Title: Twist Fields in Quantum Field Theory: Entanglement Measures and Pentagonal Amplitudes

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Abstract: <p class="MsoNormal">Branch point twist fields play an important role in the study of measures of entanglement such as the RÃ©nyi entropies and the Negativity. In 1+1 dimensions such measures can be written in terms of multi-point functions of branch point twist fields. For 1+1-dimensional integrable quantum field theories and also in conformal field theory much is known about how to compute correlation functions and, with the help of the twist field, this knowledge can be exploited in order to gain new insights into the properties of various entanglement measures. In this talk I will review some of our main results in this context.</p>

<p class="MsoNormal">I will then go on to introduce a new (related) class of fields we have recently named conical twist fields. These are fields whose two-point functions have (surprisingly) been found to describe gluon amplitudes in the strong coupling limit of super Yang-Mills theories and therefore have featured in a completely different context from that of entanglement measures. Interestingly, at critical points, branch point and conical twist fields have the same conformal dimension and beyond criticality they also have very similar form factors, however they are different in many other respects. In my talk I will discuss and justify some of their similarities and differences.</p>

Twist Fields in Quantum Field Theory: Entanglement Measures and Pentagonal Amplitudes

Olalla A. Castro-Alvaredo

School of Mathematics, Computer Science and Engineering
Department of Mathematics
City, University of London

Perimeter Institute for Theoretical Physics
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Waterloo, Ontario (Canada)

Background:

Two good introductions to the topic of branch point twist fields and their relationship with entanglement measures are:

John L. Cardy, O.C.-A. and Benjamin Doyon, *Form factors of branch-point twist fields in quantum integrable models and entanglement entropy*, J. Stat. Phys. 130 (2008) 129-168, arXiv:0706.3384.

O.C.-A. and Benjamin Doyon, *Bi-partite entanglement entropy in massive 1+1 dimensional quantum field theories*, J. Phys. A42 (2009) 504006, arXiv:0906.2946 (Review Article).

Today I will also be talking about very recent work on conical twist fields, which we have introduced in our recent paper:

O.C.-A., Benjamin Doyon and Davide Fioravanti, *Conical twist fields and null polygonal Wilson loops*, arXiv:1709.05980.

Collaborators:

I would like to thank all my collaborators on this area of research:

Davide Bianchini, Former PhD Student
Olivier Blondeau-Fournier, Université Laval (Quèbec)
John L. Cardy, University of California, Berkeley
Benjamin Doyon, King's College London
Davide Fioravanti, Università di Bologna
Emanuele Levi, Former PhD Student
Francesco Ravanini, Università di Bologna

1. Entanglement in quantum mechanics

- A quantum system is in an entangled state if performing a localised measurement (in space and time) may instantaneously affect local measurements far away.
  A typical example: a pair of opposite-spin electrons:

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)\,, \quad \langle\hat{A}\rangle = \langle\psi|\hat{A}|\psi\rangle$$

- What is special: Bell's inequality says that this cannot be described by local variables.
- A situation that looks similar to $|\psi\rangle$ but without entanglement is a factorizable state:

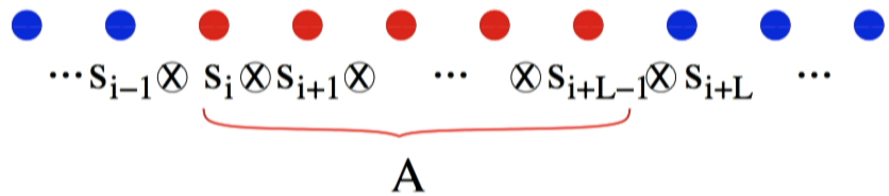
$$|\hat{\psi}\rangle = \frac{1}{2}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle + |\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle)$$

$$= \frac{1}{2}(|\uparrow\rangle + |\downarrow\rangle) \otimes (|\uparrow\rangle + |\downarrow\rangle)$$

- These examples are extremely simple but what happens in extended many-body quantum systems?
- First of all, what provides a good measure of entanglement? [Plenio & Virmani'05]
    1. Entanglement monotone: no increase under LOCC
    2. Invariant under unitary transformations
    3. Zero for separable states
    4. Non-zero for non-separable states
- Among others, the bi-partite entanglement entropy and the logarithmic negativity are good measures of entanglement according to these properties.
- Today I will talk about the entanglement entropy in the context of 1+1 dimensional QFT (at and near criticality).

