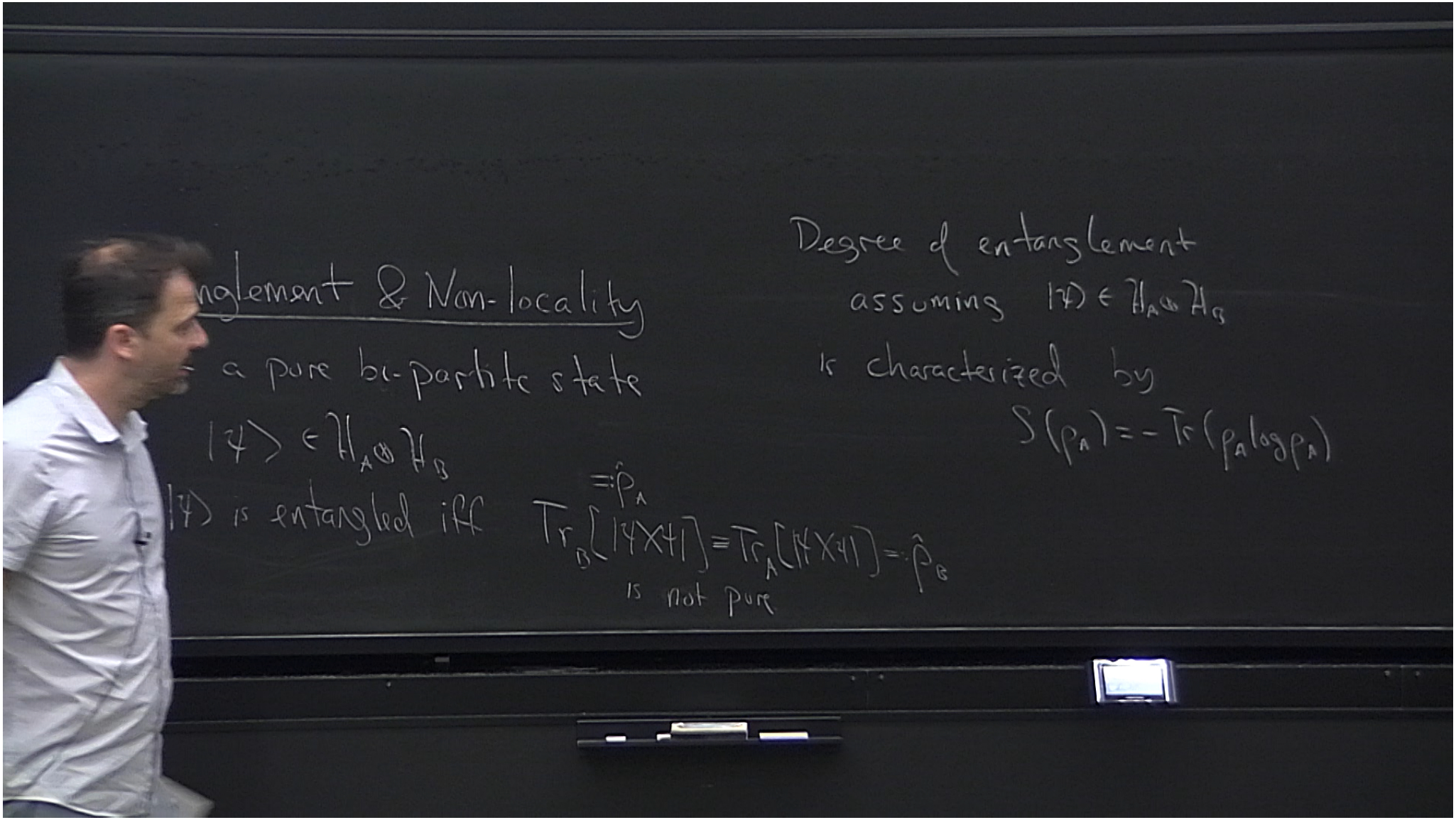
Title: PSI 17/18 - Quantum Theory - Lecture 13

Date: Sep 21, 2017 10:45 AM

URL: http://pirsa.org/17090046

Abstract:


nglement & Non-locality
a pure bi-partite state
$|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$
$|\psi\rangle$ is entangled iff $\mathrm{Tr}_B[|\psi\rangle\langle\psi|] = \mathrm{Tr}_A[|\psi\rangle\langle\psi|] = \hat{\rho}_B$
$= \hat{\rho}_A$
is not pure
Degree of entanglement
assuming $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$
is characterized by
$S(\rho_A) = -\mathrm{Tr}(\rho_A \log \rho_A)$

# Criteria for Mixed state entanglement?

$\Rightarrow$ Problem is <u>open</u> in general.

What we know so far:

Positive partial transpose (PPT) test.

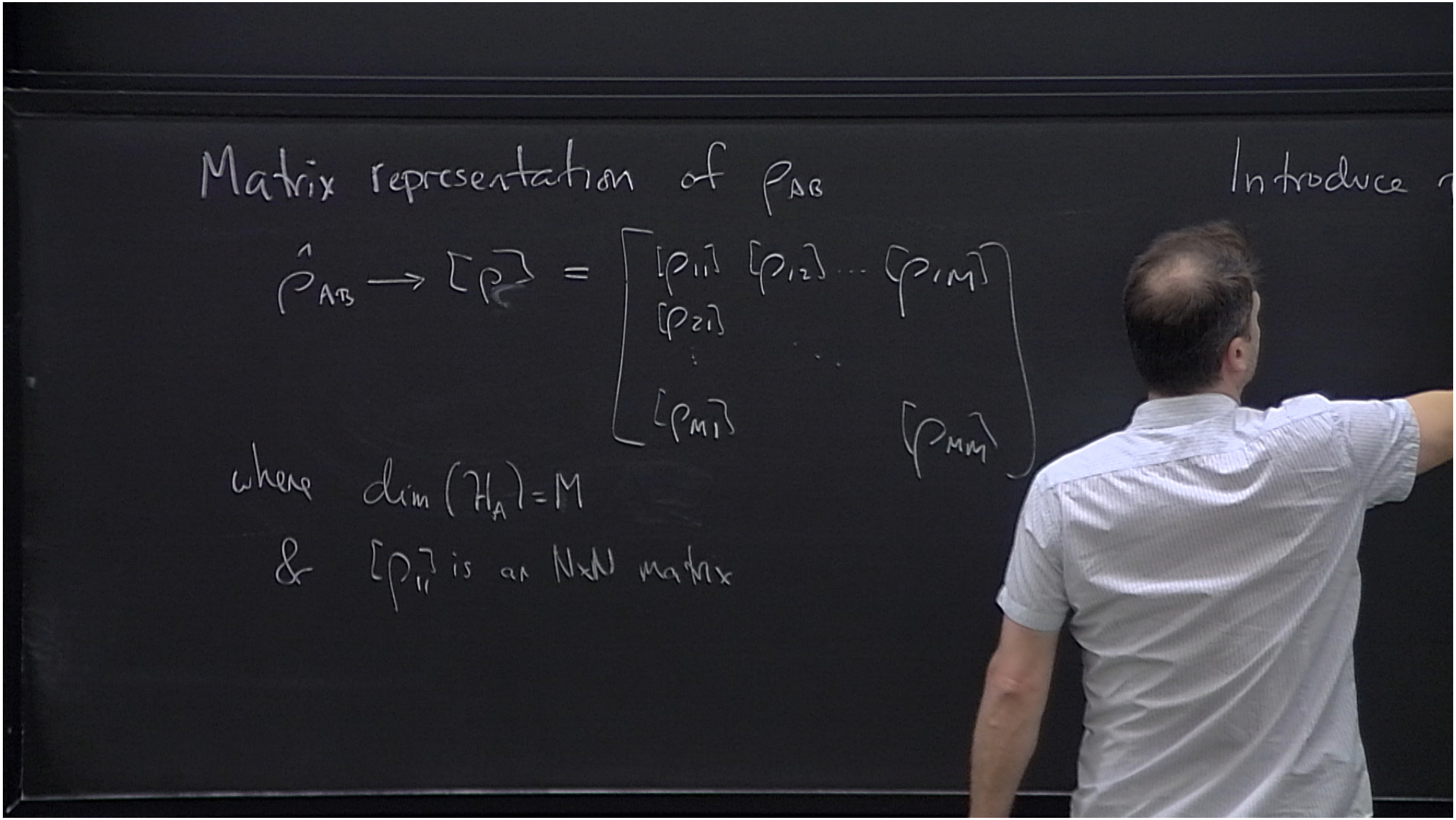
- Given $\hat{\rho}_{AB} \in D(\mathcal{H}_A \otimes \mathcal{H}_B)$
- Countable O.N. bases $\{|a_i\rangle\}, \{|b_i\rangle\}$

