

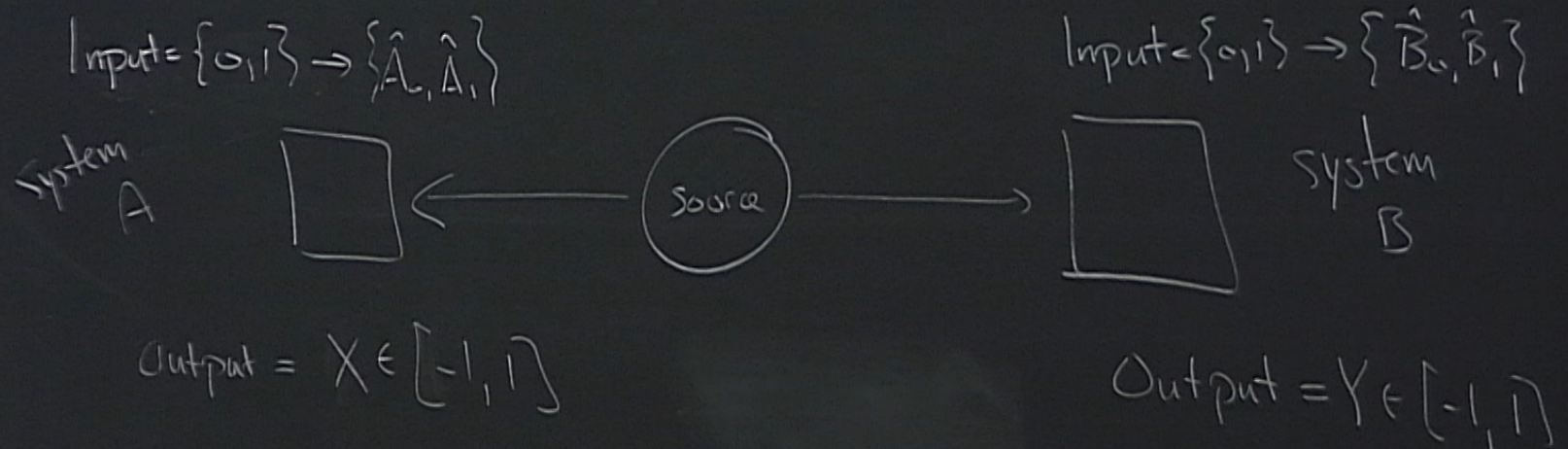
Title: PSI 17/18 - Quantum Theory - Lecture 14

Date: Sep 22, 2017 10:45 AM

URL: <http://pirsa.org/17090047>

Abstract:

Bell / CHSH Theorem & non-locality



Realist Model

to give a causal
story for observable
correlations

$$S := \langle A_0 B_0 \rangle + \langle A_0 B_1 \rangle + \langle A_1 B_0 \rangle - \langle A_1 B_1 \rangle$$

Under "Reasonable" assumptions

$$-2 \leq S \leq 2$$

Λ = space/se
of r

Λ = space/set of values
of relevant physical quantities.

$\lambda \in \Lambda$ is the values of physical
quantities in a given run.

$p(\lambda)$ is the distr^r (prob density)
allows for statistical variation
of $\lambda \in \Lambda$ over each run.

