Title: The Power of Series

Date: Sep 12, 2017 03:00 PM

URL: http://pirsa.org/17090055

Abstract: <p>After a small review on divergent series and Borel resummation I will discuss a geometric approach based on Picard-Lefschetz theory to study the interplay between perturbative and non-perturbative effects in the QM path integral.</p>

<p>Such approach can be used to characterize when the perturbative series gives the full answer and when the inclusion of non-trivial saddles--instantons--is mandatory. I will then show how a simple deformation of the original perturbation theory allows to recover the full non perturbative answer from the perturbative coefficients alone, without the need of including instanton corrections. I will illustrate this technique in examples which are known to contain non-perturbative effects, such as the (supersymmetric) double-well potential, the pure anharmonic oscillator, and the perturbative expansion around a false vacuum.</p>

$$Z(\lambda) \approx \sum_n Z_n \lambda^n + \sum_m e^{-B_m/\lambda} \sum_n Z_n^{(m)} \lambda^n + \dots$$

Borel-Le Roy transform: $\mathcal{B}_b Z(t) \equiv \sum_{n=0}^{\infty} \frac{Z_n}{\Gamma(n+1+b)} t^n$ finite radius of convergence!

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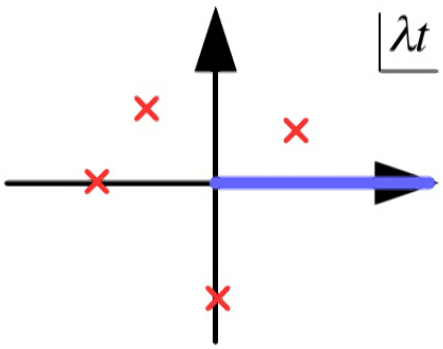

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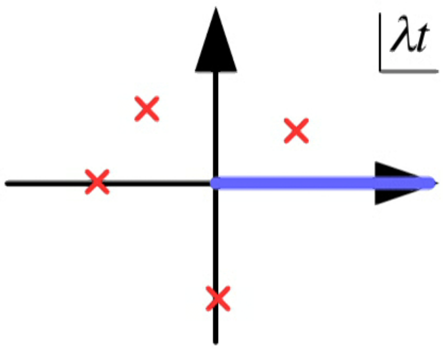

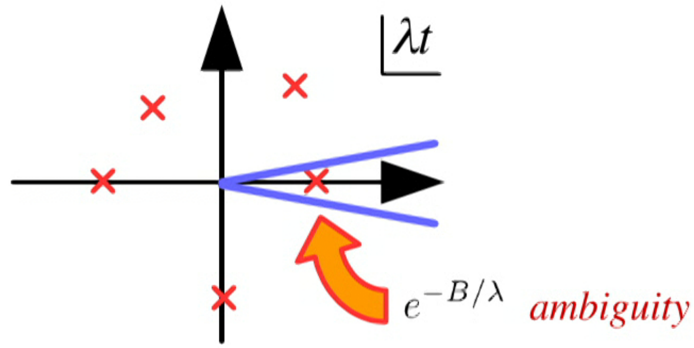

$$Z(\lambda) \equiv \frac{1}{\sqrt{\lambda}} \int_{-\infty}^{\infty} dx \, g(x) \, e^{-f(x)/\lambda}$$

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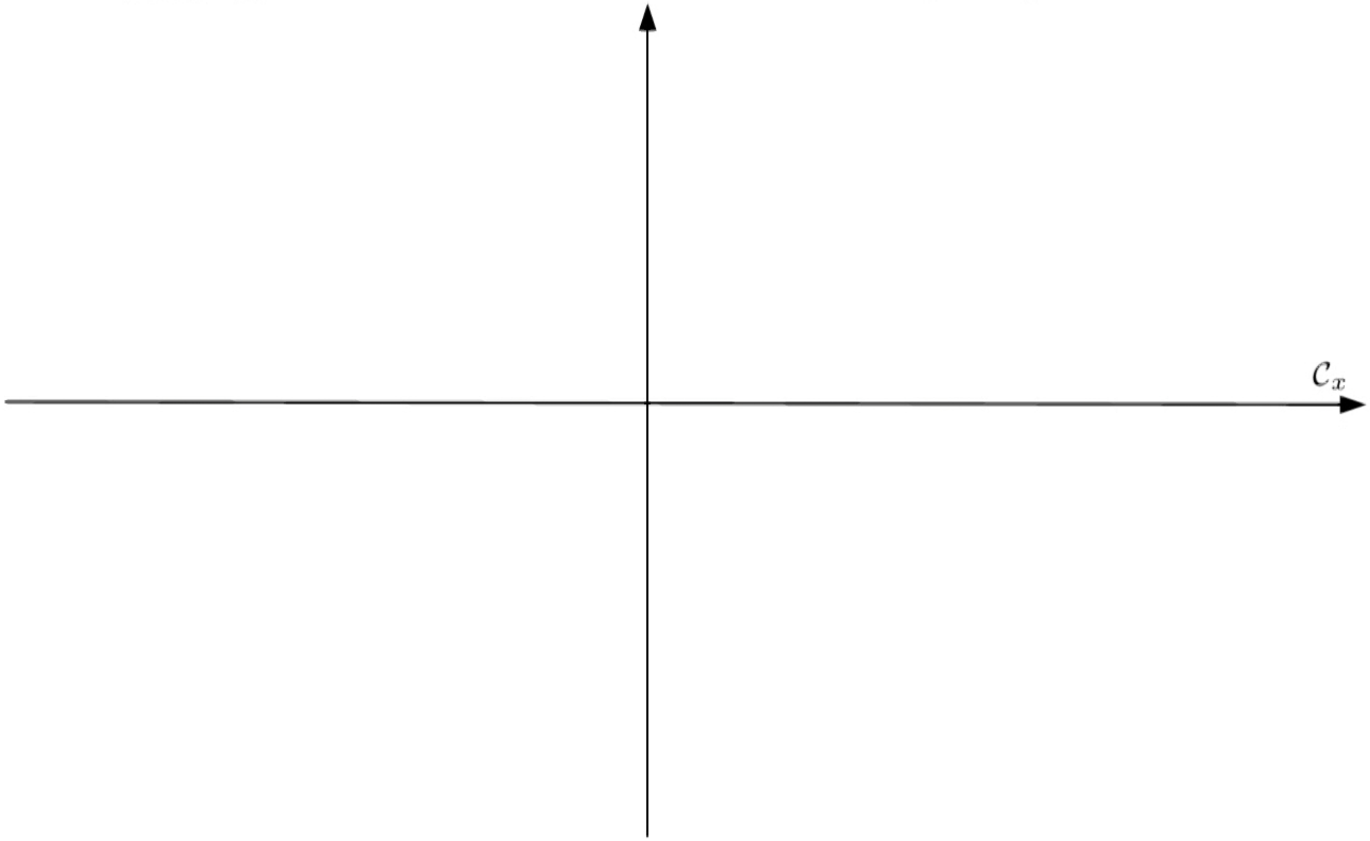

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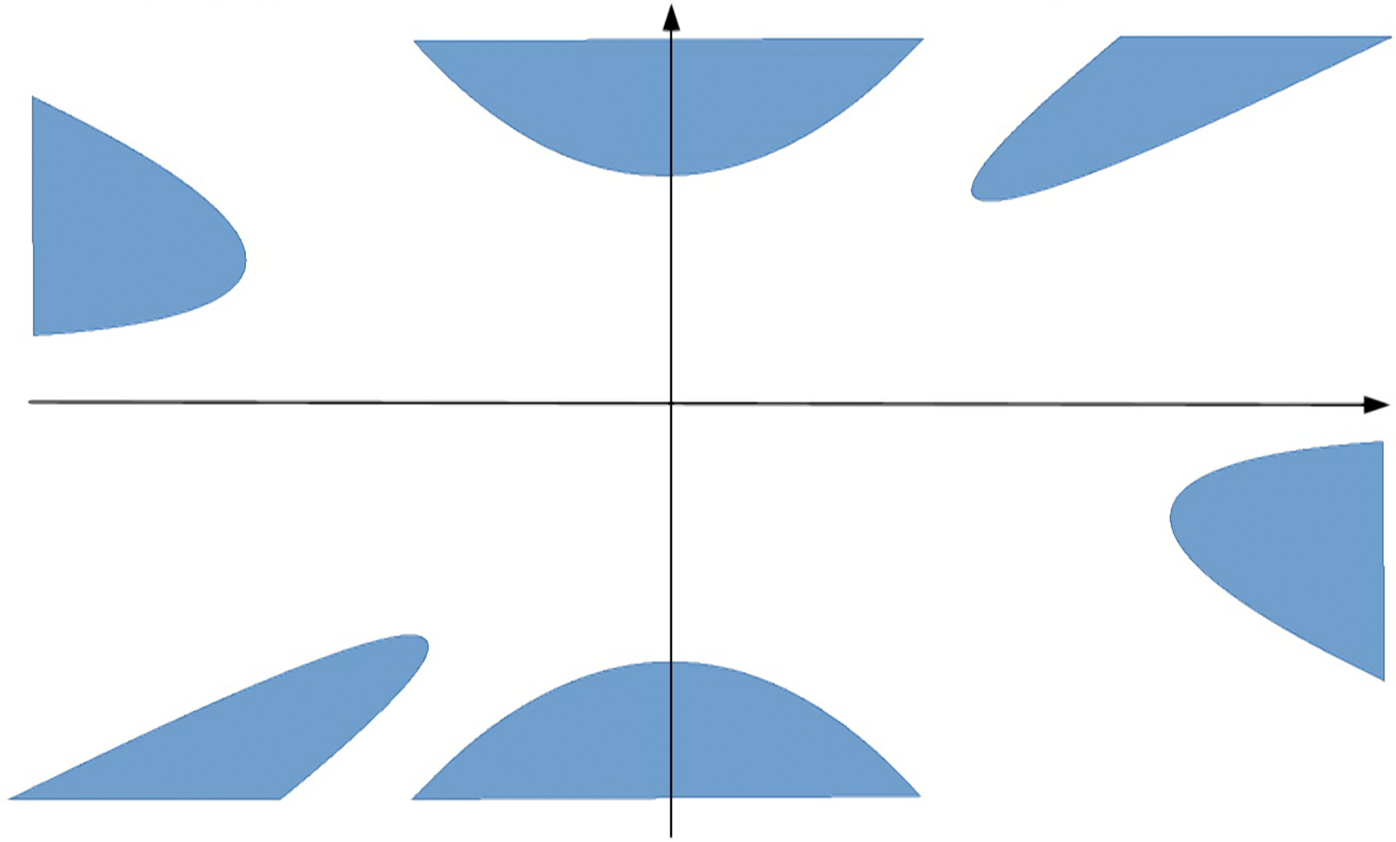

