Title: Emergent hydrodynamics in integrable systems out of equilibrium

Date: Sep 12, 2017 02:00 PM

URL: http://pirsa.org/17090056

Abstract: <p>The hydrodynamic approximation is an extremely powerful tool to describe the behavior of many-body systems such as gases. At the Euler scale (that is, when variations of densities and currents occur only on large space-time scales), the approximation is based on the idea of local thermodynamic equilibrium: locally, within fluid cells, the system is in a Galilean or relativistic boost of a Gibbs equilibrium state. This is expected to arise in conventional gases thanks to ergodicity and Gibbs thermalization, which in the quantum case is embodied by the eigenstate thermalization hypothesis. However, integrable systems are well known not to thermalize in the standard fashion. The presence of infinitely-many conservation laws preclude Gibbs thermalization, and instead generalized Gibbs ensembles emerge. In this talk I will introduce the associated theory of generalized hydrodynamics (GHD), which applies the hydrodynamic ideas to systems with infinitely-many conservation laws. It describes the dynamics from inhomogeneous states and in inhomogeneous force fields, and is valid both for quantum systems such as experimentally realized one-dimensional interacting Bose gases and quantum Heisenberg chains, and classical ones such as soliton gases and classical field theory. I will give an overview of what GHD is, how its main equations are derived, its relation to quantum and classical integrable systems, and some geometry that lies at its core. I will then explain how it reproduces the effects seen in the famous quantum Newton cradle experiment, and how it leads to exact results in transport problems such as Drude weights and non-equilibrium currents.</p>

<p>This is based on various collaborations with Alvise Bastianello, Olalla Castro Alvaredo, Jean-SÃ©bastien Caux, JÃ©rÃ´me Dubail, Robert Konik, Herbert Spohn, Gerard Watts and my student Takato Yoshimura, and strongly inspired by previous collaborations with Denis Bernard, M. Joe Bhaseen, Andrew Lucas and Koenraad Schalm.</p>

Emergent hydrodynamics in integrable systems out of equilibrium

Benjamin Doyon

Department of Mathematics, King's College London

Perimeter Institute, Waterloo, Ontario, Canada, 12 September 2017

Collaborators on the works discussed in these lectures:

Alvise Bastianello, SISSA

Olalla Castro Alvaredo, City, University of London

Jean-Sébastien Caux, University of Amsterdam

Jérôme Dubail, Université de Lorraine

Robert Konik, Brookhaven National Laboratory

Herbert Spohn, Technische Universität München

Gerard Watts, KCL

Takato Yoshimura, KCL

and on related works:

Denis Bernard, École Normale Supérieure de Paris

Joe Bhaseen, KCL

Andy Lucas, Stanford University

Koenraad Schalm, Leiden University

Main papers

O. A. Castro-Alvaredo, B. Doyon, T. Yoshimura, Emergent hydrodynamics in integrable quantum systems out of equilibrium, Phys. Rev. X 6, 041065 (2016)

B. Bertini, M. Collura, J. De Nardis, M. Fagotti, Transport in Out-of-Equilibrium XXZ Chains: Exact Profiles of Charges and Currents, Phys. Rev. Lett. 117, 207201 (2016)

Both selected for a Viewpoint in Physics written by Jérôme Dubail (`http://physics.aps.org/articles/v9/153#c1`)

Advertisement:

Les Houches summer school on Integrability in atomic physics and condensed matter

August 2017

(org.: J.-S. Caux, K. Kitanine, A. Kluemper, R. Konik)

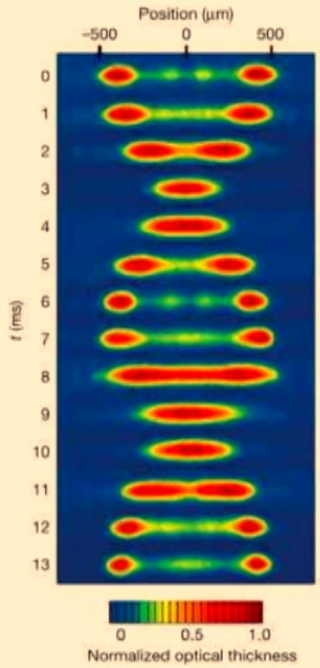
1. The problem: inhomogeneous dynamics of many-body integrable systems

1. The problem: inhomogeneous dynamics of many-body integrable systems

1. The problem: inhomogeneous dynamics of many-body integrable systems

Effects of integrability in the famous “Quantum Newton Cradle” experiment:

[Kinoshita, Wenger, Weiss 2006]



“Our results are probably explainable by the well-known fact that a homogeneous 1D Bose gas with point-like collisional interactions is integrable”

“Until now, however, the time evolution of out-of-equilibrium 1D Bose gases has been a theoretically unsettled issue, as practical factors such as harmonic trapping and imperfectly point-like interactions may compromise integrability”