$\langle A, B \rangle$

Examples

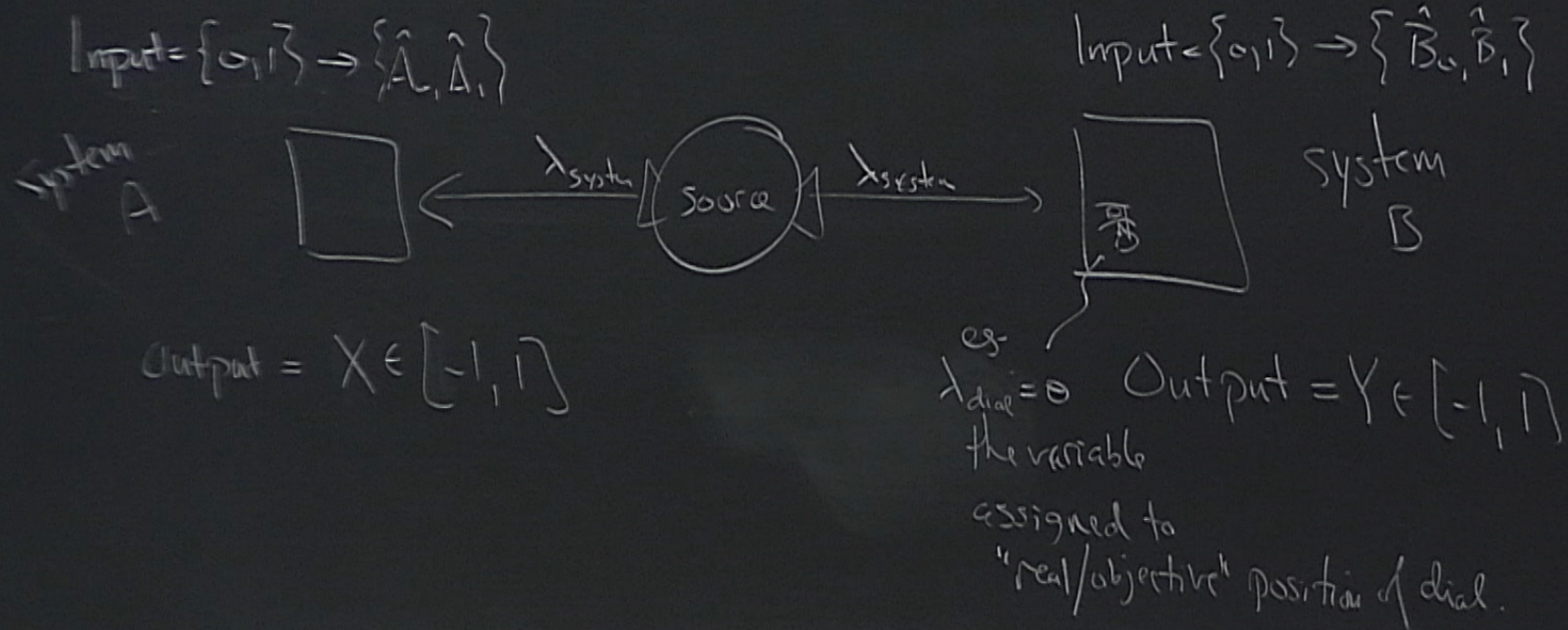
$$\lambda := |T \times T|$$

$$\text{or } \lambda = (q, p)$$

$$\text{or } \lambda = (q, |T \times T|)$$

No assumption

Bell / CHSH Theorem & non-locality



Examples

$$\lambda := \|X\|$$

$$\text{or } \lambda = (q, p)$$

$$\text{or } \lambda = (q, \|X\|)$$

No assumption

$$S = \int d\lambda \rho(\lambda) S(\lambda)$$

$$S(\lambda) = \sum_{i,j \in \{0,1\}} \langle A_i B_j \rangle_\lambda (-1)^{ij}$$

Examples

$$\lambda = \|X\|$$

$$\text{or } \lambda = (q, p)$$

$$\text{or } \lambda = (q, \|X\|)$$

No assumption

$$S = \int d\lambda p(\lambda) S(\lambda)$$

$$S(\lambda) = \sum_{i,j \in \{0,1\}} \langle A_i B_j \rangle_\lambda (-1)^{ij}$$

$$\begin{aligned} \langle A_i B_j \rangle &\equiv \mathbb{E}(XY | A_i B_j) \\ &\equiv \int d\lambda p(\lambda) \end{aligned}$$

