Title: PSI 17/18 - Quantum Theory - Lecture 12

Date: Sep 20, 2017 10:45 AM

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Abstract:

Today: Counter-example
to generalized
postulate 3: CP maps
(but not to ideal
postulate 3: Unitary maps)
& theory of entanglement

$|\psi\rangle \in \mathcal{H}_A, \ |\phi\rangle \in \mathcal{H}_B$

$|\psi(t=0)\rangle$

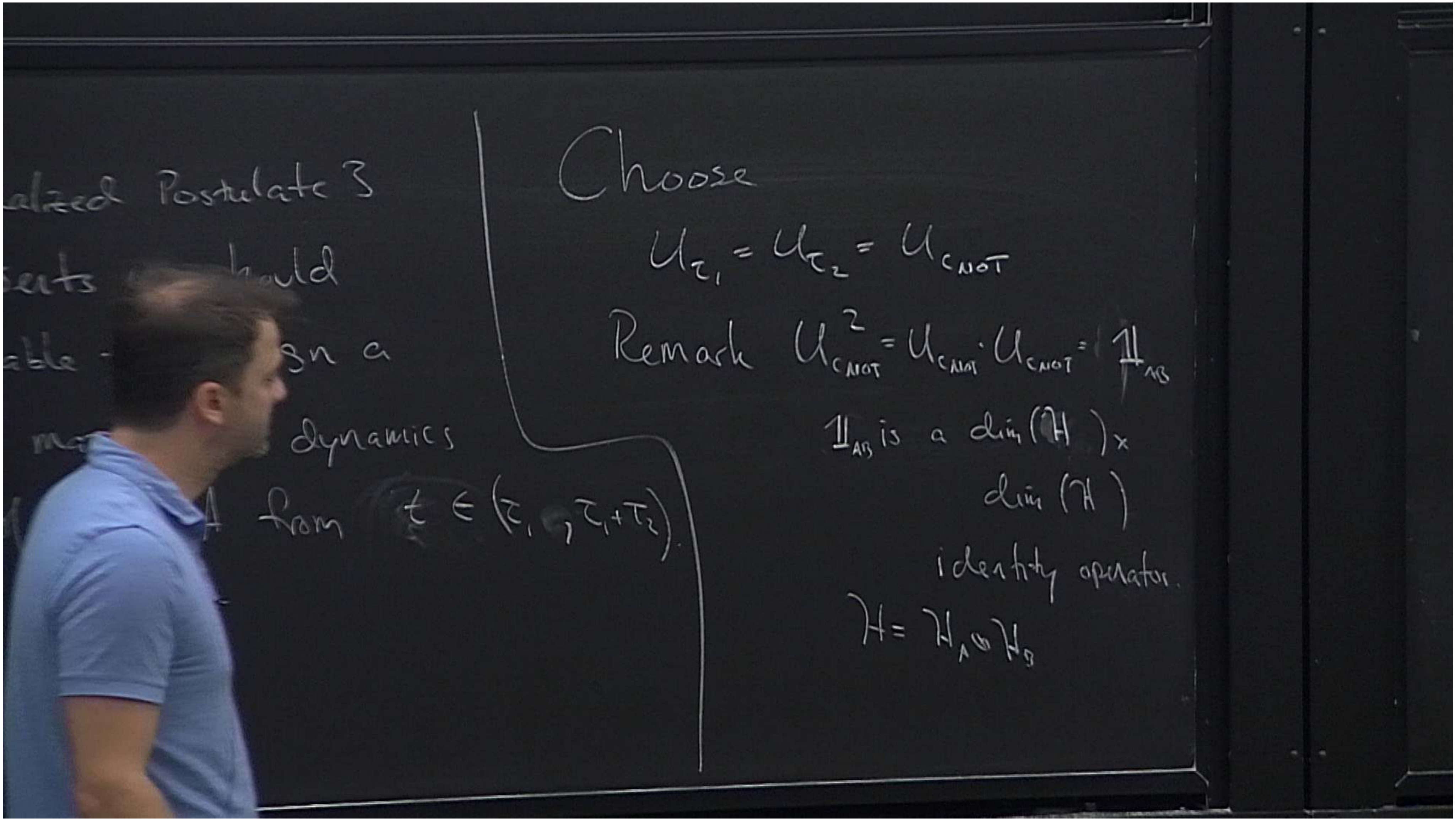
$|\phi(t=0)\rangle$

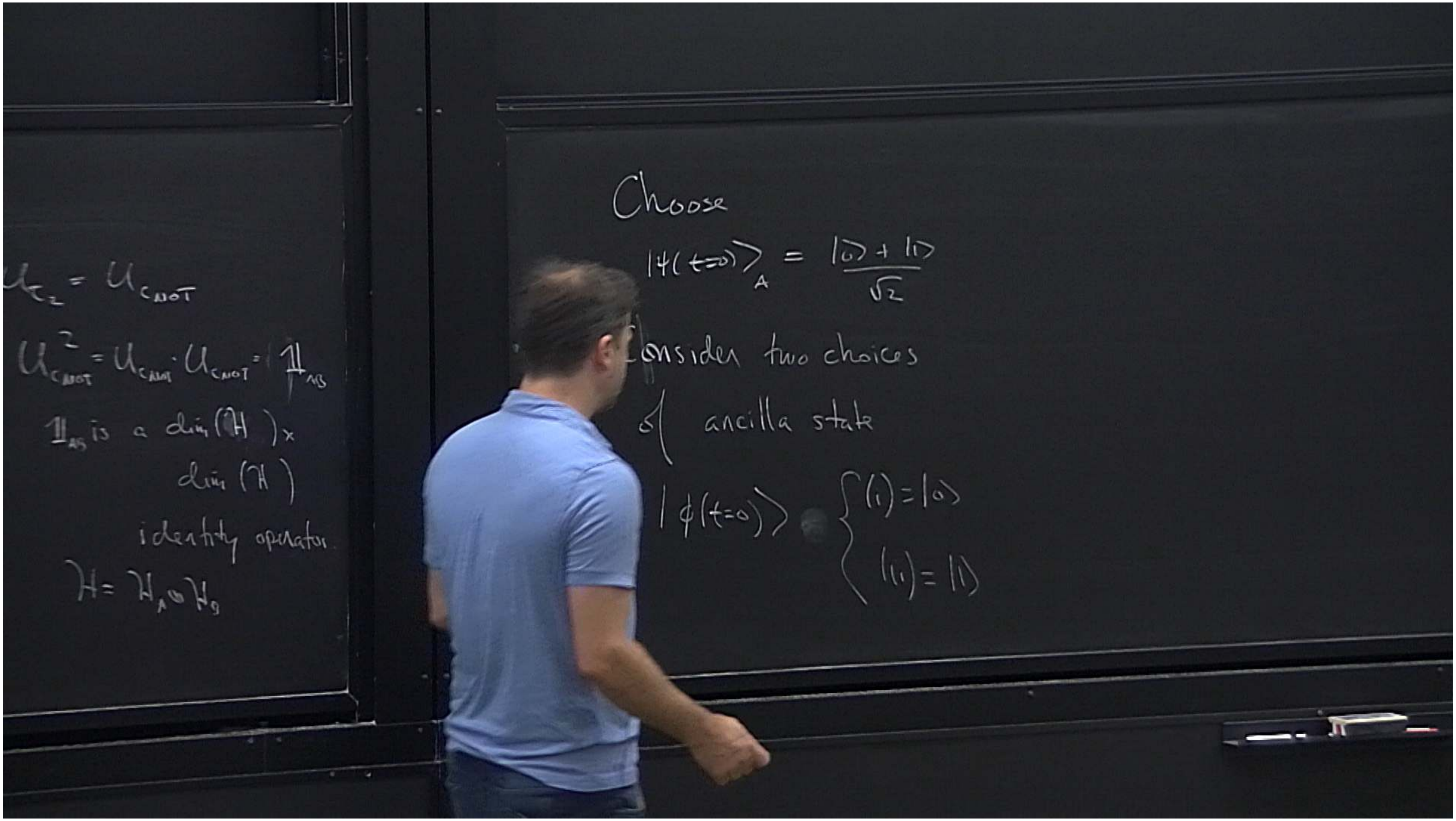
$$|\chi'\rangle = U_{\tau_2} U_{\tau_1} \left(|\psi(t=0)\rangle \otimes |\phi(t=0)\rangle \right)$$