Matrix representation of $\rho_{AB}$

$$\hat{\rho}_{AB} \longrightarrow [\rho] = \begin{bmatrix} \rho_{11} & \rho_{12} & \cdots & \rho_{1M} \\ \rho_{21} & & & \\ \vdots & & \ddots & \\ \rho_{M1} & & & \rho_{MM} \end{bmatrix}$$

where $\dim(\mathcal{H}_A) = M$

& $\rho_{11}$ is an $N \times N$ matrix

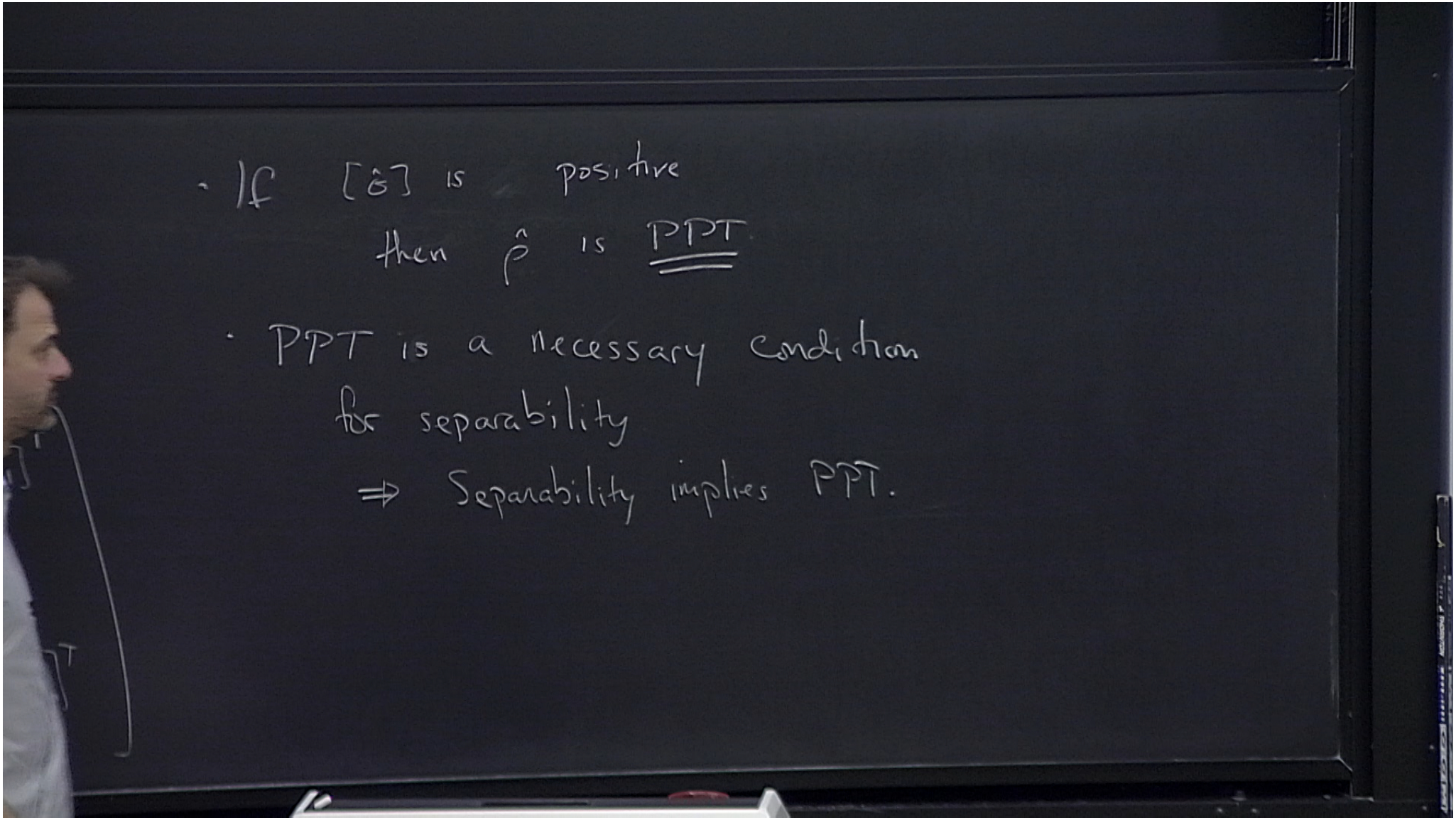

Matrix representation of $\rho_{AB}$
Introduce
$\hat{\rho}_{AB} \longrightarrow [\rho] = \begin{bmatrix} [\rho_{11}] & [\rho_{12}] & \cdots & [\rho_{1M}] \\ [\rho_{21}] & & & \\ \vdots & & \ddots & \\ [\rho_{M1}] & & & [\rho_{MM}] \end{bmatrix}$
where $\dim(\mathcal{H}_A) = M$
& $[\rho_{11}]$ is an $N \times N$ matrix

Introduce partial transpose

$$(\hat{\mathbb{1}}_A \otimes \hat{T}_B)\,\hat{\rho} = \hat{\sigma}$$

$T$ = transpose operation

$$[\hat{\sigma}] = \begin{bmatrix} [\rho_{11}]^T & [\rho_{12}]^T & \cdots & [\rho_{1M}]^T \\ [\rho_{21}]^T & & & \\ \vdots & & & \\ [\rho_{M1}]^T & \cdots & & [\rho_{MM}]^T \end{bmatrix}$$


· If $[\hat{\sigma}]$ is positive
then $\hat{\rho}$ is PPT
· PPT is a necessary condition for separability
⇒ Separability implies PPT.

Equivalently,
NPT implies entanglement.

& $\{\hat{G}_{A_i}\}$ & {

Why does this work?

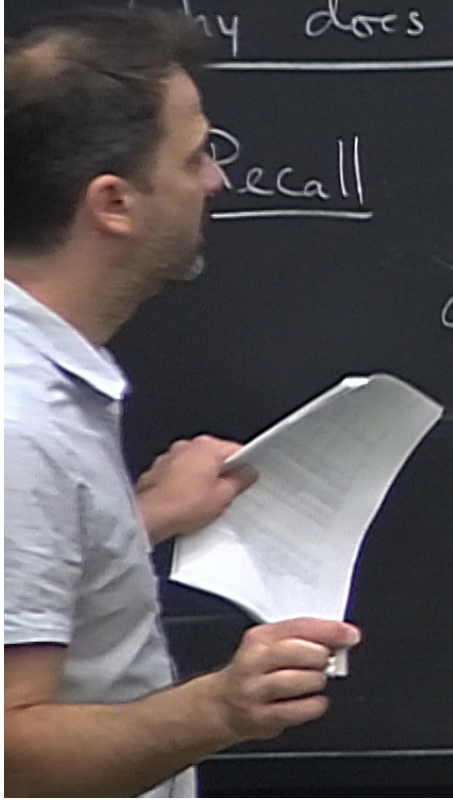
are density o

for A & B.

Recall $\hat{\rho}_{AB}$ is separable iff it admits a decomposition of the form

$$\rho_{AB} = \sum_i p_i \, \hat{\sigma}_{A_i} \otimes \hat{\sigma}_{B_i}$$

where $\{p_i\}$ is some probability distr



is not pure

Equivalently,
NPT implies entanglement.