Examples

$$\lambda = |TX|$$

$$\text{or } \lambda = (q, p)$$

$$\text{or } \lambda = (q, |TX|)$$

No assumption

$$S = \int d\lambda p(\lambda) S(\lambda)$$

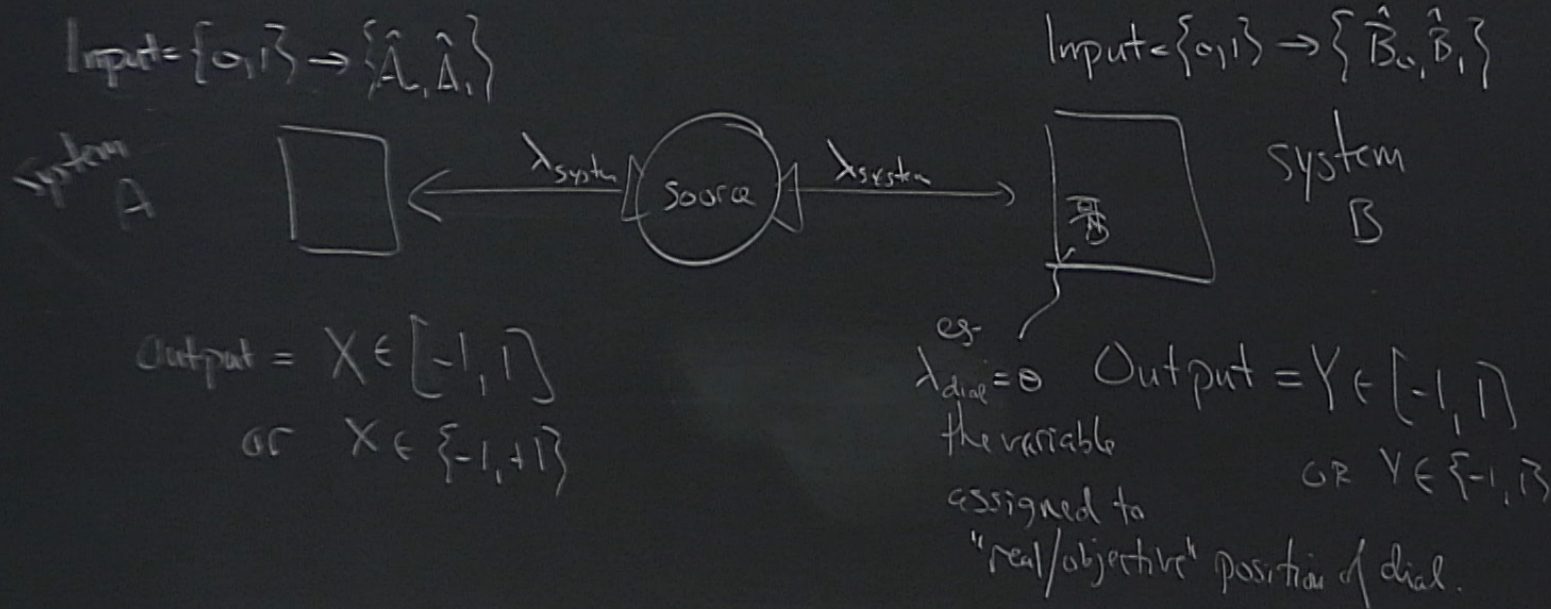
$$S(\lambda) = \sum_{i,j \in \{1,2\}} \langle A_i B_j \rangle_\lambda (-1)^{ij}$$

$$\langle A_i B_j \rangle = \mathbb{E}(XY | A_i B_j)$$

$$= \int d\lambda p(\lambda) \sum_{XY} XY P_{ij}(X, Y | A_i B_j, \lambda)$$

$$\text{or } = \int d\lambda p(\lambda) \int dX dY XY P_{ij}(X, Y | A_i B_j, \lambda)$$

Bell / CHSH Theorem & non-locality



Realist Model
to give
story
Corre
 $S := \langle$
Under

$$S = \int d\lambda p(\lambda) S(\lambda)$$

$$S(\lambda) = \sum_{i,j \in \{0,1\}} \langle A_i B_j \rangle_{\lambda} (-1)^{ij}$$

$$\langle A_i B_j \rangle = \mathbb{E}(XY | A_i B_j)$$

$$\lambda = \lambda_{\text{system}}$$

$$= \int d\lambda p(\lambda) \sum_{XY} XY \underbrace{Pr(X,Y | A_i B_j, \lambda)}$$

$$\text{OR} \quad = \int d\lambda p(\lambda) \int dx dy XY Pr(X,Y | A_i B_j, \lambda)$$

Impose: no communication b/w A & B

no causal mechanism b/w A & B

OTHER THAN λ

Disallow ^B Bob's output [&] / input
to depend on Alice's (A)
output or input
& vice versa

Factorization condition

$$P(x, y | A, B, \lambda) = P(x | A, \lambda) P(y | B, \lambda)$$

Two assumptions

• Local causality assumption

• Realist assumption : system ^{has} physical _{properties}.
 $\Rightarrow$ assign a λ .

) $P(Y|B,\lambda)$

option

has physical
properties.

