p(x,y) + \log p(y,z) = \log p(x,y,z) + \log p(y)$

$S(p(x,y)) + S(p(y,z)) = S(p(x,y,z)) + S(p(y))$

QM (we call σ Markovian if)

$\longrightarrow \sigma_{A\vee B} = e^{\log\sigma_A + \log\sigma_B - \log\sigma_{A\wedge B}}$

$\longrightarrow H_A + H_B = H_{A\vee B} + H_{A\wedge B} \qquad (H_A = -\log\rho_A \otimes 1_{\bar{A}})$

$\longrightarrow S(\sigma_A) + S(\sigma_B) = S(\sigma_{A\vee B}) + S(\sigma_{A\wedge B})$

…we are interested in this because

$$S(\rho_A||\sigma_A) + S(\rho_B||\sigma_B) \leqslant S(\rho_{A\vee B}||\sigma_{A\vee B}) + S(\rho_{A\wedge B}||\sigma_{A\wedge B})$$

This is the Strong (Super)Additivity of Relative Entropy
it holds when the second entry state is Markovian!, (not in general)

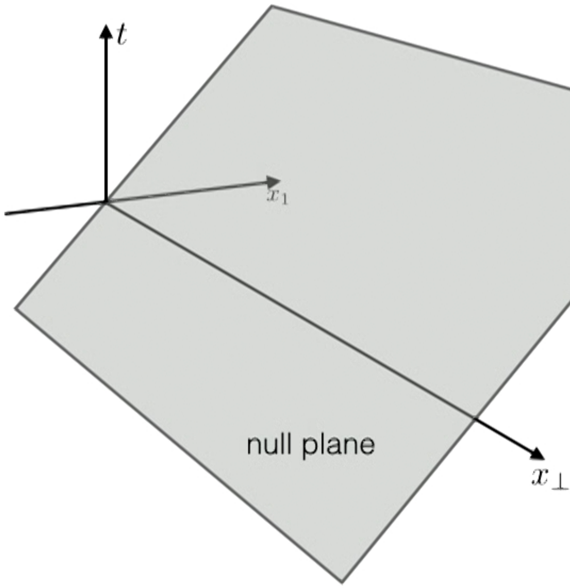
like SSA, but now each term has the monotonicity property (better defined in QFT)

Eduardo Testé
Instituto Balseiro
Centro Atómico Bariloche

The statement for the vacuum

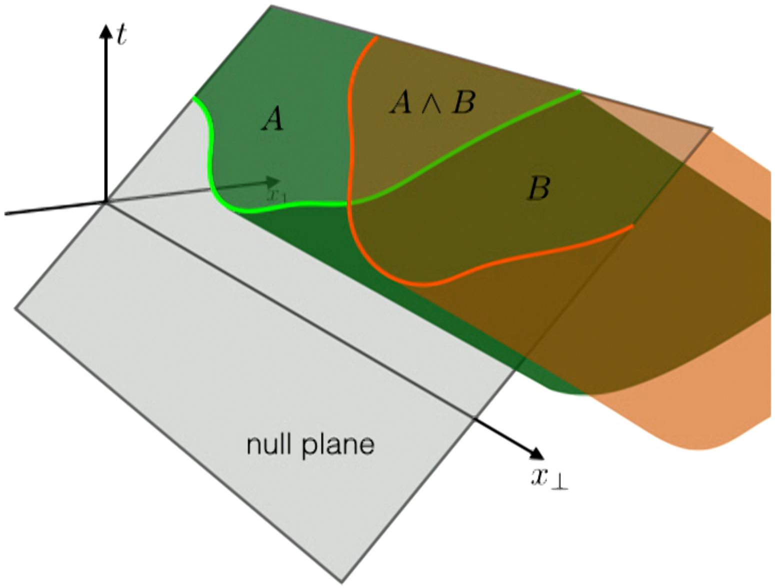
Eduardo Testé
Instituto Balseiro
Centro Atómico Bariloche

The statement for the vacuum



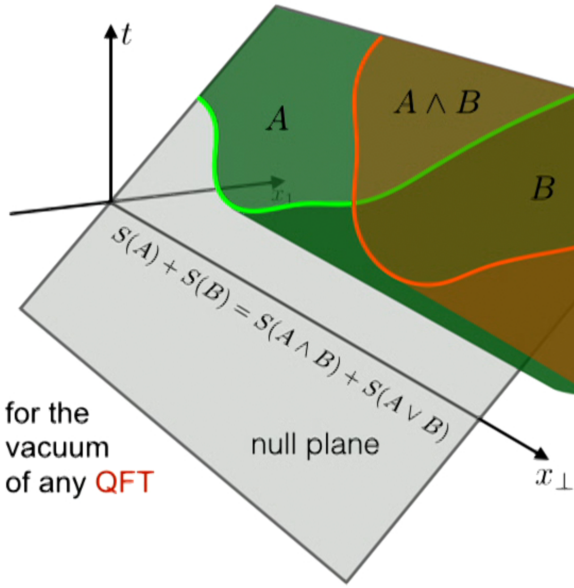
Eduardo Testé
Instituto Balseiro
Centro Atómico Bariloche

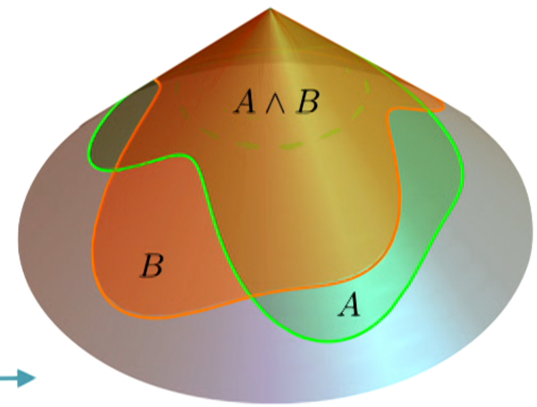
The statement for the vacuum



Eduardo Testé
Instituto Balseiro
Centro Atómico Bariloche

The statement for the vacuum





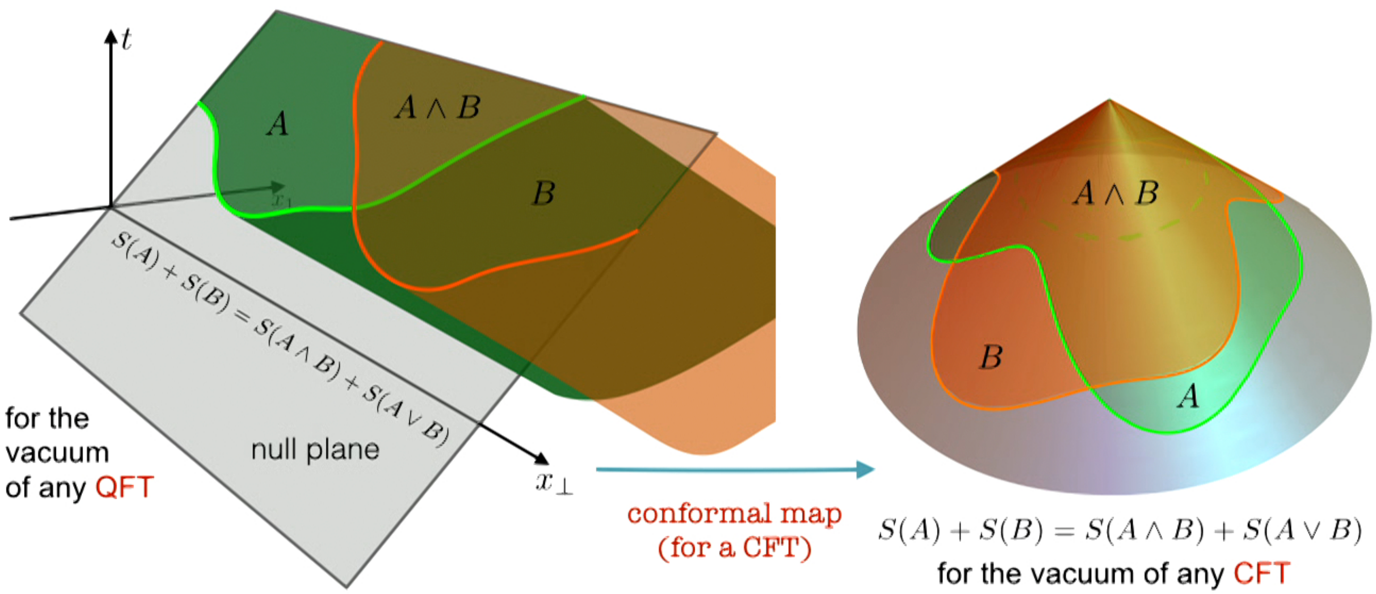
conformal map (for a CFT)

$$S(A) + S(B) = S(A \wedge B) + S(A \vee B)$$

for the vacuum of any CFT

Eduardo Testé
Instituto Balseiro
Centro Atómico Bariloche

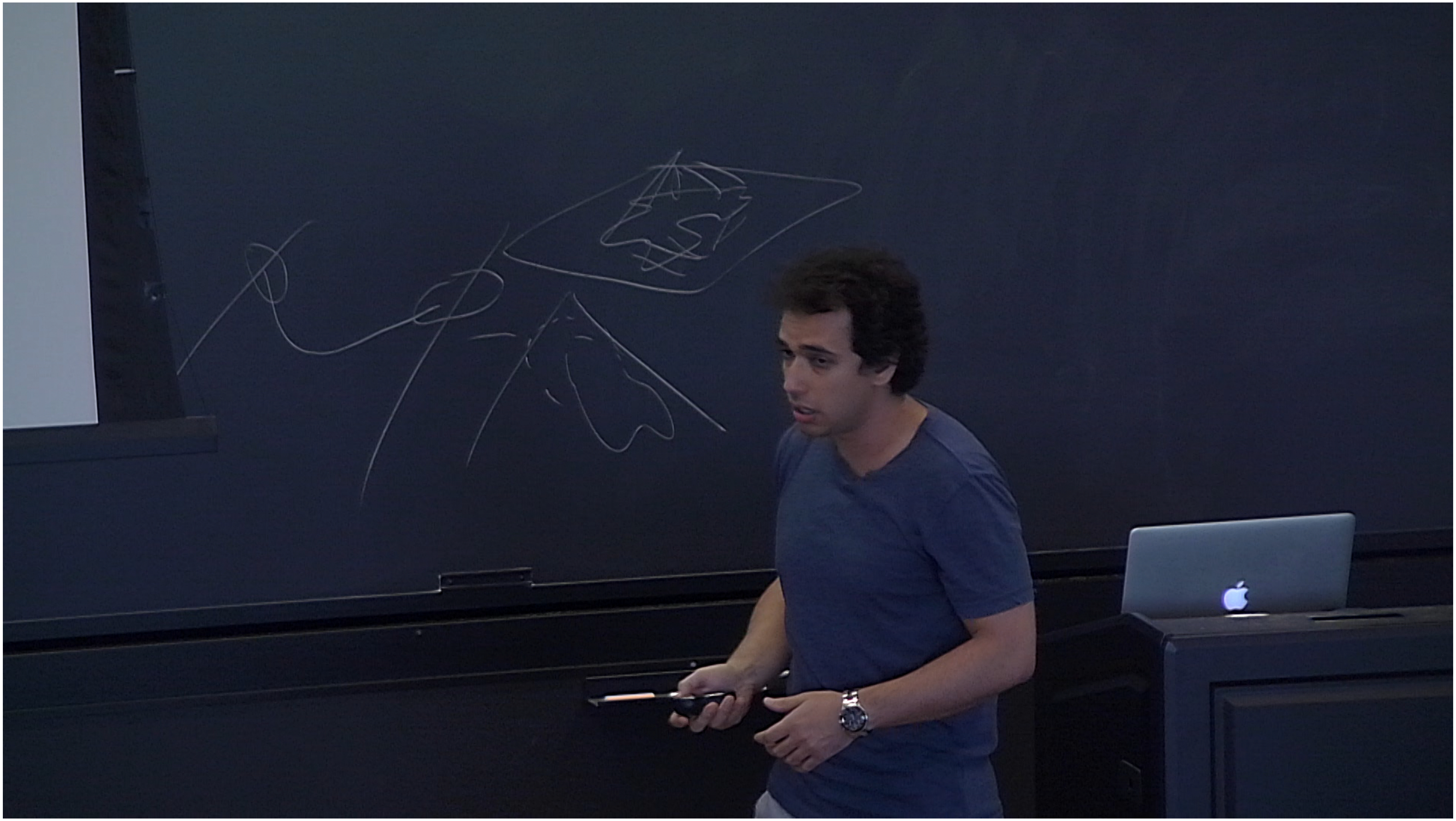
The statement for the vacuum



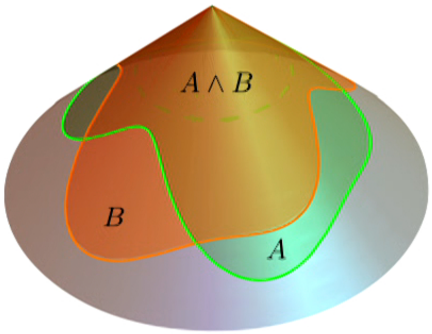
Remarks

- no relation with null quantisation
- these regions always have spacetime volume (in order to have a subalgebra of operators)
- the requirement is: the future horizon of these spacetime regions lies on a null plane

Eduardo Testé
Instituto Balseiro
Centro Atómico Bariloche

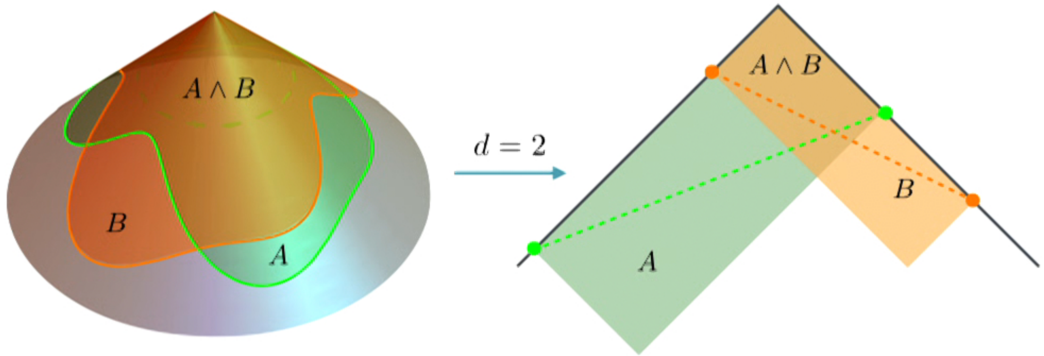


Other observations...



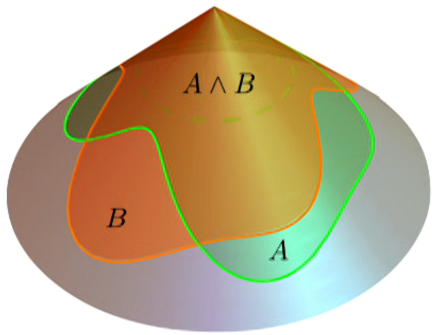
Eduardo Testé
Instituto Balseiro
Centro Atómico Bariloche

Other observations…

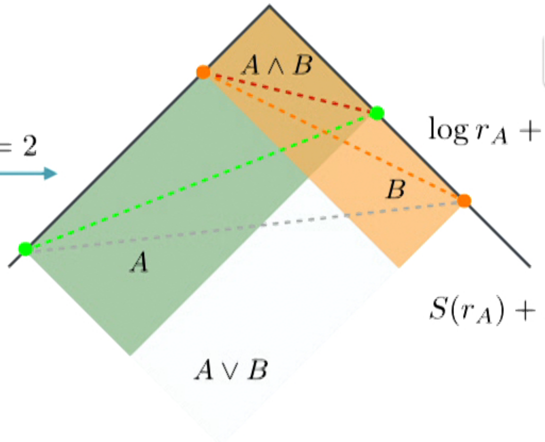


Eduardo Testé
Instituto Balseiro
Centro Atómico Bariloche

Other observations…



$d = 2$



a relativistic computation gives (for proper radius)

$$r_A \, r_B = r_{A\wedge B} \, r_{A\vee B}$$

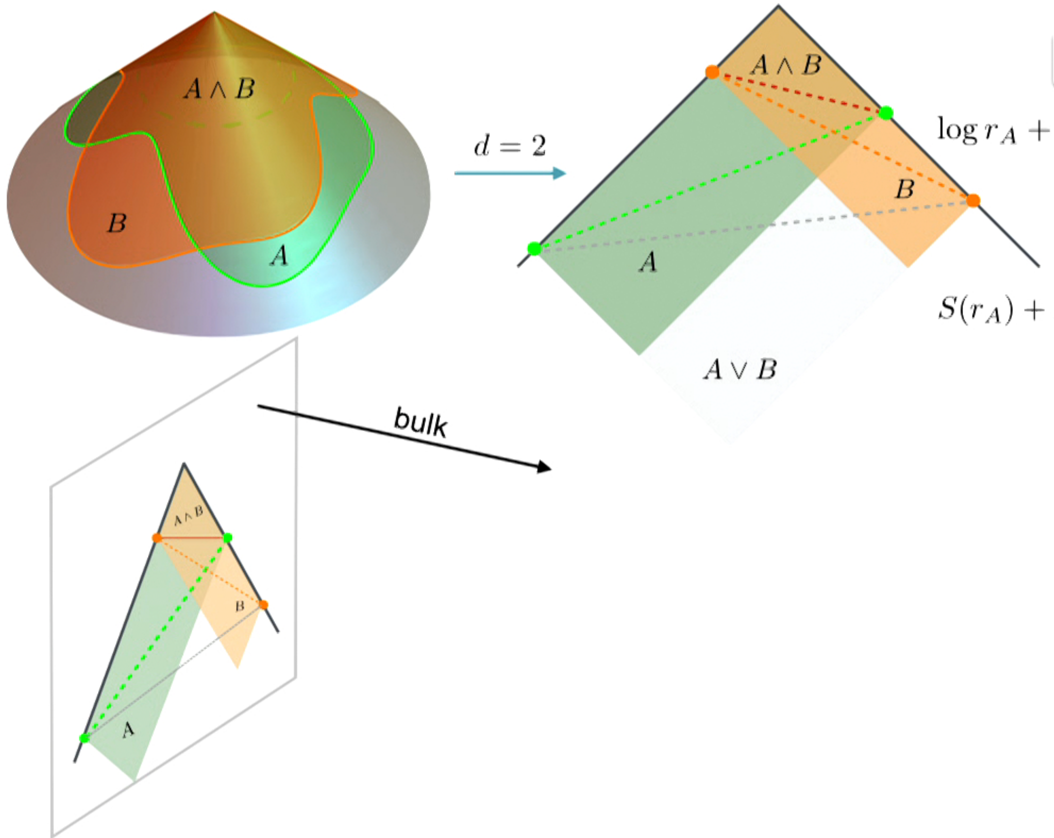
$$\log r_A + \log r_B = \log r_{A\wedge B} + \log r_{A\vee B}$$

$$S^{d=2}_{CFT}(r) = \frac{c}{3}\log(r/\epsilon)$$

$$S(r_A) + S(r_B) = S(A\wedge B) + S(A \vee B)$$

Eduardo Testé
Instituto Balseiro
Centro Atómico Bariloche

Other observations…



a relativistic computation gives
(for proper radius)

$$r_A \; r_B = r_{A\wedge B} \; r_{A\vee B}$$

$$\log r_A + \log r_B = \log r_{A\wedge B} + \log r_{A\vee B}$$

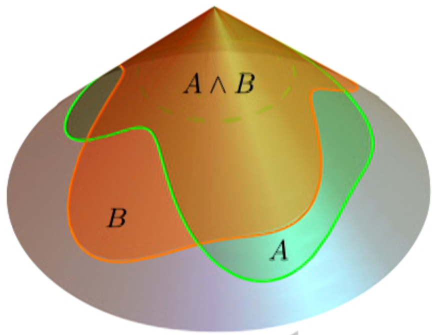
$$S^{d=2}_{CFT}(r) = \frac{c}{3}\log(r/\epsilon)$$

$$S(r_A) + S(r_B) = S(A\wedge B) + S(A\vee B)$$

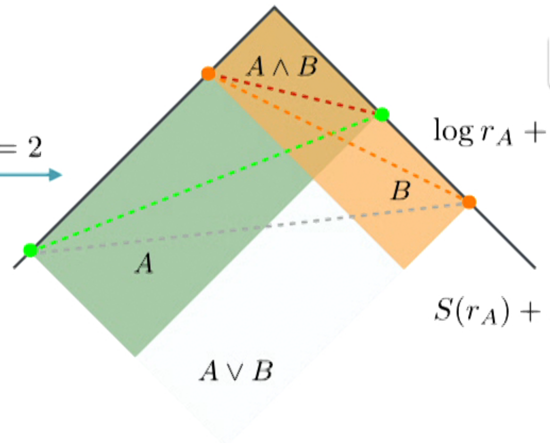
Eduardo Testé
Instituto Balseiro
Centro Atómico Bariloche