2. Bi-partite (von Neumann) Entanglement Entropy

Let us consider a spin chain of length N, subdivided into regions A and $\bar{A}$ of lengths L and $N-L$ [Bennett et al.'96]



then we define

Von Neumann Entanglement Entropy

$$S(L) = -\mathrm{Tr}_{\mathcal{A}}(\rho_A \log(\rho_A)) \quad \text{with} \quad \rho_A = \mathrm{Tr}_{\bar{\mathcal{A}}}(|\Psi\rangle\langle\Psi|)$$

$|\Psi\rangle$ ground state and ρ_A the reduced density matrix.
Other entropies may also be defined such as

Rényi Entropies & Replica Trick

$$S_n(L) = \frac{\log(\mathrm{Tr}_{\mathcal{A}}(\rho_A^n))}{1-n} \Rightarrow S(L) = -\lim_{n \to 1^+} \frac{d}{dn} \mathrm{Tr}_{\mathcal{A}}(\rho_A^n)$$

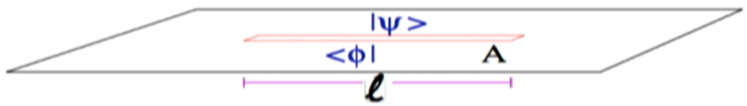
3. Replica Trick I

The object $\mathrm{Tr}_{\mathcal{A}}\rho_A^n$ with n integer is also a partition function [Callan & Wilczek'93; Holzhey, Larsen & Wiczek'94; Calabrese & Cardy'04]:

$$\mathrm{Tr}_{\mathcal{A}}\rho_A^n = \frac{Z_n}{Z_1^n},$$

but now it is defined on an n-sheeted Riemann surface:

$${}_A\langle\phi|\rho_A|\psi\rangle_A \sim$$



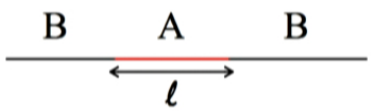
$$\mathrm{Tr}_{\mathcal{A}}(\rho_A^n) \sim Z_n = \int [d\varphi]_{\mathcal{M}_n} \exp\left[-\int_{\mathcal{M}_n} d^2x\, \mathcal{L}[\varphi](x)\right]$$

$$\mathcal{M}_3 =$$

4. Replica Trick II

- However, when computing this limit we need to extend our notion of "replica" to $n \geq 1$ and $n \in \mathbb{R}$.
- The analytic continuation problem is not solved in general although existence and uniqueness are expected and may be established under certain natural conditions (e.g. Carlson Theorem).
- Note that this is only a difficult problem when trying to obtain analytical results.

5. Rényi Entropies at and near Critical Points



At criticality ($\xi \to \infty$):

Universal scaling: [Holzhey, Larsen & Wilczek'94; Vidal, Latorre, Rico & Kitaev'03; Calabrese & Cardy'04; Bianchini,OC-A, Doyon, Levi & Ravanini'15]:

$$S_n(\ell) \sim \frac{c_{\text{eff}}(n+1)}{6n} \log \frac{\ell}{\epsilon}$$

c_{eff} is the (effective) central change which uniquely characterises the CFT. ϵ is a non-universal cut-off.

For more than one interval: information about operator content of CFT.