$\Rightarrow$ assign a λ .

$$\langle A, B \rangle = \sum_{X,Y} X Y P(X|A,\lambda) \cdot P(Y|B,\lambda)$$

1
Local causality

$$= \left(\sum_{\lambda} X P(X|A,\lambda) \right) \left(\sum_{\lambda} Y P(Y|B,\lambda) \right)$$

$$\langle A_i B_j \rangle_\lambda = \sum_{x,y} x y P(x|A_i, \lambda) \cdot P(y|B_j, \lambda)$$

↑
Local causality

$$= \left(\sum_x x P(x|A_i, \lambda) \right) \left(\sum_y y P(y|B_j, \lambda) \right)$$

" " " "

$$\equiv \langle A_i \rangle_\lambda \langle B_j \rangle_\lambda$$

$$S(\lambda) = \langle A_0 \rangle_\lambda \langle B_0 \rangle_\lambda + \langle A_0 \rangle_\lambda \langle B_1 \rangle_\lambda + \langle A_1 \rangle_\lambda \langle B_0 \rangle_\lambda - \langle A_1 \rangle_\lambda \langle B_1 \rangle_\lambda$$

Remark: We required $|X| \leq 1$, $|Y| \leq 1$,

$$\langle A_i \rangle \in [-1, +1]$$

$$\langle B_j \rangle \in [-1, +1]$$

$$\begin{aligned} S(\lambda) &= \langle A_0 \rangle_\lambda (\langle B_0 \rangle_\lambda + \langle B_1 \rangle_\lambda) + \langle A_1 \rangle_\lambda (\langle B_0 \rangle_\lambda - \langle B_1 \rangle_\lambda) \\ &\leq |\langle B_0 \rangle_\lambda + \langle B_1 \rangle_\lambda| + |\langle B_0 \rangle_\lambda - \langle B_1 \rangle_\lambda| \end{aligned}$$

$$\text{W.l.o.g., } \langle B_0 \rangle_\lambda \geq \langle B_1 \rangle_\lambda, \Rightarrow S(\lambda) \leq 2|\langle B_0 \rangle_\lambda| \leq 2$$

$$|x| \leq 1, |y| \leq 1,$$

i)

ii)

$$\langle B_0 \rangle_x + \langle B_1 \rangle_x + \langle A \rangle_x (\langle B_0 \rangle_x - \langle B_1 \rangle_x)$$

$$+ |\langle B_0 \rangle_x| + |\langle B_0 \rangle_x - \langle B_1 \rangle_x|$$

$$\Rightarrow S(x) \leq 2 |\langle B_0 \rangle_x| \leq 2$$

It follows that

$$-2 \leq S(x) \leq 2$$

$$\& S = \int dx \rho(x) S(x)$$

$$\therefore \boxed{-2 \leq S \leq 2}$$

It follows that

$$-2 \leq S(\lambda) \leq 2$$

$$\& S = \int d\lambda p(\lambda) S(\lambda)$$

$$\Rightarrow \boxed{-2 \leq S \leq 2}$$

$\Leftarrow$

Assumptions

0. Math works

1. Realism

2. Local causality (no causal mechanism).



Quantum Mechanical Prediction

$$\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2 \rightarrow \text{two photons (polarization)}$$

$\rightarrow$ two spin- $1/2$ particles

$\rightarrow$ Ground & excited state subspace
of two atoms

Define $l \in \{0,1\}$

$$A_l \Rightarrow \hat{\vec{\sigma}} \cdot \hat{\vec{a}}_l$$

$\hat{\vec{a}}_l$
 $\leftarrow$ direction
in Euclidean
space $\mathbb{R}^3$

$$\hat{\vec{\sigma}} = (X, Y, Z)$$

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Define $B_k \Rightarrow \hat{\vec{\sigma}} \cdot \hat{\vec{b}}_k$

$\hat{\vec{b}}_k$
 $\leftarrow$ spatial
direction
 $k \in \{0,1\}$

$$\equiv \hat{B}_k$$

Bob's output
is given by measuring

$$\hat{B}_k = |+b_k\rangle\langle +b_k| + |-b_k\rangle\langle -b_k|$$

$\downarrow$ $\downarrow$
outcome +1 outcome -1

$$\Pr(+b_k) = \text{Tr}(|+b_k\rangle\langle +b_k| \rho)$$

Similarly for Alice.