Other observations…

a relativistic computation gives
(for proper radius)



$d = 2$

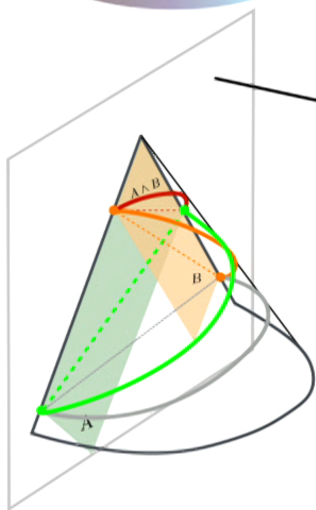


$$r_A \, r_B = r_{A\wedge B} \, r_{A\vee B}$$

$$\log r_A + \log r_B = \log r_{A\wedge B} + \log r_{A\vee B}$$

$$S^{d=2}_{CFT}(r) = \frac{c}{3}\log(r/\epsilon)$$

$$S(r_A) + S(r_B) = S(A\wedge B) + S(A\vee B)$$



bulk

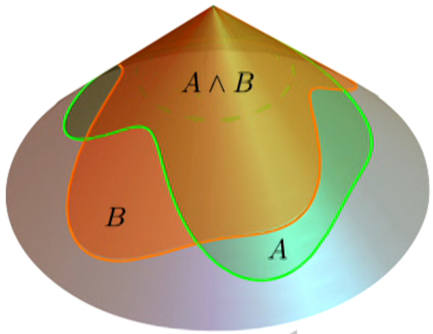
RHT surfaces are such that

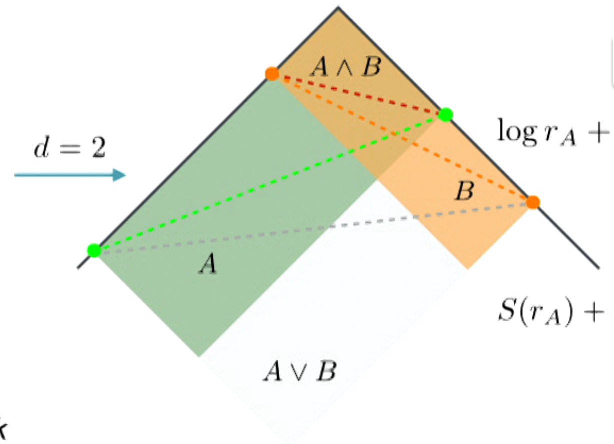
$$l + l = l + l$$

$$S(r_A) + S(r_B) = S(A\wedge B) + S(A\vee B)$$

Eduardo Testé
Instituto Balseiro
Centro Atómico Bariloche

Other observations…





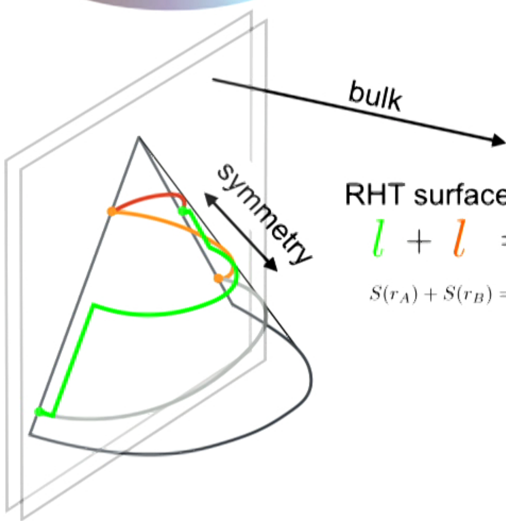
a relativistic computation gives (for proper radius)

$$r_A \; r_B = r_{A\wedge B} \; r_{A\vee B}$$

$$\log r_A + \log r_B = \log r_{A\wedge B} + \log r_{A\vee B}$$

$$S^{d=2}_{CFT}(r) = \frac{c}{3}\log(r/\epsilon)$$

$$S(r_A) + S(r_B) = S(A\wedge B) + S(A\vee B)$$



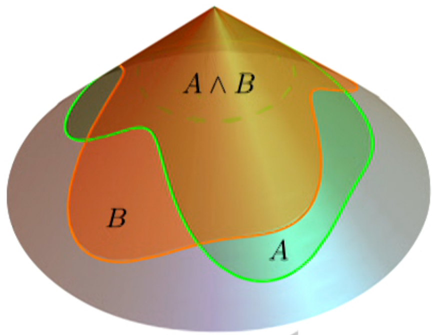
RHT surfaces are such that

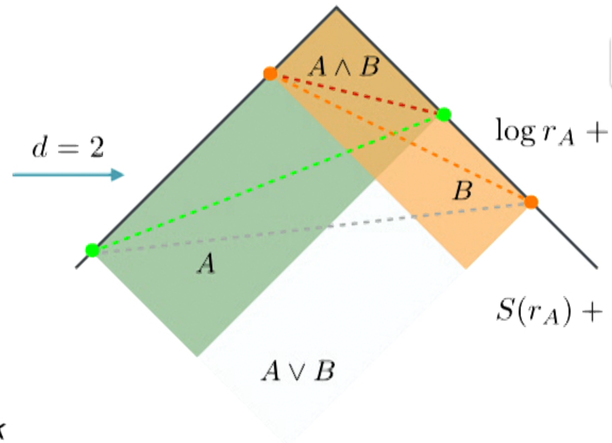
$$l + l = l + l$$

$$S(r_A) + S(r_B) = S(A\wedge B) + S(A\vee B)$$

Eduardo Testé
Instituto Balseiro
Centro Atómico Bariloche

Other observations…





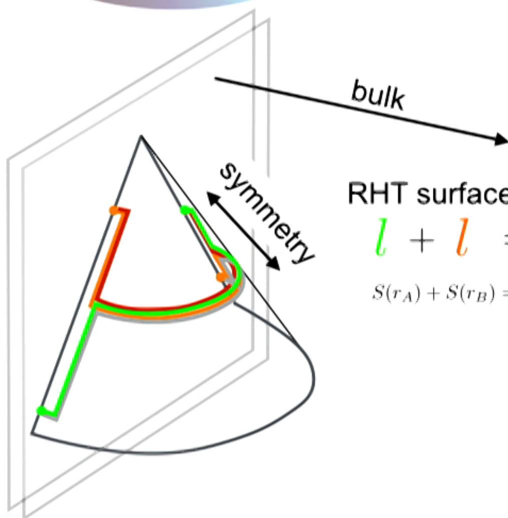
a relativistic computation gives
(for proper radius)

$$r_A \; r_B = r_{A\wedge B} \; r_{A\vee B}$$

$$\log r_A + \log r_B = \log r_{A\wedge B} + \log r_{A\vee B}$$

$$S^{d=2}_{CFT}(r) = \frac{c}{3}\log(r/\epsilon)$$

$$S(r_A) + S(r_B) = S(A\wedge B) + S(A\vee B)$$



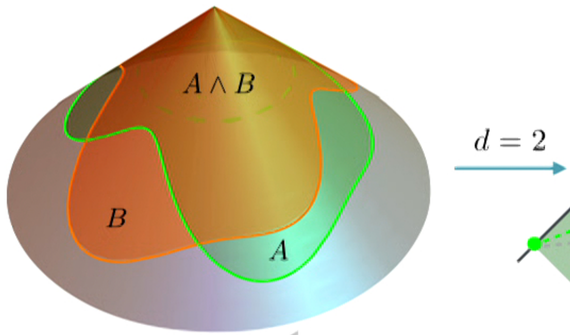
RHT surfaces are such that

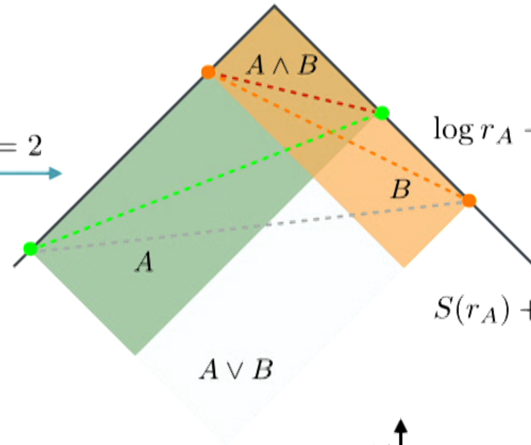
$$l + l = l + l$$

$$S(r_A) + S(r_B) = S(A\wedge B) + S(A\vee B)$$

Eduardo Testé
Instituto Balseiro
Centro Atómico Bariloche

Other observations…





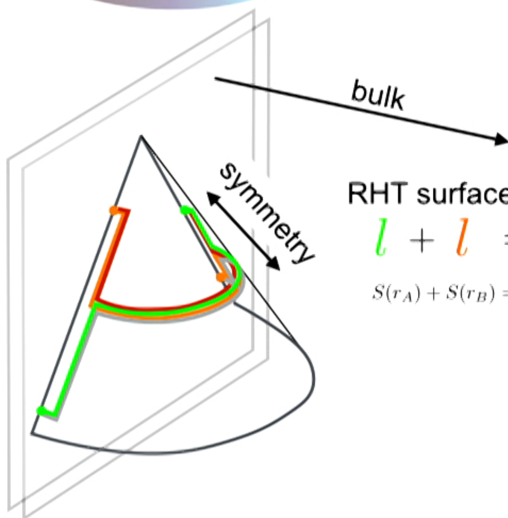
a relativistic computation gives
(for proper radius)

$$r_A \, r_B = r_{A\wedge B} \, r_{A\vee B}$$

$$\log r_A + \log r_B = \log r_{A\wedge B} + \log r_{A\vee B}$$

$$S^{d=2}_{CFT}(r) = \frac{c}{3}\log(r/\epsilon)$$

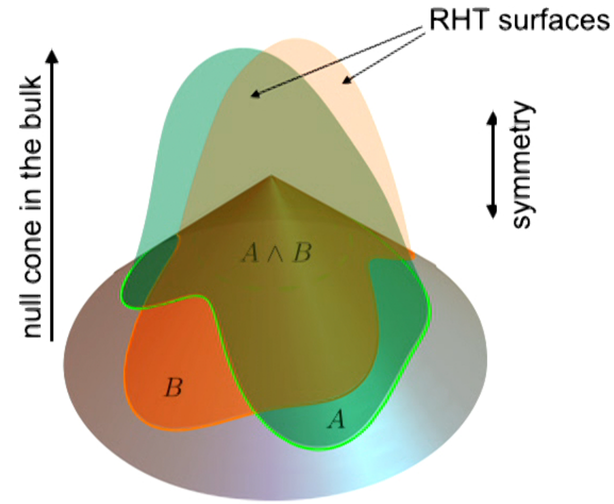
$$S(r_A) + S(r_B) = S(A\wedge B) + S(A\vee B)$$



RHT surfaces are such that

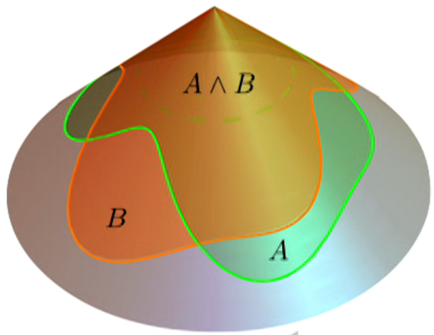
$$l + l = l + l$$

$$S(r_A) + S(r_B) = S(A\wedge B) + S(A\vee B)$$

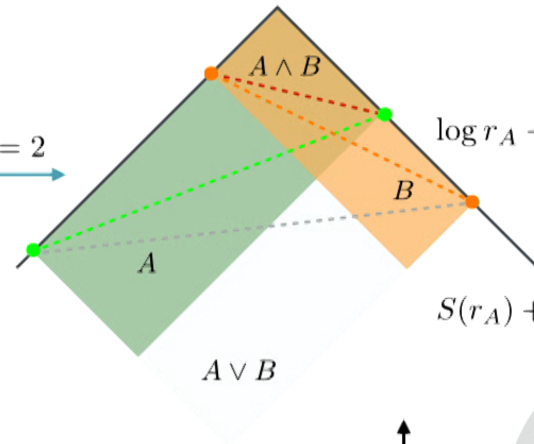


Eduardo Testé
Instituto Balseiro
Centro Atómico Bariloche

Other observations…



$d = 2$



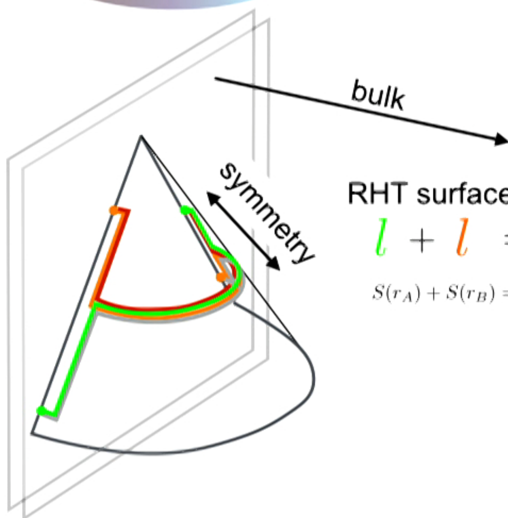
a relativistic computation gives
(for proper radius)

$$r_A \; r_B = r_{A\wedge B} \; r_{A\vee B}$$

$$\log r_A + \log r_B = \log r_{A\wedge B} + \log r_{A\vee B}$$

$$S^{d=2}_{CFT}(r) = \frac{c}{3}\log(r/\epsilon)$$

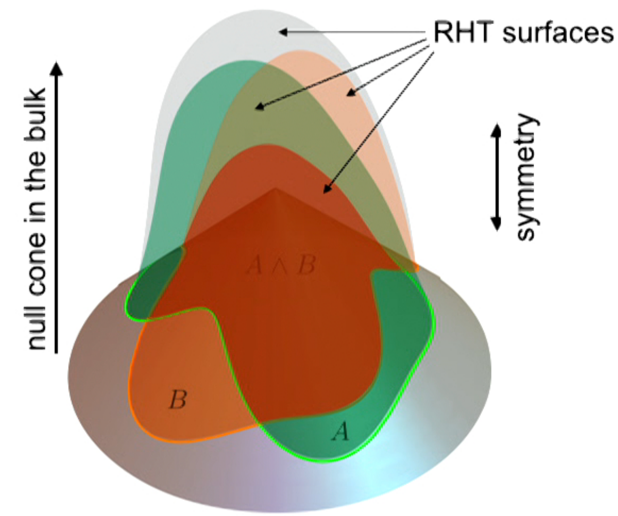
$$S(r_A) + S(r_B) = S(A\wedge B) + S(A\vee B)$$



RHT surfaces are such that

$$l + l = l + l$$

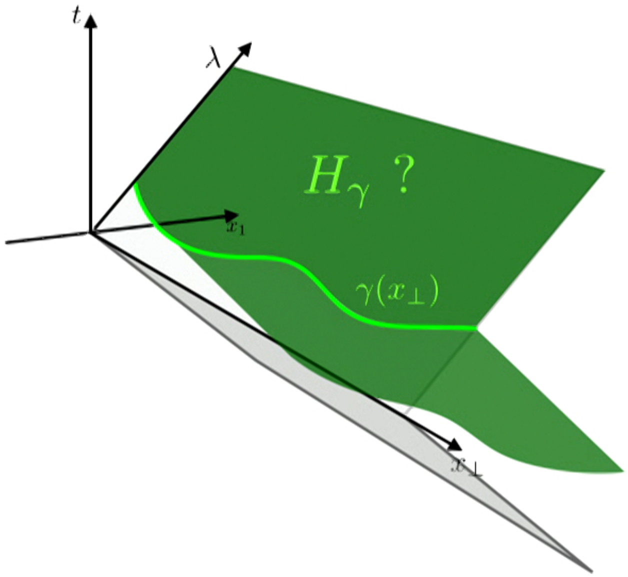
$$S(r_A) + S(r_B) = S(A\wedge B) + S(A\vee B)$$



Eduardo Testé
Instituto Balseiro
Centro Atómico Bariloche

Sketch of a general proof (modular Hamiltonians)

Idea: find the explicit form of the modular Hamiltonian of these regions:



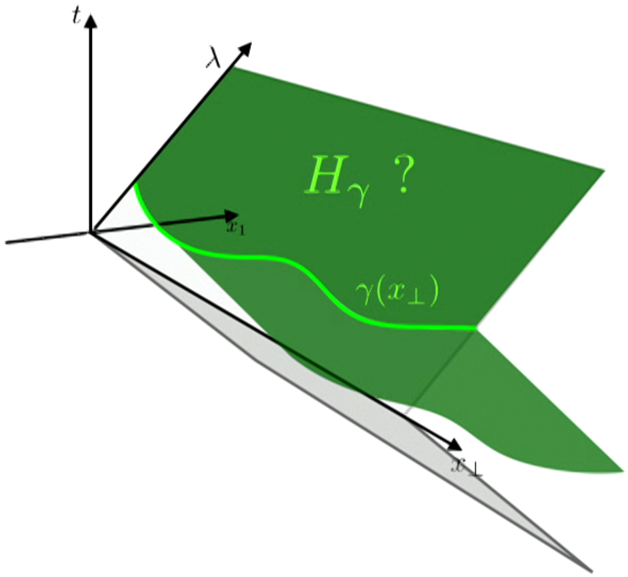
Eduardo Testé
Instituto Balseiro
Centro Atómico Bariloche

Sketch of a general proof (modular Hamiltonians)

Idea: find the explicit form of the modular Hamiltonian of these regions:

Obs:
this operator can be written as the integral of a local operator only in the null Cauchy surface

$$H_\gamma = 2\pi \int d^{d-2}x_\perp \int d\lambda\ (\lambda - \gamma(x_\perp))T_{\lambda\lambda}$$

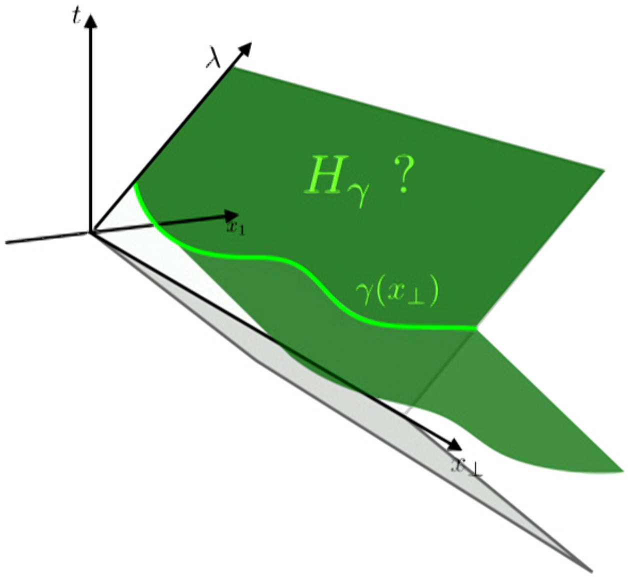


Eduardo Testé
Instituto Balseiro
Centro Atómico Bariloche

Sketch of a general proof (modular Hamiltonians)

Idea: find the explicit form of the modular Hamiltonian of these regions:

Obs:
this operator can be written as the integral of
a local operator only in the null Cauchy surface



$$H_\gamma = 2\pi \int d^{d-2}x_\perp \int d\lambda\, (\lambda - \gamma(x_\perp)) T_{\lambda\lambda}$$

(from this the it follows directly the Markov equality)

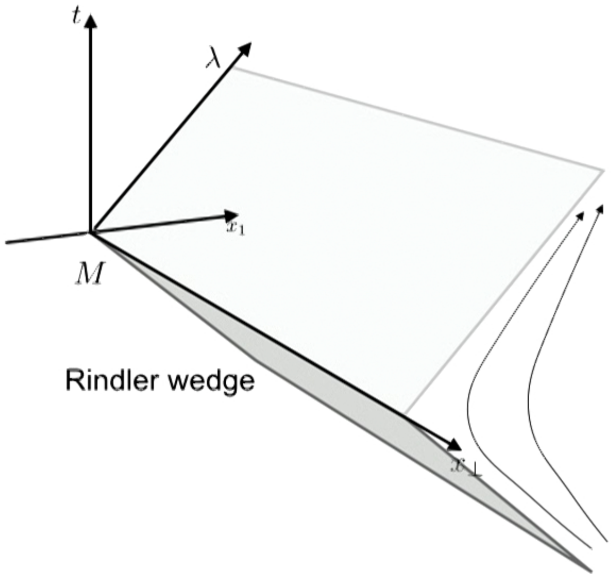
$$H_{\gamma_1} + H_{\gamma_2} = H_{\gamma_1 \cap \gamma_2} + H_{\gamma_1 \cup \gamma_2}$$

Eduardo Testé
Instituto Balseiro
Centro Atómico Bariloche

Sketch of a general proof (modular Hamiltonians)

Obs:
this operator can be written as the integral of
a local operator only in the null Cauchy surface