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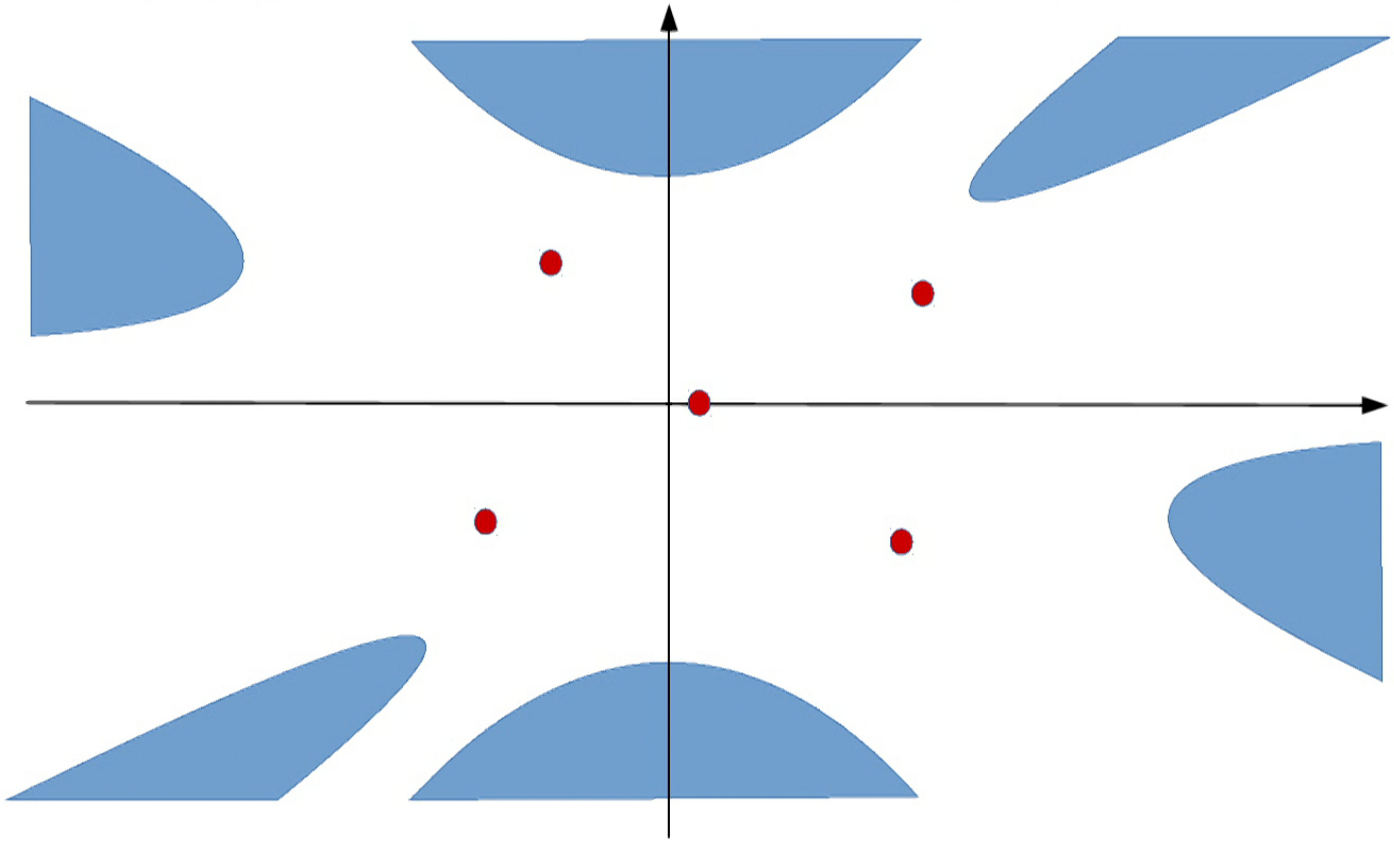

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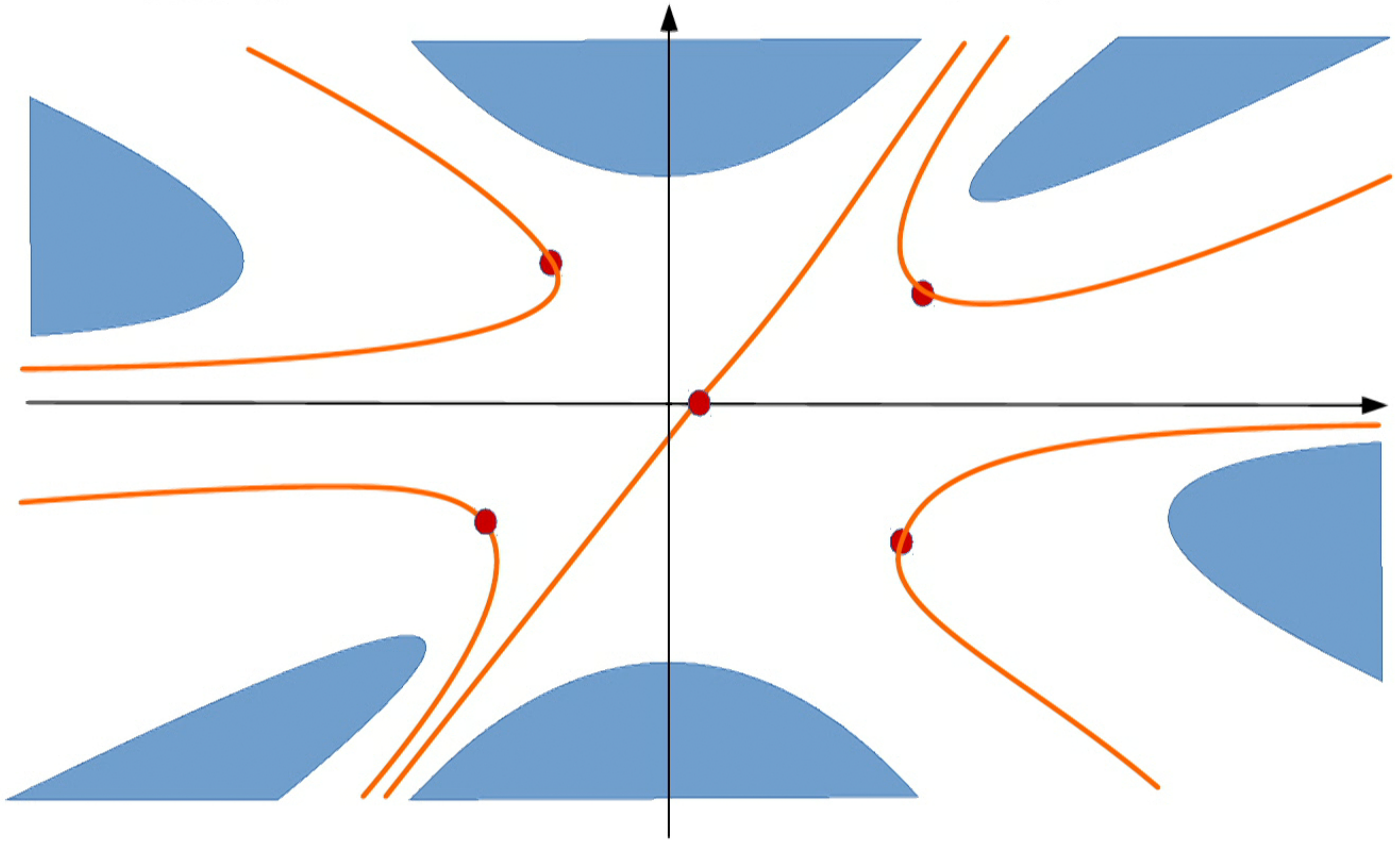

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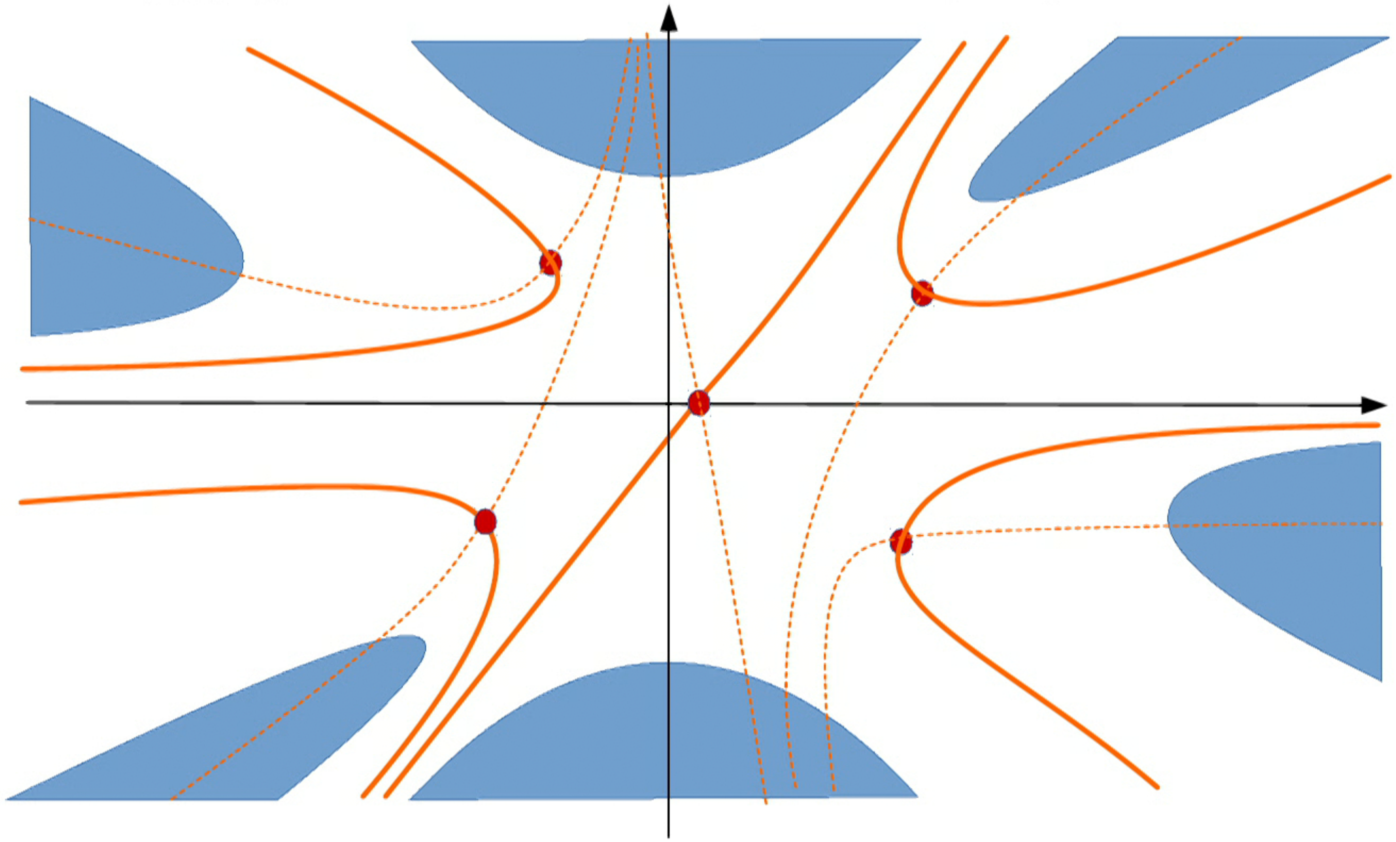