Near criticality (ξ finite):

Universal saturation [Calabrese & Cardy'04] and decay [Cardy, OC-A & Doyon'08; Doyon'09; Levi, OC-A & Doyon'11]

$$S(\ell) - \lim_{\ell \to \infty} S(\ell) = -\frac{1}{8} K_0(2m\ell) + \cdots$$

where m is the mass of the lightest particle in the spectrum. Similar corrections are found for the LN [Blondeau-Fournier, OC-A & Doyon'16]

6. Branch Point Twist Fields I

- How are the results in the previous slide obtained? At criticality, conformal invariance can be employed but near criticality a different technique is needed.
- In the context of entanglement, the idea of quantum fields associated with branch points of the Riemann surfaces $\mathcal{M}_n$ appeared first in [Calabrese & Cardy'04].
- The interpretation of these fields as symmetry fields of a QFT replica model was first given in [Cardy, OC-A & Doyon'08].
- Let us define the replica model by writing its two-particle scattering matrices as

$$S_{\mu_1\mu_2}(\theta) = (S_{ab}(\theta))^{\delta_{ij}} \quad \text{with} \quad \mu_1 = (a,i) \quad \text{and} \quad \mu_2 = (b,j)$$

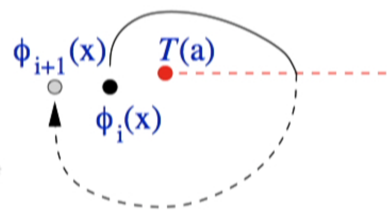
Here $a, b = 1, \ldots, k$ label particle species and $i, j = 1, \ldots, n$ label copy numbers. This represents n non-interacting copies of a QFT with diagonal scattering.

7. Branch Point Twist Fields II

- Branch Point Twist Fields are defined by the following equal-time exchange relations

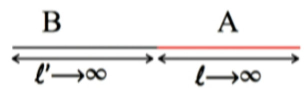
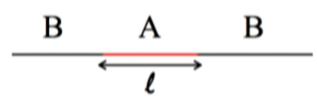
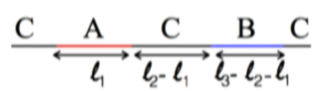
$$\begin{aligned} \Phi_i(y)\mathcal{T}(x) &= \mathcal{T}(x)\Phi_{i+1}(y) \qquad x^1 > y^1, \\ \Phi_i(y)\mathcal{T}(x) &= \mathcal{T}(x)\Phi_i(y) \qquad x^1 < y^1, \end{aligned}$$

  for $i = 1, \ldots, n$ and $n + i \equiv i$. Similarly $\tilde{\mathcal{T}} = \mathcal{T}^\dagger$ implements the symmetry $i \mapsto i - 1$.



- Twist fields have a quantum spin chain counterpart [OC-A & Doyon'11] as product of local operators on replica chains.
- Branch point twist fields were studied earlier in the context of orbifold CFT [Knizhnik'87; Dixon et al.'87] and their conformal dimension was known: $\Delta_n = \frac{c}{24}\left(n - \frac{1}{n}\right)$.

8. Rényi Entropy vs Correlation Functions

$\epsilon^{2\Delta_n}\langle \mathcal{T}\rangle_n$		EE of semi-infinite region. By CTM approach [Calabrese, Cardy, Peschel, Bianchini, Ercolessi, Franchini, Evangelisti, Ravanini,...], Töplitz determinants [Its, Jin & Korepin'04], QFT techniques [Cardy, OC-A & Doyon'08, Blondeau-Fournier & Doyon'16] or numerically [Vidal et al.'03]
$\epsilon^{4\Delta_n}\langle \mathcal{T}(0)\tilde{\mathcal{T}}(\ell)\rangle_n$		EE of finite interval. In CFT [Callan & Wilczek'93; Holzhey et al.'94; Vidal et al.'03;Calabrese & Cardy'04...]. In massive models [Cardy, OC-A, Doyon, Levi, Bianchini...]
$\epsilon^{8\Delta_n}\langle \mathcal{T}(0)\tilde{\mathcal{T}}(\ell_1)\mathcal{T}(\ell_2)\tilde{\mathcal{T}}(\ell_3)\rangle_n$		EE of 2 disconnected regions [Calabrese, Cardy, Tonni, Alba, Casini, Huerta, Furukawa, Igloi, Pasquier, Shiraishi, Caraglio, Gliozzi, Peschel, Tagliacozzo, Fagotti, Rajabpour, Datta...]