1. The problem: inhomogeneous dynamics of many-body integrable systems

As an example consider the Lieb-Liniger model, which describes point-like interactions of Galilean invariant Bose gases. Its Hamiltonian is

$$H = \int \mathrm{d}x\, \mathfrak{h}(x) = \int \mathrm{d}x \left(\frac{1}{2m} \partial_x \Psi^\dagger \partial_x \Psi + \frac{c}{2} \Psi^\dagger \Psi^\dagger \Psi \Psi \right).$$

It admits local conserved quantities Q_i:

$$Q_i = \int \mathrm{d}x\, \mathfrak{q}_i(x).$$

For instance, the number of particle N, the momentum P and the energy H:

- the particle density is $\mathfrak{q}_0(x) = \mathfrak{n}(x) = \Psi^\dagger(x)\Psi(x)$;
- the momentum density is $\mathfrak{q}_1(x) = \mathfrak{p}(x) = i\Psi^\dagger(x)\partial_x\Psi(x) + h.c.$;
- the energy density is $\mathfrak{q}_2(x) = \mathfrak{h}(x)$.

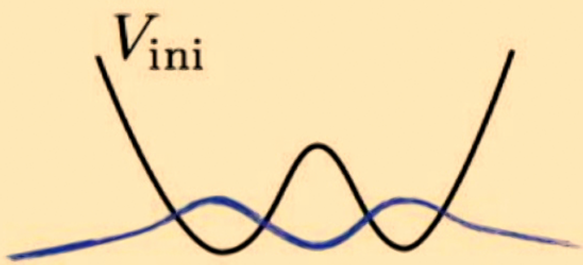
1. The problem: inhomogeneous dynamics of many-body integrable systems

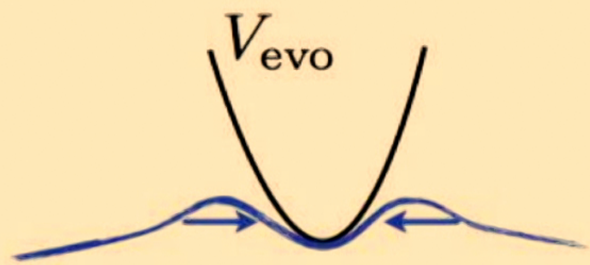
We can describe theoretically (a simplification of) the problem as follows. The initial state is an equilibrium state with a inhomogeneous potential

$$\langle A\rangle = \frac{\mathrm{Tr}\,(\rho_{\mathrm{ini}}A)}{\mathrm{Tr}\rho_{\mathrm{ini}}},\quad \rho_{\mathrm{ini}} = \exp\left[-\beta\left(H + \int \mathrm{d}x\, V_{\mathrm{ini}}(x)\mathfrak{n}(x)\right)\right].$$

Then the evolution occurs with the Hamiltonian in a different inhomogeneous potential

$$\langle A(t)\rangle = \langle e^{\mathrm{i}H_{\mathrm{evo}}t}Ae^{-\mathrm{i}H_{\mathrm{evo}}t}\rangle,\quad H_{\mathrm{evo}} = H + \int \mathrm{d}x\, V_{\mathrm{evo}}(x)\mathfrak{n}(x).$$





1. The problem: inhomogeneous dynamics of many-body integrable systems

How can we compute the evolution of such a gas?

Can we reproduce the effects seen in the quantum Newton cradle experiment?

What general theory would allow us to do so in a simple enough fashion, without using advanced computational techniques for one-dimensional quantum systems?

What are the general principles?

2. From Gibbs ensembles to conventional hydrodynamics

Hydrodynamics is the natural framework to describe inhomogeneous phenomena in many-body systems, for instance waves in water. The main idea behind hydrodynamics is what is usually referred to as **"local thermodyanamic equilibrium"**.

It says that **locally** and on **very short time scales** (in fluid cells), the many-body system **"equilibrates" or relaxes**. This means (naively) that locally we observe **Gibbs states**. Since things can be moving, then in general these will be **boosted** by the local fluid velocity. Thus, at every point x, t, the density matrix is

$$\rho_{\rm GE}(x,t) = e^{-\beta(x,t)(H-\mu(x,t)N-\nu(x,t)P)}$$

and the hydrodynamic approximation is

$$\langle \mathcal{O}(x,t)\rangle_{\rm ini} \approx \frac{{\rm Tr}\,(\rho_{\rm GE}(x,t)\mathcal{O}(0,0))}{{\rm Tr}\rho_{\rm GE}(x,t)}.$$

For instance:

$$\rho_{\rm ini} = e^{-\beta(H+\int {\rm d}x\, V_{\rm ini}(x)\mathfrak{n}(x))} \;\Rightarrow\; \rho_{\rm GE}(x,0) = e^{-\beta(H+V_{\rm ini}(x)N)}.$$