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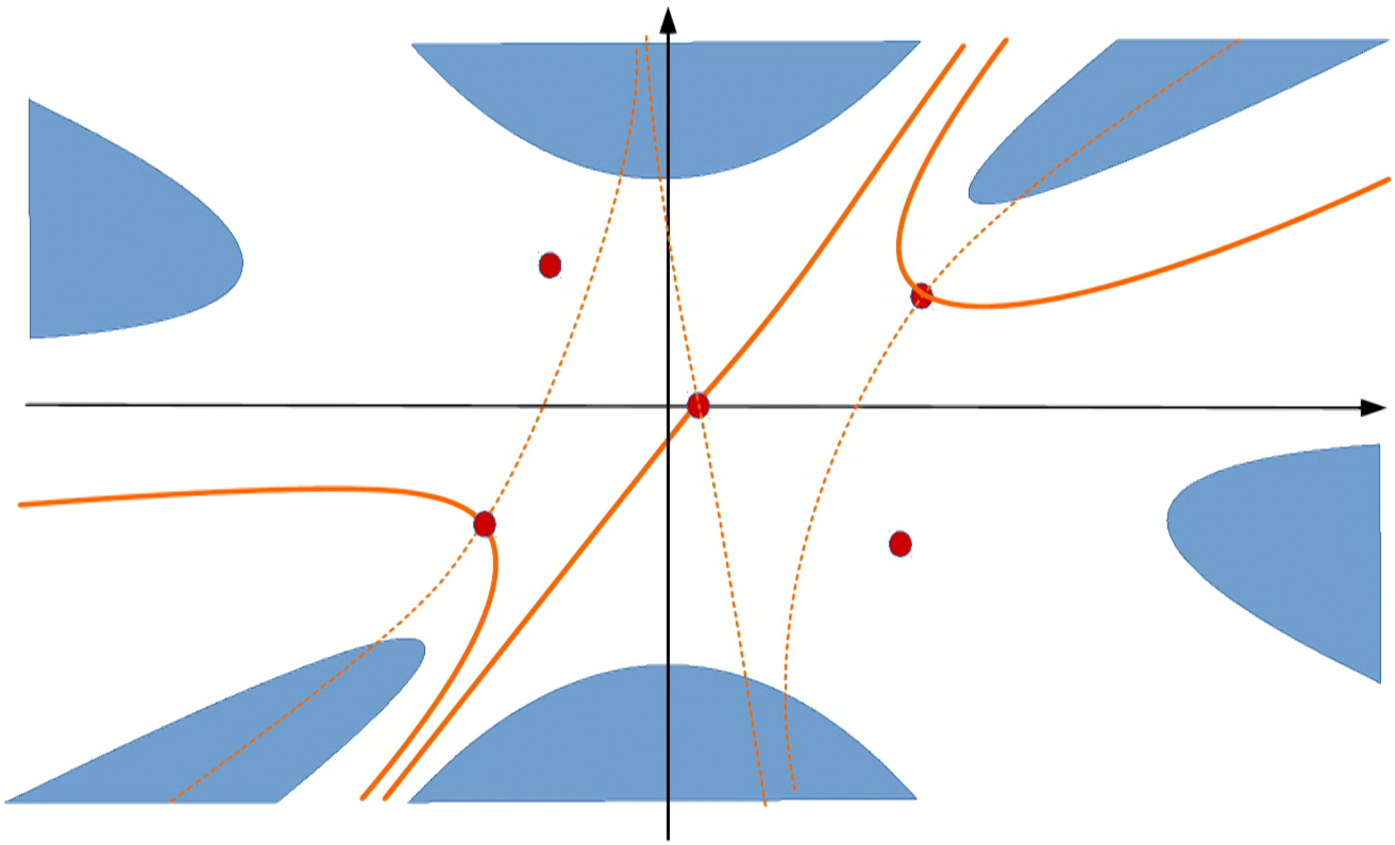

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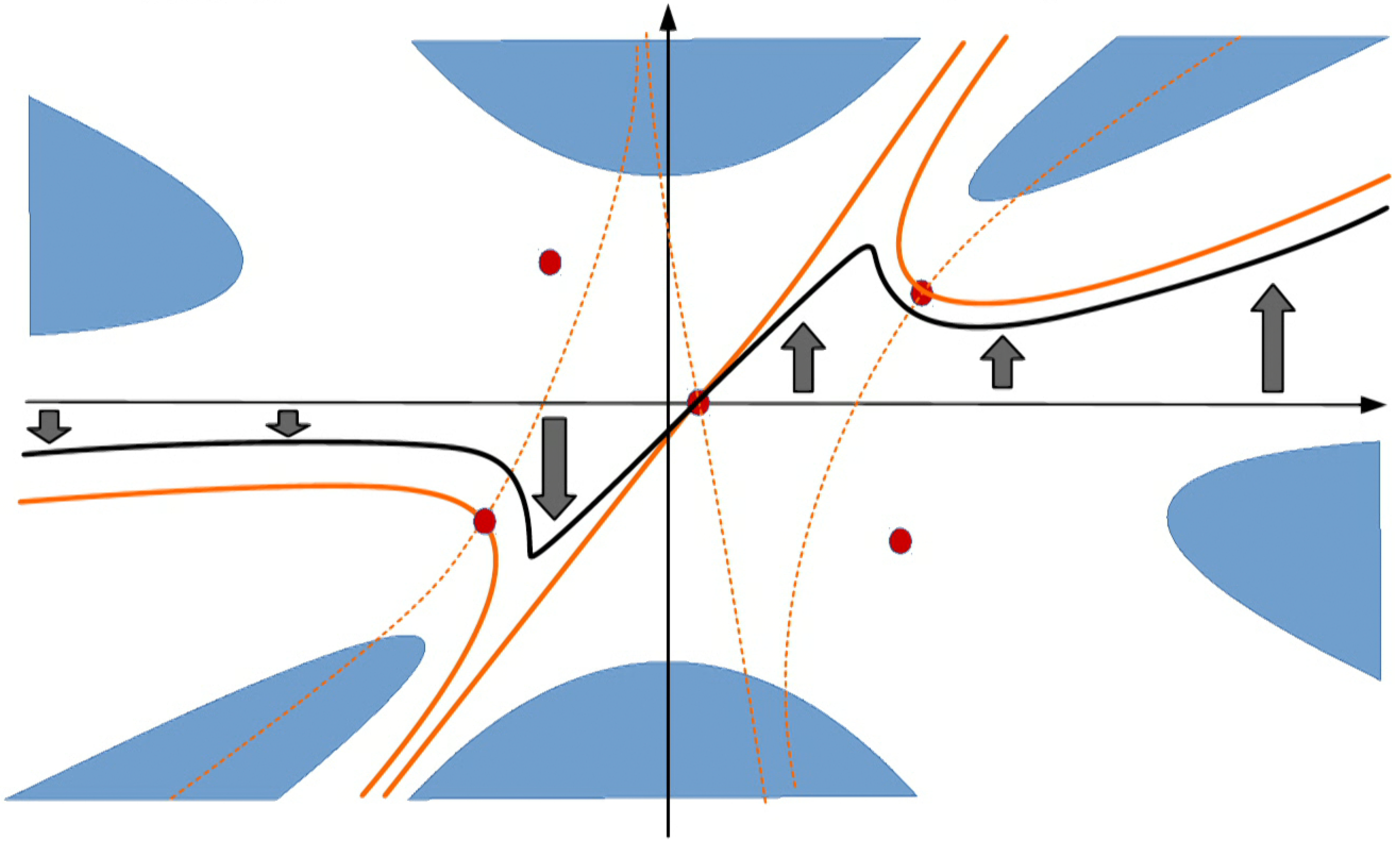

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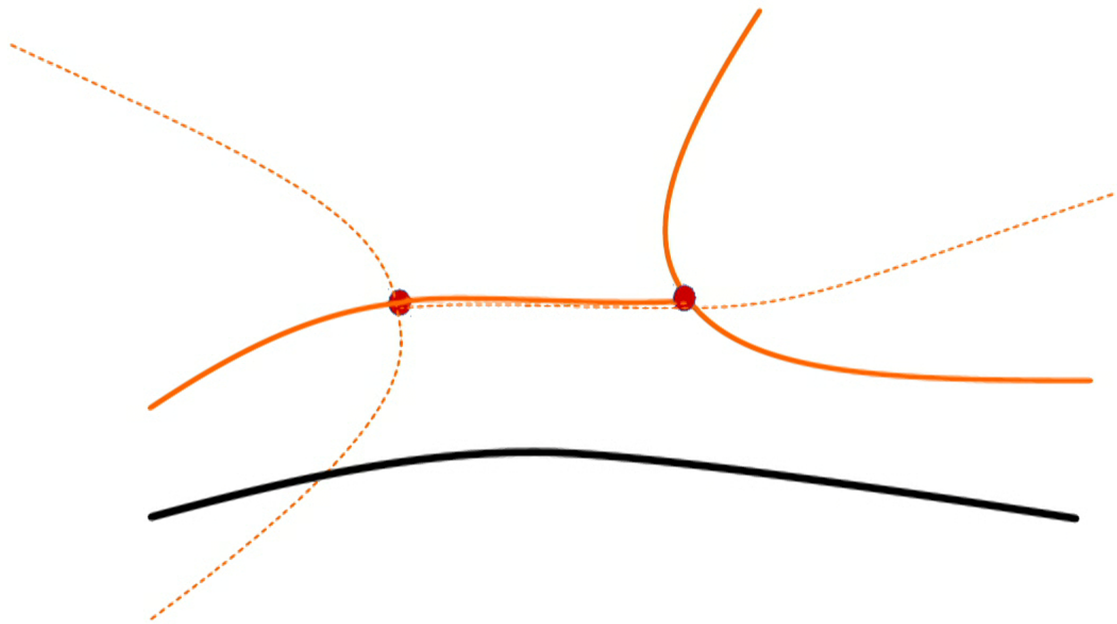
Stokes' lines