Idea: find the explicit form of the modular Hamiltonian of these regions:



$$H_\gamma = 2\pi \int d^{d-2}x_\perp \int d\lambda \, (\lambda - \gamma(x_\perp)) T_{\lambda\lambda}$$

$H_{M=\mathrm{Rindler}}$

$U_M(s) = e^{-isH_M}$ (boost): moves geometrically any local operator

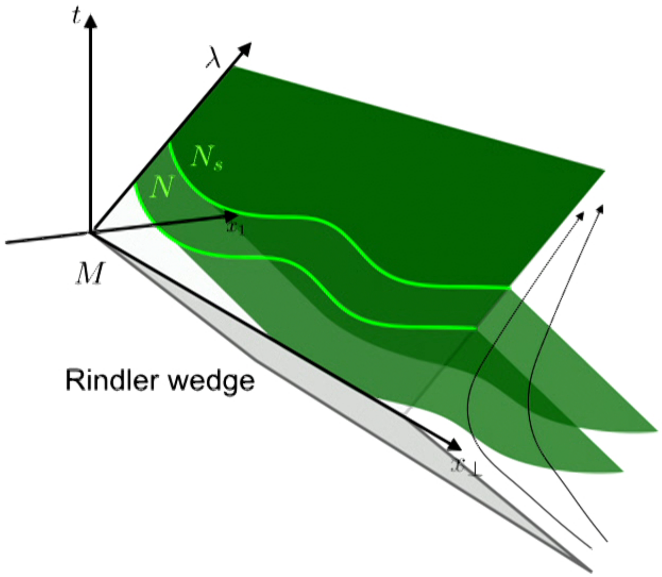
Eduardo Testé
Instituto Balseiro
Centro Atómico Bariloche

Sketch of a general proof (modular Hamiltonians)

Idea: find the explicit form of the modular Hamiltonian of these regions:

Obs:
this operator can be written as the integral of
a local operator only in the null Cauchy surface

$$H_\gamma = 2\pi \int d^{d-2}x_\perp \int d\lambda \, (\lambda - \gamma(x_\perp)) T_{\lambda\lambda}$$



$H_{M=\text{Rindler}}$

$U_M(s) = e^{-isH_M}$ (boost): moves geometrically any local operator

$$U_M(-s) N U_M(s) = N_s \subset N \quad ; \quad s > 0$$

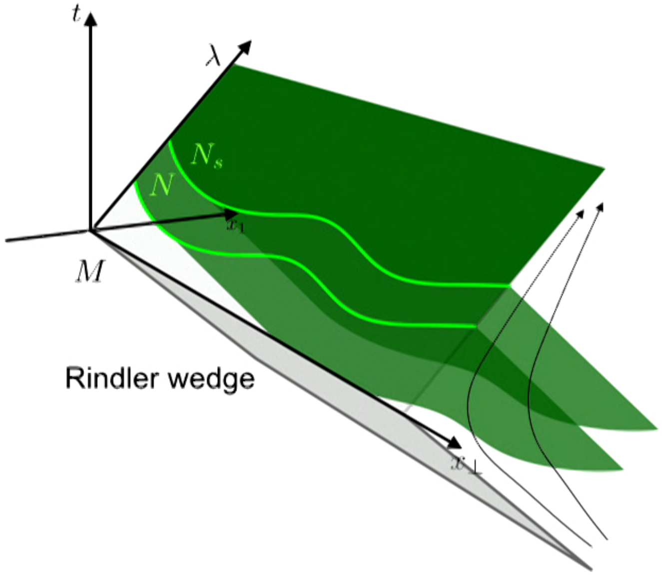
Eduardo Testé
Instituto Balseiro
Centro Atómico Bariloche

Sketch of a general proof (modular Hamiltonians)

Idea: find the explicit form of the modular Hamiltonian of these regions:

Obs:
this operator can be written as the integral of
a local operator only in the null Cauchy surface

$$H_\gamma = 2\pi \int d^{d-2}x_\perp \int d\lambda\, (\lambda - \gamma(x_\perp)) T_{\lambda\lambda}$$



$H_{M=\text{Rindler}}$

$U_M(s) = e^{-isH_M}$ (boost): moves geometrically any local operator

$$U_M(-s) N U_M(s) = N_s \subset N \quad ; \quad s > 0$$

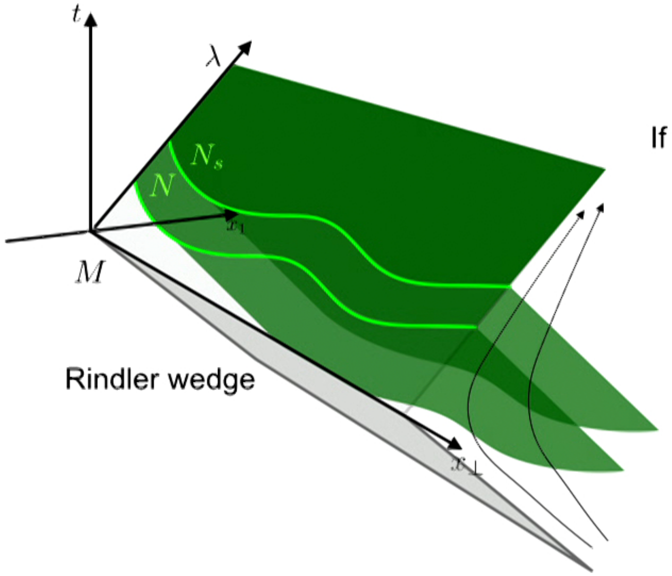
half sided modular inclusion condition

Eduardo Testé
Instituto Balseiro
Centro Atómico Bariloche

Sketch of a general proof (modular Hamiltonians)

Idea: find the explicit form of the modular Hamiltonian of these regions:

Obs:
this operator can be written as the integral of
a local operator only in the null Cauchy surface



$$H_\gamma = 2\pi \int d^{d-2}x_\perp \int d\lambda\; (\lambda - \gamma(x_\perp))T_{\lambda\lambda}$$

If (M, N) are in a half sided modular inclusion situation then:
[Borchers 93, Wiesbrok 93]

$$U_M(-s)NU_M(s) = N_s \subset N \quad ; \quad s > 0$$

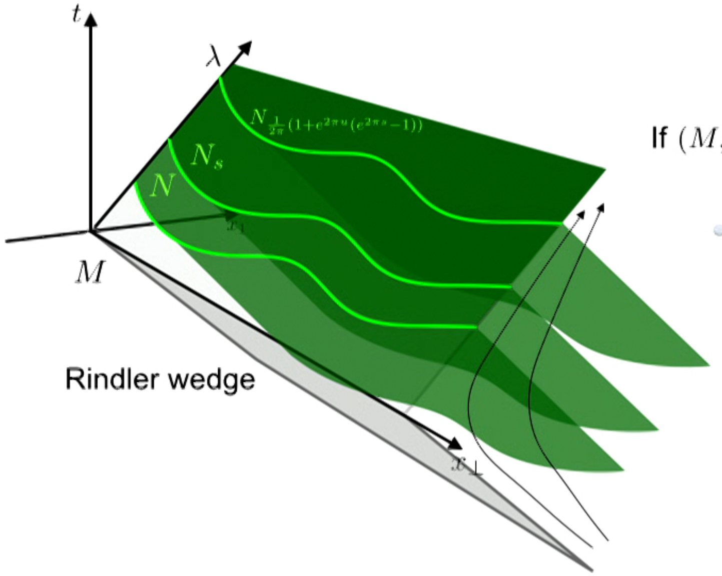
half sided modular inclusion condition

Eduardo Testé
Instituto Balseiro
Centro Atómico Bariloche

Sketch of a general proof (modular Hamiltonians)

Obs:
this operator can be written as the integral of
a local operator only in the null Cauchy surface

Idea: find the explicit form of the modular Hamiltonian of these regions:



$$H_\gamma = 2\pi \int d^{d-2}x_\perp \int d\lambda \; (\lambda - \gamma(x_\perp)) T_{\lambda\lambda}$$

If (M, N) are in a half sided modular inclusion situation then:
[Borchers 93, Wiesbrok 93]

- Thm 1: The modular Hamiltonian of N moves geometrically any region of the family N_s (to another region of the family):

$$U_N(-u) N_s U_N(u) = N_{\frac{1}{2\pi}\log(1+e^{2\pi u}(e^{2\pi s}-1))}$$

$$U_M(-s) N U_M(s) = N_s \subset N \quad ; \quad s > 0$$

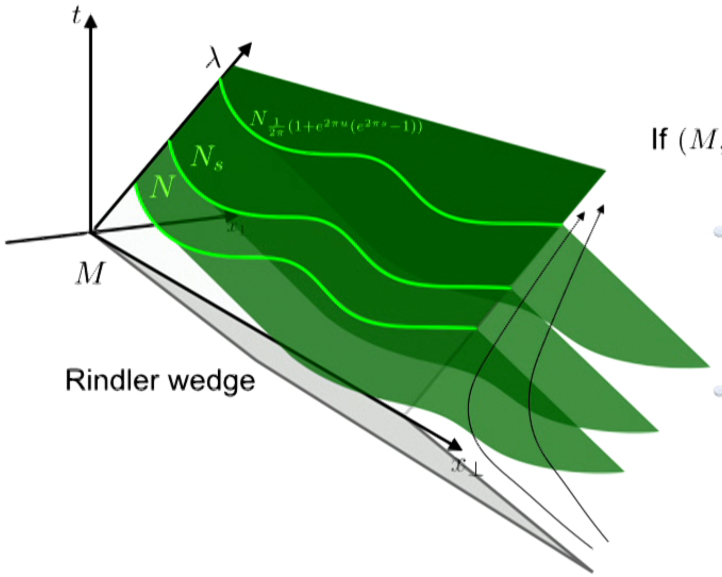
half sided modular inclusion condition

Eduardo Testé
Instituto Balseiro
Centro Atómico Bariloche

Sketch of a general proof (modular Hamiltonians)

Idea: find the explicit form of the modular Hamiltonian of these regions:

Obs:
this operator can be written as the integral of
a local operator only in the null Cauchy surface



$$H_\gamma = 2\pi \int d^{d-2}x_\perp \int d\lambda \, (\lambda - \gamma(x_\perp)) T_{\lambda\lambda}$$

If (M, N) are in a half sided modular inclusion situation then:
[Borchers 93, Wiesbrok 93]

- Thm 1: The modular Hamiltonian of N moves geometrically any region of the family N_s (to another region of the family):
$$U_N(-u) N_s U_N(u) = N_{\frac{1}{2\pi}\log(1+e^{2\pi u}(e^{2\pi s}-1))}$$
- Thm 2: $\hat{H}_M - \hat{H}_N = G \geqslant 0$
$$[\hat{H}_M, \hat{H}_n] = 2\pi i(\hat{H}_M - \hat{H}_N) = 2\pi i G$$

$$U_M(-s) N U_M(s) = N_s \subset N \quad ; \quad s > 0$$
half sided modular inclusion condition

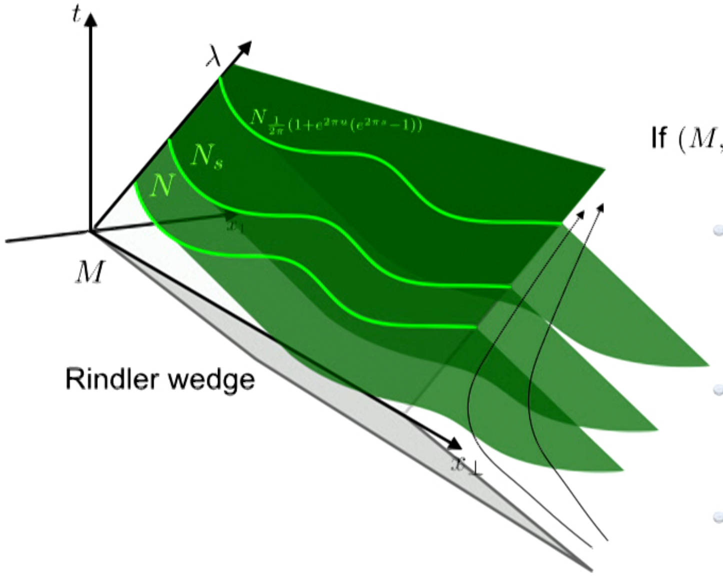
Eduardo Testé
Instituto Balseiro
Centro Atómico Bariloche

Sketch of a general proof (modular Hamiltonians)

Idea: find the explicit form of the modular Hamiltonian of these regions:

Obs:
this operator can be written as the integral of
a local operator only in the null Cauchy surface

$$H_\gamma = 2\pi \int d^{d-2}x_\perp \int d\lambda\; (\lambda - \gamma(x_\perp)) T_{\lambda\lambda}$$



$$U_M(-s) N U_M(s) = N_s \subset N \quad ; \quad s > 0$$

half sided modular inclusion condition

If (M, N) are in a half sided modular inclusion situation then:
[Borchers 93, Wiesbrok 93]

- Thm 1: The modular Hamiltonian of N moves geometrically any region of the family N_s (to another region of the family):

  $$U_N(-u) N_s U_N(u) = N_{\frac{1}{2\pi}\log(1+e^{2\pi u}(e^{2\pi s}-1))}$$

- Thm 2: $\hat{H}_M - \hat{H}_N = G \geqslant 0$

  $$[\hat{H}_M, \hat{H}_n] = 2\pi i(\hat{H}_M - \hat{H}_N) = 2\pi i G$$

- Thm 3: $T_{M,N}(\tau) = e^{-iG\tau}$, the modular translation (relative to (M, N)) moves the N_s geometrically:

  $$T(-\tau) N_s T(\tau) = N_{\frac{1}{2\pi}\log(e^{2\pi s}+2\pi\tau)}$$

  intuition: when M and N are both Rindler, $T_{M,N}$ is the null translation.

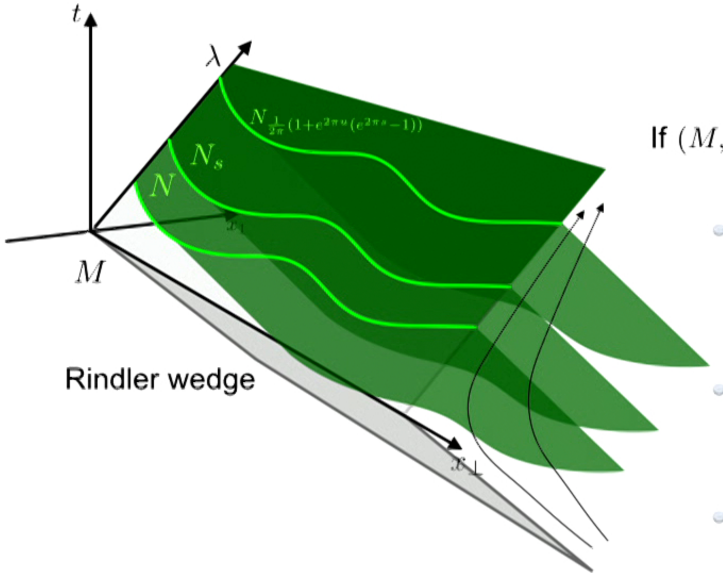
Eduardo Testé
Instituto Balseiro
Centro Atómico Bariloche

Sketch of a general proof (modular Hamiltonians)

Idea: find the explicit form of the modular Hamiltonian of these regions:

Obs:
this operator can be written as the integral of
a local operator only in the null Cauchy surface

$$H_\gamma = 2\pi \int d^{d-2}x_\perp \int d\lambda \; (\lambda - \gamma(x_\perp))T_{\lambda\lambda}$$



$$U_M(-s)NU_M(s) = N_s \subset N \quad ; \quad s > 0$$

half sided modular inclusion condition

If (M, N) are in a half sided modular inclusion situation then:
[Borchers 93, Wiesbrok 93]

- Thm 1: The modular Hamiltonian of N moves geometrically any region of the family N_s (to another region of the family):

$$U_N(-u)N_sU_N(u) = N_{\frac{1}{2\pi}\log(1+e^{2\pi u}(e^{2\pi s}-1))}$$

- Thm 2: $\hat{H}_M - \hat{H}_N = G \geqslant 0$

$$[\hat{H}_M, \hat{H}_n] = 2\pi i(\hat{H}_M - \hat{H}_N) = 2\pi i G$$

- Thm 3: $T_{M,N}(\tau) = e^{-iG\tau}$, the modular translation (relative to (M, N)) moves the N_s geometrically:

$$T(-\tau)N_sT(\tau) = N_{\frac{1}{2\pi}\log(e^{2\pi s}+2\pi\tau)}$$

intuition: when M and N are both Rindler, $T_{M,N}$ is the null translation.

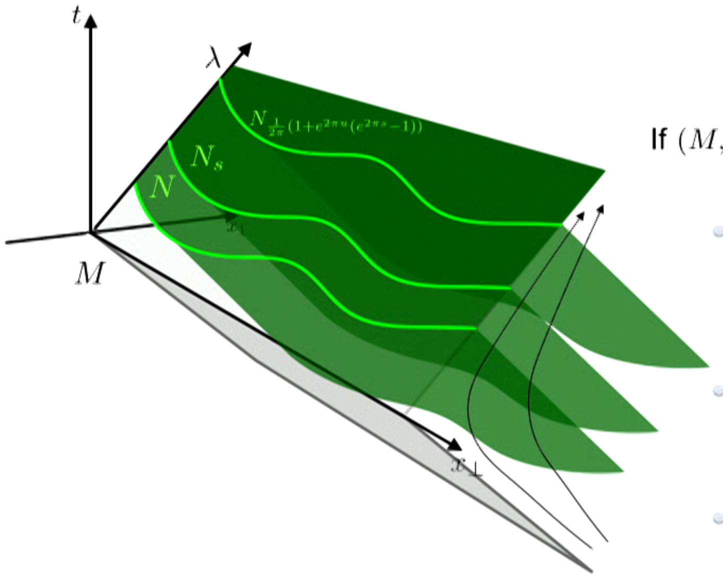
- Thm 4: Any unitary operator that moves N_s like $T_{M,N}$ and has a generator that is positive and annihilate the vacuum is unique.

Eduardo Testé
Instituto Balseiro
Centro Atómico Bariloche

Sketch of a general proof (modular Hamiltonians)

Idea: find the explicit form of the modular Hamiltonian of these regions:

Obs:
this operator can be written as the integral of
a local operator only in the null Cauchy surface



$$H_\gamma = 2\pi \int d^{d-2}x_\perp \int d\lambda\ (\lambda - \gamma(x_\perp))T_{\lambda\lambda}$$

If (M, N) are in a half sided modular inclusion situation then:
[Borchers 93, Wiesbrok 93]

- Thm 1: The modular Hamiltonian of N moves geometrically any region of the family N_s (to another region of the family):

$$U_N(-u)N_sU_N(u) = N_{\frac{1}{2\pi}\log(1+e^{2\pi u}(e^{2\pi s}-1))}$$

- Thm 2: $\hat{H}_M - \hat{H}_N = G \geqslant 0$

$$[\hat{H}_M, \hat{H}_n] = 2\pi i(\hat{H}_M - \hat{H}_N) = 2\pi i G$$

- Thm 3: $T_{M,N}(\tau) = e^{-iG\tau}$, the modular translation (relative to (M,N)) moves the N_s geometrically:

$$T(-\tau)N_sT(\tau) = N_{\frac{1}{2\pi}\log(e^{2\pi s}+2\pi\tau)}$$

intuition: when M and N are both Rindler, $T_{M,N}$ is the null translation.

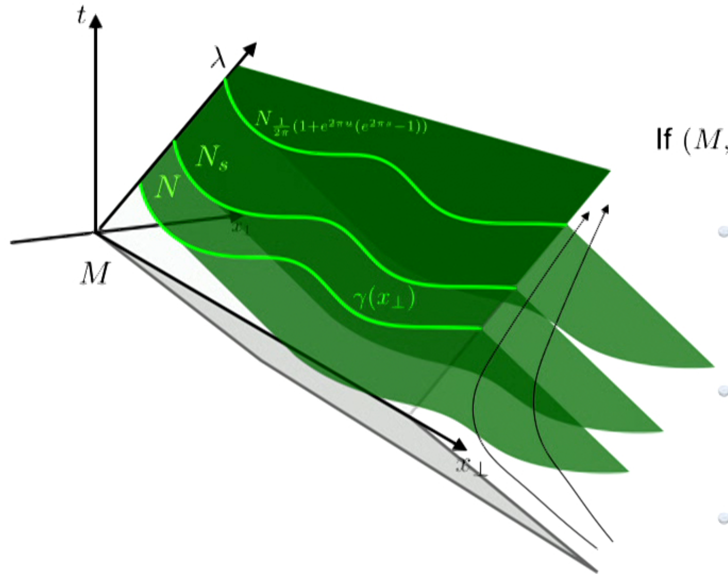
- Thm 4: Any unitary operator that moves N_s like $T_{M,N}$ and has a generator that is positive and annihilate the vacuum is unique.