O.A. Castro-Alvaredo, City, University of London — Twist Fields in 1+1D QFT

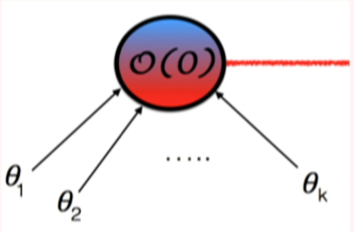
9. Form Factors of Local Fields: Definition

- Let $|\theta_1, \ldots, \theta_k\rangle_{\mu_1 \ldots \mu_k}$ a k-particle *in*-state. The particles have rapidities $\theta_1 > \cdots > \theta_k$ and quantum numbers $\mu_1 \ldots \mu_k$. Let $\mathcal{O}(0)$ be a local field located at the origin of space-time. Let $|0\rangle = (\langle 0|)^\dagger$ be the ground state (vacuum).

Form Factor

$$F_k^{\mathcal{O}|\mu_1 \ldots \mu_k}(\theta_1, \ldots, \theta_k) \quad := \quad \langle 0|\mathcal{O}(0)|\theta_1, \ldots, \theta_k\rangle_{\mu_1 \ldots \mu_k}$$

$$=$$

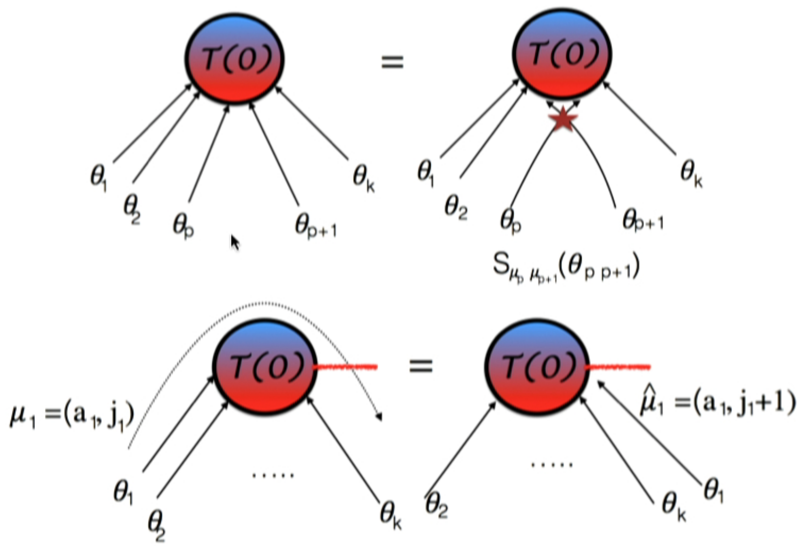


- Form factors are the building blocks of correlation functions.
- For local fields in integrable QFT, FFs are solutions of a Riemann-Hilbert problem and have been computed for many models [Karowski & Weisz'78; Smirnov'90s]

10. Twist Field Watson's Equations

$$F_k^{\mathcal{T}|\dots\mu_p\mu_{p+1}\dots}(\dots\theta_p,\theta_{p+1}\dots) = S_{\mu_p\mu_{p+1}}(\theta_{p,p+1})F_k^{\mathcal{T}|\dots\mu_{p+1}\mu_p\dots}(\dots,\theta_{p+1},\theta_p,\dots)$$

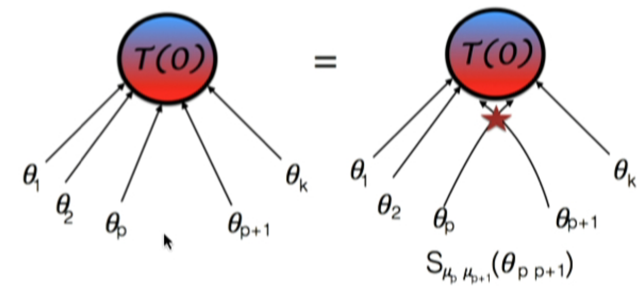
$$F_k^{\mathcal{T}|\mu_1\dots\mu_k}(\theta_1+2\pi i,\dots,\theta_k) = F_k^{\mathcal{T}|\mu_2\dots\mu_k\hat{\mu}_1}(\theta_2,\dots,\theta_k,\theta_1)$$