Input state

$$\rho = |\psi^-\rangle\langle\psi^-|$$

$$|\psi^-\rangle = \frac{|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle}{\sqrt{2}}$$

$$= \frac{|-1,+1\rangle - |+1,-1\rangle}{\sqrt{2}}$$

$$= \frac{|01\rangle - |10\rangle}{\sqrt{2}}$$

$$|\downarrow\uparrow\rangle$$

$$= |\downarrow\rangle_A \otimes |\uparrow\rangle_B$$

$|\psi^-\rangle$
 $\{$
 output - 1

ρ

"Singlet state"

"Bell state"

Maximally entangled across $\mathbb{C}^2 \otimes \mathbb{C}^2$

$$\rho_A = \frac{11}{2} = \text{Tr}_B[\rho \cdot X \cdot \rho^{-1}]$$

$$S(\rho_A) = \log(2)$$

Joint measurement

$$\hat{\sigma}_{\vec{a}_\ell} \otimes \hat{\sigma}_{\vec{b}_k}$$

Experimental quantity

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N X_n^{\vec{a}_\ell} Y_n^{\vec{b}_k}$$

$$= \text{Tr}[\hat{\rho} \hat{\sigma}_{\vec{a}_\ell} \otimes \hat{\sigma}_{\vec{b}_k}]$$

Joint measurement

$$\hat{\sigma}_{\vec{a}_x} \otimes \hat{\sigma}_{\vec{b}_k}$$

Experimental quantity

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N x_n^{\vec{a}_x} y_n^{\vec{b}_k}$$

$$= \text{Tr}[\hat{\rho} \hat{\sigma}_{\vec{a}_x} \otimes \hat{\sigma}_{\vec{b}_k}]$$

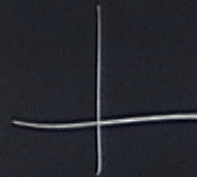
Directly compute

$$\text{Tr}[(|\psi\rangle\langle\psi| \hat{\sigma}_{\vec{a}_x} \otimes \hat{\sigma}_{\vec{b}_k})] = -\vec{a}_x \cdot \vec{b}_k$$

Optimal choice of $\vec{a}_x$ & $\vec{b}_k \forall x, k \in \{0,1\}$

$$S = 2\sqrt{2}$$

Geometrically



First widely-accepted
experimental verification
(validation of QM prediction)

Alain Aspect (1982, 1981)

⇒ assign a λ

$$\equiv \langle A_i \rangle_\lambda \langle B_j \rangle_\lambda$$

$$S(\lambda) = \langle A_0 \rangle_\lambda \langle B_0 \rangle_\lambda + \langle A_0 \rangle_\lambda \langle B_1 \rangle_\lambda + \langle A_1 \rangle_\lambda \langle B_0 \rangle_\lambda - \langle A_1 \rangle_\lambda \langle B_1 \rangle_\lambda$$

Contextuality (one incarnation of it)

⇒ Peres-Mermin Square

$$H = \sigma_x^2 \otimes \sigma_y^2$$

$$\sigma_x \otimes I \quad I \otimes \sigma_x \quad \sigma_x \otimes \sigma_x$$

$$I \otimes \sigma_y \quad \sigma_y \otimes I \quad \sigma_y \otimes \sigma_y$$

$$\sigma_x \otimes \sigma_y \quad \sigma_y \otimes \sigma_x \quad \sigma_z \otimes \sigma_z$$