Eduardo Testé
Instituto Balseiro
Centro Atómico Bariloche

Sketch of a general proof (modular Hamiltonians)

Idea: find the explicit form of the modular Hamiltonian of these regions:

Obs:
this operator can be written as the integral of
a local operator only in the null Cauchy surface



$$H_\gamma = 2\pi \int d^{d-2}x_\perp \int d\lambda\ (\lambda - \gamma(x_\perp))T_{\lambda\lambda}$$

If (M, N) are in a half sided modular inclusion situation then:
[Borchers 93, Wiesbrok 93]

- Thm 1: The modular Hamiltonian of N moves geometrically any region of the family N_s (to another region of the family):

$$U_N(-u)N_sU_N(u) = N_{\frac{1}{2\pi}\log(1+e^{2\pi u}(e^{2\pi s}-1))}$$

- Thm 2: $\hat{H}_M - \hat{H}_N = G \geqslant 0$

$$[\hat{H}_M, \hat{H}_n] = 2\pi i(\hat{H}_M - \hat{H}_N) = 2\pi i G$$

- Thm 3: $T_{M,N}(\tau) = e^{-iG\tau}$, the modular translation (relative to (M, N)) moves the N_s geometrically:

$$T(-\tau)N_sT(\tau) = N_{\frac{1}{2\pi}\log(e^{2\pi s}+2\pi\tau)}$$

intuition: when M and N are both Rindler, $T_{M,N}$ is the null translation.

- Thm 4: Any unitary operator that moves N_s like $T_{M,N}$ and has a generator that is positive and annihilate the vacuum is unique.

$G = \hat{H}_{\text{Rindler}} - \hat{H}_\gamma$ generates null translation of magnitude γ

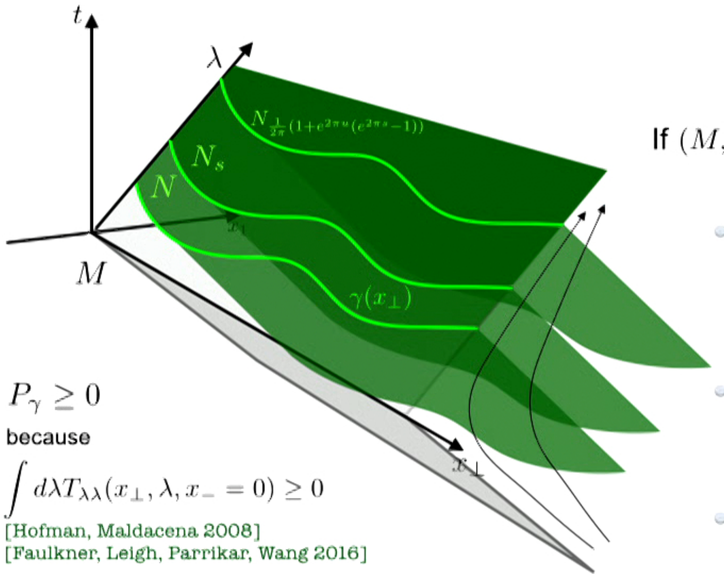
Eduardo Testé
Instituto Balseiro
Centro Atómico Bariloche

Sketch of a general proof (modular Hamiltonians)

Idea: find the explicit form of the modular Hamiltonian of these regions:

Obs:
this operator can be written as the integral of
a local operator only in the null Cauchy surface

$$H_\gamma = 2\pi \int d^{d-2}x_\perp \int d\lambda \; (\lambda - \gamma(x_\perp))T_{\lambda\lambda}$$



$P_\gamma \geq 0$

because

$$\int d\lambda T_{\lambda\lambda}(x_\perp, \lambda, x_- = 0) \geq 0$$

[Hofman, Maldacena 2008]
[Faulkner, Leigh, Parrikar, Wang 2016]

$G = \hat{H}_{\text{Rindler}} - \hat{H}_\gamma$ generates null translation of magnitude γ

||

$$P_\gamma = 2\pi \int d^{d-2}x_\perp \gamma(x_\perp) \int d\lambda T_{\lambda\lambda}(x_\perp, \lambda, x_- = 0)$$

generates null translation of magnitude γ

If (M, N) are in a half sided modular inclusion situation then:
[Borchers 93, Wiesbrok 93]

- Thm 1: The modular Hamiltonian of N moves geometrically any region of the family N_s (to another region of the family):
$$U_N(-u)N_sU_N(u) = N_{\frac{1}{2\pi}\log(1+e^{2\pi u}(e^{2\pi s}-1))}$$
- Thm 2: $\hat{H}_M - \hat{H}_N = G \geqslant 0$
$$[\hat{H}_M, \hat{H}_n] = 2\pi i(\hat{H}_M - \hat{H}_N) = 2\pi i G$$
- Thm 3: $T_{M,N}(\tau) = e^{-iG\tau}$, the modular translation (relative to (M, N)) moves the N_s geometrically:
$$T(-\tau)N_sT(\tau) = N_{\frac{1}{2\pi}\log(e^{2\pi s}+2\pi\tau)}$$
intuition: when M and N are both Rindler, $T_{M,N}$ is the null translation.
- Thm 4: Any unitary operator that moves N_s like $T_{M,N}$ and has a generator that is positive and annihilate the vacuum is unique.

Eduardo Testé
Instituto Balseiro
Centro Atómico Bariloche

Some intuitions about being Markovian

The condition $S(\rho_{12}) + S(\rho_{23}) = S(\rho_{123}) + S(\rho_2)$ is strong enough to fix great part of the structure of the state

Eduardo Testé
Instituto Balseiro
Centro Atómico Bariloche

Some intuitions about being Markovian

The condition $S(\rho_{12}) + S(\rho_{23}) = S(\rho_{123}) + S(\rho_2)$ is strong enough to fix great part of the structure of the state

↓ [Hayden,Jozsa, Petz,Winter: 04]

$\exists$ a decomposition of $\mathcal{H}_2 = \oplus_k \mathcal{H}_{2L}^k \otimes \mathcal{H}_{2R}^k$

such that $\rho_{123} = \sum_k p_k \; \rho_{1L}^k \otimes \rho_{R3}^k$

Eduardo Testé
Instituto Balseiro
Centro Atómico Bariloche

Some intuitions about being Markovian

The condition $S(\rho_{12}) + S(\rho_{23}) = S(\rho_{123}) + S(\rho_2)$ is strong enough to fix great part of the structure of the state

↓ [Hayden,Jozsa, Petz,Winter: 04]

$\exists$ a decomposition of $\mathcal{H}_2 = \oplus_k \mathcal{H}^k_{2L} \otimes \mathcal{H}^k_{2R}$

such that $\rho_{1\bullet 3} = \sum_k p_k \; \rho^k_{1\bullet} \otimes \rho^k_{\bullet 3}$ separable

Eduardo Testé
Instituto Balseiro
Centro Atómico Bariloche

Some intuitions about being Markovian

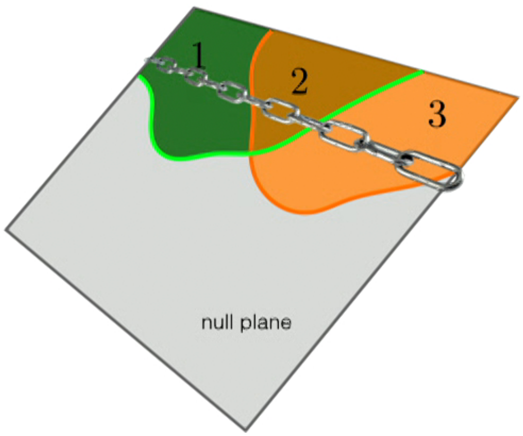
The condition $S(\rho_{12}) + S(\rho_{23}) = S(\rho_{123}) + S(\rho_2)$ is strong enough to fix great part of the structure of the state

↓ [Hayden,Jozsa, Petz,Winter: 04]

$\exists$ a decomposition of $\mathcal{H}_2 = \oplus_k \mathcal{H}^k_{2L} \otimes \mathcal{H}^k_{2R}$

such that $\rho_{1\bullet 3} = \sum_k p_k \; \rho^k_{1\bullet} \otimes \rho^k_{\bullet 3}$ separable

The entanglement structure of a Markov state is like a chain:
If we cut a link (take trace on the intersection algebra) the state becomes separable



the vacuum is like this (with respect to null cut subalgefbras)

Eduardo Testé
Instituto Balseiro
Centro Atómico Bariloche

Some intuitions about being Markovian

The condition $S(\rho_{12}) + S(\rho_{23}) = S(\rho_{123}) + S(\rho_2)$ is strong enough to fix great part of the structure of the state

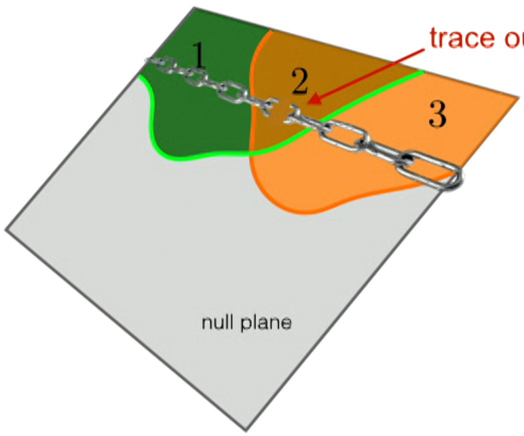
↓ [Hayden,Jozsa, Petz,Winter: 04]

$$\exists \text{ a decomposition of } \mathcal{H}_2 = \oplus_k \mathcal{H}_{2L}^k \otimes \mathcal{H}_{2R}^k$$

$$\text{such that } \rho_{1\bullet 3} = \sum_k p_k \, \rho_{1\bullet}^k \otimes \rho_{\bullet 3}^k$$

separable

The entanglement structure of a Markov state is like a chain:
If we cut a link (take trace on the intersection algebra) the state becomes separable



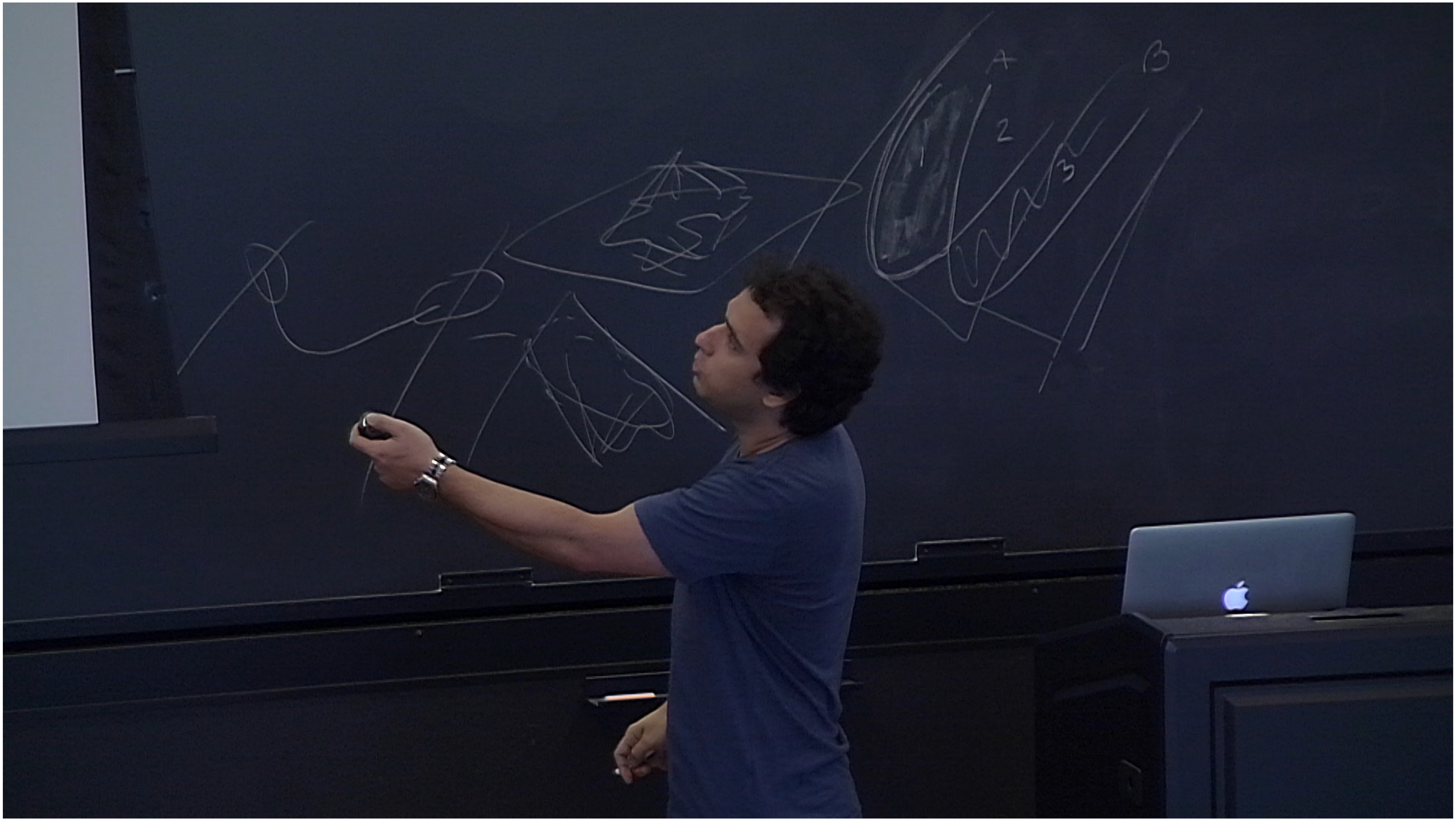
the vacuum is like this
(with respect to null cut
subalgefbras)

Eduardo Testé
Instituto Balseiro
Centro Atómico Bariloche

Entropic proof of the a-theorem

Eduardo Testé
Instituto Balseiro
Centro Atómico Bariloche


A
B


A
B
2
3

Some intuitions about being Markovian

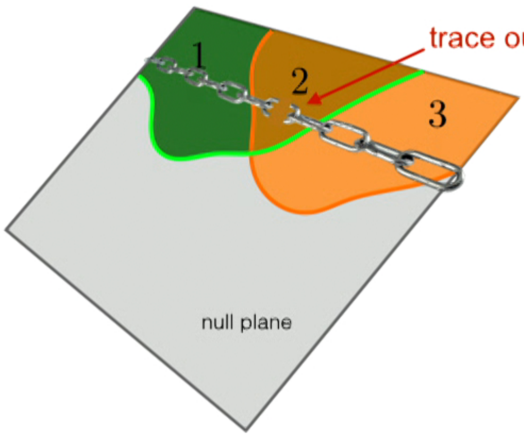
The condition $S(\rho_{12}) + S(\rho_{23}) = S(\rho_{123}) + S(\rho_2)$ is strong enough to fix great part of the structure of the state

↓ [Hayden,Jozsa, Petz,Winter: 04]

$$\exists \text{ a decomposition of } \mathcal{H}_2 = \oplus_k \mathcal{H}^k_{2L} \otimes \mathcal{H}^k_{2R}$$

$$\text{such that } \rho_{1\bullet 3} = \sum_k p_k \; \rho^k_{1\bullet} \otimes \rho^k_{\bullet 3}$$ separable

The entanglement structure of a Markov state is like a chain:
If we cut a link (take trace on the intersection algebra) the state becomes separable



trace out the intersection

the vacuum is like this
(with respect to null cut
subalgefbras)

Eduardo Testé
Instituto Balseiro
Centro Atómico Bariloche

Some intuitions about being Markovian

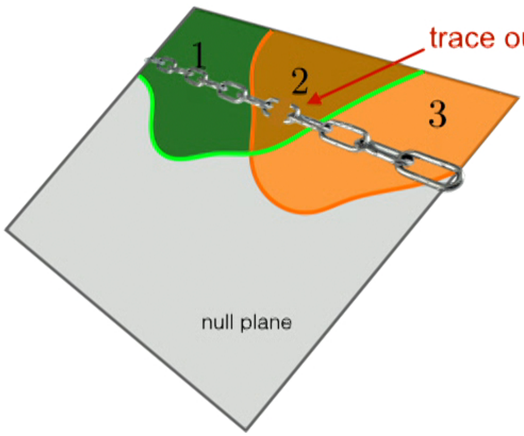
The condition $S(\rho_{12}) + S(\rho_{23}) = S(\rho_{123}) + S(\rho_2)$ is strong enough to fix great part of the structure of the state

↓ [Hayden,Jozsa, Petz,Winter: 04]

$$\exists \text{ a decomposition of } \mathcal{H}_2 = \oplus_k \mathcal{H}_{2L}^k \otimes \mathcal{H}_{2R}^k$$

$$\text{such that } \rho_{1\bullet3} = \sum_k p_k \, \rho_{1\bullet}^k \otimes \rho_{\bullet3}^k$$ separable

The entanglement structure of a Markov state is like a chain:
If we cut a link (take trace on the intersection algebra) the state becomes separable



the vacuum is like this (with respect to null cut subalgefbras)

Eduardo Testé
Instituto Balseiro
Centro Atómico Bariloche

Entropic proof of the a-theorem

Ingredients for the entropic c and F theorems [Casini, Huerta, 2004, 2012], [Casini, Huerta, Myers,Yale, 2015]

1. strong subadditivity (SSA) of von Neumann entropy (entanglement entropy EE)
2. Lorentz invariance of the vacuum
3. unitarity and causality
4. relation between EE and intrinsic quantities of the theory at fixed point

$$S^d_{CFT}(r) = \mu_{d-2}\, r^{d-2} + \mu_{d-4}\, r^{d-4} + \cdots + \begin{cases} (-)^{\frac{d}{2}-1} 4\, A\, \log(R/\epsilon) & d \text{ even}\,. \\ (-)^{\frac{d-1}{2}}\, F & d \text{ odd}\,. \end{cases}$$

monotonic under RG flows? [Myers,Sinha 2010]

Eduardo Testé
Instituto Balseiro
Centro Atómico Bariloche

Entropic proof of the a-theorem

Ingredients for the entropic c and F theorems [Casini, Huerta, 2004, 2012], [Casini, Huerta, Myers,Yale, 2015]

1. strong subadditivity (SSA) of von Neumann entropy (entanglement entropy EE)
2. Lorentz invariance of the vacuum
3. unitarity and causality
4. relation between EE and intrinsic quantities of the theory at fixed point

Review of the entropic c theorem $d = 2$ [Casini, Huerta, 2004]

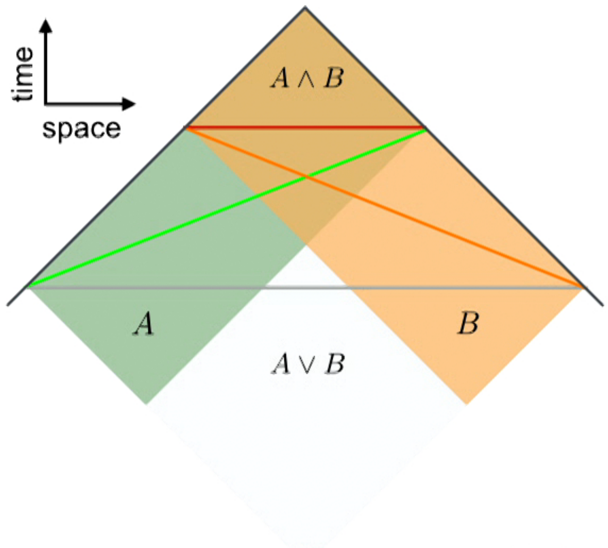
Eduardo Testé
Instituto Balseiro
Centro Atómico Bariloche

Entropic proof of the a-theorem

Ingredients for the entropic c and F theorems [Casini, Huerta, 2004, 2012], [Casini, Huerta, Myers,Yale, 2015]

1. strong subadditivity (SSA) of von Neumann entropy (entanglement entropy EE)
2. Lorentz invariance of the vacuum
3. unitarity and causality
4. relation between EE and intrinsic quantities of the theory at fixed point

Review of the entropic c theorem $d = 2$ [Casini, Huerta, 2004]



SSA: $S(A) + S(B) \geqslant S(A \wedge B) + S(A \vee B)$

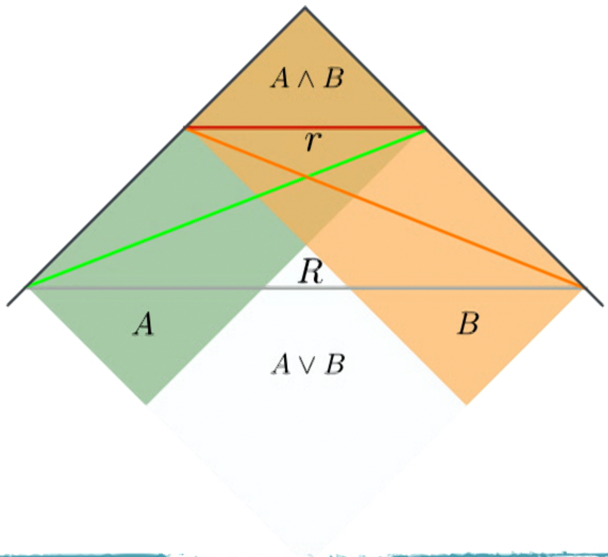
Eduardo Testé
Instituto Balseiro
Centro Atómico Bariloche