2. From Gibbs ensembles to conventional hydrodynamics

The same holds with the momentum current $\mathtt{j_p}$ and the particle current $\mathtt{j_n} = \mathtt{p}$ (equal to the momentum density in Galilean systems), giving the macroscopic conservation laws:

$$\partial_t \mathtt{h}(x,t) + \partial_x \mathtt{j_h}(x,t) = 0$$
$$\partial_t \mathtt{p}(x,t) + \partial_x \mathtt{j_p}(x,t) = 0$$
$$\partial_t \mathtt{n}(x,t) + \partial_x \mathtt{p}(x,t) = 0.$$

But also, since there are only three parameters μ, ν, β to determine a boosted Gibbs state, there must be two relations:

$$\mathtt{j_h} = F(\mathtt{h},\mathtt{p},\mathtt{n}), \quad \mathtt{j_p} = G(\mathtt{h},\mathtt{p},\mathtt{n}).$$

These are the **equations of state of the gas** (which are highly model-dependent), and combined with the above give the **hydrodynamic equations** for $\mathtt{h},\ \mathtt{p},\ \mathtt{n}$.

Note that $\mathtt{h},\ \mathtt{p},\ \mathtt{n}$ fix the potentials $\beta,\ \mu,\ \nu$. Thus hydrodynamics fixes the local space-time dependent state.

2. From Gibbs ensembles to conventional hydrodynamics

Remarks:

- This is valid at the **Euler scale**: all variations in space and time must be very smooth. Beyond this scale, there are higher derivative corrections, such as viscosity terms. But at large scales, such higher derivative terms are scaled out.
- These equations can be re-written in standard hydrodynamic form. Defining a velocity v via $\mathtt{p} = \mathtt{n}\, v$, the $\mathtt{n}$ and $\mathtt{p}$ conservation laws imply

$$\partial_t v + v\partial_x v = -\frac{1}{\mathtt{n}}\partial_x \mathcal{P}$$

  where the pressure is $\mathcal{P} = \mathtt{j_p} - \mathtt{n}v^2$. This is the usual Euler equation. Combined with $\partial_t \mathtt{n} + \partial_x(v\mathtt{n}) = 0$ these are the usual hydrodynamic equations (without viscosity).
- The pressure $\mathcal{P}$ in a boosted Gibbs state can be evaluated in the Lieb-Liniger model using Bethe ansatz. This Lieb-Liniger conventional hydrodynamics has been used, with partial success.

2. From Gibbs ensembles to conventional hydrodynamics

The same holds with the momentum current $\mathtt{j}_\mathtt{p}$ and the particle current $\mathtt{j}_\mathtt{n} = \mathtt{p}$ (equal to the momentum density in Galilean systems), giving the macroscopic conservation laws:

$$\partial_t \mathtt{h}(x,t) + \partial_x \mathtt{j}_\mathtt{h}(x,t) = 0$$
$$\partial_t \mathtt{p}(x,t) + \partial_x \mathtt{j}_\mathtt{p}(x,t) = 0$$
$$\partial_t \mathtt{n}(x,t) + \partial_x \mathtt{p}(x,t) = 0.$$

But also, since there are only three parameters μ, ν, β to determine a boosted Gibbs state, there must be two relations:

$$\mathtt{j}_\mathtt{h} = F(\mathtt{h},\mathtt{p},\mathtt{n}), \quad \mathtt{j}_\mathtt{p} = G(\mathtt{h},\mathtt{p},\mathtt{n}).$$

These are the **equations of state of the gas** (which are highly model-dependent), and combined with the above give the **hydrodynamic equations** for $\mathtt{h},\ \mathtt{p},\ \mathtt{n}$.

Note that $\mathtt{h},\ \mathtt{p},\ \mathtt{n}$ fix the potentials $\beta,\ \mu,\ \nu$. Thus hydrodynamics fixes the local space-time dependent state.

2. From Gibbs ensembles to conventional hydrodynamics

Thus there is an unambiguous procedure to go from homogeneous, stationary states to inhomogeneous, dynamical states, describing the large (Euler) scale space-time variations.

homogeneous stationary	inhomogeneous dynamical
(boosted) Gibbs ensembles	hydrodynamics
$e^{-\beta(H-\mu N-\nu P)}$	$e^{-\beta(x,t)(H-\mu(x,t)N\ldots}$ $\partial_t \mathtt{h} + \partial_x \mathtt{j}_\mathtt{h} = 0$ $\partial_t \mathtt{p} + \partial_x \mathtt{j}_\mathtt{p} = 0$ $\partial_t \mathtt{n} + \partial_x \mathtt{p} = 0$

3. From Gibbs ensembles to Generalized Gibbs ensembles (GGEs)

But **the Lieb-Liniger model is integrable**. Integrable models possess an **infinite number of local conserved quantities**

$$Q_i = \int \mathrm{d}x \, \mathfrak{q}_i(x), \qquad \partial_t \mathfrak{q}_i + \partial_x \mathfrak{j}_i = 0, \qquad i = 0, 1, 2, 3, \ldots \text{ unboundedly.}$$

Because of the presence of all these conserved quantities, integrable models **do not generically relax to Gibbs ensembles**.