10. Twist Field Watson's Equations

$$F_k^{\mathcal{T}|\ldots\mu_p\mu_{p+1}\ldots}(\ldots\theta_p,\theta_{p+1}\ldots) = S_{\mu_p\mu_{p+1}}(\theta_{p,p+1})F_k^{\mathcal{T}|\ldots\mu_{p+1}\mu_p\ldots}(\ldots,\theta_{p+1},\theta_p,\ldots)$$

$$F_k^{\mathcal{T}|\mu_1\ldots\mu_k}(\theta_1+2\pi i,\ldots,\theta_k) = F_k^{\mathcal{T}|\mu_2\ldots\mu_k\hat{\mu}_1}(\theta_2,\ldots,\theta_k,\theta_1)$$

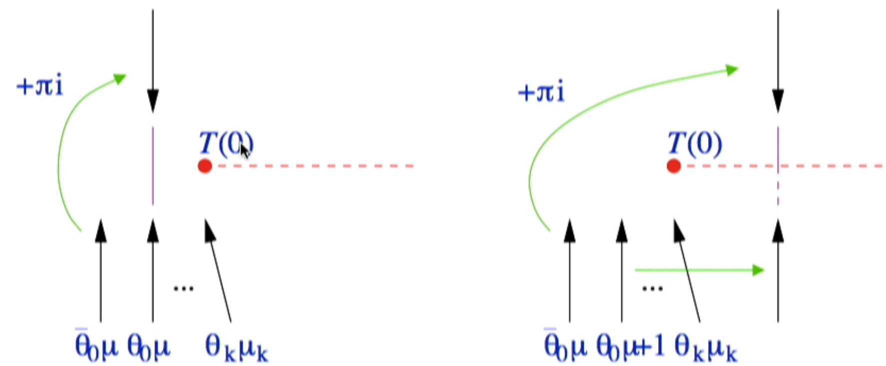


11. Twist Field Kinematic Residue Equations

- For twist fields, the kinematic residue equation splits into two equations:

$$\lim_{\bar{\theta}_0\to\theta_0}(\bar{\theta}_0-\theta_0)F_{k+2}^{\mathcal{T}|\bar{\mu}\mu\mu_1\ldots\mu_k}(\bar{\theta}_0+i\pi,\theta_0,\theta_1,\ldots,\theta_k) = iF_k^{\mathcal{T}|\mu_1\ldots\mu_k}(\theta_1,\ldots,\theta_k)$$

$$\lim_{\bar{\theta}_0\to\theta_0}(\bar{\theta}_0-\theta_0)F_{k+2}^{\mathcal{T}|\bar{\hat{\mu}}\hat{\mu}\mu_1\ldots\mu_k}(\bar{\theta}_0+i\pi,\theta_0,\theta_1,\ldots,\theta_k) = -i\prod_{j=1}^{k}S_{\hat{\mu}\mu_j}(\theta_{0j})F_k^{\mathcal{T}|\mu_1\ldots\mu_k}(\theta_1,\ldots,\theta_k)$$



- There is an additional equation in the presence of bound states.

12. Branch Point Twist Fields Summary

- By systematically computing the form factors of branch point twist fields we can compute their correlation functions and learn about measures of entanglement.
- In order to compute the von Neumann EE we can use the replica trick and study the analytic continuation of these correlation functions as $n \to 1^+$.
- Since we work on replica theories, where the number of particles in the spectrum is a multiple of n, this analytic continuation is quite tricky but can be done at least in some cases, leading for instance to the universal Bessel function decay described earlier.