Entropic proof of the a-theorem

Ingredients for the entropic c and F theorems [Casini, Huerta, 2004, 2012], [Casini, Huerta, Myers,Yale, 2015]

1. strong subadditivity (SSA) of von Neumann entropy (entanglement entropy EE)
2. Lorentz invariance of the vacuum
3. unitarity and causality
4. relation between EE and intrinsic quantities of the theory at fixed point

Review of the entropic c theorem $d = 2$ [Casini, Huerta, 2004]



SSA: $S(A) + S(B) \geqslant S(A \wedge B) + S(A \vee B)$

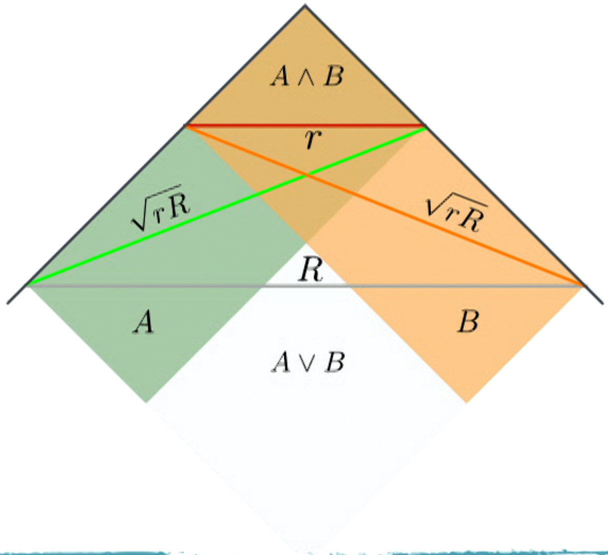
Eduardo Testé
Instituto Balseiro
Centro Atómico Bariloche

Entropic proof of the a-theorem

Ingredients for the entropic c and F theorems [Casini, Huerta, 2004, 2012], [Casini, Huerta, Myers,Yale, 2015]

1. strong subadditivity (SSA) of von Neumann entropy (entanglement entropy EE)
2. Lorentz invariance of the vacuum
3. unitarity and causality
4. relation between EE and intrinsic quantities of the theory at fixed point

Review of the entropic c theorem $d = 2$ [Casini, Huerta, 2004]



SSA: $S(A) + S(B) \geqslant S(A \wedge B) + S(A \vee B)$

$$S(\sqrt{rR}) + S(\sqrt{rR}) \geqslant S(r) + S(R)$$

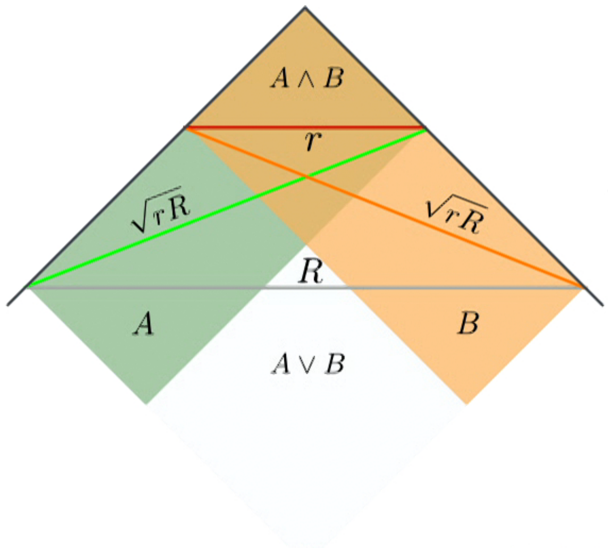
Eduardo Testé
Instituto Balseiro
Centro Atómico Bariloche

Entropic proof of the a-theorem

Ingredients for the entropic c and F theorems [Casini, Huerta, 2004, 2012], [Casini, Huerta, Myers,Yale, 2015]

1. strong subadditivity (SSA) of von Neumann entropy (entanglement entropy EE)
2. Lorentz invariance of the vacuum
3. unitarity and causality
4. relation between EE and intrinsic quantities of the theory at fixed point

Review of the entropic c theorem $d = 2$ [Casini, Huerta, 2004]



SSA: $S(A) + S(B) \geqslant S(A \wedge B) + S(A \vee B)$

$$S(\sqrt{rR}) + S(\sqrt{rR}) \geqslant S(r) + S(R)$$

take $R = r + \varepsilon$ and $\varepsilon \to 0$

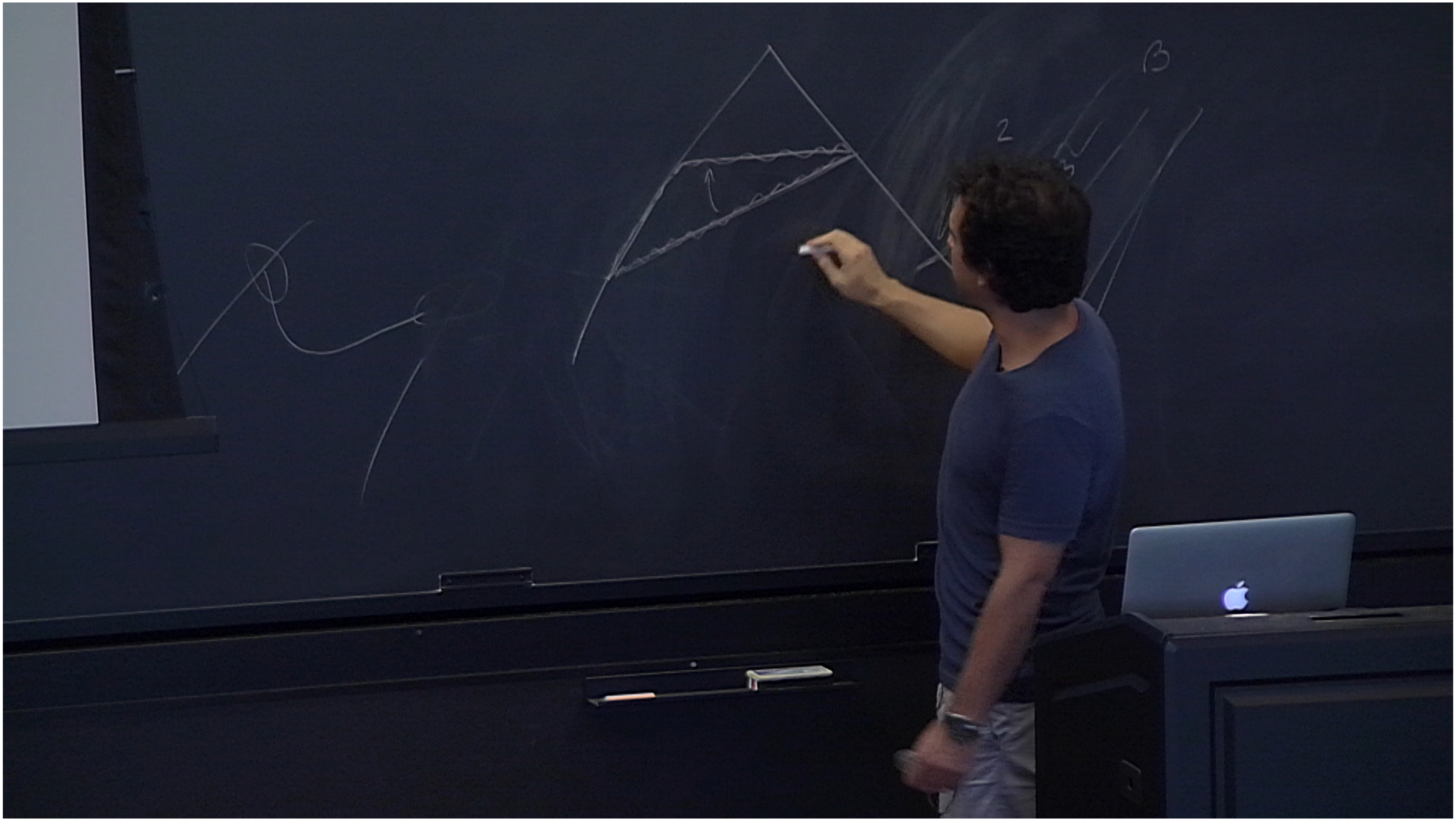
$$(rS'(r))' \equiv c'(r) \leqslant 0$$

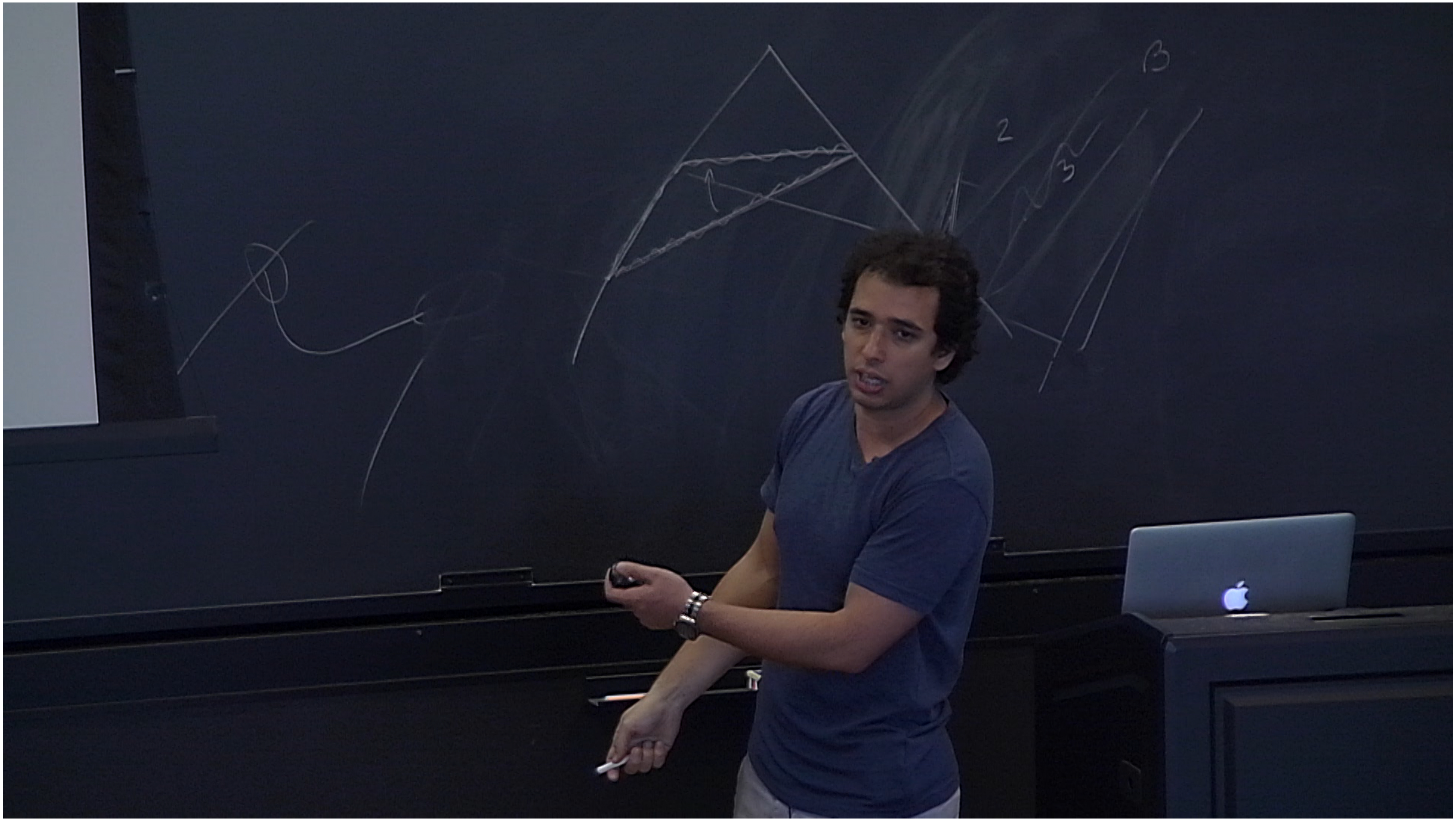
by $S^{d=2}_{CFT}(r) = \frac{c}{3} \log(r/\epsilon)$

this $c(r)$ function equals the Virasoro central charges at the fixed points. Then

$$c_{UV} \geqslant c_{IR}$$

Eduardo Testé
Instituto Balseiro
Centro Atómico Bariloche


2
3
B



Entropic proof of the a-theorem

Ingredients for the entropic c and F theorems [Casini, Huerta, 2004, 2012], [Casini, Huerta, Myers,Yale, 2015]

1. strong subadditivity (SSA) of von Neumann entropy (entanglement entropy EE)
2. Lorentz invariance of the vacuum
3. unitarity and causality
4. relation between EE and intrinsic quantities of the theory at fixed point

Review of the entropic F theorem $d = 3$ [Casini, Huerta, 2012]

Eduardo Testé
Instituto Balseiro
Centro Atómico Bariloche

Entropic proof of the a-theorem

Ingredients for the entropic c and F theorems [Casini, Huerta, 2004, 2012], [Casini, Huerta, Myers,Yale, 2015]

1. strong subadditivity (SSA) of von Neumann entropy (entanglement entropy EE)
2. Lorentz invariance of the vacuum
3. unitarity and causality
4. relation between EE and intrinsic quantities of the theory at fixed point

Review of the entropic F theorem $d = 3$ [Casini, Huerta, 2012]

in $d = 2$ the intersection of two boosted segments is a segment, but in $d = 3$ the intersection of two boosted circles is not a circle. Instead of SSA, apply a symmetric version:

SSA: $S(X_1) + S(X_2) \geq S(X_1 \cup X_2) + S(X_1 \cap X_2)$

symmetric SSA

$$\sum_i S(X_i) \geq S(\cup_i X_i) + S(\cup_{\{ij\}}(X_i \cap X_j)) + S(\cup_{\{ijk\}}(X_i \cap X_j \cap X_k)) + ... + S(\cap_i X_i)$$

Eduardo Testé
Instituto Balseiro
Centro Atómico Bariloche

Entropic proof of the a-theorem

Ingredients for the entropic c and F theorems [Casini, Huerta, 2004, 2012], [Casini, Huerta, Myers,Yale, 2015]

1. strong subadditivity (SSA) of von Neumann entropy (entanglement entropy EE)
2. Lorentz invariance of the vacuum
3. unitarity and causality
4. relation between EE and intrinsic quantities of the theory at fixed point

Review of the entropic F theorem $d = 3$ [Casini, Huerta, 2012]

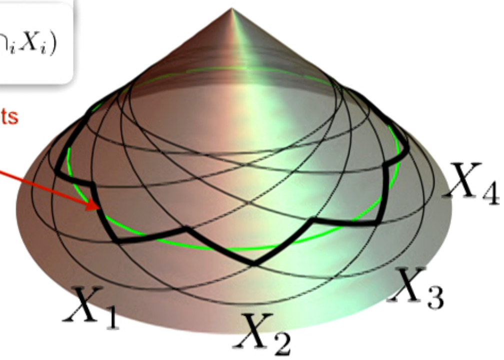
in $d = 2$ the intersection of two boosted segments is a segment, but in $d = 3$ the intersection of two boosted circles is not a circle. Instead of SSA, apply a symmetric version:

SSA: $S(X_1) + S(X_2) \geq S(X_1 \cup X_2) + S(X_1 \cap X_2)$

symmetric SSA

$$\sum_i S(X_i) \geq S(\cup_i X_i) + S(\cup_{\{ij\}}(X_i \cap X_j)) + S(\cup_{\{ijk\}}(X_i \cap X_j \cap X_k)) + ... + S(\cap_i X_i)$$

wiggly sets



Eduardo Testé
Instituto Balseiro
Centro Atómico Bariloche

Entropic proof of the a-theorem

Ingredients for the entropic c and F theorems [Casini, Huerta, 2004, 2012], [Casini, Huerta, Myers,Yale, 2015]

1. strong subadditivity (SSA) of von Neumann entropy (entanglement entropy EE)
2. Lorentz invariance of the vacuum
3. unitarity and causality
4. relation between EE and intrinsic quantities of the theory at fixed point

Review of the entropic F theorem $d = 3$ [Casini, Huerta, 2012]

in $d = 2$ the intersection of two boosted segments is a segment, but in $d = 3$ the intersection of two boosted circles is not a circle. Instead of SSA, apply a symmetric version:

SSA: $S(X_1) + S(X_2) \geq S(X_1 \cup X_2) + S(X_1 \cap X_2)$

symmetric SSA

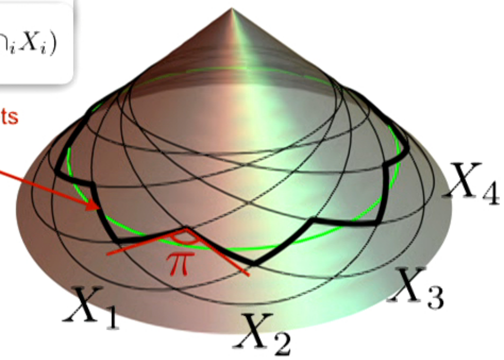
$$\sum_i S(X_i) \geq S(\cup_i X_i) + S(\cup_{\{ij\}}(X_i \cap X_j)) + S(\cup_{\{ijk\}}(X_i \cap X_j \cap X_k)) + ... + S(\cap_i X_i)$$

wiggly sets

$$S(\sqrt{rR}) \geq \frac{1}{N}\sum_{k=1}^{N} \tilde{S}_k \approx \int_r^R dl\, \beta(l)\, \tilde{S}(l)$$

no angle contribution

$$S(\sqrt{rR}) \geqslant \int_r^R dl\, \beta(l)\, S(l)$$



Eduardo Testé
Instituto Balseiro
Centro Atómico Bariloche

Entropic proof of the a-theorem

Ingredients for the entropic c and F theorems [Casini, Huerta, 2004, 2012], [Casini, Huerta, Myers,Yale, 2015]

1. strong subadditivity (SSA) of von Neumann entropy (entanglement entropy EE)
2. Lorentz invariance of the vacuum
3. unitarity and causality
4. relation between EE and intrinsic quantities of the theory at fixed point

Review of the entropic F theorem $d = 3$ [Casini, Huerta, 2012]

in $d = 2$ the intersection of two boosted segments is a segment, but in $d = 3$ the intersection of two boosted circles is not a circle. Instead of SSA, apply a symmetric version:

SSA: $S(X_1) + S(X_2) \geq S(X_1 \cup X_2) + S(X_1 \cap X_2)$

symmetric SSA

$$\sum_i S(X_i) \geq S(\cup_i X_i) + S(\cup_{\{ij\}}(X_i \cap X_j)) + S(\cup_{\{ijk\}}(X_i \cap X_j \cap X_k)) + ... + S(\cap_i X_i)$$

wiggly sets

$$S(\sqrt{rR}) \geq \frac{1}{N}\sum_{k=1}^{N} \tilde{S}_k \approx \int_r^R dl\, \beta(l)\, \tilde{S}(l)$$

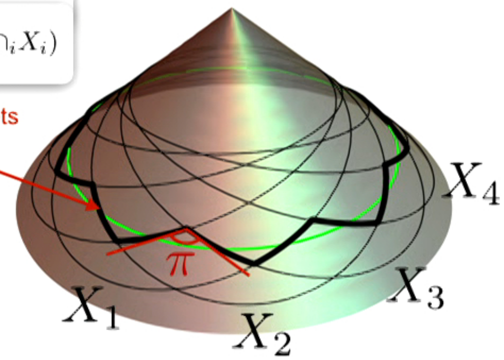
no angle contribution

$$S(\sqrt{rR}) \geqslant \int_r^R dl\, \beta(l)\, S(l)$$

monotonic

take $R = r + \varepsilon$ and $\varepsilon \to 0:\quad S''(r) \leqslant 0 \rightarrow F(r) = rS'(r) - S(r)$

$$S^{d=3}_{CFT}(r) = \mu_1 r - F \rightarrow F_{UV} \geqslant F_{IR}$$



Eduardo Testé
Instituto Balseiro
Centro Atómico Bariloche

Entropic proof of the a-theorem

Ingredients for the entropic c and F theorems [Casini, Huerta, 2004, 2012], [Casini, Huerta, Myers,Yale, 2015]

1. strong subadditivity (SSA) of von Neumann entropy (entanglement entropy EE)
2. Lorentz invariance of the vacuum
3. unitarity and causality
4. relation between EE and intrinsic quantities of the theory at fixed point

Review of the entropic F theorem $d=3$ [Casini, Huerta, 2012]

in $d=2$ the intersection of two boosted segments is a segment, but in $d=3$ the intersection of two boosted circles is not a circle. Instead of SSA, apply a symmetric version:

SSA: $S(X_1)+S(X_2)\geq S(X_1\cup X_2)+S(X_1\cap X_2)$

symmetric SSA

$$\sum_i S(X_i)\geq S(\cup_i X_i)+S(\cup_{\{ij\}}(X_i\cap X_j))+S(\cup_{\{ijk\}}(X_i\cap X_j\cap X_k))+\ldots+S(\cap_i X_i)$$

$$S(\sqrt{rR})\geq \frac{1}{N}\sum_{k=1}^{N}\tilde{S}_k\approx\int_r^R dl\,\beta(l)\,\tilde{S}(l)$$

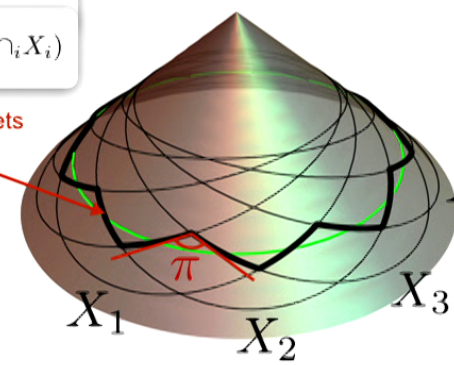
no angle contribution

$$S(\sqrt{rR})\geqslant\int_r^R dl\,\beta(l)\,S(l)$$

monotonic

take $R=r+\varepsilon$ and $\varepsilon\to 0$: $S''(r)\leqslant 0$ → $F(r)=rS'(r)-S(r)$

$S^{d=3}_{CFT}(r)=\mu_1 r-F$ → $F_{UV}\geqslant F_{IR}$



Eduardo Testé
Instituto Balseiro
Centro Atómico Bariloche

Entropic proof of the a-theorem

$$S(\sqrt{rR}) \geqslant \int_r^R dl\, \beta(l)\, \tilde{S}(l)$$

always true, but useless

?

$$S(\sqrt{rR}) \geqslant \int_r^R dl\, \beta(l)\, S(l)$$

$$\beta(l) = \frac{2^{d-3}\Gamma[(d-1)/2]}{\sqrt{\pi}\Gamma[(d-2)/2]} \frac{(rR)^{\frac{d-2}{2}} ((l-r)(R-l))^{\frac{d-4}{2}}}{l^{d-2}(R-r)^{d-3}}$$

Eduardo Testé
Instituto Balseiro
Centro Atómico Bariloche

Entropic proof of the a-theorem

$$S(\sqrt{rR}) \geqslant \int_r^R dl\, \beta(l)\, \tilde{S}(l)$$

always true, but useless

$$\beta(l) = \frac{2^{d-3}\Gamma[(d-1)/2]}{\sqrt{\pi}\Gamma[(d-2)/2]} \frac{(rR)^{\frac{d-2}{2}} ((l-r)(R-l))^{\frac{d-4}{2}}}{l^{d-2}(R-r)^{d-3}}$$

?

$$S(\sqrt{rR}) \geqslant \int_r^R dl\, \beta(l)\, S(l)$$

If applied in more dimension (taking the wiggly sets as smooth spheres) you will get a wrong result.
For example, in $d = 4$, when applied to a fixed point you get $A \leqslant 0$.

wrong

Eduardo Testé
Instituto Balseiro
Centro Atómico Bariloche

Entropic proof of the a-theorem

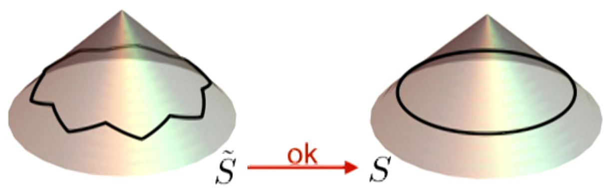
$$S(\sqrt{rR}) \geqslant \int_r^R dl\, \beta(l)\, \tilde{S}(l)$$ always true, but useless

?

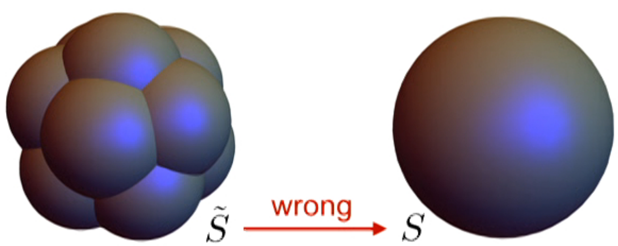
$$S(\sqrt{rR}) \geqslant \int_r^R dl\, \beta(l)\, S(l)$$

$$\beta(l) = \frac{2^{d-3}\Gamma[(d-1)/2]}{\sqrt{\pi}\Gamma[(d-2)/2]} \frac{(rR)^{\frac{d-2}{2}}((l-r)(R-l))^{\frac{d-4}{2}}}{l^{d-2}(R-r)^{d-3}}$$

If applied in more dimension (taking the wiggly sets as smooth spheres) you will get a wrong result.
For example, in $d=4$, when applied to a fixed point you get $A \leqslant 0$.
wrong



in $d=3$ only corners



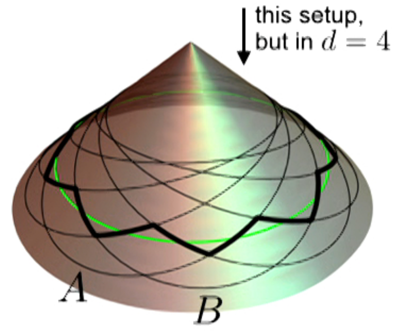
in $d=4$ new features appear:
- dihedral and trihedral angles
- different local intrinsic curvatures

Eduardo Testé
Instituto Balseiro
Centro Atómico Bariloche

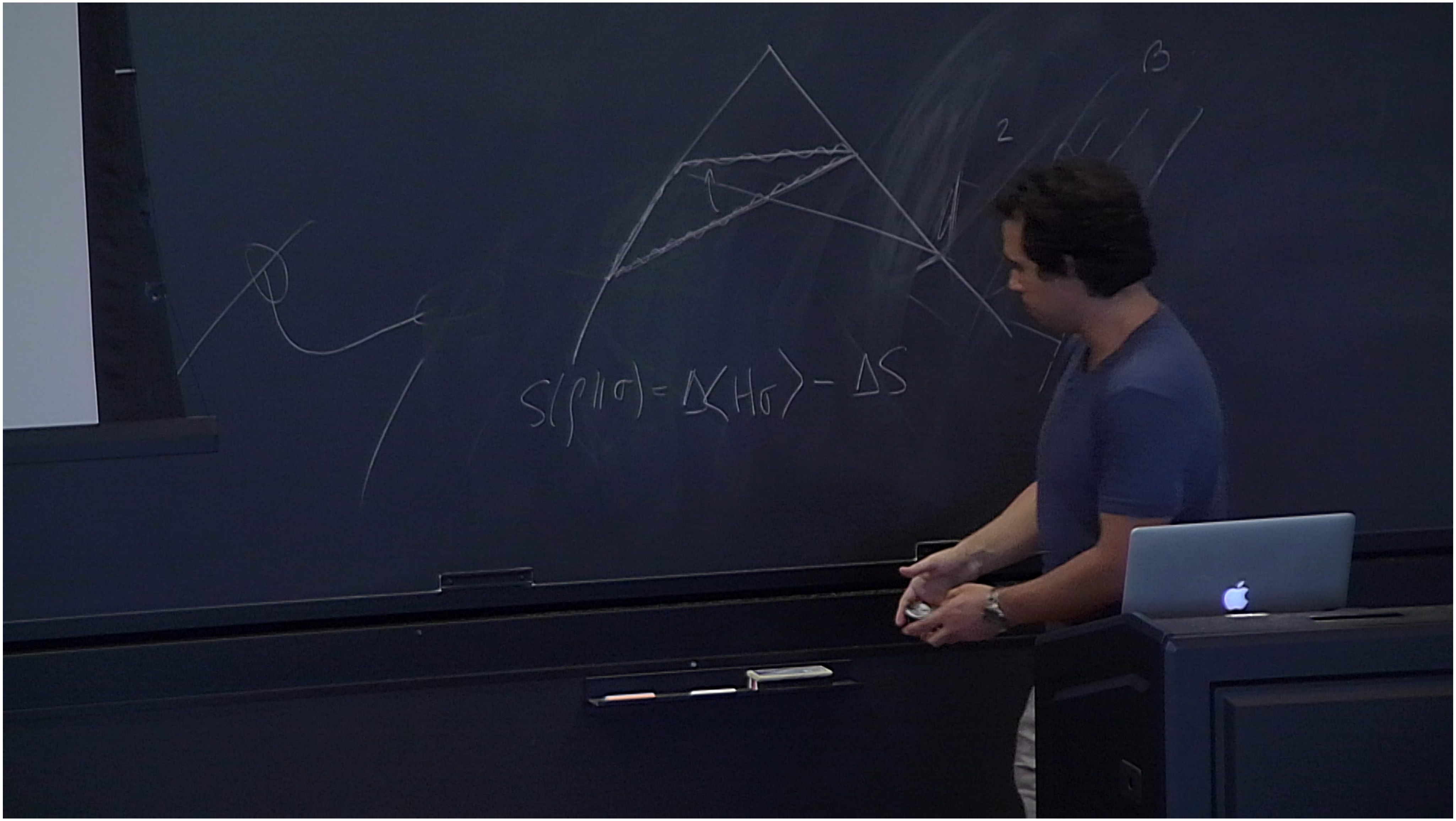
Entropic proof of the a-theorem

$S(\rho_A) + S(\rho_B) \geqslant S(\rho_{A\wedge B}) + S(\rho_{A\vee B})$ ρ vacuum of $S_{CFT_{UV}} + \int g\mathcal{O}$

$S(\sigma_A) + S(\sigma_B) = S(\sigma_{A\wedge B}) + S(\sigma_{A\vee B})$ σ vacuum of $S_{CFT_{UV}}$



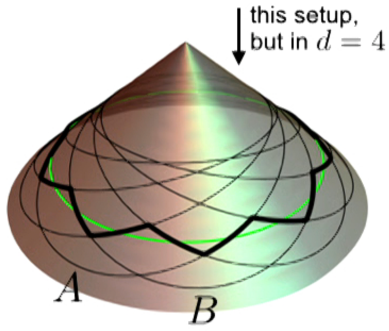
Eduardo Testé
Instituto Balseiro
Centro Atómico Bariloche


S(ρ||σ) = Δ⟨Hσ⟩ − ΔS
1
2
3

Entropic proof of the a-theorem

$S(\rho_A) + S(\rho_B) \geqslant S(\rho_{A\wedge B}) + S(\rho_{A\vee B})$ ρ vacuum of $S_{CFT_{UV}} + \int g\mathcal{O}$

$S(\sigma_A) + S(\sigma_B) = S(\sigma_{A\wedge B}) + S(\sigma_{A\vee B})$ σ vacuum of $S_{CFT_{UV}}$



Because σ is Markov state relative to these boosted spheres A and B:

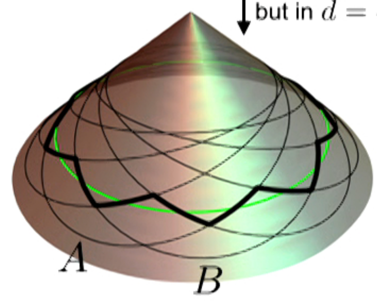
$$S(\rho_A||\sigma_A) + S(\rho_B||\sigma_B) \leqslant S(\rho_{A\vee B}||\sigma_{A\vee B}) + S(\rho_{A\wedge B}||\sigma_{A\wedge B})$$

$$\Updownarrow \; H_{\sigma_A} + H_{\sigma_B} = H_{\sigma_{A\wedge B}} + H_{\sigma_{A\vee B}}$$

$$\Delta S(A) + \Delta S(B) \geqslant \Delta S(A\vee B) + \Delta S(A\wedge B)$$

Eduardo Testé
Instituto Balseiro
Centro Atómico Bariloche

Entropic proof of the a-theorem



$$S(\rho_A) + S(\rho_B) \geqslant S(\rho_{A\wedge B}) + S(\rho_{A\vee B}) \qquad \rho \text{ vacuum of } S_{CFT_{UV}} + \int g\mathcal{O}$$

$$S(\sigma_A) + S(\sigma_B) = S(\sigma_{A\wedge B}) + S(\sigma_{A\vee B}) \qquad \sigma \text{ vacuum of } S_{CFT_{UV}}$$

Because σ is Markov state relative to these boosted spheres A and B:

$$S(\rho_A||\sigma_A) + S(\rho_B||\sigma_B) \leqslant S(\rho_{A\vee B}||\sigma_{A\vee B}) + S(\rho_{A\wedge B}||\sigma_{A\wedge B})$$

$$\Updownarrow \; H_{\sigma_A} + H_{\sigma_B} = H_{\sigma_{A\wedge B}} + H_{\sigma_{A\vee B}}$$

$$\Delta S(A) + \Delta S(B) \geqslant \Delta S(A\vee B) + \Delta S(A\wedge B)$$

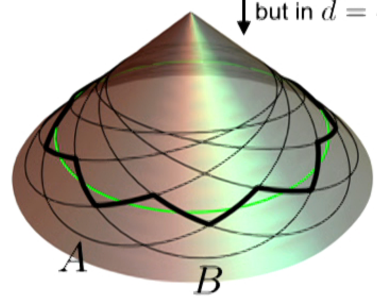
$$\Delta S(\sqrt{rR}) \geq\approx \int_r^R dl\, \beta(l)\, \Delta\tilde{S}(l)$$

Eduardo Testé
Instituto Balseiro
Centro Atómico Bariloche

Entropic proof of the a-theorem

$S(\rho_A) + S(\rho_B) \geqslant S(\rho_{A\wedge B}) + S(\rho_{A\vee B})$ ρ vacuum of $S_{CFT_{UV}} + \int g\mathcal{O}$

$S(\sigma_A) + S(\sigma_B) = S(\sigma_{A\wedge B}) + S(\sigma_{A\vee B})$ σ vacuum of $S_{CFT_{UV}}$



Because σ is Markov state relative to these boosted spheres A and B:

$$S(\rho_A||\sigma_A) + S(\rho_B||\sigma_B) \leqslant S(\rho_{A\vee B}||\sigma_{A\vee B}) + S(\rho_{A\wedge B}||\sigma_{A\wedge B})$$

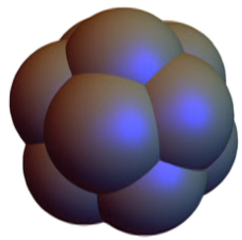
$$\Updownarrow \; H_{\sigma_A} + H_{\sigma_B} = H_{\sigma_{A\wedge B}} + H_{\sigma_{A\vee B}}$$

$$\Delta S(A) + \Delta S(B) \geqslant \Delta S(A\vee B) + \Delta S(A\wedge B)$$

$$\Delta S(\sqrt{rR}) \geq\approx \int_r^R dl\, \beta(l)\, \Delta\tilde{S}(l)$$

$$\downarrow$$

$$\Delta S(\sqrt{rR}) \geq\approx \int_r^R dl\, \beta(l)\, \Delta S(l)$$

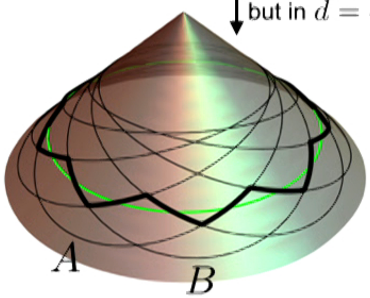


Eduardo Testé
Instituto Balseiro
Centro Atómico Bariloche

Entropic proof of the a-theorem

$S(\rho_A) + S(\rho_B) \geqslant S(\rho_{A\wedge B}) + S(\rho_{A\vee B})$ ρ vacuum of $S_{CFT_{UV}} + \int g\mathcal{O}$

$S(\sigma_A) + S(\sigma_B) = S(\sigma_{A\wedge B}) + S(\sigma_{A\vee B})$ σ vacuum of $S_{CFT_{UV}}$



Because σ is Markov state relative to these boosted spheres A and B:

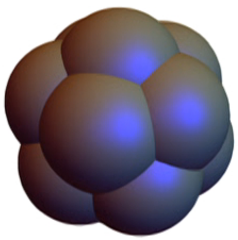
$$S(\rho_A||\sigma_A) + S(\rho_B||\sigma_B) \leqslant S(\rho_{A\vee B}||\sigma_{A\vee B}) + S(\rho_{A\wedge B}||\sigma_{A\wedge B})$$

$$\Updownarrow \; H_{\sigma_A} + H_{\sigma_B} = H_{\sigma_{A\wedge B}} + H_{\sigma_{A\vee B}}$$

$$\Delta S(A) + \Delta S(B) \geqslant \Delta S(A\vee B) + \Delta S(A\wedge B)$$

$$\Delta S(\sqrt{rR}) \gtrsim\approx \int_r^R dl\, \beta(l)\, \Delta\tilde{S}(l)$$

$$\Delta S(\sqrt{rR}) \gtrsim\approx \int_r^R dl\, \beta(l)\, \Delta S(l)$$



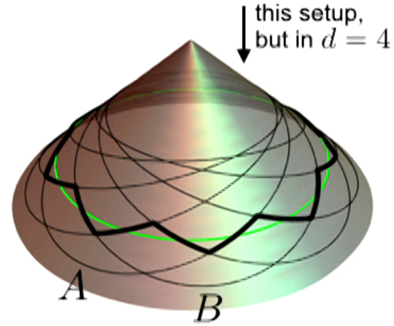
$$r\Delta S''(r) - (d-3)\Delta S'(r) \leqslant 0$$

Eduardo Testé
Instituto Balseiro
Centro Atómico Bariloche

Entropic proof of the a-theorem

$$S(\rho_A)+S(\rho_B)\geqslant S(\rho_{A\wedge B})+S(\rho_{A\vee B})$$ ρ vacuum of $S_{CFT_{UV}}+\int g\mathcal{O}$

$$S(\sigma_A)+S(\sigma_B)=S(\sigma_{A\wedge B})+S(\sigma_{A\vee B})$$ σ vacuum of $S_{CFT_{UV}}$



Because σ is Markov state relative to these boosted spheres A and B:

$$S(\rho_A||\sigma_A)+S(\rho_B||\sigma_B)\leqslant S(\rho_{A\vee B}||\sigma_{A\vee B})+S(\rho_{A\wedge B}||\sigma_{A\wedge B})$$

$$\Updownarrow\; H_{\sigma_A}+H_{\sigma_B}=H_{\sigma_{A\wedge B}}+H_{\sigma_{A\vee B}}$$

$$\Delta S(A)+\Delta S(B)\geqslant \Delta S(A\vee B)+\Delta S(A\wedge B)$$

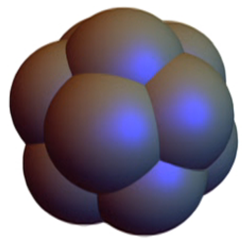
$$r\Delta S''(r)-(d-3)\Delta S'(r)\leqslant 0$$

apply this in the IR $r>>m^{-1}$

$$S^{d=4}_{CFT}(r)=\mu_2 r^2-4a\log(r/\epsilon)$$

$$\Delta S(\sqrt{rR})\geq\approx\int_r^R dl\,\beta(l)\,\Delta\tilde{S}(l)$$

$$\Delta S(\sqrt{rR})\geq\approx\int_r^R dl\,\beta(l)\,\Delta S(l)$$



$$a_{UV}\geqslant a_{IR}$$

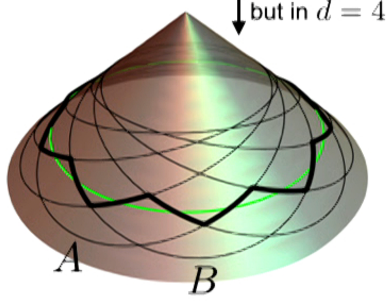
$$r\Delta S''(r)-(d-3)\Delta S'(r)\leqslant 0$$

Eduardo Testé
Instituto Balseiro
Centro Atómico Bariloche

Entropic proof of the a-theorem

$S(\rho_A)+S(\rho_B)\geqslant S(\rho_{A\wedge B})+S(\rho_{A\vee B})$ ρ vacuum of $S_{CFT_{UV}}+\int g\mathcal{O}$

$S(\sigma_A)+S(\sigma_B)=S(\sigma_{A\wedge B})+S(\sigma_{A\vee B})$ σ vacuum of $S_{CFT_{UV}}$