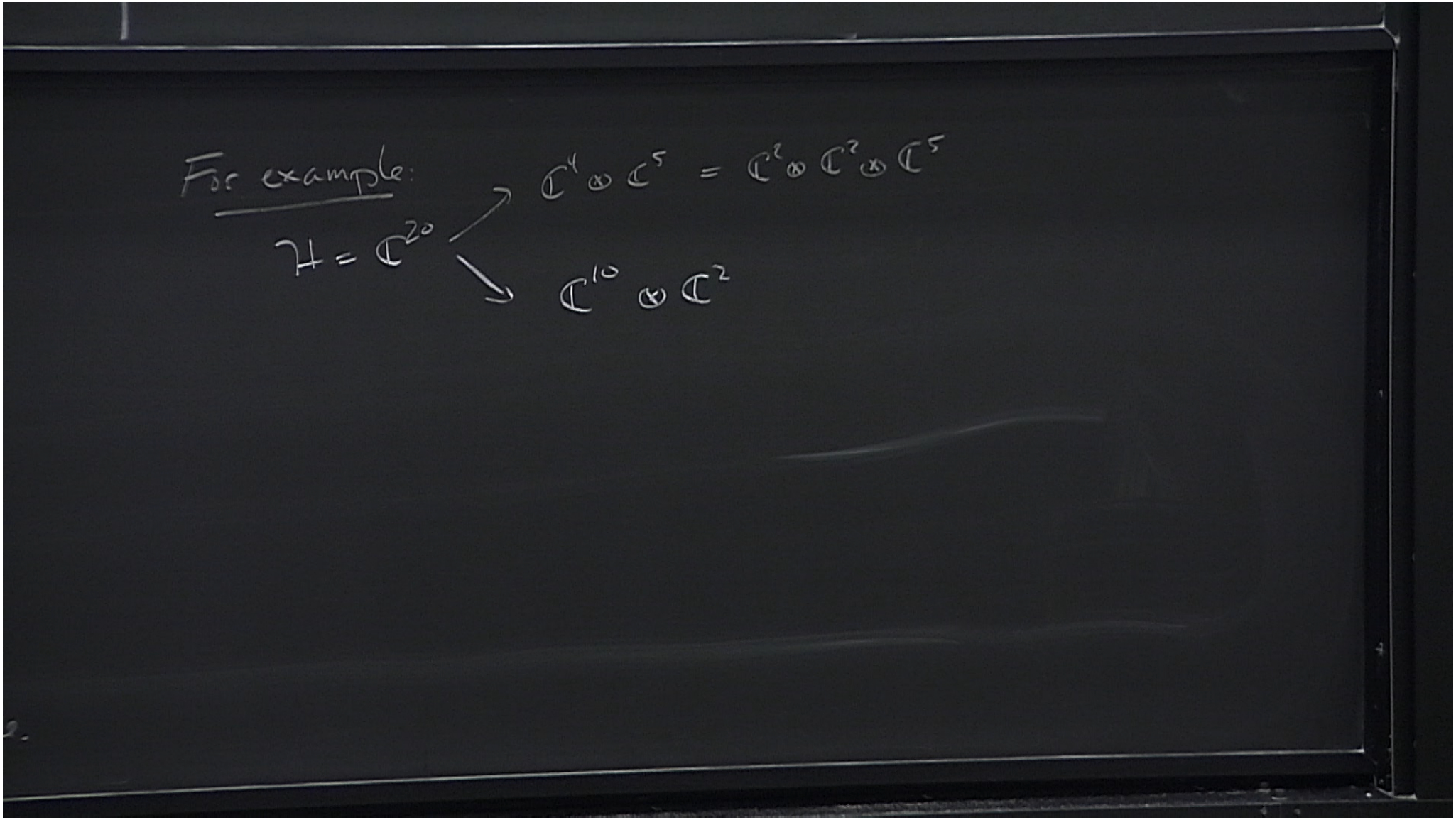
Why does this work?

Recall $\hat{\rho}_{AB}$ is separable iff (= unentangled) it admits a decomposition of the form

$$\rho_{AB} = \sum_i p_i \, \hat{\sigma}_{A,i} \otimes \hat{\sigma}_{B,i}$$

where $\{p_i\}$ is some probability distr$^{n}$ & $\{\hat{\sigma}_{A,i}\}$ & $\{\hat{\sigma}_{B,i}\}$ are density operators for A & B.

Note: Entanglement is only defined in relation some particular factorization of the Hilbert space.

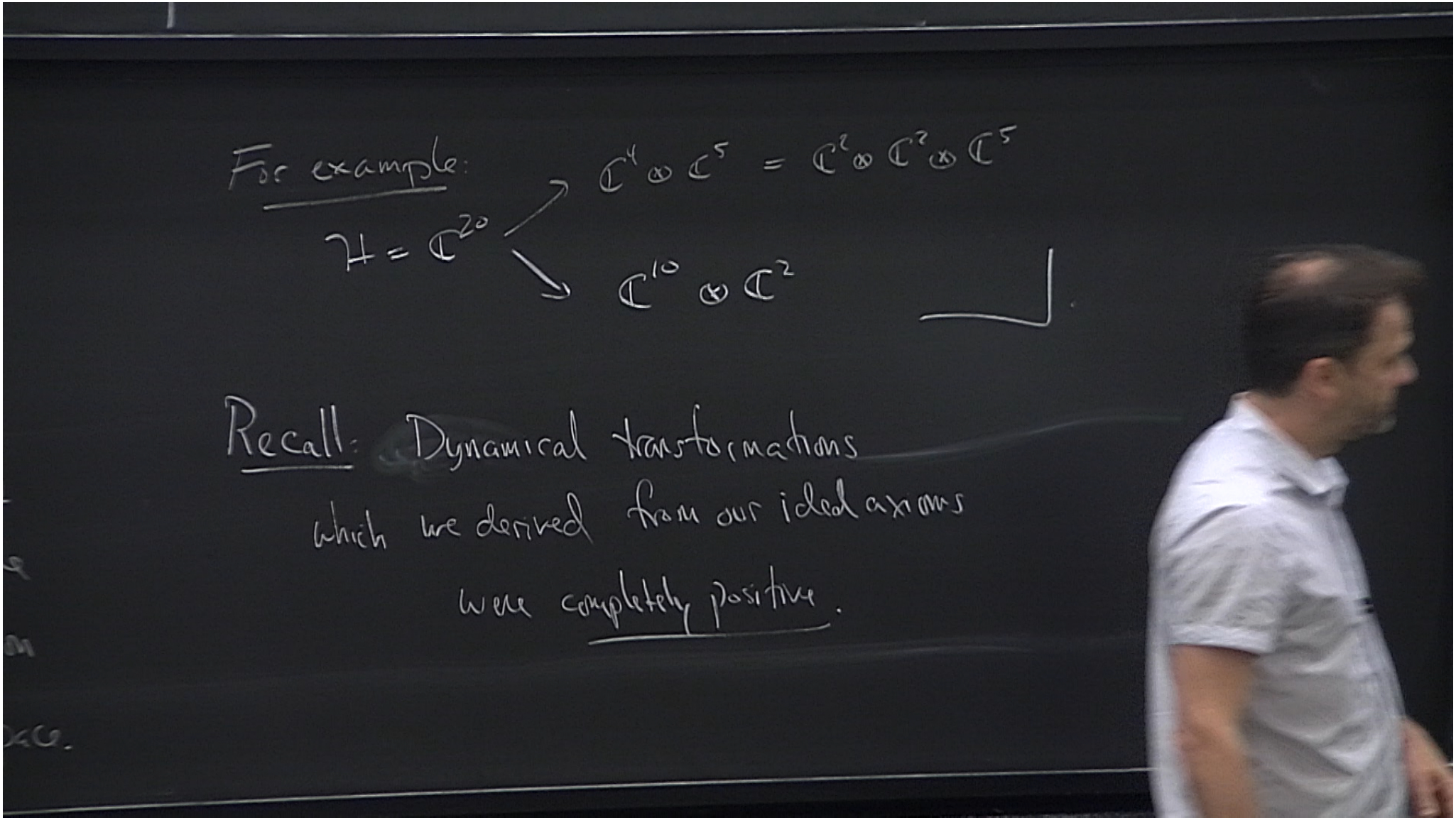

For example:
$\mathcal{H} = \mathbb{C}^{20}$
$\mathbb{C}^4 \otimes \mathbb{C}^5 = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^5$
$\mathbb{C}^{10} \otimes \mathbb{C}^2$

For example:

$\mathcal{H} = \mathbb{C}^{20}$ → $\mathbb{C}^4 \otimes \mathbb{C}^5 = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^5$

$\mathcal{H} = \mathbb{C}^{20}$ → $\mathbb{C}^{10} \otimes \mathbb{C}^2$

Recall: Dynamical transformations which we derived from our ideal axioms


For example:
$\mathcal{H} = \mathbb{C}^{20}$
$\mathbb{C}^4 \otimes \mathbb{C}^5 = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^5$
$\mathbb{C}^{10} \otimes \mathbb{C}^2$
Recall: Dynamical transformations
which we derived from our ideal axioms
were completely positive.

- If $[\hat{\sigma}]$ is positive then $\hat{\rho}$ is PPT

$\langle\psi|\hat{\sigma}|\psi\rangle \geq 0 \quad \forall |\psi\rangle \in \mathcal{H}$

$\hat{\sigma} = |\sigma\rangle\langle\sigma| \in \mathcal{D}(\mathbb{C}^2)$

$\langle\sigma|\sigma\rangle = 0.$

- PPT is a necessary condition for separability

$\Rightarrow$ Separability implies PPT.

For example:

$\mathcal{H} = \mathbb{C}^{20} \rightarrow \mathbb{C}^4 \otimes \mathbb{C}^5 = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^5$

$\mathcal{H} = \mathbb{C}^{20} \rightarrow \mathbb{C}^{10} \otimes \mathbb{C}^2$



Recall: Dynamical transformations which we derived from our ideal axioms were completely positive.

Complete-positivity: requirement that, for $(\Lambda \otimes \mathbb{I})\rho = \sigma$, $\sigma$ is positive $\forall\ \rho$.