3. From Gibbs ensembles to Generalized Gibbs ensembles (GGEs)

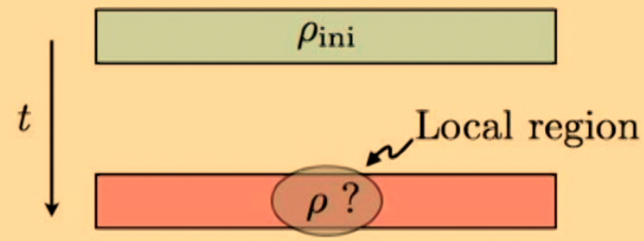
Let us explain what replaces Gibbs ensembles using the following "quench protocol".

A system is in some homogeneous initial state ρ_{ini}. Then this state is evolved with a homogeneous Hamiltonian H_{evo}. Consider a local observable $\mathcal{O}(x)$ in the evolved state:

$$\langle \mathcal{O}(x,t) \rangle_{\text{ini}} = \langle e^{\mathrm{i}H_{\text{evo}}t} \mathcal{O}(x) e^{-\mathrm{i}H_{\text{evo}}t} \rangle_{\text{ini}}.$$

What is the limit $\lim_{t\to\infty} \langle \mathcal{O}(x,t) \rangle_{\text{ini}}$?

Although the state of a closed quantum system itself cannot relax as a whole, **it does from the viewpoint of local observables** in infinite volume.



3. From Gibbs ensembles to Generalized Gibbs ensembles (GGEs)

Remarks

- The set of conserved charges $\{Q_i\}$ and the infinite series $\sum_i \beta_i Q_i$ must be defined carefully. The correct definition is that of **pseudolocal charges**, which form a Hilbert space in which $\sum_i \beta_i Q_i$ is interpreted as a basis decomposition. The space of all GGEs with respect to a given integrable H is **infinite-dimensional**. It is probably an infinite-dimensional Riemannian manifold [BD 2017].
- According to standard results of the operator algebra approach to quantum statistical mechanics, any extremal $H_{\rm evol}$-stationary state is a Gibbs state (or Kubo-Martin-Schwinger (KMS) state) with respect to a conserved "Hamiltonian" $H_{\rm sta}$ (not necessarily local, but generating a one-parameter group of unitaries).
- There is the related **eigenstate thermalization hypothesis**: in the large-volume limit, for generic quantum lattices, $\langle E|\mathcal{O}|E\rangle = {\rm Tr}\left(e^{-\beta H}\mathcal{O}\right)/{\rm Tr}\, e^{-\beta H}$. This is generalized to integrable systems with the replacement $e^{-\beta H} \mapsto e^{-\sum_i \beta_i Q_i}$.

Combining: inhomogeneous dynamics of integrable systems:

$$\partial_t \mathtt{q}_i + \partial_x \mathtt{j}_i = 0, \quad \mathtt{q}_i, \mathtt{j}_i = \frac{\mathrm{Tr}\,[\rho_{\mathrm{GGE}}(x,t)\,\mathtt{q}_i, \mathtt{j}_i]}{\mathrm{Tr}\,\rho_{\mathrm{GGE}}(x,t)}, \quad i = 0, 1, 2, \ldots$$

This is **generalized hydrodynamics (GHD)**.

- The hydrodynamic principle is the emergence of local entropy maximization with respect to all available conserved charges, valid when variation lengths are large enough.
- Local averages are fixed by space-time dependent (generalized) Gibbs ensembles,

$$\langle \mathcal{O}(x,t) \rangle_{\mathrm{ini}} \approx \frac{\mathrm{Tr}\,[\rho_{\mathrm{GGE}}(x,t)\,\mathcal{O}]}{\mathrm{Tr}\,\rho_{\mathrm{GGE}}(x,t)}$$

- There are equations of states: $\mathtt{j}_i = F_i(\{\mathtt{q}_j\})$, and a bijective relation $\mathtt{q}_i \leftrightarrow \beta_i$.
- Equations of conservations give dynamical equations determining the space-time dependent (generalized) Gibbs ensembles.

3. From Gibbs ensembles to Generalized Gibbs ensembles (GGEs)

Thus there is an unambiguous procedure to construct homogeneous, stationary states of integrable models: a "generalization" to infinitely-many conservation laws.