Because σ is Markov state relative to these boosted spheres A and B:

$$S(\rho_A||\sigma_A)+S(\rho_B||\sigma_B)\leqslant S(\rho_{A\vee B}||\sigma_{A\vee B})+S(\rho_{A\wedge B}||\sigma_{A\wedge B})$$

$$\Updownarrow\; H_{\sigma_A}+H_{\sigma_B}=H_{\sigma_{A\wedge B}}+H_{\sigma_{A\vee B}}$$

$$\Delta S(A)+\Delta S(B)\geqslant\Delta S(A\vee B)+\Delta S(A\wedge B)$$

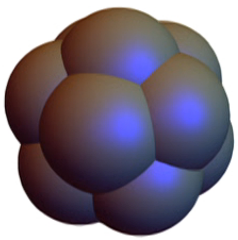
$$r\Delta S''(r)-(d-3)\Delta S'(r)\leqslant 0$$

apply this in the IR $r>>m^{-1}$

$S^{d=4}_{CFT}(r)=\mu_2 r^2-4a\log(r/\epsilon)$

$$\Delta S(\sqrt{rR})\geq\approx\int_r^R dl\,\beta(l)\,\Delta\tilde{S}(l)$$

$$\Delta S(\sqrt{rR})\geq\approx\int_r^R dl\,\beta(l)\,\Delta S(l)$$



$$a_{UV}\geqslant a_{IR}$$

$$r\Delta S''(r)-(d-3)\Delta S'(r)\leqslant 0$$

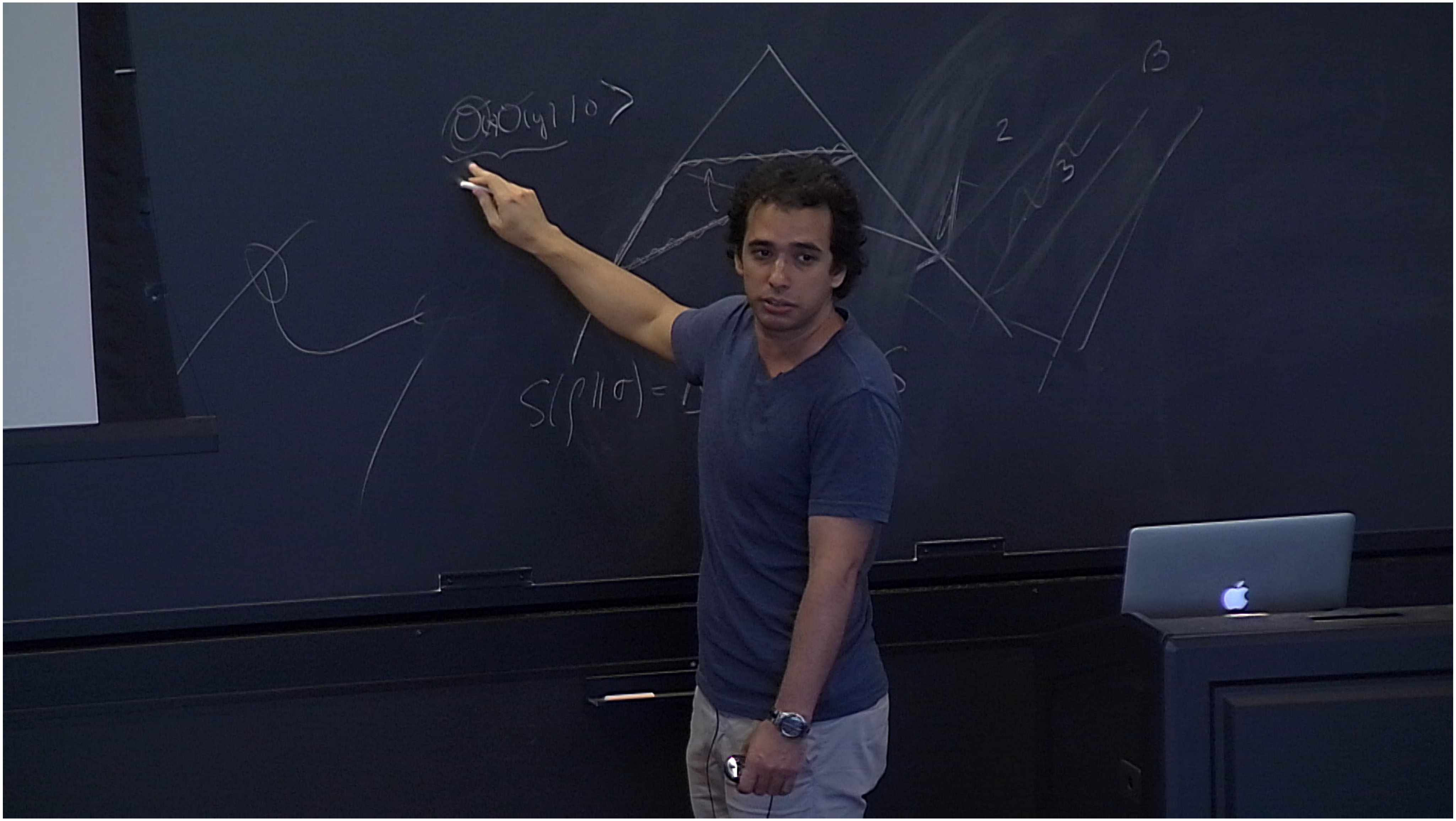
This inequality unifies all the known c theorems under a statement about vacuum entanglement entropy !

Eduardo Testé
Instituto Balseiro
Centro Atómico Bariloche

Another application

Relation between SSA and unitarity…
Unitarity is an inequality over all the vectors of the Hilbert space
Vacuum SSA is an inequality for one state over many regions

Eduardo Testé
Instituto Balseiro
Centro Atómico Bariloche



Another application

Relation between SSA and unitarity…
Unitarity is an inequality over all the vectors of the Hilbert space
Vacuum SSA is an inequality for one state over many regions

unitarity

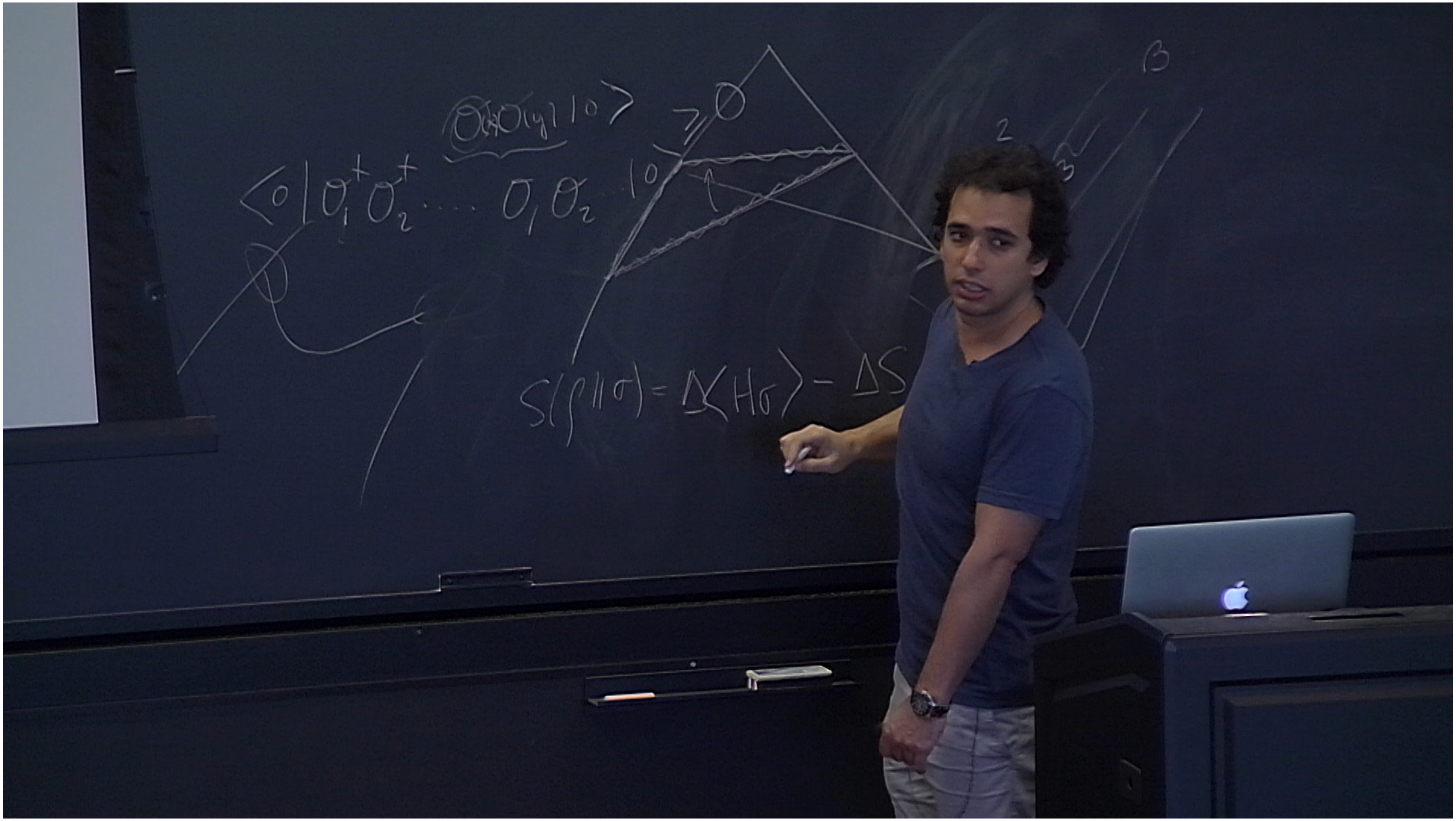
$\Downarrow$ (reflexion positivity)

$$\Delta_s \geqslant \frac{d-2}{2}$$

$$\Delta_f \geqslant \frac{d-1}{2}$$

$$\Delta_j \geqslant d+j-2$$

Eduardo Testé
Instituto Balseiro
Centro Atómico Bariloche



S(ρ||σ) = Δ⟨Hσ⟩ − ΔS

Another application

Relation between SSA and unitarity…
Unitarity is an inequality over all the vectors of the Hilbert space
Vacuum SSA is an inequality for one state over many regions

unitarity

$\Downarrow$ (reflexion positivity)

$$\Delta_s \geqslant \frac{d-2}{2}$$

$$\Delta_f \geqslant \frac{d-1}{2} \qquad \overset{?}{\Leftarrow} \quad \text{SSA}$$

$$\Delta_j \geqslant d + j - 2$$

Eduardo Testé
Instituto Balseiro
Centro Atómico Bariloche

Another application

Relation between SSA and unitarity…
Unitarity is an inequality over all the vectors of the Hilbert space
Vacuum SSA is an inequality for one state over many regions

unitarity

$\Downarrow$ (reflexion positivity)

$$\Delta_s \geqslant \frac{d-2}{2}$$

$$\Delta_f \geqslant \frac{d-1}{2} \quad \overset{?}{\Leftarrow} \quad \text{SSA}$$

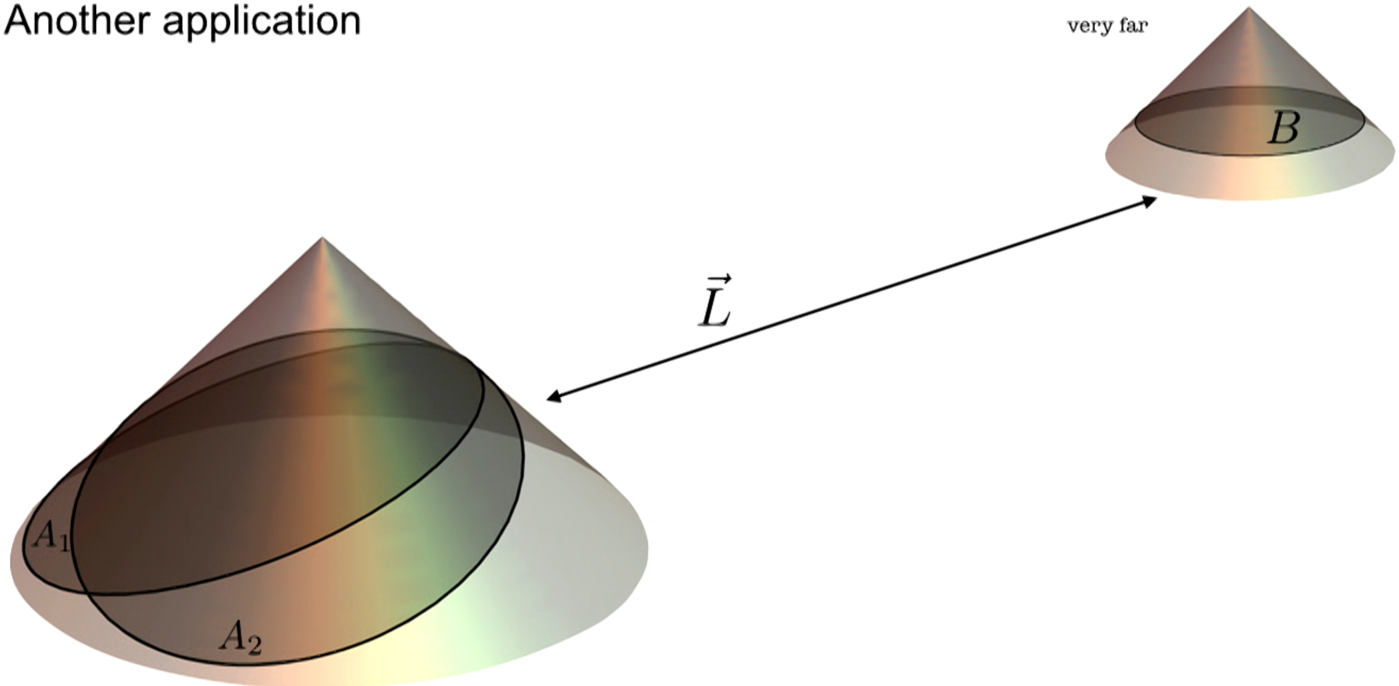
$$\Delta_j \geqslant d+j-2$$

Answer: yes, but with the help of the Markov property

(we need to know where the saturation of SSA is)
(or the same, we need SSuperA of Relative entropy)

Eduardo Testé
Instituto Balseiro
Centro Atómico Bariloche

Another application

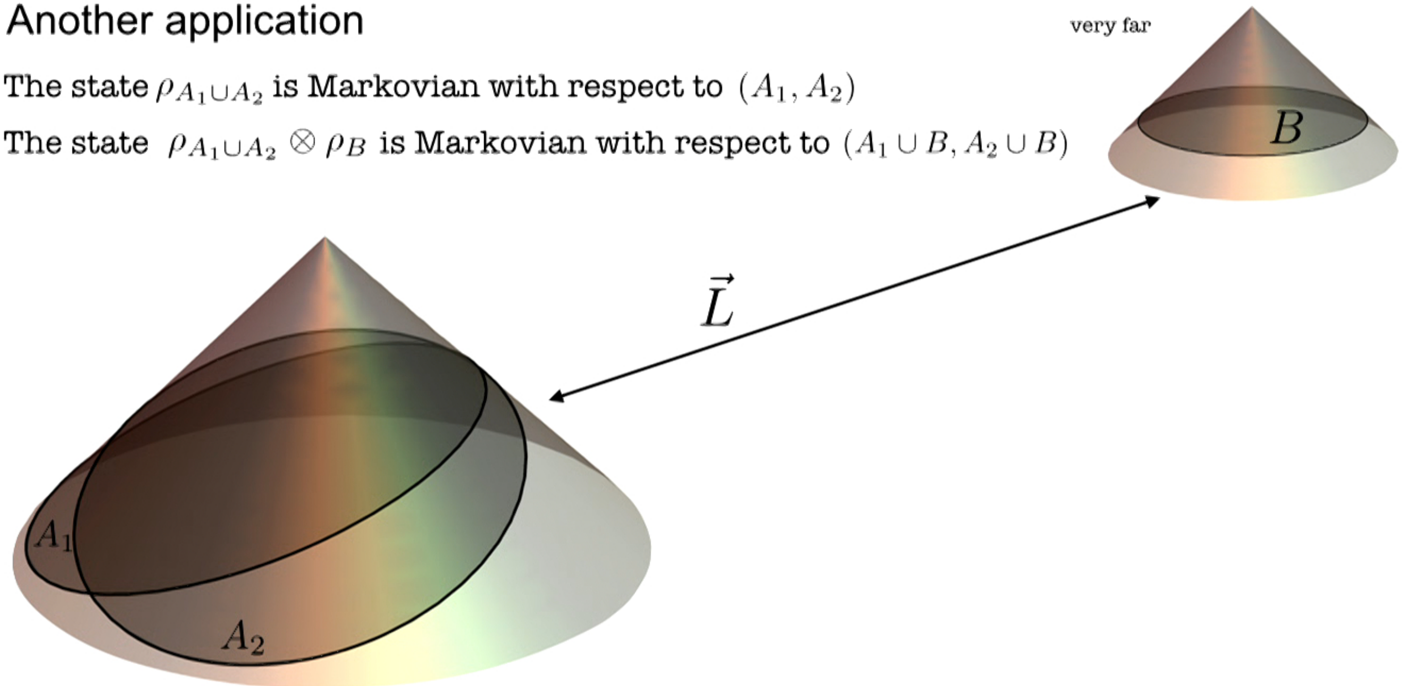


Eduardo Testé
Instituto Balseiro
Centro Atómico Bariloche

Another application

The state $\rho_{A_1 \cup A_2}$ is Markovian with respect to (A_1, A_2)

The state $\rho_{A_1 \cup A_2} \otimes \rho_B$ is Markovian with respect to $(A_1 \cup B, A_2 \cup B)$



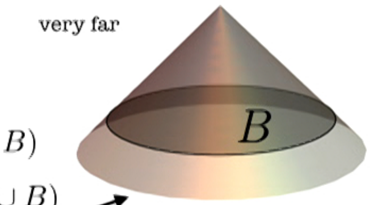
Eduardo Testé
Instituto Balseiro
Centro Atómico Bariloche

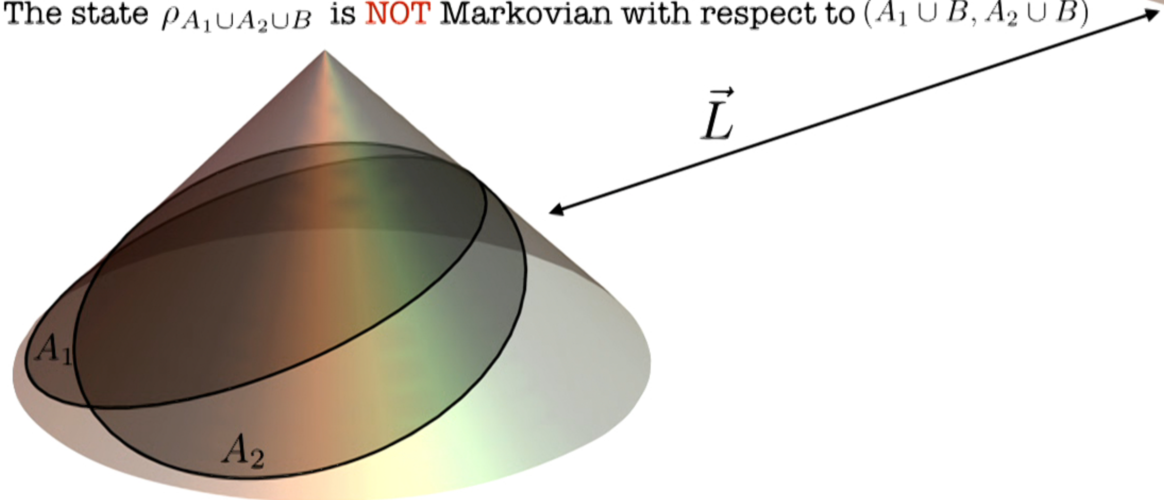
Another application

The state $\rho_{A_1 \cup A_2}$ is Markovian with respect to (A_1, A_2)

The state $\rho_{A_1 \cup A_2} \otimes \rho_B$ is Markovian with respect to $(A_1 \cup B, A_2 \cup B)$

The state $\rho_{A_1 \cup A_2 \cup B}$ is NOT Markovian with respect to $(A_1 \cup B, A_2 \cup B)$