If I measure $\sigma_a \Rightarrow$ ^{binary} outcome $\{-1, +1\}$

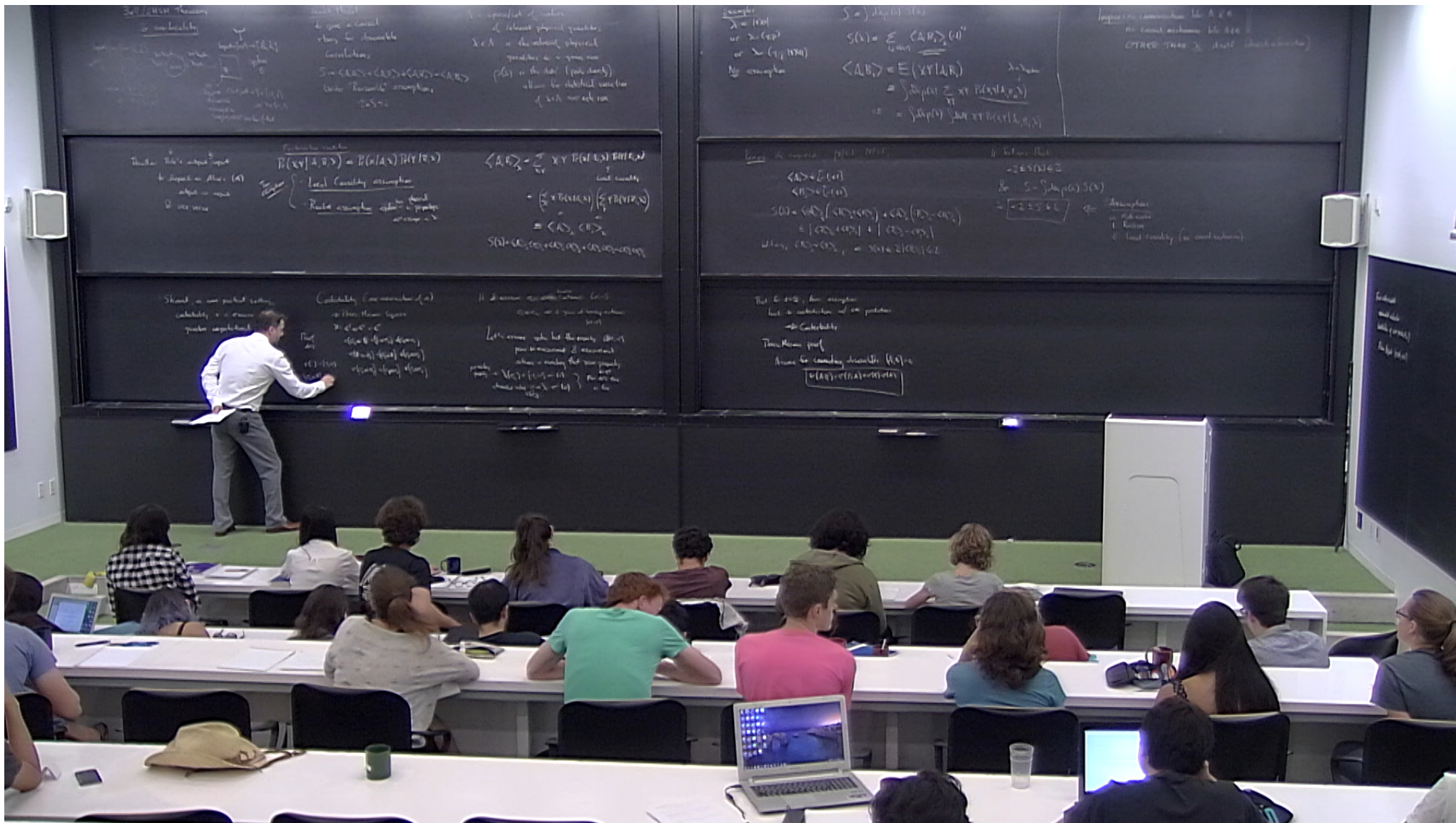
$\sigma_a \otimes \sigma_b \Rightarrow$ A pair of binary outcomes $\{-1, +1\}$

Let's assume system had the property $\{-1, +1\}$

prior to measurement & measurement

outcome is revealing that prior property.

pre-existing property $\rightarrow \lambda(\sigma_a) \in \{-1, +1\} \leftarrow (i)$ } For $d=2$ this is fine
observed value $v(\sigma_a) = \lambda \leftarrow (ii)$



$$S(\lambda) = \langle A_0 \rangle_\lambda$$

- Showed, in some practical setting, contextuality is a resource for quantum computational speedup.

Contextuality (one incarnation of it)

⇒ Peres-Mermin Square

$$I = C^2 \otimes C^2 = C^4$$

Proof
d=4

For each row

$$v(\cdot) \in \{-1, 1\}$$

$$(C_x \otimes I)(I \otimes C_x)$$

$$(C_x \otimes C_x) = I, \quad v(I) = 1, \quad 1, \quad -1$$

$v(C_x \otimes I)$	$v(I \otimes C_x)$	$v(C_x \otimes C_x)$	+1
$v(I \otimes C_y)$	$v(C_y \otimes I)$	$v(C_y \otimes C_y)$	+1
$v(C_x \otimes C_y)$	$v(C_y \otimes C_x)$	$v(C_z \otimes C_z)$	+1