(boosted) Gibbs ensembles

finite number
of conservation laws

$$e^{-\beta(H-\mu N-\nu P)}$$

"generalization"

generalized
Gibbs ensembles (GGEs)

infinite number
of conservation laws

$$e^{-\sum_{i=0}^{\infty}\beta_i Q_i}$$

4. GGEs via quasi-particles

Gibbs ensembles and generalized Gibbs ensembles can be described, in Bethe ansatz integrable models, by using the **(generalized) thermodynamic Bethe ansatz**.

This is based on the fact that there emerge **quasi-particles**. The set of their momenta and other quantum numbers is preserved under scattering, thus giving good quantum numbers used to describe GGEs.

These quantum numbers are gathered into a **spectral parameter** θ characterizing the quasi-particle, and we imagine states to be of the form

$$|\theta_1, \theta_2, \ldots\rangle.$$

A model is fully defined by giving the space of spectral parameters, the momentum $p(\theta)$ and energy $E(\theta)$ functions, and the **differential scattering phase** $\varphi(\theta, \theta')$.

4. GGEs via quasi-particles

For instance, in many models with a single-particle spectrum we may take $\theta \in \mathbb{R}$ and

- with Galilean invariance $p(\theta) = m\theta,\ E(\theta) = m\theta^2/2$
- with relativistic invariance $p(\theta) = m\sinh(\theta),\ E(\theta) = m\cosh\theta$

with in both case $\varphi(\theta - \theta') = -\mathrm{id}\log S(\theta - \theta')/\mathrm{d}\theta$ where $S(\theta - \theta')$ is the two-particle scattering amplitude.

4. GGEs via quasi-particles

Each quasi-particle θ carry a quantity $h_i(\theta)$ of the conserved charge Q_i. That is, conserved charges act as

$$Q_i|\theta_1, \theta_2, \ldots\rangle = \left(\sum_k h_i(\theta_k)\right)|\theta_1, \theta_2, \ldots\rangle.$$

A GGE can be seen as a single state with infinitely-many quasi-particles of a given density. It is fully characterized by the number $L\rho_{\rm p}(\theta)\mathrm{d}\theta$ of quasiparticles in the element $[\theta, \theta + \mathrm{d}\theta]$ (where L is the infinite volume).

In a GGE (where integral symbol includes sum over quasi-particle species),

$$\mathsf{q}_i = \frac{\mathrm{Tr}\left[\rho_{\rm GGE}\,\mathsf{q}_i\right]}{\mathrm{Tr}\rho_{\rm GGE}} = \frac{1}{L}\frac{\mathrm{Tr}\left[\rho_{\rm GGE}\,Q_i\right]}{\mathrm{Tr}\rho_{\rm GGE}} = \int \mathrm{d}\theta\, h_i(\theta)\rho_{\rm p}(\theta).$$

The set of functions $h_i(\theta)$ is assumed to be “complete” in some sense.

Thus **the set $\{\mathsf{q}_i\}$ and the function $\rho_{\rm p}(\theta)$ are both complete characterization of a GGE**.

4. GGEs via quasi-particles

Each quasi-particle θ carry a quantity $h_i(\theta)$ of the conserved charge Q_i. That is, conserved charges act as

$$Q_i|\theta_1,\theta_2,\ldots\rangle = \left(\sum_k h_i(\theta_k)\right)|\theta_1,\theta_2,\ldots\rangle.$$

A GGE can be seen as a single state with infinitely-many quasi-particles of a given density. It is fully characterized by the number $L\rho_{\rm p}(\theta){\rm d}\theta$ of quasiparticles in the element $[\theta,\theta+{\rm d}\theta]$ (where L is the infinite volume).

In a GGE (where integral symbol includes sum over quasi-particle species),

$$\mathtt{q}_i = \frac{{\rm Tr}\left[\rho_{\rm GGE}\,\mathtt{q}_i\right]}{{\rm Tr}\rho_{\rm GGE}} = \frac{1}{L}\frac{{\rm Tr}\left[\rho_{\rm GGE}\,Q_i\right]}{{\rm Tr}\rho_{\rm GGE}} = \int {\rm d}\theta\, h_i(\theta)\rho_{\rm p}(\theta).$$

The set of functions $h_i(\theta)$ is assumed to be “complete” in some sense.

Thus **the set $\{\mathtt{q}_i\}$ and the function $\rho_{\rm p}(\theta)$ are both complete characterization of a GGE**.

4. GGEs via quasi-particles

The relation to the Lagrange parameters β_i is obtained as follows. Here we use **fermionic statistics**, for instance as used for the quasi-particles of the Lieb-Liniger model.