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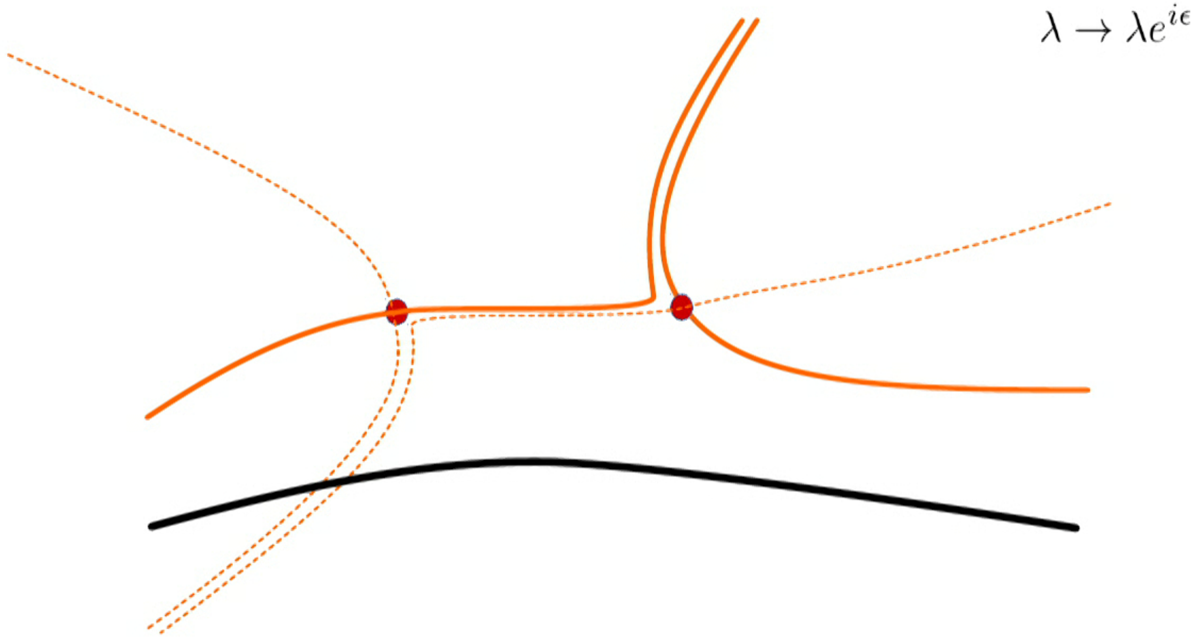

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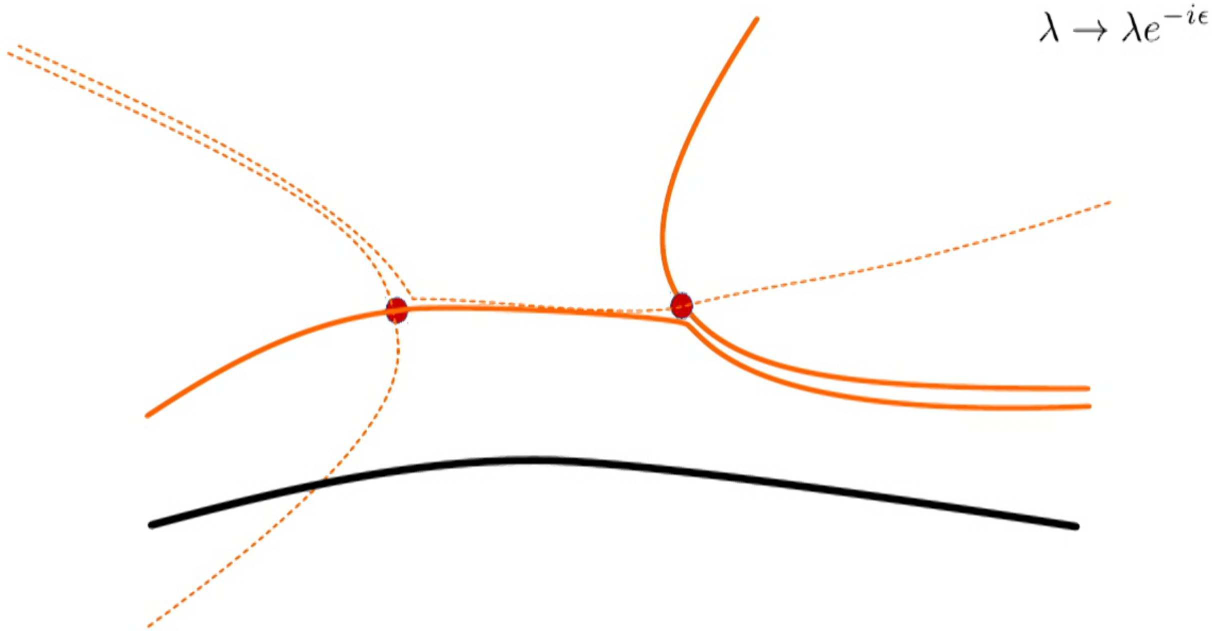
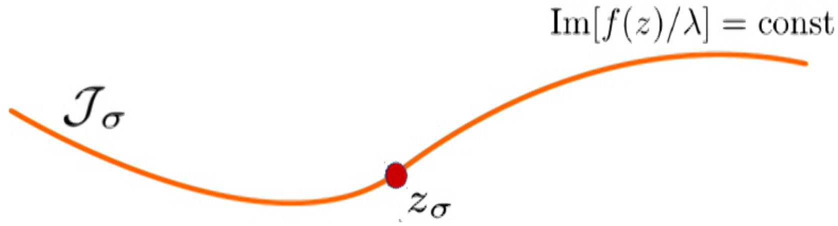


Borel summability



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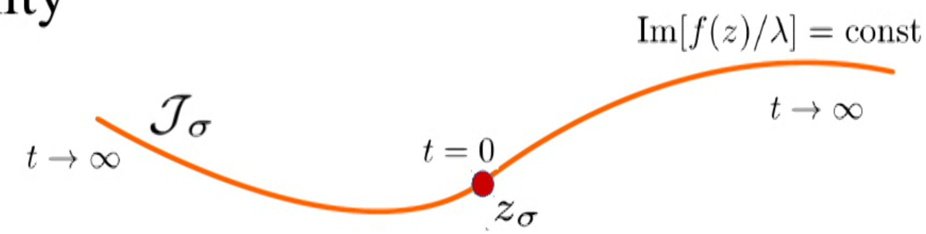

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$z_\pm(\lambda t)$

$$= e^{-f(z_\sigma)/\lambda} \int_0^\infty dt\, e^{-t} t^{-1/2} \sum_\pm \pm \frac{g(z_\pm(\lambda t))}{f'(z_\pm(\lambda t))} \sqrt{\lambda t}$$

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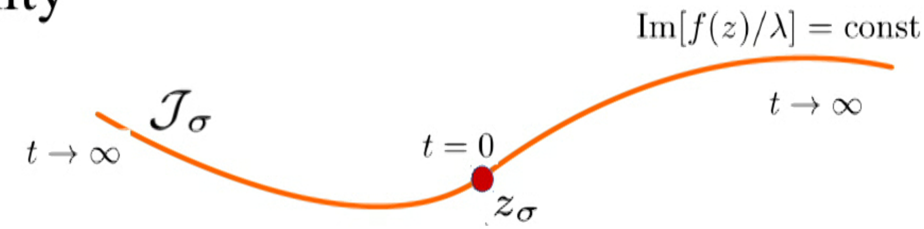

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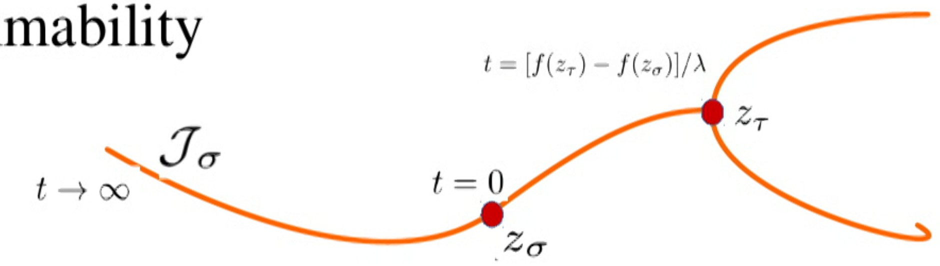

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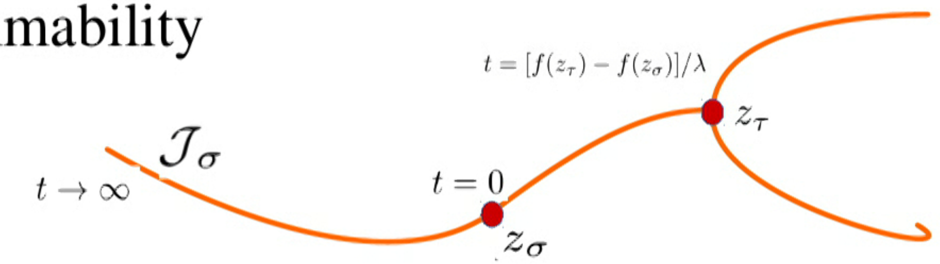

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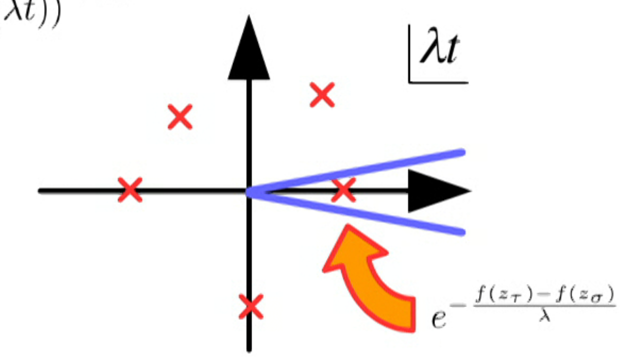

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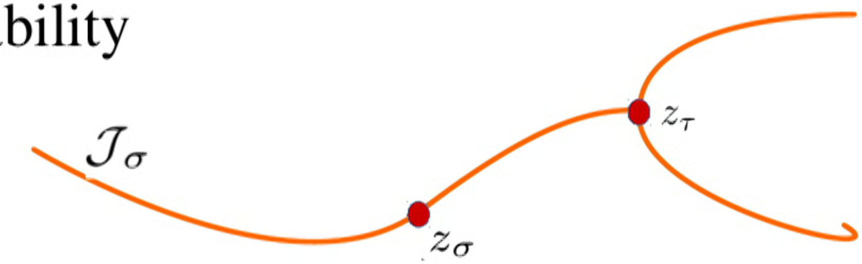

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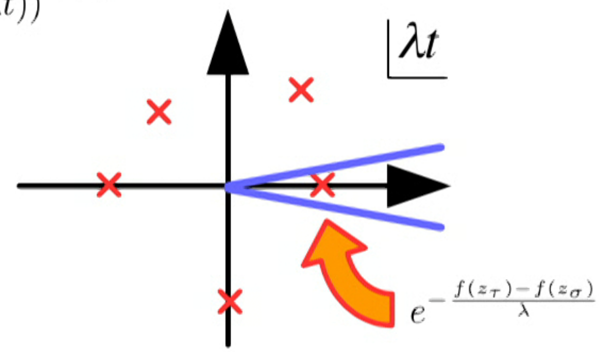

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Common relevant cases:

1. $Z(\lambda)$ – **Borel summable to exact result**

2. $Z(\lambda)$ – **Borel summable but not exact result**

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   → only 1 saddle contributes - no Stokes line – no transseries

   e.g. real integral with one single real saddle

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   e.g. most real integrals with multiple real saddles *n*-dimensional case:

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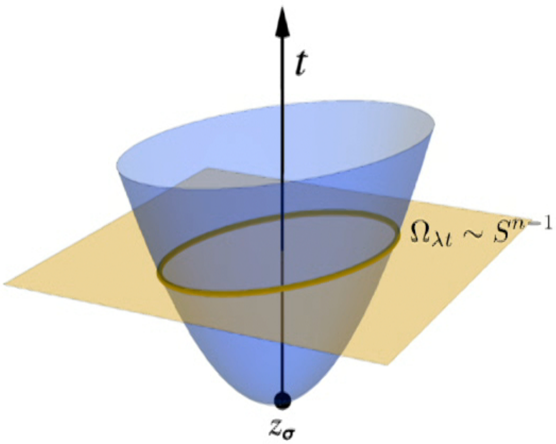
$$t = \frac{f(\boldsymbol{z}) - f(\boldsymbol{z}_\sigma)}{\lambda}$$

$$Z_\sigma(\lambda) = e^{-f(\boldsymbol{z}_\sigma)/\lambda} \int_0^\infty dt\, t^{n/2-1}\, e^{-t}\, \underbrace{B_\sigma(\lambda t)}$$

$$(\lambda t)^{1-n/2} \int_{\mathcal{J}_\sigma} d\boldsymbol{z}\, g(\boldsymbol{z})\, \delta[f(\boldsymbol{z}) - f(\boldsymbol{z}_\sigma) - \lambda t]$$



QM Path Integral:

$$S[x] = \int d\tau \left[\frac{1}{2}\dot{x}^2 + V(x)\right]$$

$$\mathcal{Z}(\lambda) = \int \mathcal{D}x(\tau)\, G[x(\tau)] e^{-S[x(\tau)]/\lambda}$$

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$$\mathcal{B}^{(N)}_{-1/2}\mathcal{Z}(\lambda t) = \sqrt{\lambda t}\; \partial^N_{\lambda t} \int \mathcal{D}^{(N)}_{\lambda=1} x(\tau)\, G^{(N)}[x(\tau)]\, \delta[S^{(N)}[x(\tau)] - S^{(N)}[x_0(\tau)] - \lambda t]\,,$$

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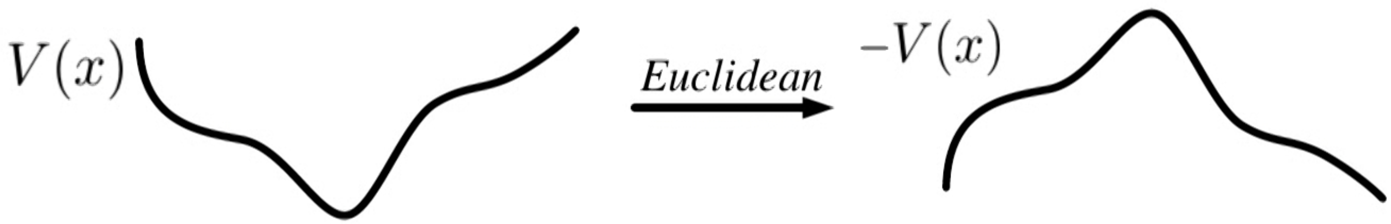

e.g. The anharmonic oscillator $V(x) = \frac{1}{2}x^2 + \frac{\lambda}{2}x^4$


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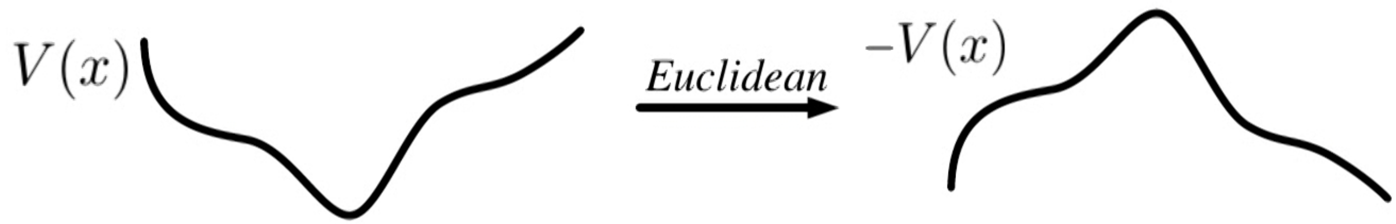

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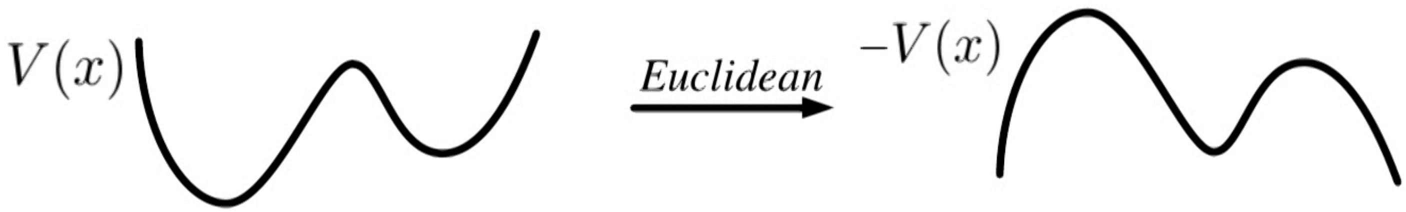
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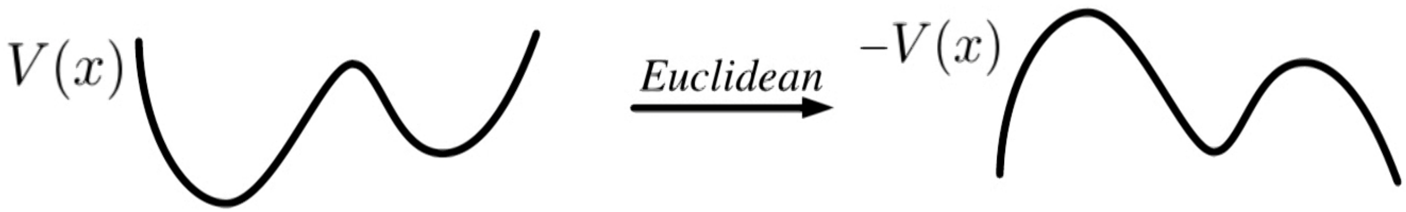
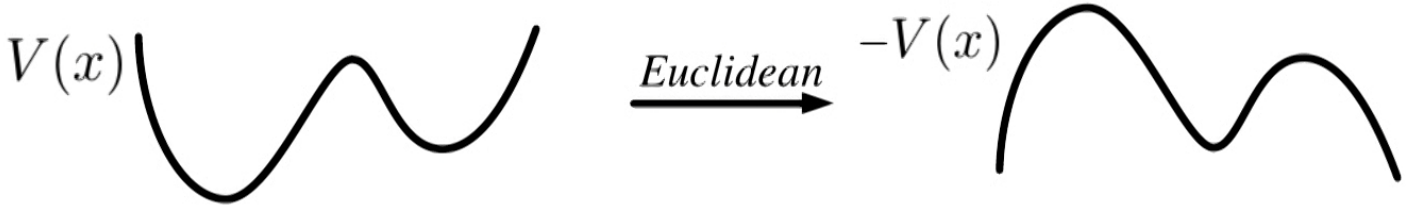


$$W_\beta(\lambda) = \int_{x(0)=x_0,\, x(\beta)=x_0} \mathcal{D}x\ e^{-S[x]/\lambda} = \sum_{n=0}^{\infty} |\psi_n(x_0)|^2 e^{-E_n \beta}$$



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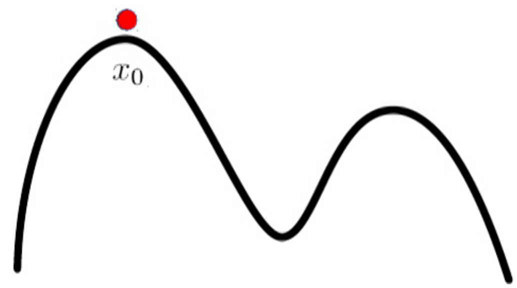

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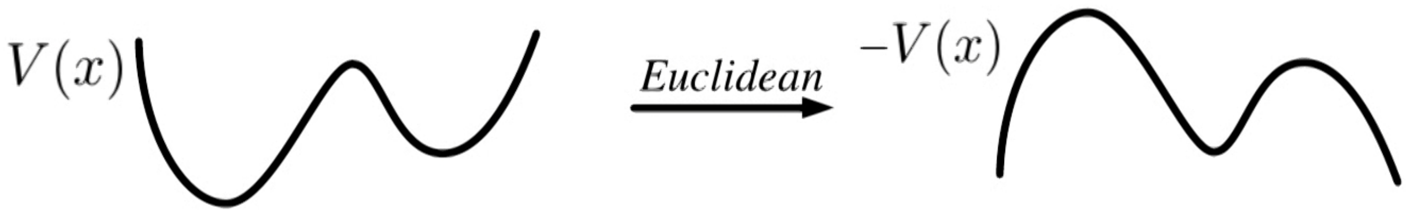

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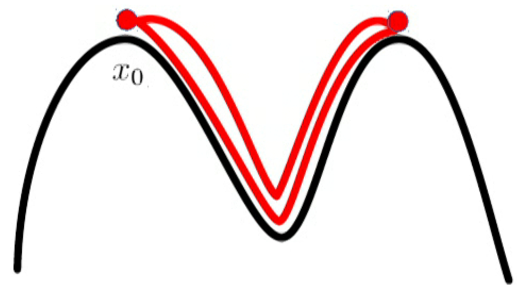

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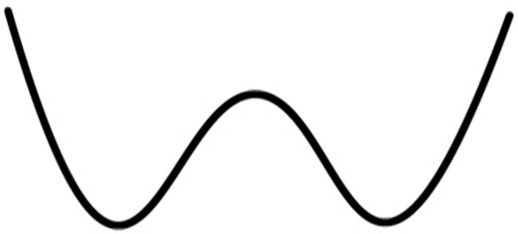
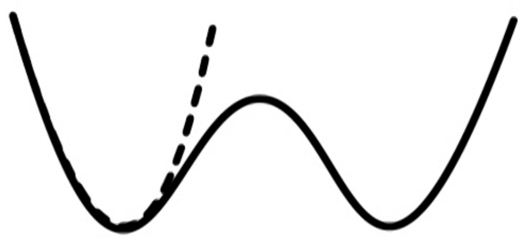
Exact Perturbation Theory:

$$Z(\lambda) = \int \mathcal{D}x \, e^{-\frac{S[x]}{\lambda}} G[x]$$



Exact Perturbation Theory:

$$\hat{Z}(\lambda, \lambda_0) = \int \mathcal{D}x \, e^{-\frac{S_0[x]}{\lambda}} \underbrace{e^{-\frac{\Delta S[x]}{\lambda_0}} G[x]}_{\hat{G}[x, \lambda_0]}$$



$$\begin{cases} \hat{Z}(\lambda, \lambda) = Z(\lambda) \\ \hat{Z}(\lambda, \lambda_0) \approx \sum_{n=0}^{\infty} \hat{Z}_n(\lambda_0) \lambda^n \end{cases}$$