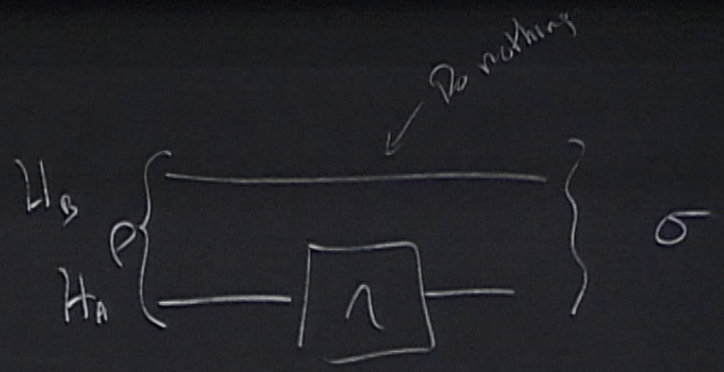
$\Lambda$ is CPTP on $\mathcal{H}_A$,

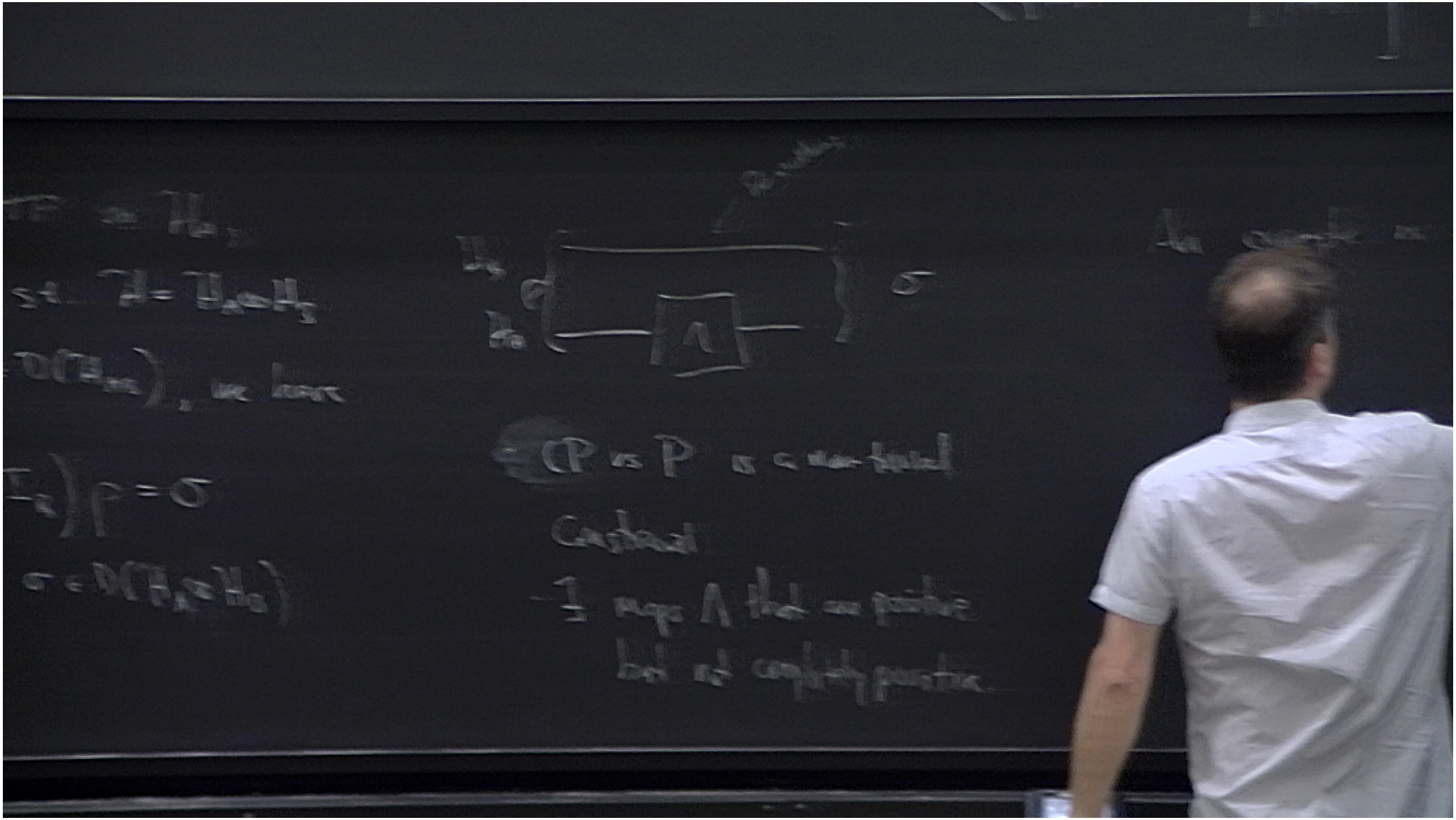
$\forall$ $\mathcal{H}_B$, s.t. $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$

$\forall$ $\rho \in D(\mathcal{H}_{AB})$, we have

$$\left(\Lambda_A \otimes I_B\right)\rho = \sigma$$

s.t $\sigma \in D(\mathcal{H}_A \otimes \mathcal{H}_D)$





$\left[\rho_{MM}\right]^T$ ]

An example is the Transpose map!

$T : D(\mathcal{H}_A) \longrightarrow D(\mathcal{H}_A)$

Given any $\tau \in D(\mathcal{H}_A)$

$T\tau = \sigma_A \in D(\mathcal{H}_A)$

$\left[\rho_{MN}\right]^T \cdots \left[\rho_{MN}\right]^T$

An example is the Transpose map!

$T : D(\mathcal{H}_A) \to D(\mathcal{H}_A)$

Given any $\tau \in D(\mathcal{H}_A)$

$T\tau = \sigma_A \in D(\mathcal{H}_A)$

But $T$ is not CP because

$\exists\ \rho \in D(\mathcal{H}_A \otimes \mathcal{H}_B)$ s.t

$(T \otimes I)\rho = \sigma \notin D(\mathcal{H}_A \otimes \mathcal{H}_B)$

$\Rightarrow \rho$ must be entangled

if $\sigma \notin D(\mathcal{H}_A \otimes \mathcal{H}_B)$.

Proof:

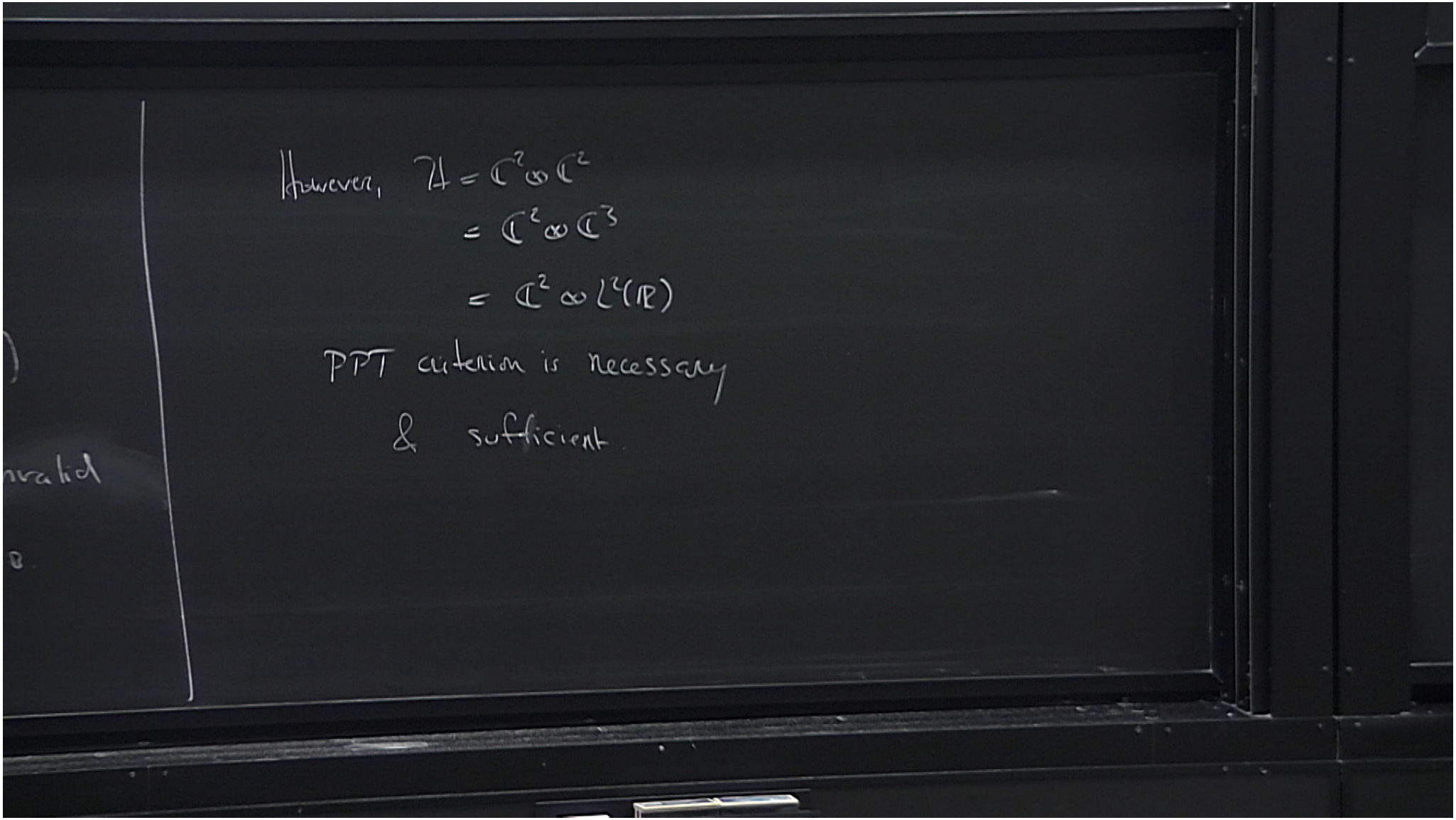
$$T \otimes I \left( \sum_i p_i \, \hat{\sigma}_{A,i} \otimes \hat{\sigma}_{B,i} \right)$$