Average of local conserved densities are evaluated using a **free energy**:

$$\mathsf{q}_i = -\frac{\partial}{\partial \beta_i} F$$

where

$$F = \int \mathrm{d}p(\theta) \log(1 + e^{-\epsilon(\theta)})$$

with **pseudo-energy**

$$\epsilon(\theta) = \sum_i \beta_i h_i(\theta) - \int \frac{\mathrm{d}\alpha}{2\pi} \varphi(\theta, \alpha) \log(1 + e^{-\epsilon(\alpha)}).$$

5. GGE equations of states

For GHD, we need the **GGE equations of state**: the expressions of the currents.

Since ρ_{p} fully determines the GGE, one can always write, for some $v^{\mathrm{eff}}(\theta) = v^{\mathrm{eff}}_{[\rho_{\mathrm{p}}]}(\theta)$,

$$\mathrm{j}_i = \int \mathrm{d}\theta\, h_i(\theta) v^{\mathrm{eff}}(\theta)\rho_{\mathrm{p}}(\theta).$$

Using crossing symmetry in relativistic QFT, checking with form factor expansions of GGE averages, and with numerical verifications in the XXZ chain, one finds

$$v^{\mathrm{eff}}(\theta) = \frac{E'(\theta)}{p'(\theta)} + \int \mathrm{d}\alpha\, \frac{\varphi(\theta,\alpha)\,\rho_{\mathrm{p}}(\alpha)}{p'(\theta)}\,(v^{\mathrm{eff}}(\alpha) - v^{\mathrm{eff}}(\theta))$$

This can be seen as the GGE equations of state.

5. GGE equations of states

Define the **occupation function**:

$$n(\theta) = \frac{\rho_{\rm p}(\theta)}{\rho_{\rm s}(\theta)}, \qquad 2\pi\rho_{\rm s}(\theta) = p'(\theta) + \int {\rm d}\alpha\, \varphi(\theta,\alpha)\rho_{\rm p}(\alpha).$$

Here $\rho_{\rm s}$ as the interpretation as a **density of states**: the "availabilities" for quasi-particles.

Define the all-important **"dressing" operation**:

$$h^{\rm dr}(\theta) = h(\theta) + \int \frac{{\rm d}\alpha}{2\pi}\varphi(\theta,\alpha)n(\alpha)h^{\rm dr}(\alpha).$$

Then $2\pi\rho_{\rm s} = (p')^{\rm dr}(\theta)$ where $p'(\theta) = {\rm d}p(\theta)/{\rm d}\theta$.

The mapping $n(\theta) \leftrightarrow \rho_{\rm p}(\theta)$ is a change of coordinate in the space of GGEs.

6. GHD in the quasi-particle language

We now make GGEs space-time dependent. This means we promote

$$\rho_{\rm p}(\theta) \mapsto \rho_{\rm p}(x,t;\theta) \qquad \text{or equivalently} \qquad n(\theta) \mapsto n(x,t;\theta).$$

The quantity $\rho_{\rm p}(x,t;\theta){\rm d}x{\rm d}\theta$ is the number of quasi-particles in the "phase-space" element $[\theta,\theta+{\rm d}\theta]\times[x,x+{\rm d}x]$.

We use

$$\mathsf{q}_i(x,t) = \int {\rm d}\theta\, h_i(\theta)\rho_{\rm p}(x,t;\theta), \qquad \mathsf{j}_i(x,t) = \int {\rm d}\theta\, h_i(\theta)v^{\rm eff}(\theta)\rho_{\rm p}(x,t;\theta)$$

and **completeness of** $\{h_i(\theta)\}$ inside the fundamental GHD equations

$$\partial_t \mathsf{q}_i + \partial_x \mathsf{j}_i = 0.$$

6. GHD in the quasi-particle language

This gives

$$\partial_t \rho_{\mathrm{p}}(x,t;\theta) + \partial_x \left[v^{\mathrm{eff}}(x,t;\theta)\rho_{\mathrm{p}}(x,t;\theta) \right] = 0.$$

These are the **GHD equations in the quasi-particle language**.

6. GHD in the quasi-particle language

All of this generalizes to the presence of **force fields, temperature fields, etc.**

It is the energy function that controls the time evolution. Assume that it is explicitly space dependent $E(\theta) = E(x;\theta)$. For instance in the repulsive Lieb-Liniger model,

$$H_{\rm evo} = H + \int {\rm d}x\, V_{\rm evo}(x)\mathfrak{n}(x) \quad \Rightarrow \quad E(x;\theta) = m\theta^2/2 + V_{\rm evo}(x).$$