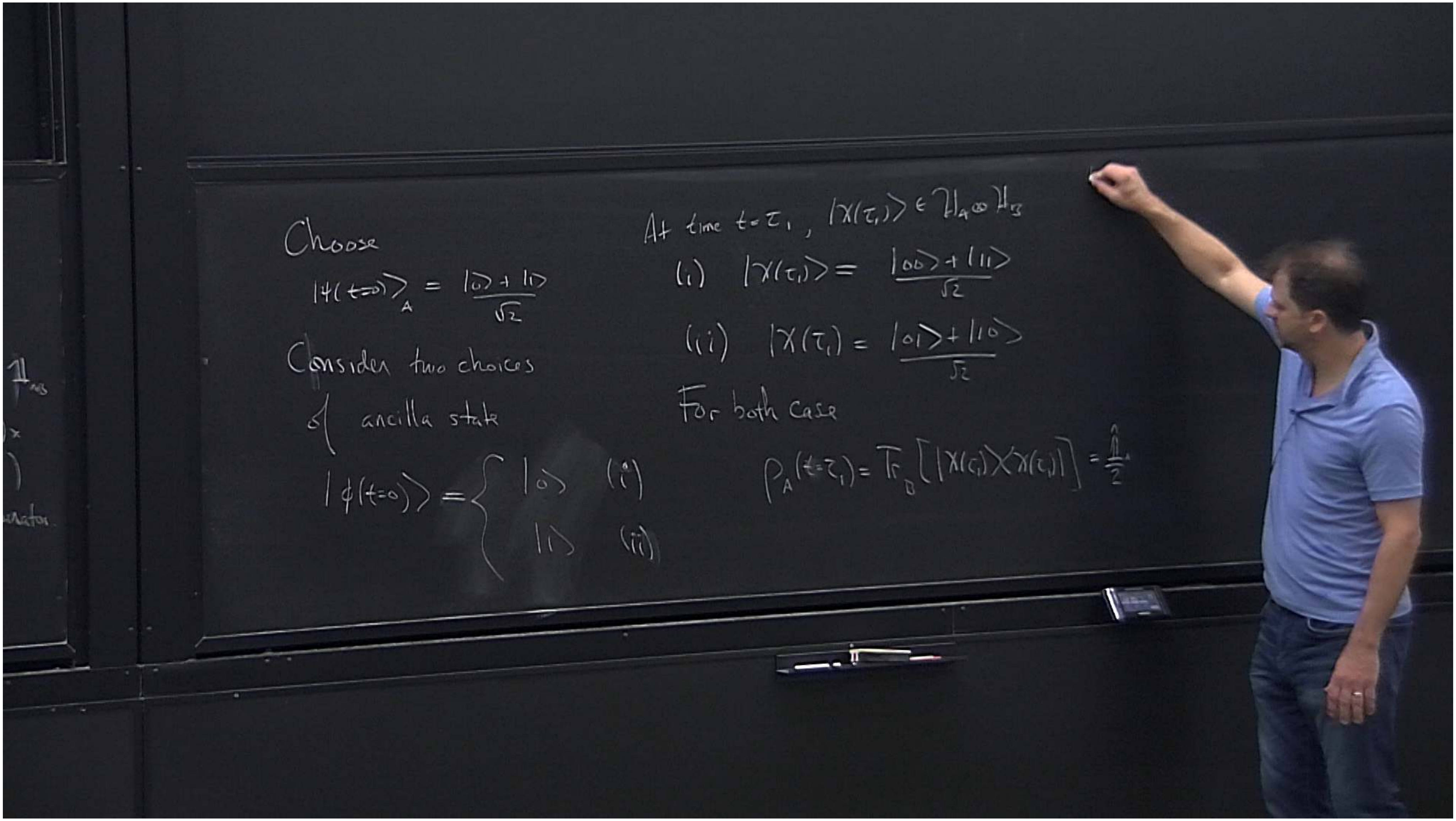
$|\chi'\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$