$$= \sum_i p_i \left( \left( T \hat{\sigma}_{A,i} \right) \otimes \left( I \hat{\sigma}_{B,i} \right) \right)$$

$$= \sum_i p_i \, \hat{\tau}_{A,i} \otimes \hat{\sigma}_{B,i} \in D(H_A \otimes H_B)$$

where $\tau_{A,i} \in D(H_A)$

- PPT is not a complete test because there exists non-separable (entangled) input states to $T \otimes I$ that do not produce invalid q states in $H_A \otimes H_B$


However, $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2$
$= \mathbb{C}^2 \otimes \mathbb{C}^3$
$= \mathbb{C}^2 \otimes L^2(\mathbb{R})$
PPT criterion is necessary
& sufficient

**Proof:** only entangled states can be NPT

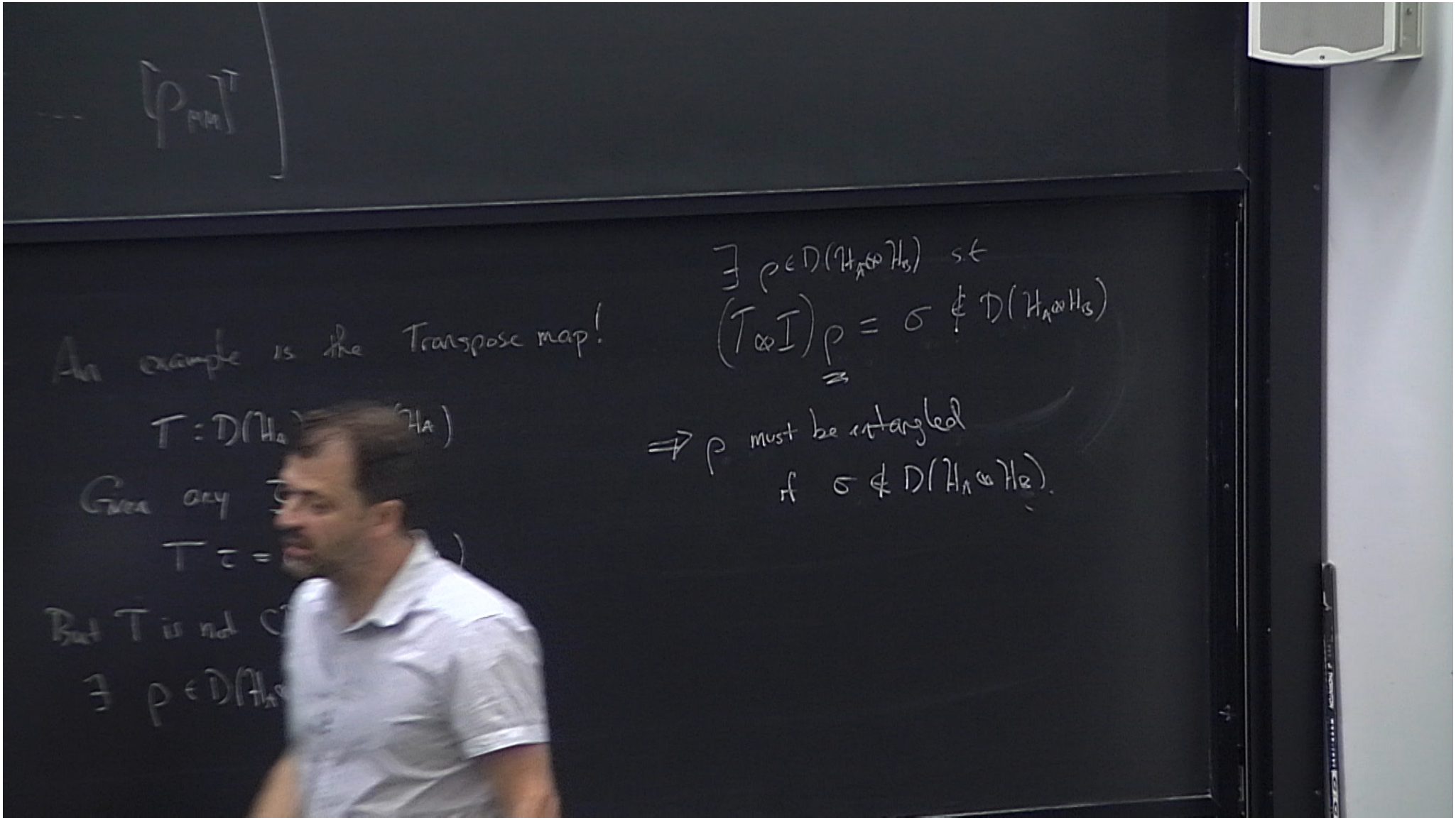
$$T \otimes I \left( \sum_i p_i \, \hat{\sigma}_{A,i} \otimes \hat{\sigma}_{B,i} \right)$$

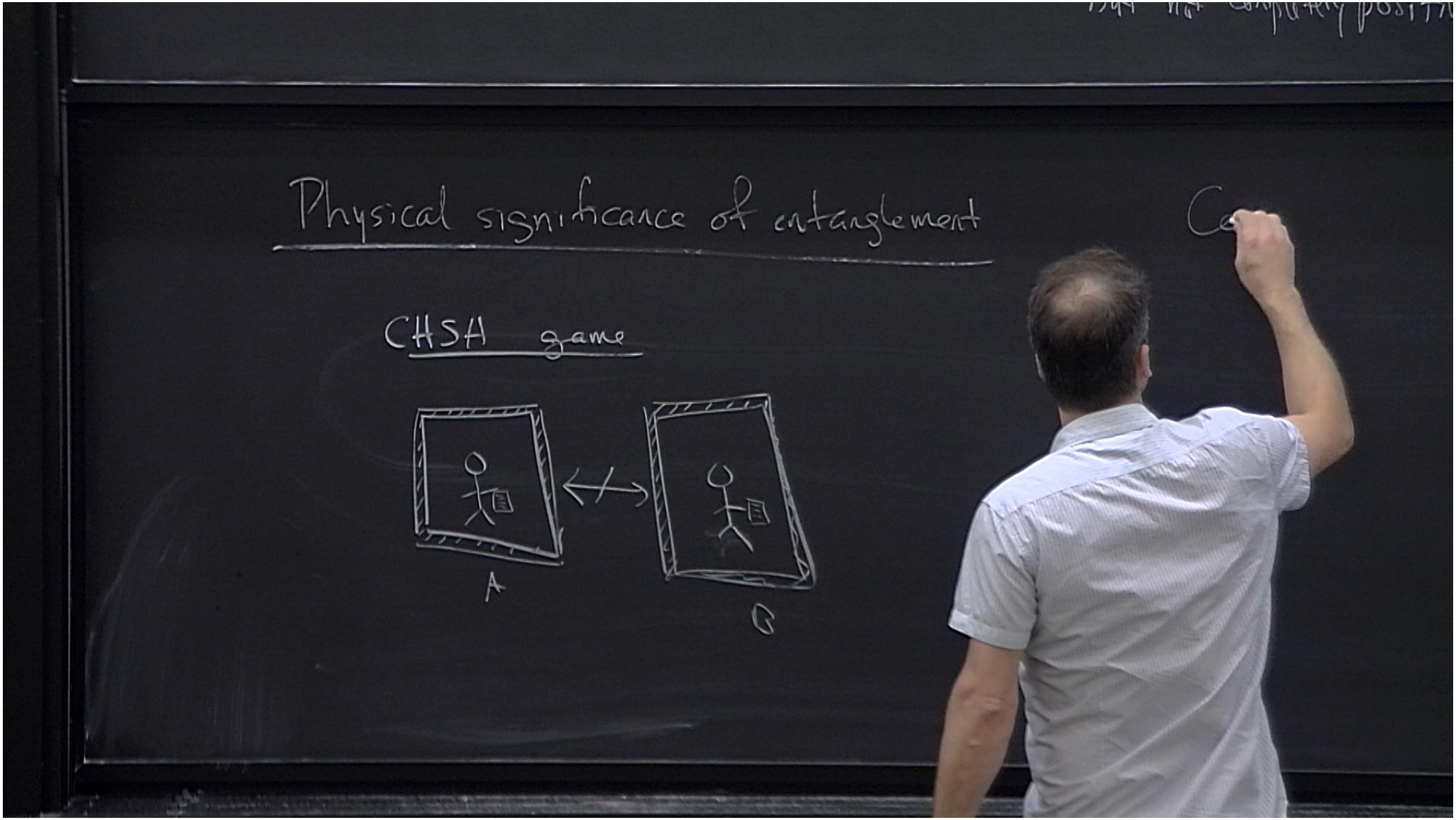
$$= \sum_i p_i \left( (T \hat{\sigma}_{A,i}) \otimes (I \hat{\sigma}_{B,i}) \right)$$