If I measure $\sigma_a = \sigma_x \otimes \sigma_y$

Let's assume system prior to measurement outcome is

pre-existing property $\rightarrow \lambda(\sigma_a) \in \{-1, 1\}$
observed value $v = \lambda(\sigma_a)$

$$\text{W.l.o.g., } \langle B_0 \rangle_\lambda > \langle B_1 \rangle_\lambda, \Rightarrow S(\lambda) \leq 2 |\langle B_0 \rangle_\lambda| \leq 2$$

But for $d \geq 2$, these assumptions
lead to contradiction w/ QM predictions

$\Rightarrow$ Contextuality

Peter Mermin proof

Assume for commuting observables $[A, B] = 0$

$$\boxed{v(AB) = v(BA) = v(B)v(A)}$$

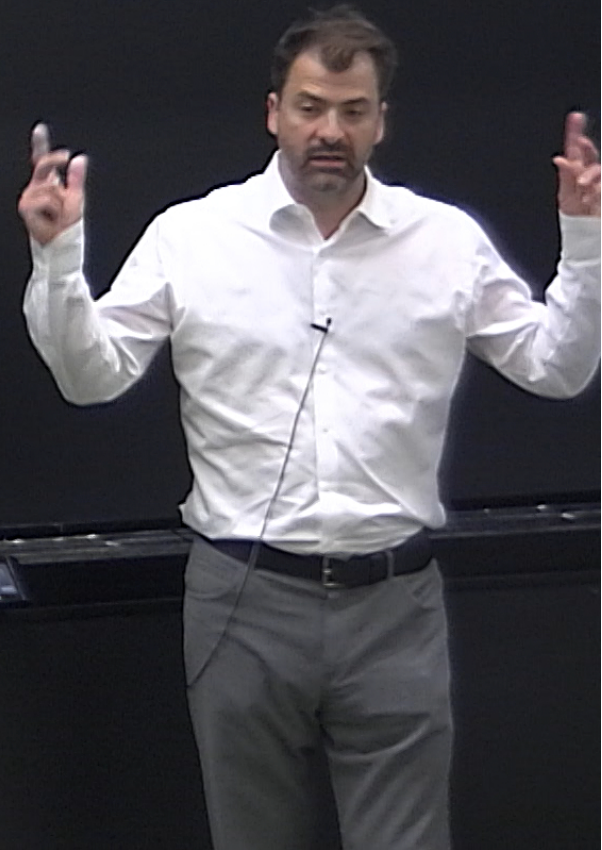
Multiplying v 's for rows

$\rightarrow +1$

Multiplying v 's for columns

$\rightarrow -1$

$\rightarrow$ Contradiction



$$S(\lambda) = \langle A_0 \rangle_{\lambda} \langle B_0 \rangle_{\lambda} + \langle A_0 \rangle_{\lambda} \langle B_1 \rangle_{\lambda} + \langle A_1 \rangle_{\lambda} \langle B_0 \rangle_{\lambda} - \langle A_1 \rangle_{\lambda} \langle B_1 \rangle_{\lambda}$$

Contextuality (one incarnation of it)

⇒ Peres-Mermin Square

$$I = \sigma_x^2 \otimes \sigma_y^2 = \sigma^4$$

Proof
d=4

$U(\sigma_x \otimes \mathbb{1})$	$U(\mathbb{1} \otimes \sigma_y)$	$U(\sigma_x \otimes \sigma_y)$	identity +1
$U(\mathbb{1} \otimes \sigma_y)$	$U(\sigma_y \otimes \mathbb{1})$	$U(\sigma_y \otimes \sigma_y)$	+1
$U(\sigma_x \otimes \sigma_y)$	$U(\sigma_y \otimes \sigma_x)$	$U(\sigma_z \otimes \sigma_z)$	+1

$$U(\cdot) \in \{-1, 1\}$$

$$(\sigma_x \otimes \mathbb{1})(\mathbb{1} \otimes \sigma_y)$$

$$(\sigma_x \otimes \sigma_x) = \mathbb{1}, \quad U(\mathbb{1}) = +1$$

If I measure $\sigma_a \Rightarrow$ ^{binary} outcome $\{-1, 1\}$

$\sigma_a \otimes \sigma_b \Rightarrow$ A pair of binary outcomes $\{-1, 1\}$

Let's assume system had the property $\{-1, 1\}$

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