Then the two following equivalent equations hold (here suppressing $x, t; \theta$ dependence):

$$\partial_t \rho_{\rm p} + \partial_x \left[v^{\rm eff}\rho_{\rm p}\right] + \partial_\theta \left[a^{\rm eff}\rho_{\rm p}\right] = 0$$
$$\partial_t n + v^{\rm eff}\partial_x n + a^{\rm eff}\partial_\theta n = 0$$

where the **effective acceleration** is

$$a^{\rm eff} = \frac{F^{\rm dr}}{(p')^{\rm dr}}, \qquad F = -\partial_x E.$$

6. GHD in the quasi-particle language

Remarks:

- This is the full Euler-scale hydrodynamics with force or external fields. It is valid assuming both that the fluid variables and the external fields vary only on large distances. Beyond the Euler scale, there are higher-derivative terms (such as viscosity).
- The equations look a little bit like Boltzmann equations if we interpret $v^{\rm eff}$ as giving rise to collision terms. However, they are not of Boltxmann type. The GHD equations are rather Euler-type hydrodynamic equations: they are time-reversal invariant, and their validity necessitates the assumption of local entropy maximization, which Boltzmann equations are not / do not.
- The state density $\rho_{\rm s}$ satisfies the same equation $\partial_t\rho_{\rm s} + \partial_x\left[v^{\rm eff}\rho_{\rm s}\right] + \partial_\theta\left[a^{\rm eff}\rho_{\rm s}\right] = 0.$
- The Yang-Yang entropy of thermodynamic Bethe ansatz is also conserved.

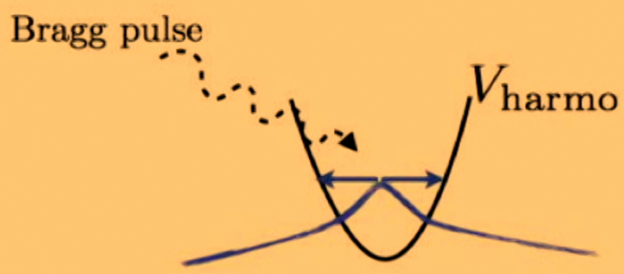
6. GHD in the quasi-particle language

- Since external space-dependent fields generically break integrability, in their presence, beyond the Euler scale, there are also integrability-breaking terms. These will eventually cause the system to relax towards the Gibbs ensemble of the evolution Hamiltonian. Writing $E(x;\theta) = \sum_i h_i(\theta)V_i(x)$, at very large times, after corrections to Euler hydrodynamics accumulate, the system relaxes to the Gibbs state of the corresponding Hamiltonian, $\exp\left[-\beta\sum_i\int \mathrm{d}x\, V_i(x)\mathfrak{q}_i(x)\right]$. In the hydrodynamic approximation, this is

$$\exp\left[-\beta\sum_i\int \mathrm{d}x\, V_i(x)\mathfrak{q}_i(x)\right] \quad\Rightarrow\quad \exp\left[-\beta\sum_i V_i(x)Q_i\right].$$

One can show that this is a stationary solution of the GHD equation in the external inhomogeneous fields.

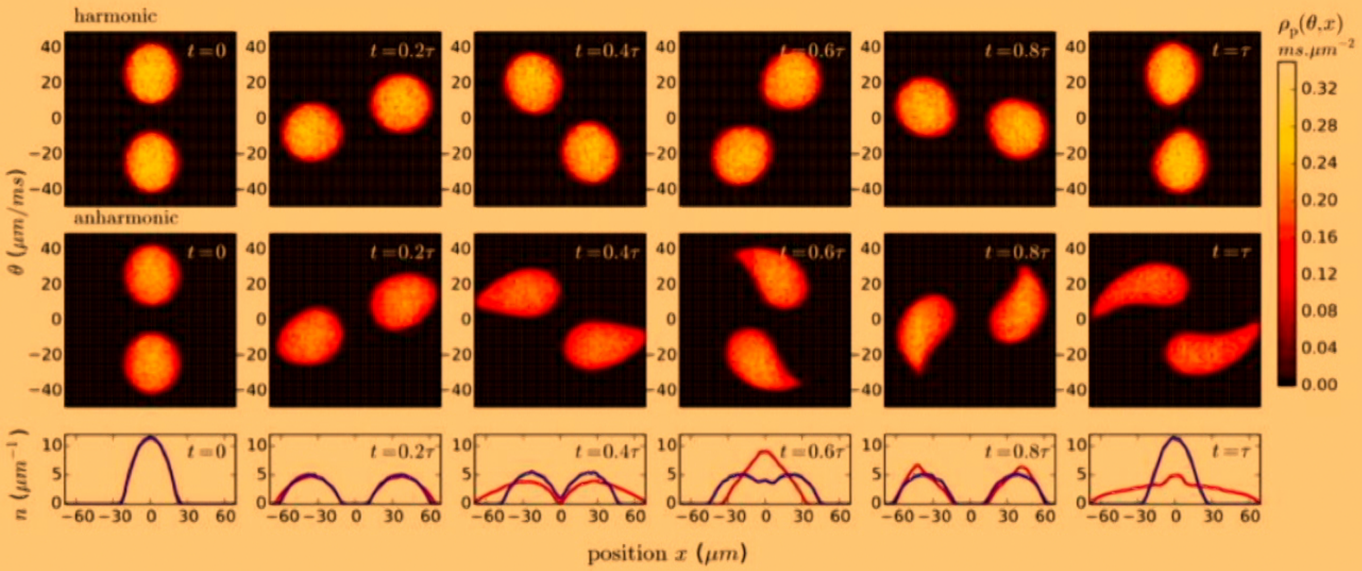
7. An example: quantum Newton cradle-like setups