$$= \sum_i p_i \, \hat{\tau}_{A,i} \otimes \hat{\sigma}_{B,i} \in D(H_A \otimes H_B)$$

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- PPT is not a complete test because there exists non-separable (entangled) input states to $T \otimes I$ that do not produce invalid q. states on $H_A \otimes H_B$


An example is the Transpose map!
$\exists\ \rho \in D(\mathcal{H}_A \otimes \mathcal{H}_B)$ s.t
$(T \otimes I)\rho = \sigma \notin D(\mathcal{H}_A \otimes \mathcal{H}_B)$
$\Rightarrow \rho$ must be entangled
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Physical significance of entanglement
CHSH game
A
G

completely positive.

...anglement

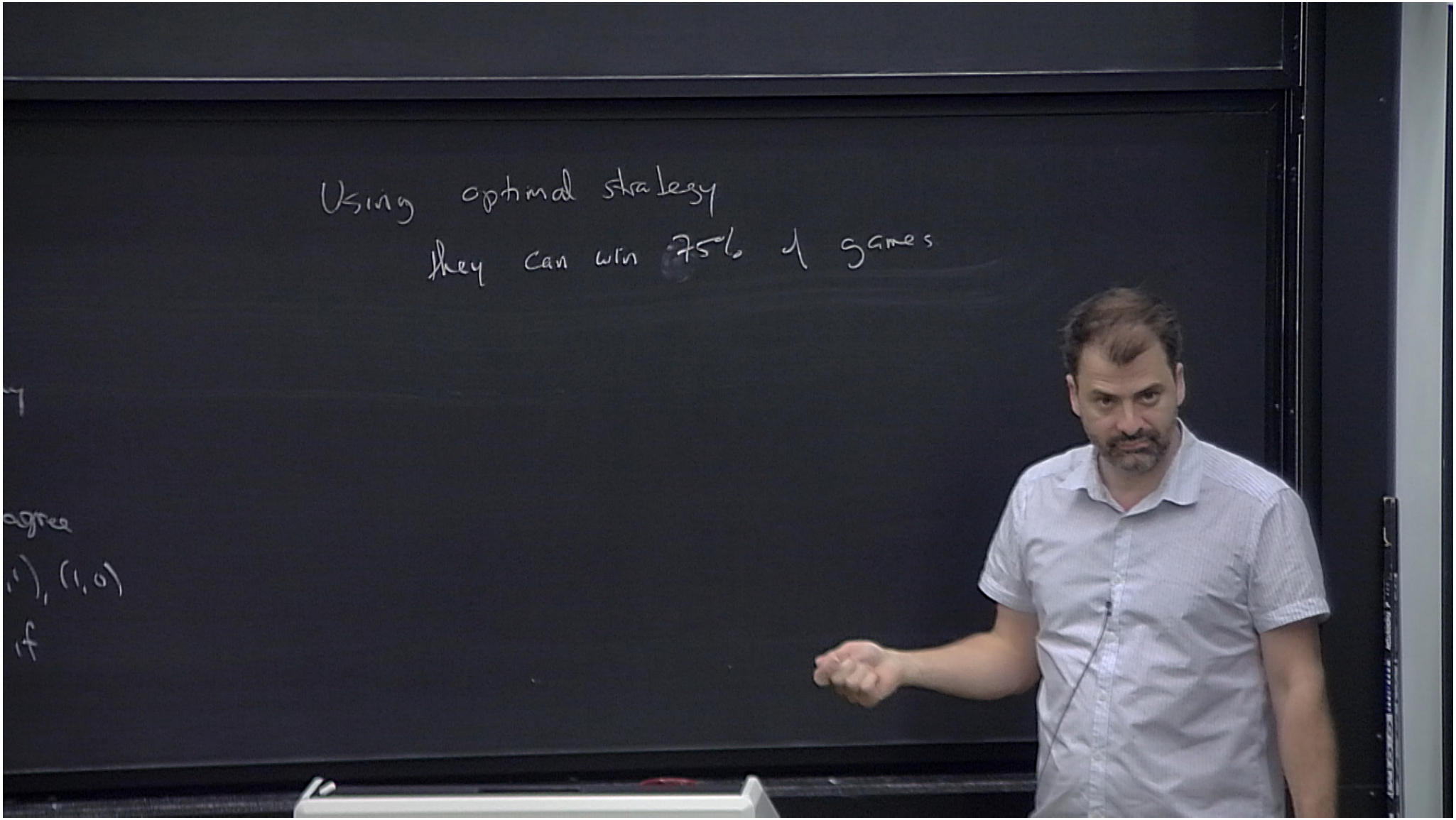
## Correlations with no causal explanation