13. Twist Fields and Pentagonal Amplitudes

- In [Basso, Sever & Vieira'14] found a surprising result: in their own words...
  "we consider the collinear limit of gluon scattering amplitudes in planar $\mathcal{N} = 4$ SYM theory at strong coupling. We argue that in this limit scattering amplitudes map into correlators of twist fields in the two dimensional non-linear $O(6)$ sigma model, similar to those appearing in recent studies of entanglement entropy"

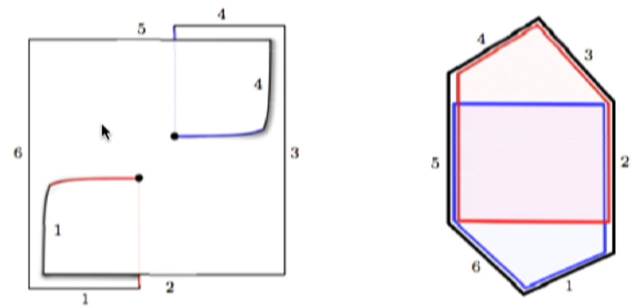


FIG. 4. At strong coupling, the hexagon Wilson loop in the collinear limit is given by a correlator of two twist operators in the $O(6)$ sigma model (on the left), corresponding to the two pentagons in its decomposition (on the right).

14. Pentagonal Amplitudes

- In [Basso, Sever & Vieira'14] it was observed that the "picture" above is translated into the OPE (at criticality)

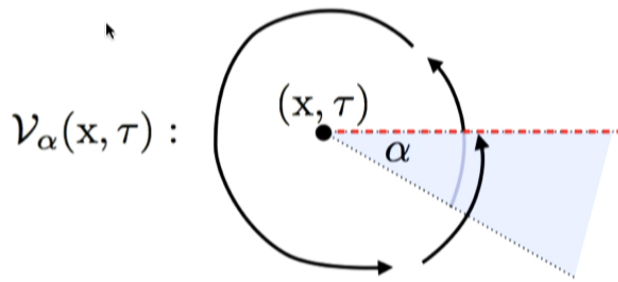
$$\phi_{\text{pen}}(0)\phi_{\text{pen}}(z) \sim \frac{(\log z)^B}{z^A}\phi_{\text{hex}}(0)$$

where $A = 4\Delta_{\frac{5}{4}} - 2\Delta_{\frac{3}{2}}$, where $\Delta_n = \frac{c}{24}\left(n - \frac{1}{n}\right)$ is the same dimension as for the branch point twist field (this was checked numerically also by [Bonini et al.'17]).

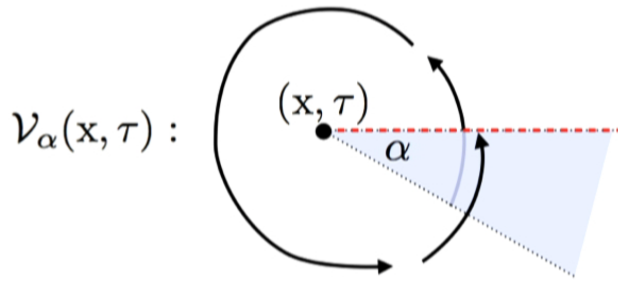
- This provided strong support for the idea that these "polygon" fields should be related to branch point twist fields.
- The values of n associated to these polygonal Wilson loops are generally fractional.
- The value of n is related to the so-called "excess angle" without connection to any notion of replica.
- Despite this, the form factor expansion of the two point function of pentagonal fields has a similar structure as for branch point twist fields.

15. Conical Twist Fields: Definition

- The fields emerging in this context are not branch point twist fields: they are not associated to an internal symmetry of the theory and they do not live in a replica model.
- Instead, they are twist fields associated to an space-time external symmetry. The symmetry in question is rotation symmetry about a point.
- They are fields that induce a conical singularity of arbitrary excess angle $\alpha = 2\pi(n-1)$, where n is not necessarily integer.