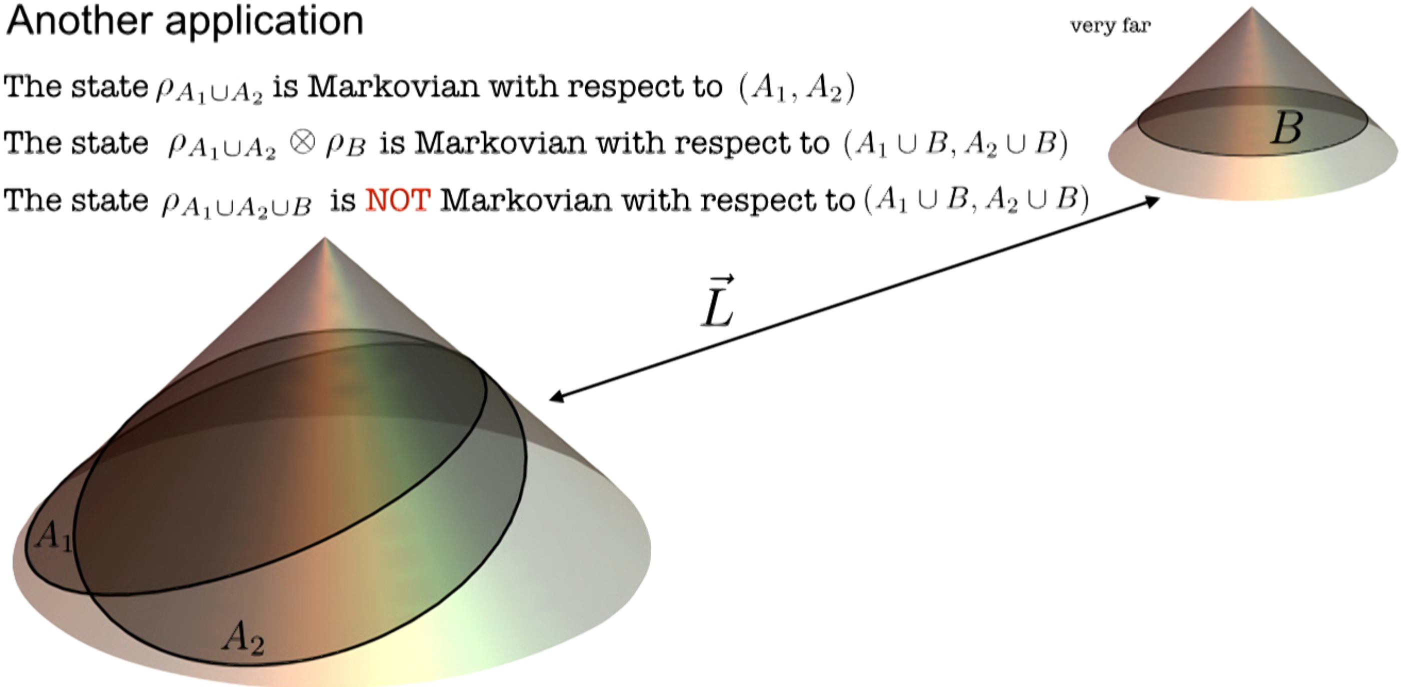
Eduardo Testé
Instituto Balseiro
Centro Atómico Bariloche

Another application

The state $\rho_{A_1 \cup A_2}$ is Markovian with respect to (A_1, A_2)

The state $\rho_{A_1 \cup A_2} \otimes \rho_B$ is Markovian with respect to $(A_1 \cup B, A_2 \cup B)$

The state $\rho_{A_1 \cup A_2 \cup B}$ is NOT Markovian with respect to $(A_1 \cup B, A_2 \cup B)$

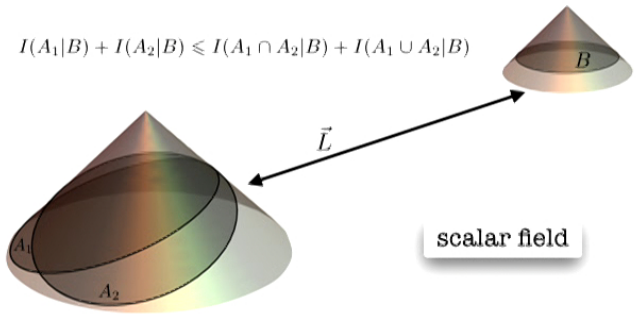


$$S(\rho_{A_1 \cup B} || \rho_{A_1} \otimes \rho_B) + S(\rho_{A_2 \cup B} || \rho_{A_2} \otimes \rho_B) \leqslant S(\rho_{(A_1 \cap A_2) \cup B} || \rho_{A_1 \cap A_1} \otimes \rho_B) + S(\rho_{(A_1 \cup A_2) \cup B} || \rho_{A_1 \cup A_2} \otimes \rho_B)$$

$$I(A_1|B) + I(A_2|B) \leqslant I(A_1 \cap A_2|B) + I(A_1 \cup A_2|B)$$

Eduardo Testé
Instituto Balseiro
Centro Atómico Bariloche

$$I(A_1|B) + I(A_2|B) \leqslant I(A_1 \cap A_2|B) + I(A_1 \cup A_2|B)$$

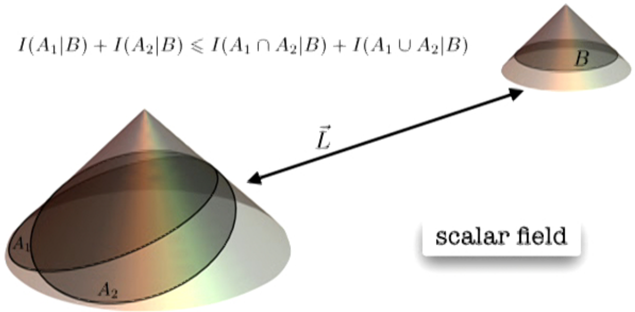


construct a symmetric version

$$\frac{1}{N}\sum_i I(A_i|B) \leqslant \int_r^R dl\ \beta(l) I(A_l|B)$$

Eduardo Testé
Instituto Balseiro
Centro Atómico Bariloche

$$I(A_1|B) + I(A_2|B) \leqslant I(A_1 \cap A_2|B) + I(A_1 \cup A_2|B)$$



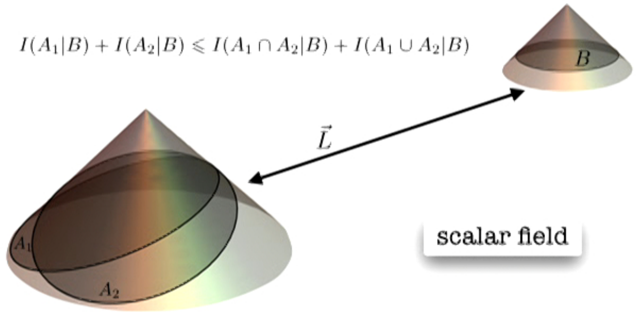
construct a symmetric version

$$\frac{1}{N}\sum_i I(A_i|B) \leqslant \int_r^R dl\ \beta(l) I(A_l|B)$$

from [Cardy, 13] $\quad I(A_i|B) \sim \frac{F(r_{A_i})F(r_B)}{L^{4\Delta}}$

Eduardo Testé
Instituto Balseiro
Centro Atómico Bariloche

$I(A_1|B) + I(A_2|B) \leqslant I(A_1 \cap A_2|B) + I(A_1 \cup A_2|B)$



construct a symmetric version

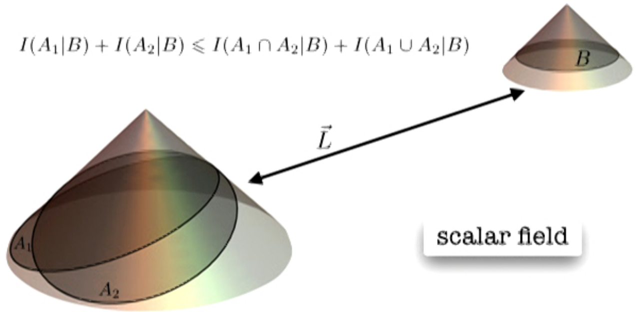
$$\frac{1}{N}\sum_i I(A_i|B) \leqslant \int_r^R dl\ \beta(l) I(A_l|B)$$

from [Cardy, 13] $\quad I(A_i|B) \sim \frac{F(r_{A_i})F(r_B)}{L^{4\Delta}}$

$$F(\sqrt{rR}) \leqslant \int_r^R dl\ \beta(l) F(A_l)$$

Eduardo Testé
Instituto Balseiro
Centro Atómico Bariloche

$I(A_1|B) + I(A_2|B) \leqslant I(A_1 \cap A_2|B) + I(A_1 \cup A_2|B)$



scalar field

construct a symmetric version

$$\frac{1}{N}\sum_i I(A_i|B) \leqslant \int_r^R dl\ \beta(l) I(A_l|B)$$

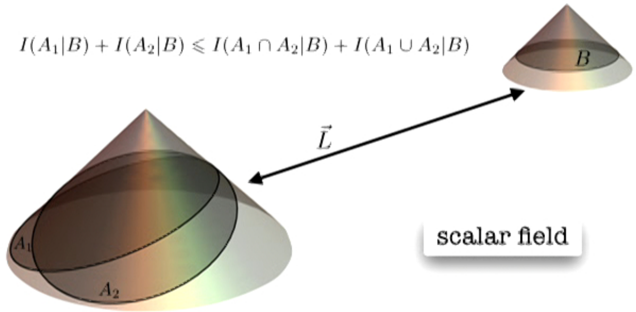
from [Cardy, 13] $\quad I(A_i|B) \sim \frac{F(r_{A_i})F(r_B)}{L^{4\Delta}}$

$$F(\sqrt{rR}) \leqslant \int_r^R dl\ \beta(l) F(A_l)$$ and take the $R \to r$ limit

$$rF''(r) - (d-3)F'(r) \geqslant 0$$

Eduardo Testé
Instituto Balseiro
Centro Atómico Bariloche

$I(A_1|B) + I(A_2|B) \leqslant I(A_1 \cap A_2|B) + I(A_1 \cup A_2|B)$



scalar field

construct a symmetric version

$$\frac{1}{N}\sum_i I(A_i|B) \leqslant \int_r^R dl\ \beta(l) I(A_l|B)$$

from [Cardy, 13] $\quad I(A_i|B) \sim \frac{F(r_{A_i})F(r_B)}{L^{4\Delta}}$

$$F(\sqrt{rR}) \leqslant \int_r^R dl\ \beta(l) F(A_l)$$ and take the $R \to r$ limit

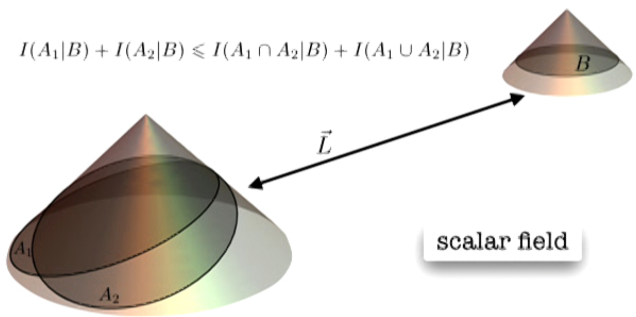
$$rF''(r) - (d-3)F'(r) \geqslant 0 \qquad F(r) \sim r^{2\Delta}$$

$$2\Delta(2\Delta - 1) - (d-3)2\Delta \geqslant 0 \quad \Rightarrow \quad \Delta \geqslant \frac{d-2}{2}$$

scalar
unitarity bound

Eduardo Testé
Instituto Balseiro
Centro Atómico Bariloche

$I(A_1|B) + I(A_2|B) \leqslant I(A_1 \cap A_2|B) + I(A_1 \cup A_2|B)$



scalar field

construct a symmetric version

$$\frac{1}{N}\sum_i I(A_i|B) \leqslant \int_r^R dl\ \beta(l) I(A_l|B)$$

from [Cardy, 13] $\quad I(A_i|B) \sim \dfrac{F(r_{A_i})F(r_B)}{L^{4\Delta}}$

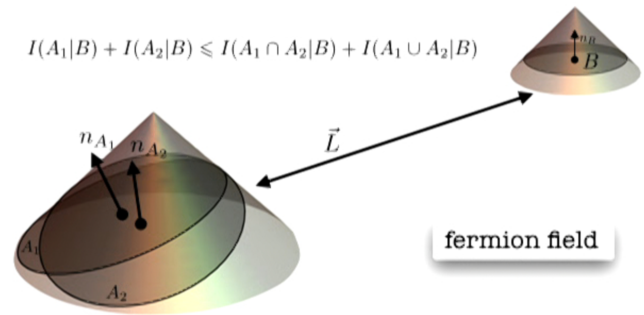
$$F(\sqrt{rR}) \leqslant \int_r^R dl\ \beta(l) F(A_l)$$ and take the $R \to r$ limit

$$rF''(r) - (d-3)F'(r) \geqslant 0 \qquad F(r) \sim r^{2\Delta}$$

$$2\Delta(2\Delta - 1) - (d-3)2\Delta \geqslant 0 \quad \Rightarrow \quad \Delta \geqslant \frac{d-2}{2}$$

scalar
unitarity bound

$I(A_1|B) + I(A_2|B) \leqslant I(A_1 \cap A_2|B) + I(A_1 \cup A_2|B)$



fermion field

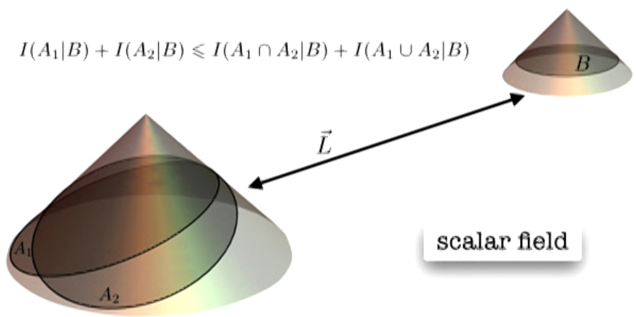
construct a symmetric version

$$\frac{1}{N}\sum_i I(A_i|B) \leqslant \int_r^R dl\ \beta(l) I(A_l|B)$$

$$I(A_i|B) \sim \frac{F(r_{A_i})F(r_B)}{L^{4\Delta}}\ n_{A_i} \cdot n_B$$

Eduardo Testé
Instituto Balseiro
Centro Atómico Bariloche

$I(A_1|B) + I(A_2|B) \leqslant I(A_1 \cap A_2|B) + I(A_1 \cup A_2|B)$



construct a symmetric version

$$\frac{1}{N}\sum_i I(A_i|B) \leqslant \int_r^R dl\ \beta(l) I(A_l|B)$$

from [Cardy, 13] $\quad I(A_i|B) \sim \dfrac{F(r_{A_i})F(r_B)}{L^{4\Delta}}$

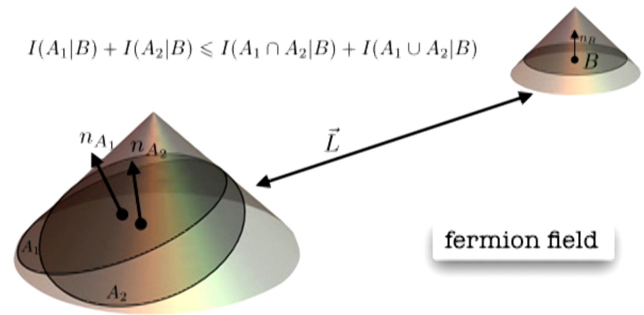
$$F(\sqrt{rR}) \leqslant \int_r^R dl\ \beta(l) F(A_l)$$ and take the $R \to r$ limit

$$rF''(r) - (d-3)F'(r) \geqslant 0 \qquad F(r) \sim r^{2\Delta}$$

$$2\Delta(2\Delta - 1) - (d-3)2\Delta \geqslant 0 \quad \Rightarrow \quad \Delta \geqslant \frac{d-2}{2}$$

scalar
unitarity bound

$I(A_1|B) + I(A_2|B) \leqslant I(A_1 \cap A_2|B) + I(A_1 \cup A_2|B)$



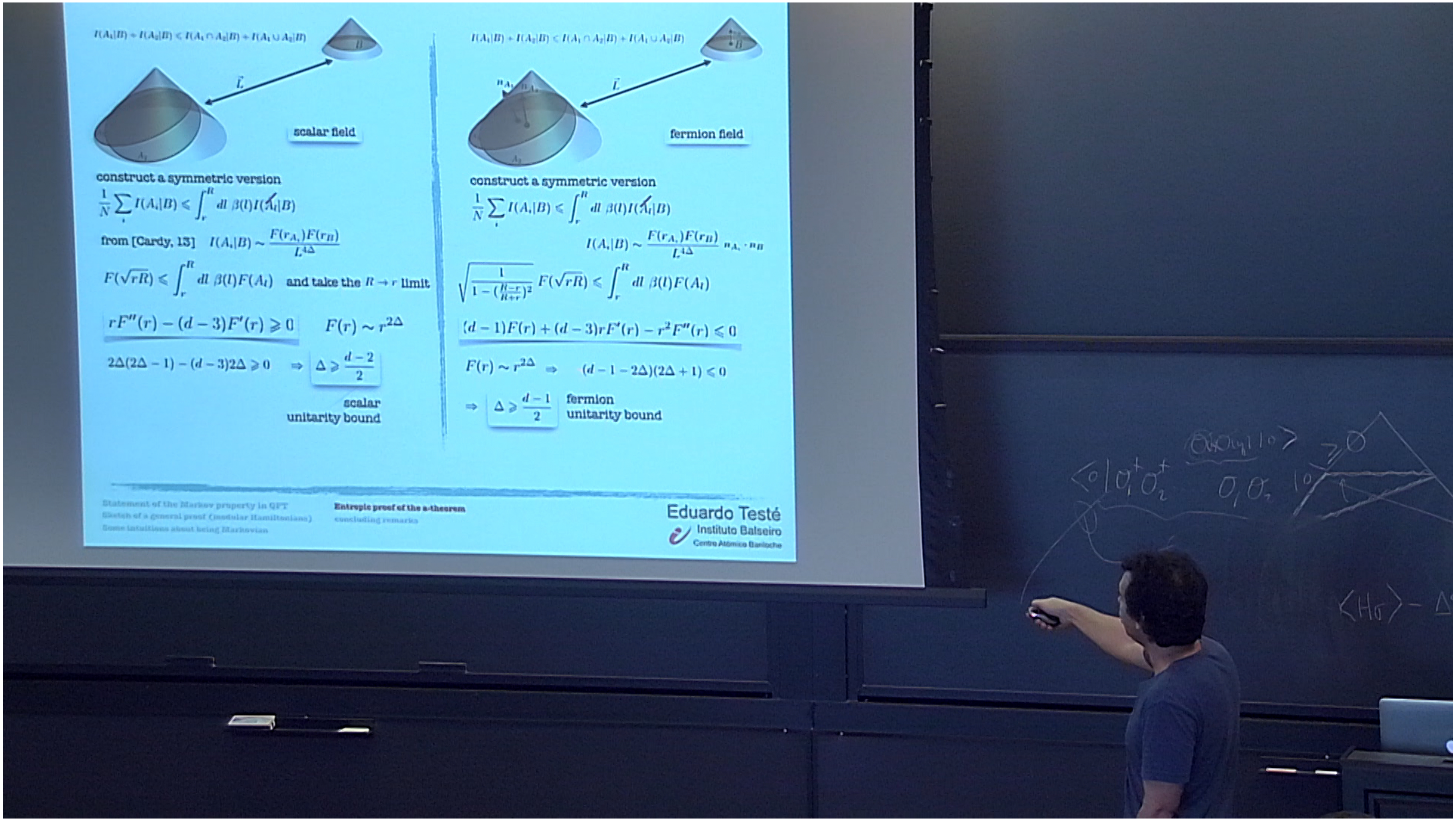
construct a symmetric version

$$\frac{1}{N}\sum_i I(A_i|B) \leqslant \int_r^R dl\ \beta(l) I(A_l|B)$$

$$I(A_i|B) \sim \frac{F(r_{A_i})F(r_B)}{L^{4\Delta}}\ n_{A_i} \cdot n_B$$

$$\sqrt{\frac{1}{1-(\frac{R-r}{R+r})^2}}\ F(\sqrt{rR}) \leqslant \int_r^R dl\ \beta(l) F(A_l)$$

Eduardo Testé
Instituto Balseiro
Centro Atómico Bariloche


scalar field
construct a symmetric version
from [Cardy, 13]
and take the R → r limit
scalar
unitarity bound
fermion field
construct a symmetric version
fermion
unitarity bound
Eduardo Testé
Instituto Balseiro
Centro Atómico Bariloche

concluding remarks

- $r\Delta S''(r) - (d-3)\Delta S'(r) \leqslant 0$ unifies all the known c theorems $d = 2, 3, 4$
- We have obtained an explicit form for the modular Hamiltonian of general null cut regions
- The vacuum is a Markov state relative to some space time regions subalgebras: it has a simpler entanglement structure than the expected (almost a product state).
- This Markov property play a role in other applications of information theory in QFT: for example it is needed to get the unitarity bound from SSA.
- Holographically, the Markov property is very easy to check.
- With a Markov state we have at our disposal the strong super additivity of relative entropy (stronger than SSA)

...

- hope the Markov property serve to discover new aspects of QFT and CFTs!

Eduardo Testé
Instituto Balseiro
Centro Atómico Bariloche

concluding remarks

- $r\Delta S''(r)-(d-3)\Delta S'(r)\leqslant 0$ unifies all the known c theorems $d=2,3,4$
- We have obtained an explicit form for the modular Hamiltonian of general null cut regions
- The vacuum is a Markov state relative to some space time regions subalgebras: it has a simpler entanglement structure than the expected (almost a product state).
- This Markov property play a role in other applications of information theory in QFT: for example it is needed to get the unitarity bound from SSA.
- Holographically, the Markov property is very easy to check.
- With a Markov state we have at our disposal the strong super additivity of relative entropy (stronger than SSA)

…

- hope the Markov property serve to discover new aspects of QFT and CFTs!

Thank you

Eduardo Testé
Instituto Balseiro
Centro Atómico Bariloche