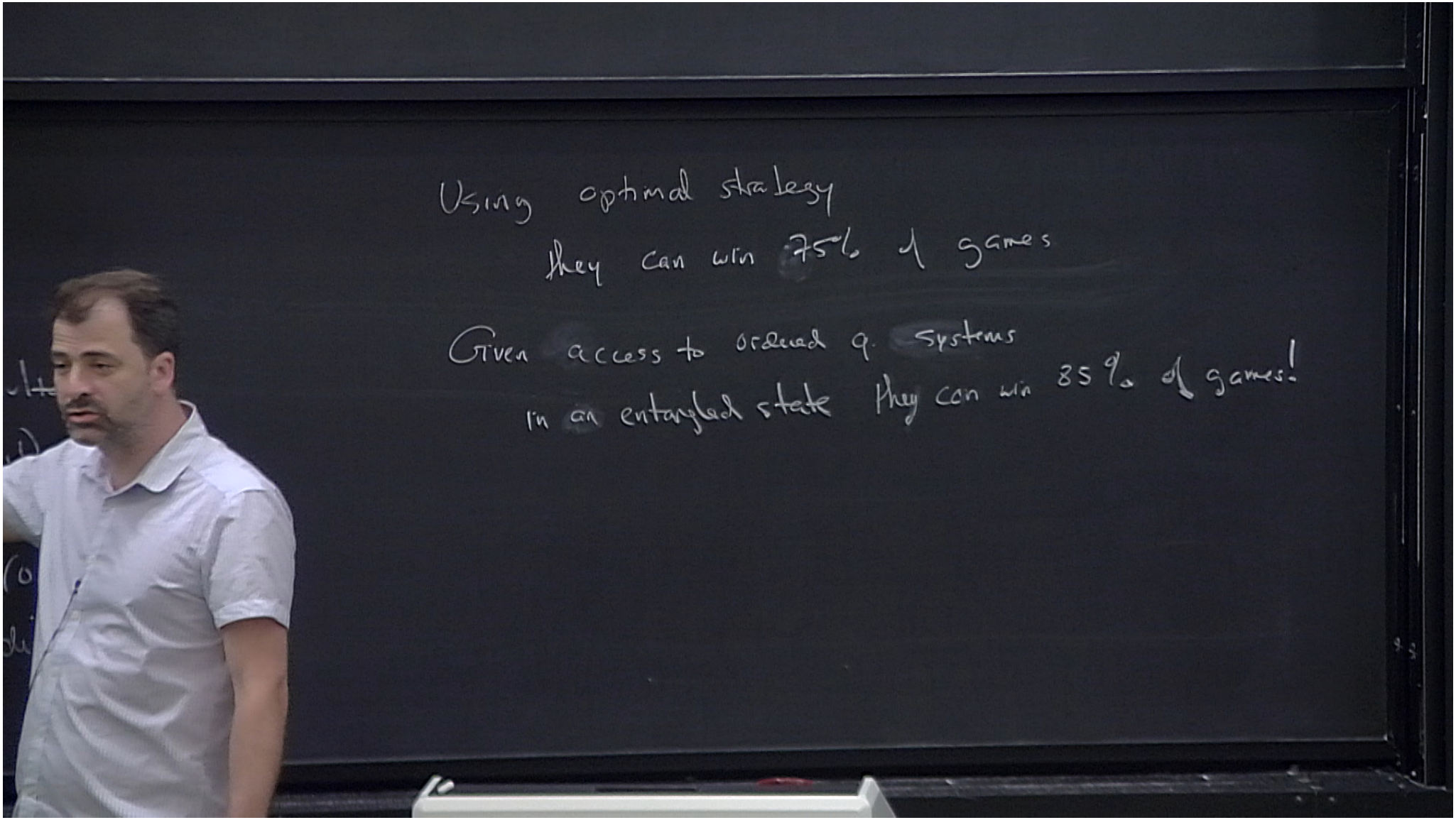
Asked question pairs simultaneously

$(0,0), (0,1), (1,0), (1,1)$

Goal: Answers (yes/no) should agree for question pairs $(0,0), (0,1), (1,0)$

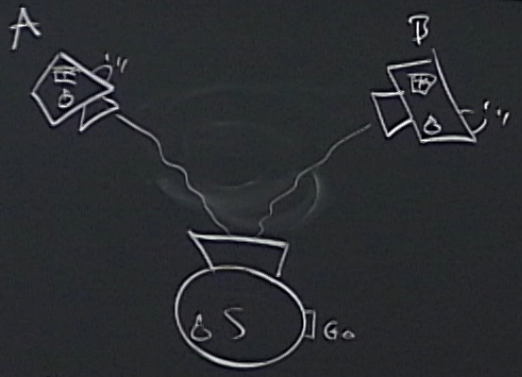
BUT answers should disagree if asked $(1,1)$


Using optimal strategy
they can win 75% of games


Using optimal strategy
they can win 75% of games
Given access to ordered q. systems
in an entangled state they can win 85% of games!


$\rho \in D(H_{AB})$, we have
$(\Lambda_A \otimes I_B)\rho = \sigma$
s.t. $\sigma \in D(H_A \otimes H_B)$
CP vs P is a non-trivial
Constraint
$\exists$ maps $\Lambda$ that are positive
but not completely positive
Given any $\tau \in D(H_A)$
$T\tau = \sigma_A \in D(H_A)$
But T is not CP because
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Physical significance of entanglement
CHSH game
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# Experimental Set-up




Input
$\{0,1\}$
↓
$\{\hat{A}_0, \hat{A}_1\}$

Measurement operators

Output
$x(\hat{A}_k) \in [-1,1]$
or $\in \{-1,1\}$

Input
$\{0,1\}$
↓
$\{\hat{B}_1, \hat{B}_2\}$

Output
$y(\hat{B}_k) \in [-1,1]$
OR $\in \{-1,1\}$

Success criterion over runs $n \in \{1, \ldots$
$= \#$ (

Success criterion over n

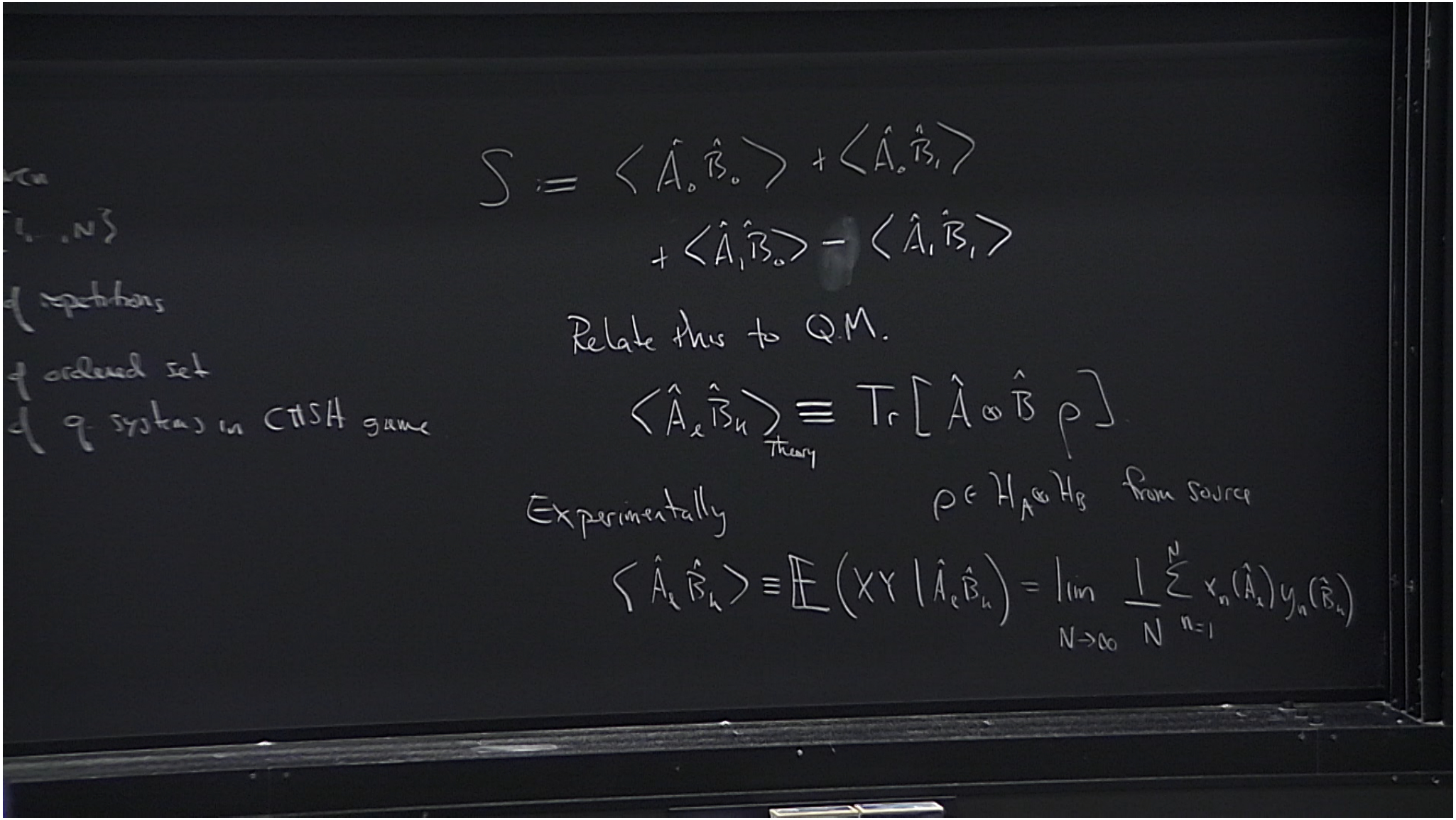
many runs $n \in \{1, \ldots, N\}$

$=$ # of repetitions

$=$ # of ordered set of $q$ systems in CHSH game

$\in [-1, 1]$

$\in \{-1, 1\}$


$S := \langle \hat{A}_0 \hat{B}_0 \rangle + \langle \hat{A}_0 \hat{B}_1 \rangle + \langle \hat{A}_1 \hat{B}_0 \rangle - \langle \hat{A}_1 \hat{B}_1 \rangle$
Relate this to Q.M.
$\langle \hat{A}_x \hat{B}_y \rangle_{\text{theory}} \equiv \text{Tr}[\hat{A} \otimes \hat{B}\, \rho]$
Experimentally
$\rho \in \mathcal{H}_A \otimes \mathcal{H}_B$ from source
$\langle \hat{A}_x \hat{B}_y \rangle \equiv \mathbb{E}(XY \mid \hat{A}_x \hat{B}_y) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} x_n(\hat{A}_x)\, y_n(\hat{B}_y)$

$$\overline{\langle \hat{A}_e \hat{B}_h \rangle}^{Exp} = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} X_n(\hat{A}_e)\, y_n(\hat{B}_h)$$

Expt

$$\langle \hat{A}_e \hat{B}_h \rangle_{expt} = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} X_n(\hat{A}_e)\, y_n(\hat{B}_h)$$

---

Bell (1964) → CHSH (1972?)