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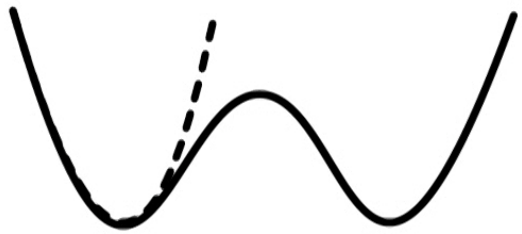

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e.g.

$$V = \frac{1}{2}x^2 + \frac{\lambda}{2}x^4 + \sqrt{\lambda}x^3$$

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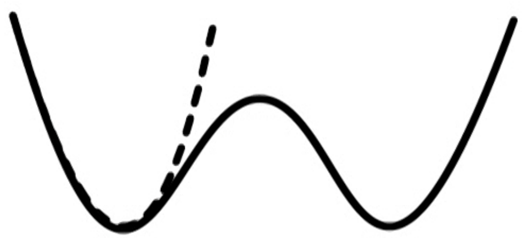

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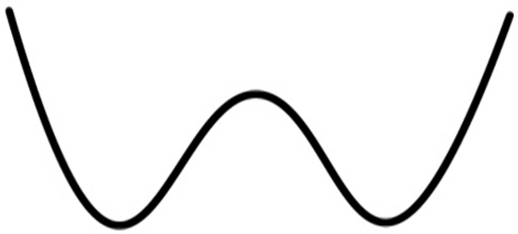
$$V = \frac{1}{2}x^2 + \frac{\lambda}{2}x^4 + \sqrt{\lambda}x^3 \to \frac{1}{2}x^2 + \frac{\lambda}{2}x^4 + \frac{\lambda^{3/2}}{\lambda_0}x^3$$

double-well:



$$E_0^{\text{inst}} \approx \frac{2}{\sqrt{\pi\lambda}} e^{-\frac{1}{6\lambda}}$$

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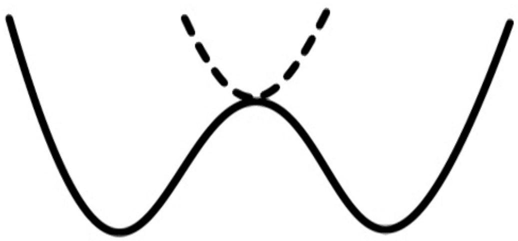


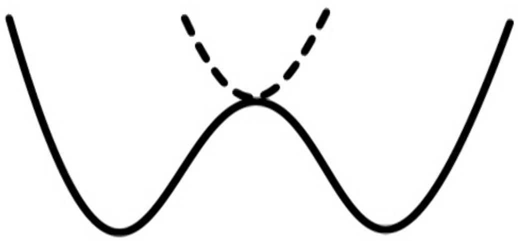
$$E_0^{\text{inst}} \approx \frac{2}{\sqrt{\pi\lambda}} e^{-\frac{1}{6\lambda}} \approx 0.087$$

e.g.

for $\lambda = 1/25$ and $N = 200$

E_0, E_1 with $10^{-8} \div 10^{-14}$ precision

double-well:

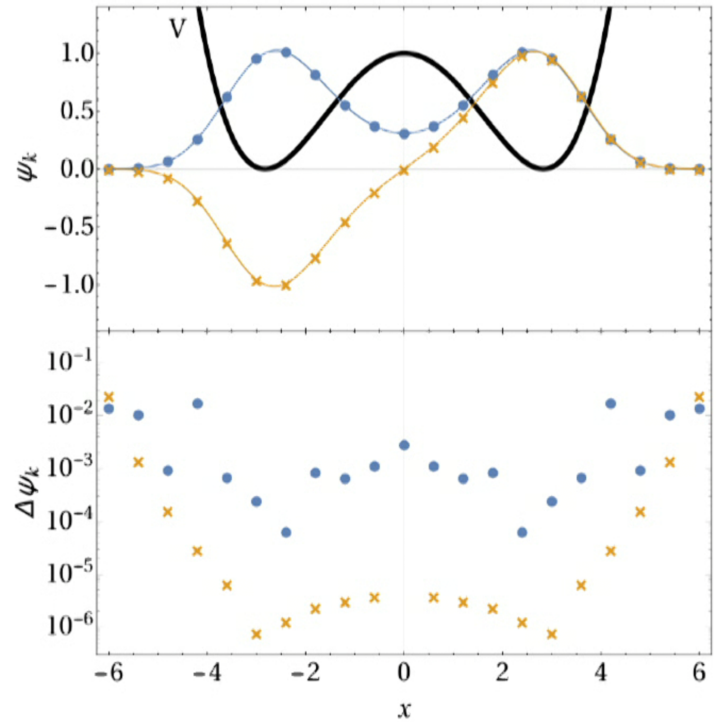


$$E_0^{\text{inst}} \approx \frac{2}{\sqrt{\pi\lambda}} e^{-\frac{1}{6\lambda}} \approx 0.087$$

e.g.

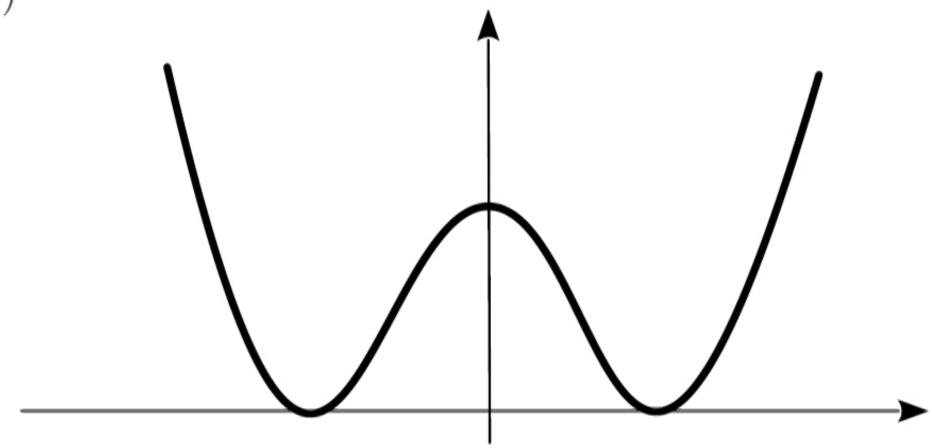
for $\lambda = 1/25$ and $N = 200$

E_0, E_1 with $10^{-8} \div 10^{-14}$ precision

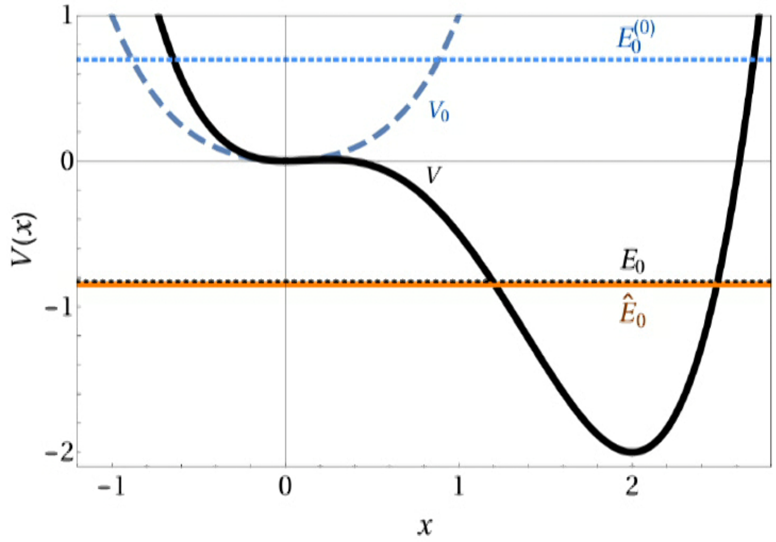


supersymmetric double-well:

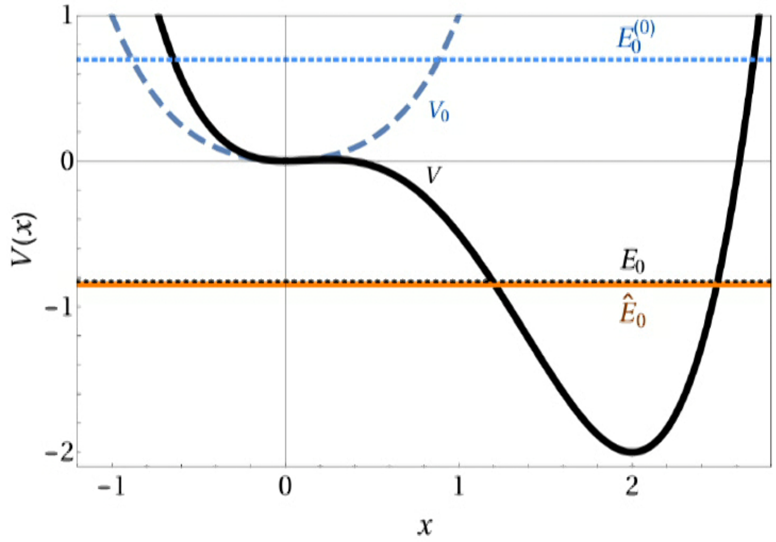
$$V(x;\lambda) = \frac{\lambda}{2}\left(x^2 - \frac{1}{4\lambda}\right)^2$$



False vacuum:



False vacuum:



Conclusions:

– Resummation of PT in QM from Lefschetz-thimbles

– EPT reconstructs non-pert. saddles
(transseries, Stokes-lines, Borel non-summability)

– Allows to recover full results at strong coupling from PT alone

Outlook:

QM $\rightarrow$ QFT

– UV limit? renormalization?
  non-semiclassical NP effects (renormalons?)

Outlook:

QM $\rightarrow$ QFT

– UV limit? renormalization?
  non-semiclassical NP effects (renormalons?)

– IR limit potentially more delicate (phase transitions?)