16. Conical Twist Fields: Explicit Expression

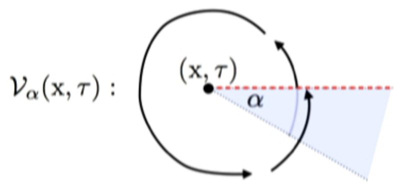


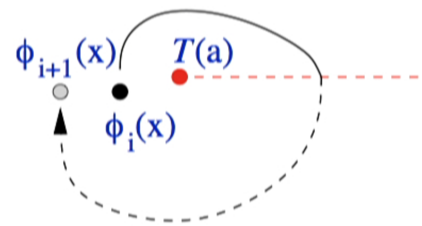
- In QFT such fields can (at least formally) be written in terms of the corresponding Noether current, that is

$$\mathcal{V}_\alpha(0,0) = \lim_{\epsilon\to 0} \epsilon^{-2\Delta_n} \exp\left[-\alpha \int_0^\infty ds \frac{dy^\mu(s)}{ds} \varepsilon_{\mu\nu} R^\nu(s)\right]_\epsilon$$

  where $R^\nu(s)$ is the rotations current and s parametrizes a curve from $(0,0)$ to infinity. ϵ characterizes the regularization scheme.
- In CFT the rotation current R^ν can be explicitly written and it is possible to show that this field has indeed the correct conformal dimension.

17. OPEs





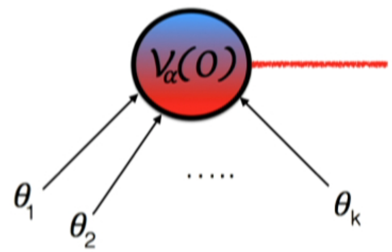
- Whereas branch point twist fields are defined for a particular integer value of n, conical twist fields are characterized by the excess angle α and it is possible to define OPEs and correlators involving various fields with different values of α.
- Thus the conformal OPEs (in non-logarithmic CFT) of the two types of fields are quite different. For instance:

Conformal OPEs

$$\begin{aligned}
\mathcal{V}_\alpha(0)\mathcal{V}_{\alpha'}(z,\bar{z}) &\sim C_{\alpha\,\alpha'}^{\alpha+\alpha'}|z|^{2\Delta_{\alpha+\alpha'}-2\Delta_\alpha-2\Delta_{\alpha'}}\mathcal{V}_{\alpha+\alpha'}(0)\\
\mathcal{T}(0)\tilde{\mathcal{T}}(z,\bar{z}) &\sim |z|^{-4\Delta_\alpha}\mathbf{1} \quad \text{with} \quad \alpha = 2\pi(n-1)\\
\mathcal{T}(0)\mathcal{T}(z,\bar{z}) &\sim |z|^{2\Delta-4\Delta_\alpha}\mathcal{T}^2(0)
\end{aligned}$$

18. Form Factors

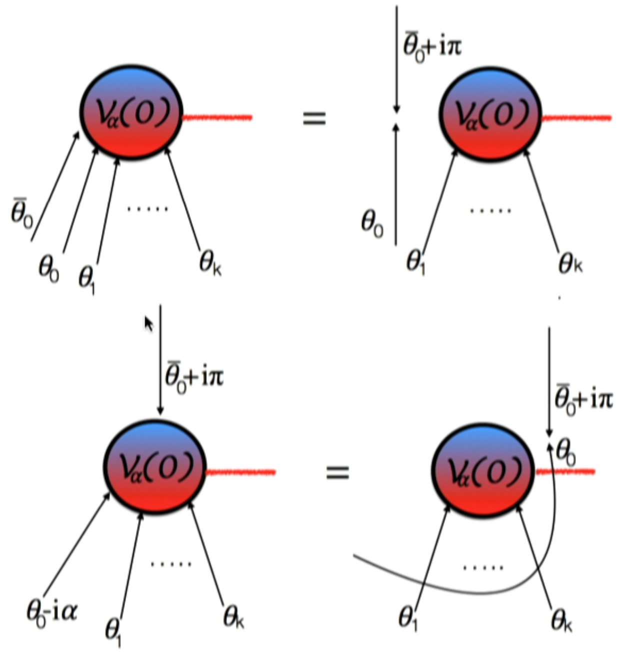
- In the context of pentagonal amplitudes, the two-point function $\langle \mathcal{V}_\alpha(0)\mathcal{V}_{\alpha'}(\tau, x)\rangle$ was computed for the (massive) $O(6)$ NLSM and its short distance behaviour was identified for $\alpha = \alpha' = \frac{\pi}{2}$ [Basso, Sever & Vieira'14; Bonini, Piscaglia, Rossi & Fioravanti'17].
- By deriving form factor equations for the conical twist field, it is possible to compute the two-point function for generic values of α and α' and study its short distance behaviour for any QFT of our choosing.
- Let us represent the k-particle form factor of $\mathcal{V}_\alpha$ as:



20. Kinematic Residue Equations

$$\operatorname*{Res}_{\bar\theta_0=\theta_0} F_{k+2}^{\mathcal{V}_\alpha|\bar\mu\mu\mu_1\ldots\mu_k}(\bar\theta_0+i\pi,\theta_0,\theta_1\ldots,\theta_k) = i\,F_k^{\mathcal{V}_\alpha|\mu_1\ldots\mu_k}(\theta_1,\ldots,\theta_k)$$

$$\operatorname*{Res}_{\bar\theta_0=\theta_0} F_{k+2}^{\mathcal{V}_\alpha|\bar\mu\mu\mu_1\ldots\mu_k}(\bar\theta_0+\pi,\theta_0-i\alpha,\theta_1\ldots,\theta_k) = -i\prod_{i=1}^{k} S_{\mu\mu_i}(\theta_{0i}-i\alpha)F_k^{\mathcal{V}_\alpha|\mu_1\ldots\mu_k}(\theta_1,\ldots,\theta_k)$$

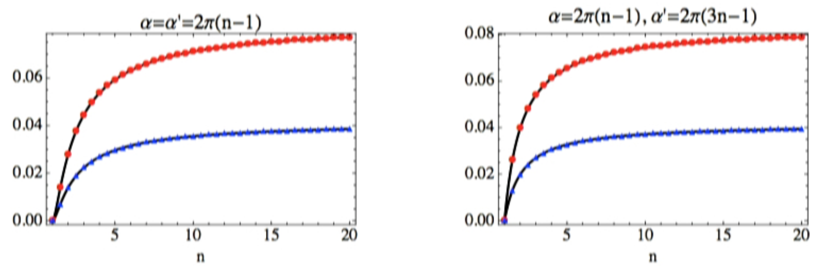


21. Some Numerical Results

- These equations can be solved for particular theories and it is not difficult to show that their solutions are identical as those for a branch point twist field $\mathcal{T}$ where only particles in one copy are considered and $\alpha = 2\pi(n-1)$.
- For massive free theories with $S(\theta) = \pm 1$ these form factors are all known and so it is possible to compute

$$\log\left(\frac{\langle \mathcal{V}_\alpha(0)\mathcal{V}_{\alpha'}(0,x)\rangle}{\langle \mathcal{V}_\alpha\rangle\langle \mathcal{V}'_\alpha\rangle}\right)_{m|x|\ll 1} \sim -x_{\alpha\alpha'} \log|x|$$

and study its short-distance behaviour.
- For free theories this short-distance behaviour is accessible exactly from form factors and we find for instance



Conclusions & Open Problems

- Twist fields provide a powerful tool for the computation of partition functions/scattering amplitudes in non-trivial geometries. In my talk we have seen two applications: to measures of entanglement and to the study of scattering amplitudes in SYM theory.
- There are many interesting open problems to be addressed within both areas:
  - On the entanglement side, we are currently looking at applying twist fields to the study of measures of entanglement in excited states of massive QFT. We would also like to eventually generalize the twist field idea to higher dimensions to study measures of entanglement in 2+1 dimensions.
  - For conical twist fields we would like to reach a better understanding of their correlators and of how they may be applied to the study of other types of amplitudes, such as the structure constants of single-trace operators in the $\mathcal{N} = 4$ SYM theory at large N which have been studied employing "hexagonal form factors" [Basso, Komatsu & Vieira'15].