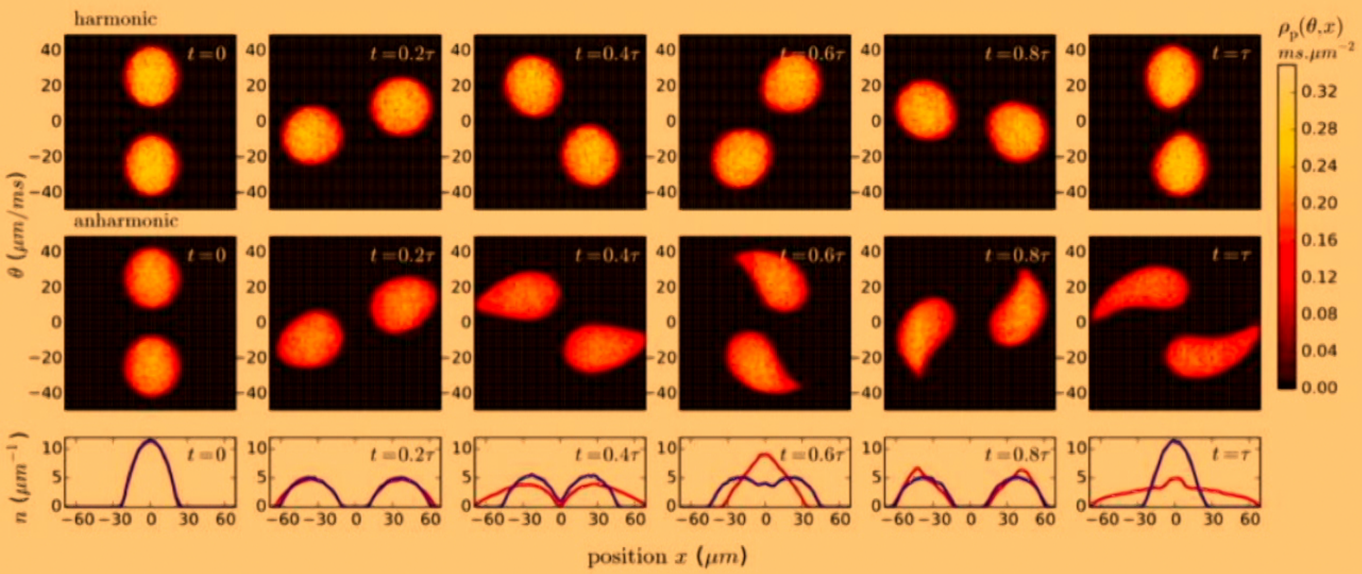
Another way of initializing would be by representing the Bragg pulse used in experiment. Here we calculate $\rho_{\mathrm{p}}(\theta)$ associated to $e^{-\beta(H+V_{\mathrm{harmo}}N)}$, and then set, as the effect of the Bragg pulse followed by fast local entropy maximization,

$$\rho_{\mathrm{p}}(x,0;\theta) = \frac{1}{2}\left[\rho_{\mathrm{p}}(\theta+\theta_{\mathrm{Bragg}}) + \rho_{\mathrm{p}}(\theta-\theta_{\mathrm{Bragg}})\right].$$

7. An example: quantum Newton cradle-like setups



7. An example: quantum Newton cradle-like setups



Some open questions

- Most important question of all: higher-derivative corrections. This includes viscosity terms and associated diffusive effects (analyzed in an extensive numerical study in XXZ [Ljubotina, Znidaric, Prosen (2017)]), and integrability terms when force field is present. Time scale of integrablity breaking?
- Second most important question of all: large deviation theory of charge transfer, fluctuation relations, macroscopic fluctuation theory. Our result for $\int \mathrm{d}t \, \langle \mathrm{j}_i(0,t)\mathrm{j}_j(0,0)\rangle^{\mathrm{c}}$ is the first "second cumulant" in any nontrivial integrable quantum model, but we need infinitely many more...
- Proving emergence of hydrodynamics using integrability techniques?
- Generalizing to time-dependent external field, GHD of classical field theory, ... (works in progress by various people).