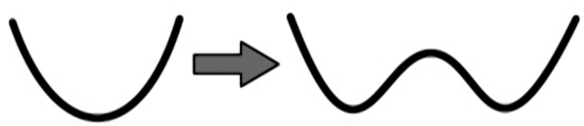
Work in Progress:

$\lambda\phi^4$ theory in 2D

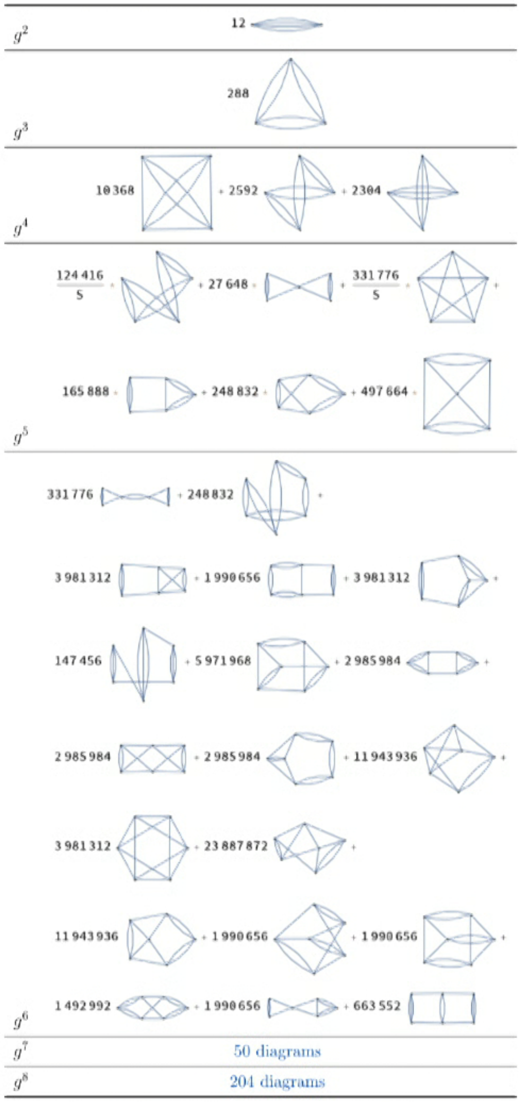
– Renormalized by normal ordering

– Borel resummable ($m^2 > 0$)

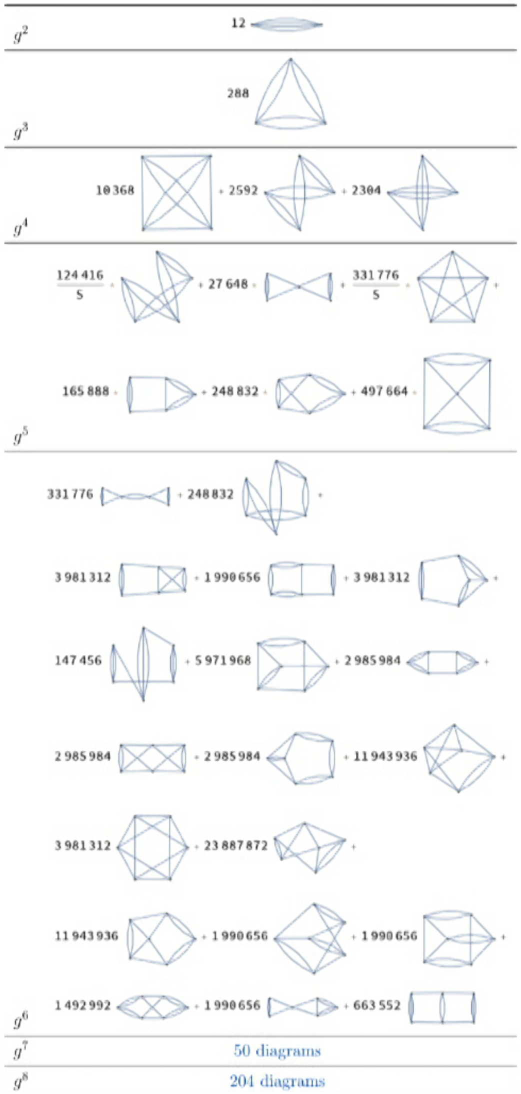
– Rich physics: phase-transition – duality – Ising



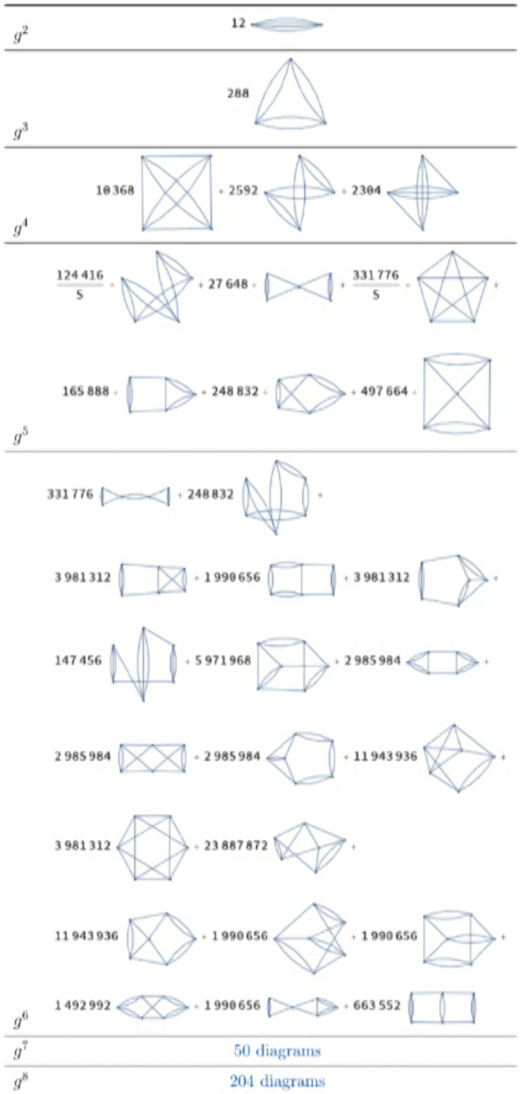
– Quantitatively studied with other methods



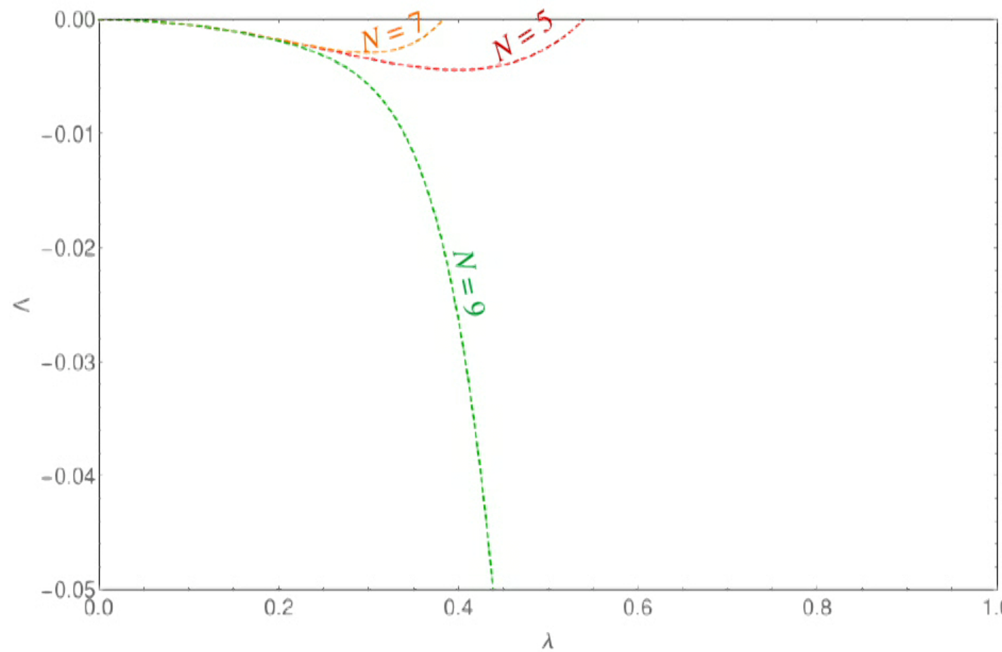
g^2
12
g^3
288
g^4
10368 + 2592 + 2304
g^5
124416/5 + 27648 + 331776/5 +
165888 + 248832 + 497664
g^6
331776 + 248832 +
3981312 + 1990656 + 3981312 +
147456 + 5971968 + 2985984 +
2985984 + 2985984 + 11943936 +
3981312 + 23887872 +
11943936 + 1990656 + 1990656 +
1492992 + 1990656 + 663552
g^7
50 diagrams
g^8
204 diagrams

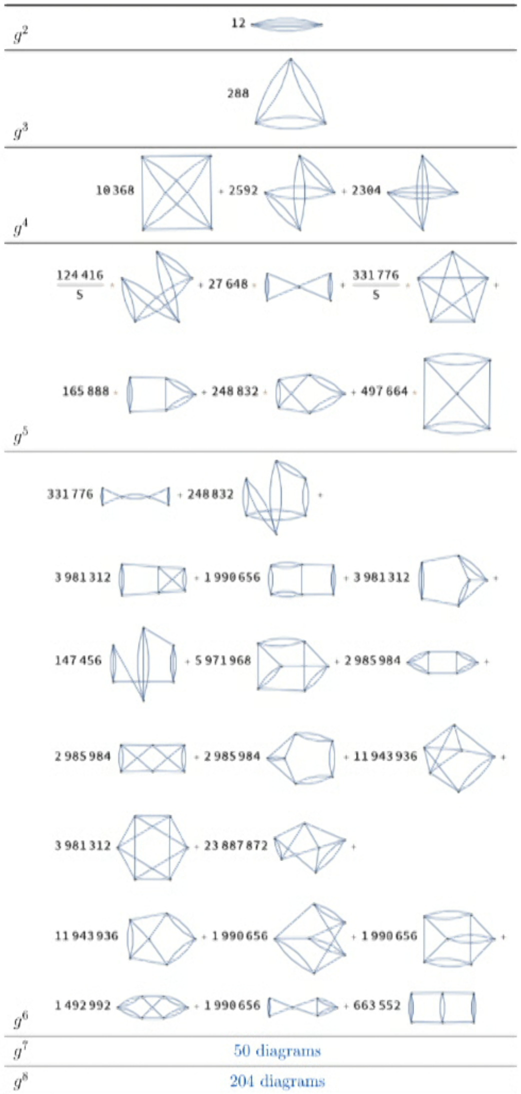


$$
\begin{aligned}
\mathcal{E}_0|_m = &-\Lambda - \frac{21\zeta(3)}{16\pi^3}\frac{g^2}{m^2} + 0.0416484872(94)\,\frac{g^3}{m^4} - 0.11612577(22)\,\frac{g^4}{m^6} \\
&+ 0.3949494(81)\,\frac{g^5}{m^8} - 1.62958(12)\,\frac{g^6}{m^{10}} + 7.8496(30)\,\frac{g^7}{m^{12}} - 42.712(90)\,\frac{g^8}{m^{14}} + \dots
\end{aligned}
$$