CHSH inequality

$$-2 \leq S \leq 2$$

based on physically plausible assumptions.

Exp$^{\underline{s}}$

$$\overline{\langle \hat{A}_e \hat{B}_h \rangle}_{exp^t} = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} X_n(\hat{A}_e)\, y_n(\hat{B}_h)$$

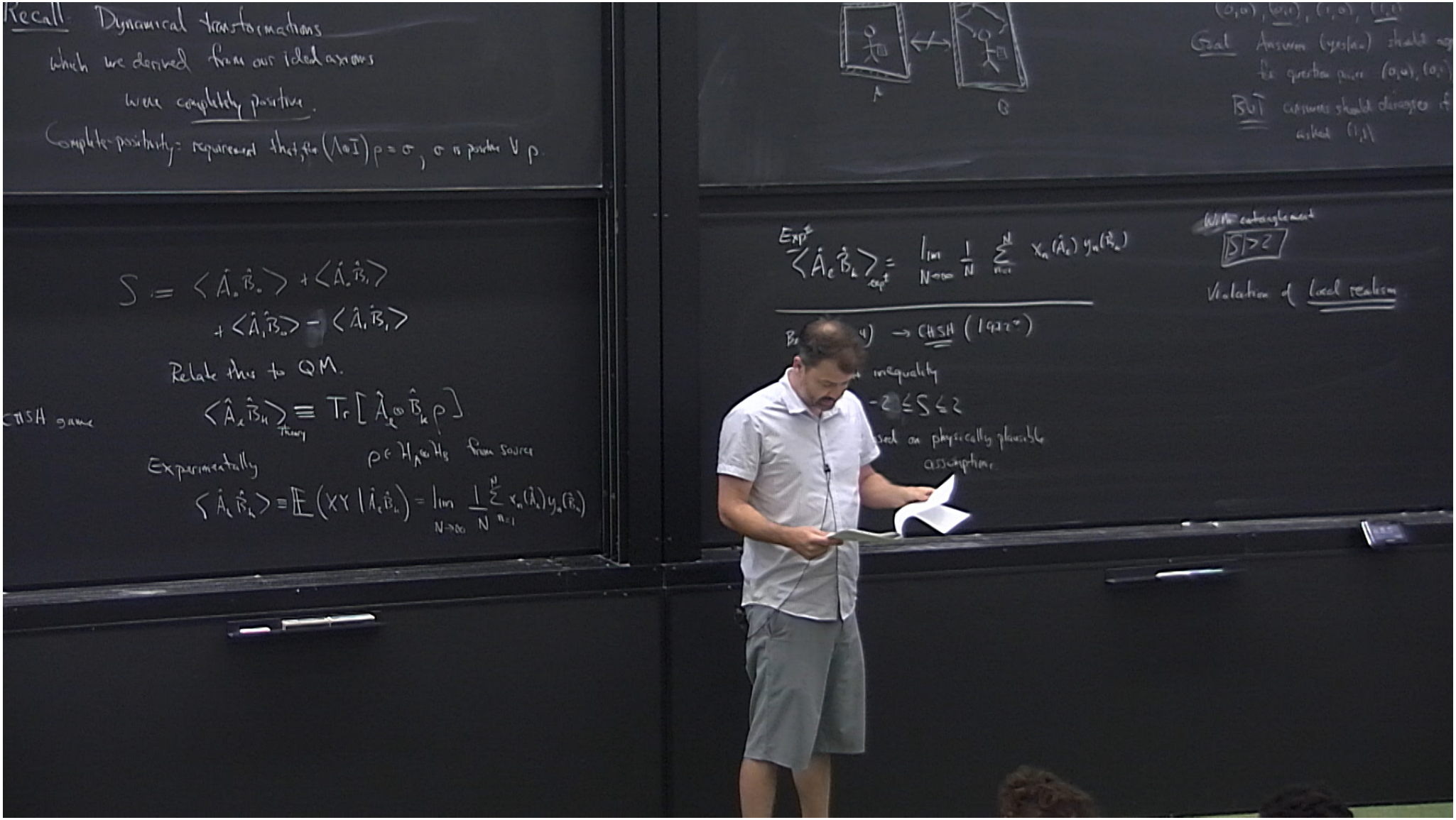
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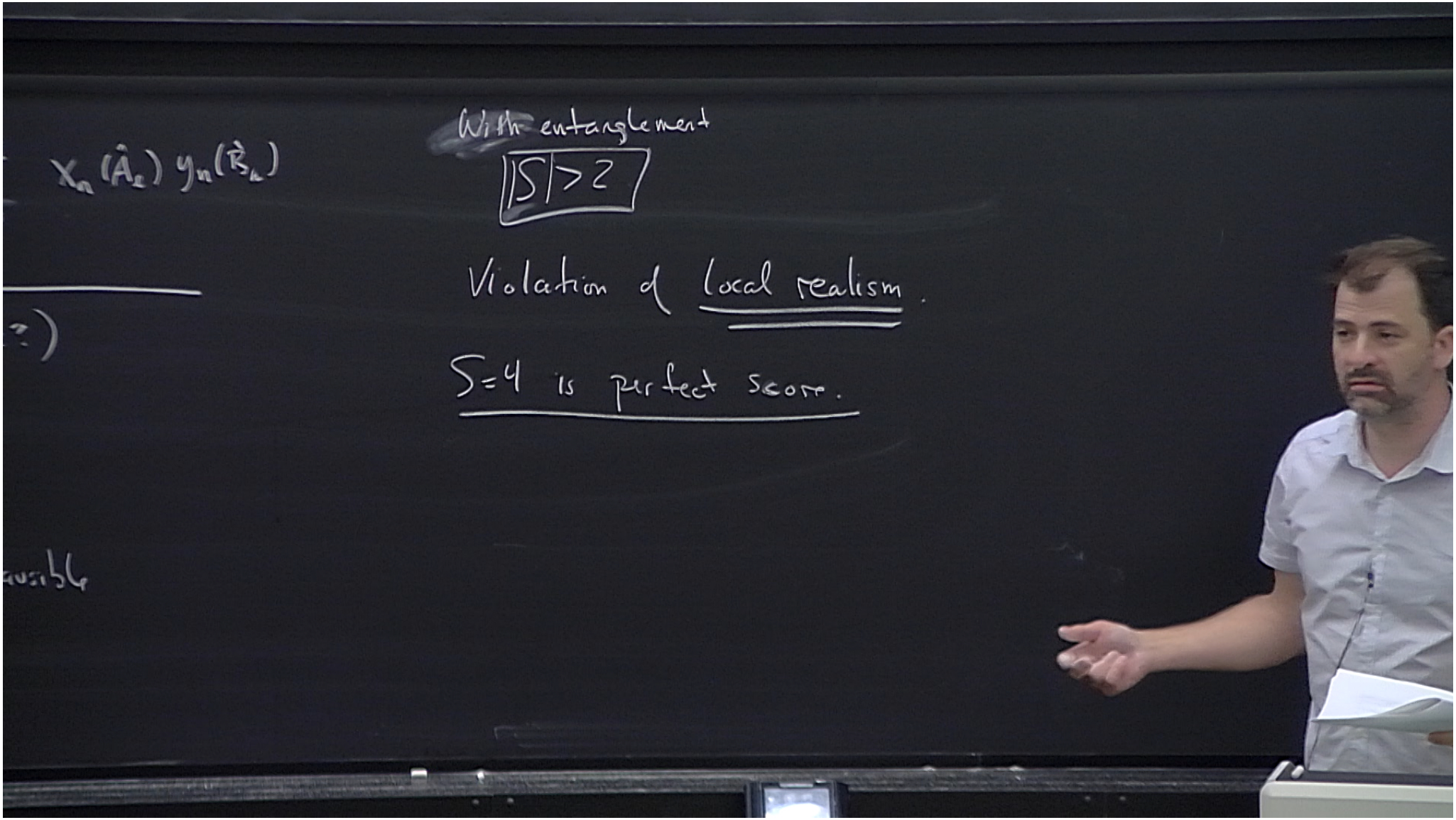
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Relate this to QM.
CHSH game
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A
B
Goal
But
asked (1,1)
Exp'
$\langle \hat{A}_x \hat{B}_y \rangle_{\text{exp}} = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} x_n(\hat{A}_x) y_n(\hat{B}_y)$
With entanglement
$|S| > 2$
Violation of Local realism
CHSH (1972)
inequality
$-2 \leq S \leq 2$
based on physically plausible
assumptions


$X_n(\hat{A}_\perp)\, y_n(\hat{B}_\perp)$
With entanglement
$|S| > 2$
Violation of local realism.
$S = 4$ is perfect score.