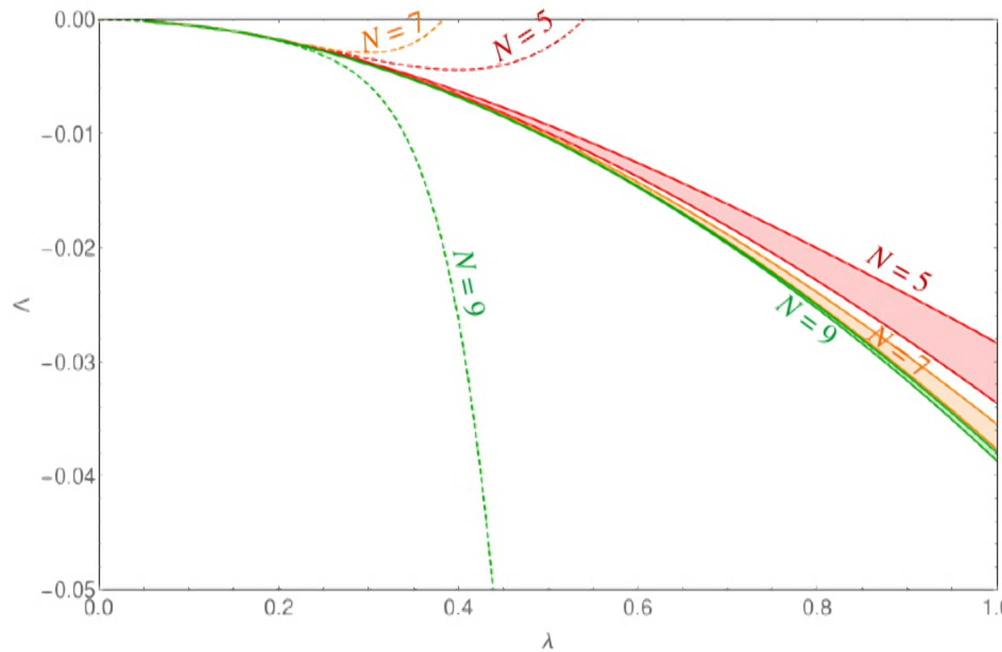


$$\mathcal{E}_0|_m = -\Lambda - \frac{21\zeta(3)}{16\pi^3}\frac{g^2}{m^2} + 0.0416484872(94)\frac{g^3}{m^4} - 0.11612577(22)\frac{g^4}{m^6} + 0.3949494(81)\frac{g^5}{m^8} - 1.62958(12)\frac{g^6}{m^{10}} + 7.8496(30)\frac{g^7}{m^{12}} - 42.712(90)\frac{g^8}{m^{14}} + \ldots$$





$$\mathcal{E}_0|_m = -\Lambda - \frac{21\zeta(3)}{16\pi^3}\frac{g^2}{m^2} + 0.0416484872(94)\frac{g^3}{m^4} - 0.11612577(22)\frac{g^4}{m^6} + 0.3949494(81)\frac{g^5}{m^8} - 1.62958(12)\frac{g^6}{m^{10}} + 7.8496(30)\frac{g^7}{m^{12}} - 42.712(90)\frac{g^8}{m^{14}} + \dots$$



The Power of Series

Giovanni Villadoro

The Abdus Salam
International Centre
for Theoretical Physics

based on:

1612.04376 & 1702.04148

M. Serone and G. Spada

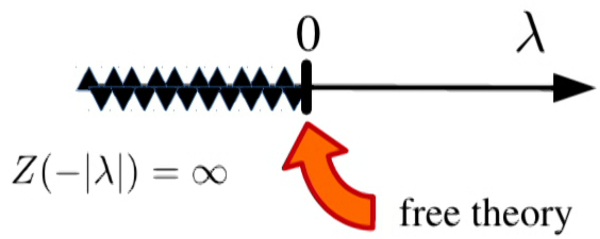
Perturbation Theory

$$Z(\lambda) \approx \sum_{n=0}^{\infty} Z_n \lambda^n$$

Perturbation Theory

$$Z(\lambda) \approx \sum_{n=0}^{\infty} Z_n \lambda^n \longrightarrow \infty$$

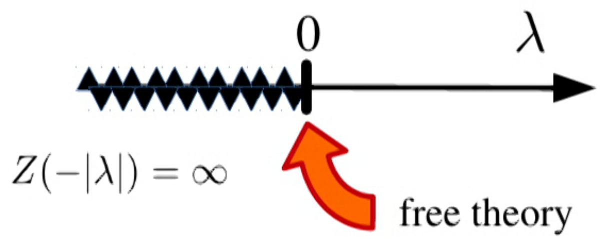
Dyson argument:



Perturbation Theory

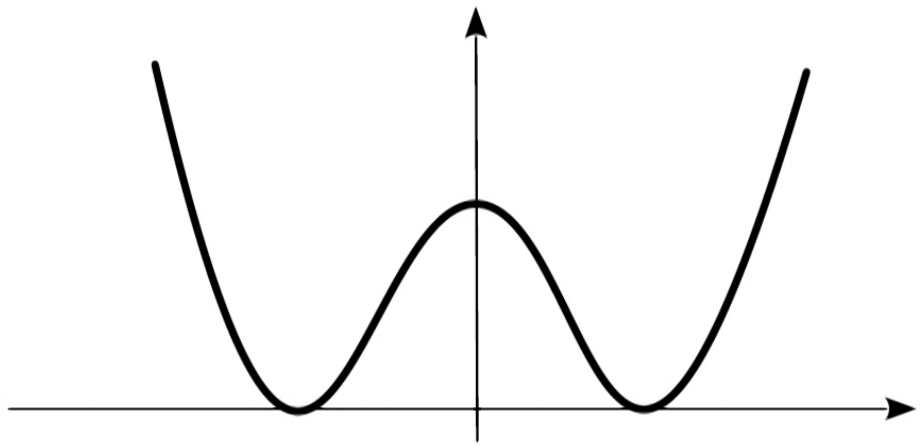
$$Z(\lambda) \approx \sum_{n=0}^{\infty} Z_n \lambda^n \longrightarrow \infty$$

Dyson argument:



Transseries $$Z(\lambda) \approx \sum_n Z_n \lambda^n + \sum_m e^{-B_m/\lambda} \sum_n Z_n^{(m)} \lambda^n + \ldots$$

e.g. double-well in QM



$$E_{\pm} = \sum_n e_n \lambda^n \pm \frac{e^{-\frac{1}{6\lambda}}}{\sqrt{\lambda}} \sum_n e_n^{(1)} \lambda^n + \dots$$

$$Z(\lambda) \approx \sum_n Z_n \lambda^n + \sum_m e^{-B_m/\lambda} \sum_n Z_n^{(m)} \lambda^n + \dots$$

$$Z_n \sim a^n n^c n!$$

$$Z(\lambda) \approx \sum_n Z_n \lambda^n + \sum_m e^{-B_m/\lambda} \sum_n Z_n^{(m)} \lambda^n + \dots$$

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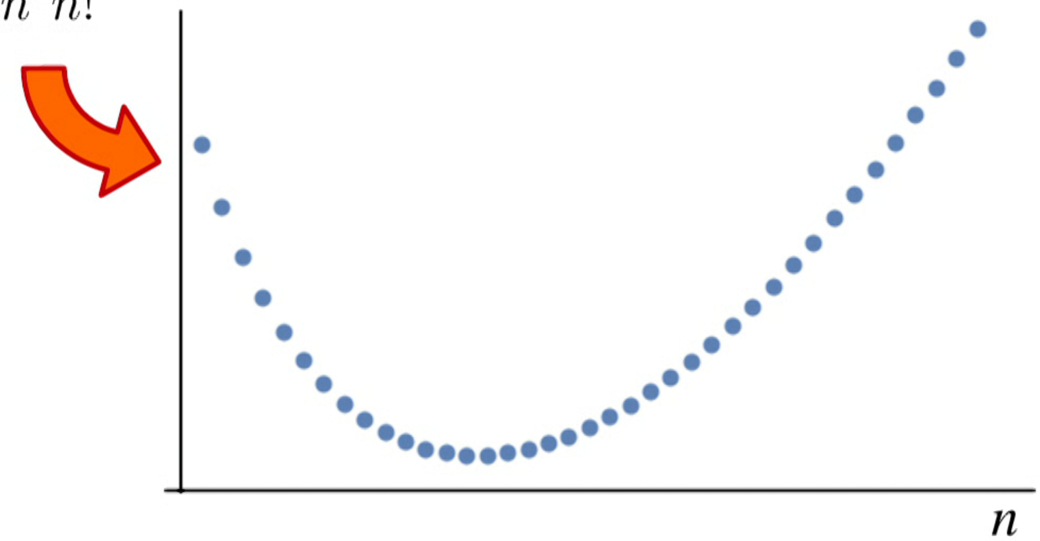

$$Z(\lambda) \approx \sum_n Z_n \lambda^n + \sum_m e^{-B_m/\lambda} \sum_n Z_n^{(m)} \lambda^n + \dots$$

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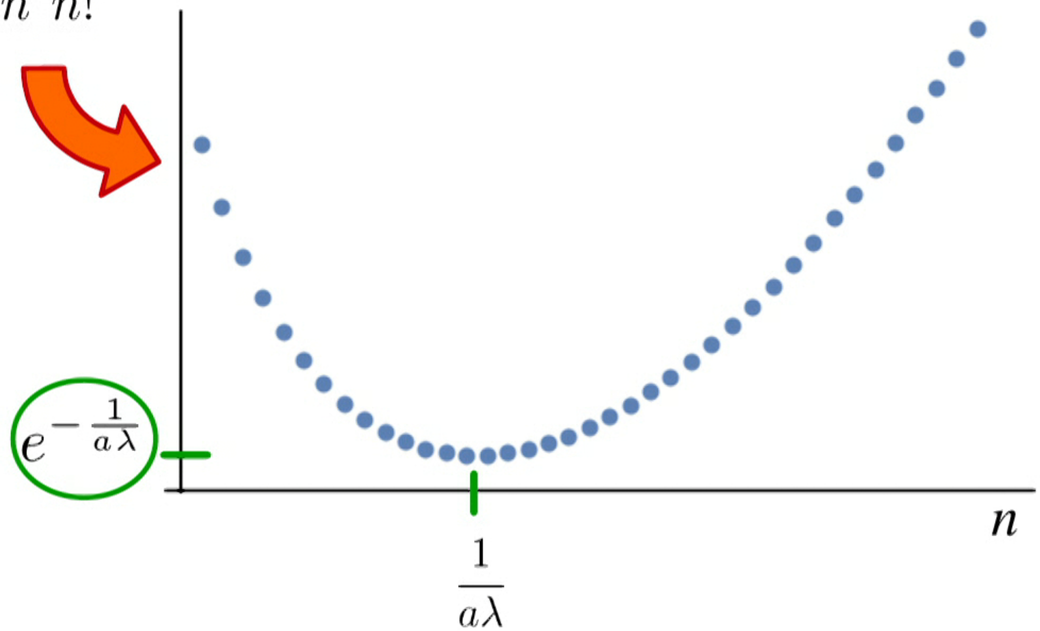

$$Z(\lambda) \approx \sum_n Z_n \lambda^n + \sum_m e^{-B_m/\lambda} \sum_n Z_n^{(m)} \lambda^n + \ldots$$

$$\sum_{n=0}^{\infty} \frac{Z_n}{\Gamma(n+1+b)} t^n$$

finite radius of convergence!