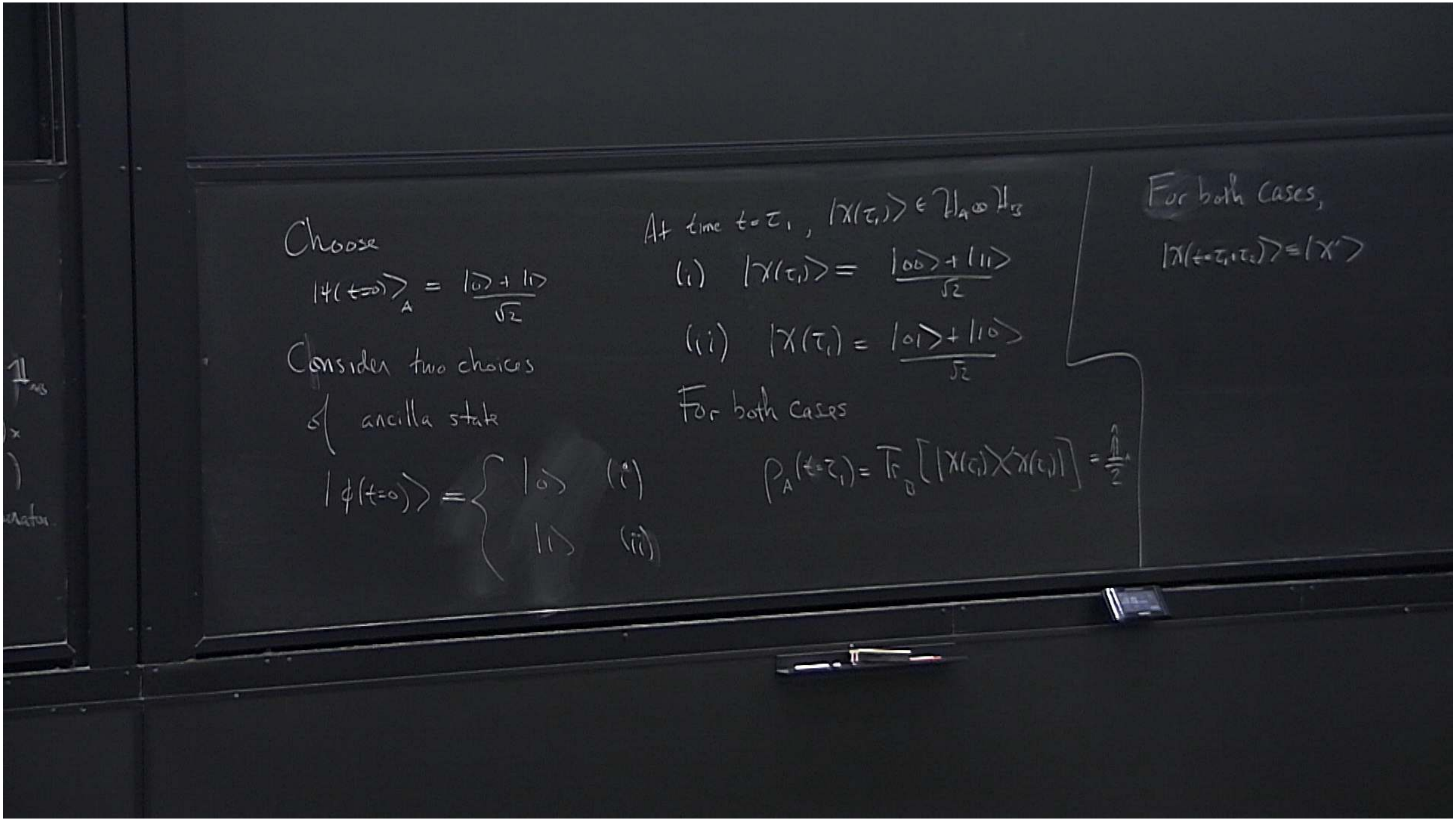
at time $t = \tau_1 + \tau_2$

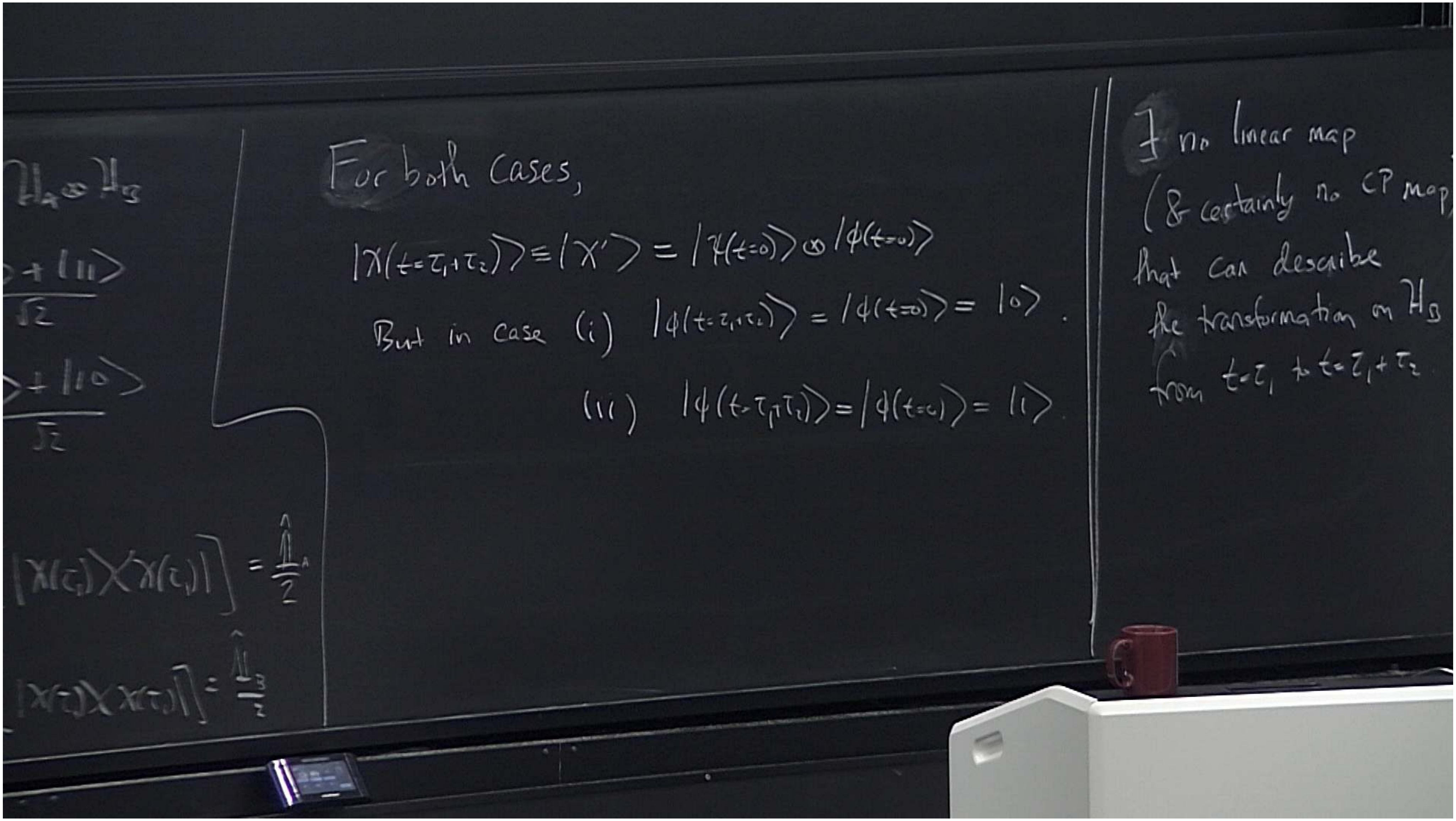
Generalized Postulate 3 asserts we should be able to assign a CP map for the dynamics of system A from $t \in (\tau_1, \tau_1 + \tau_2)$

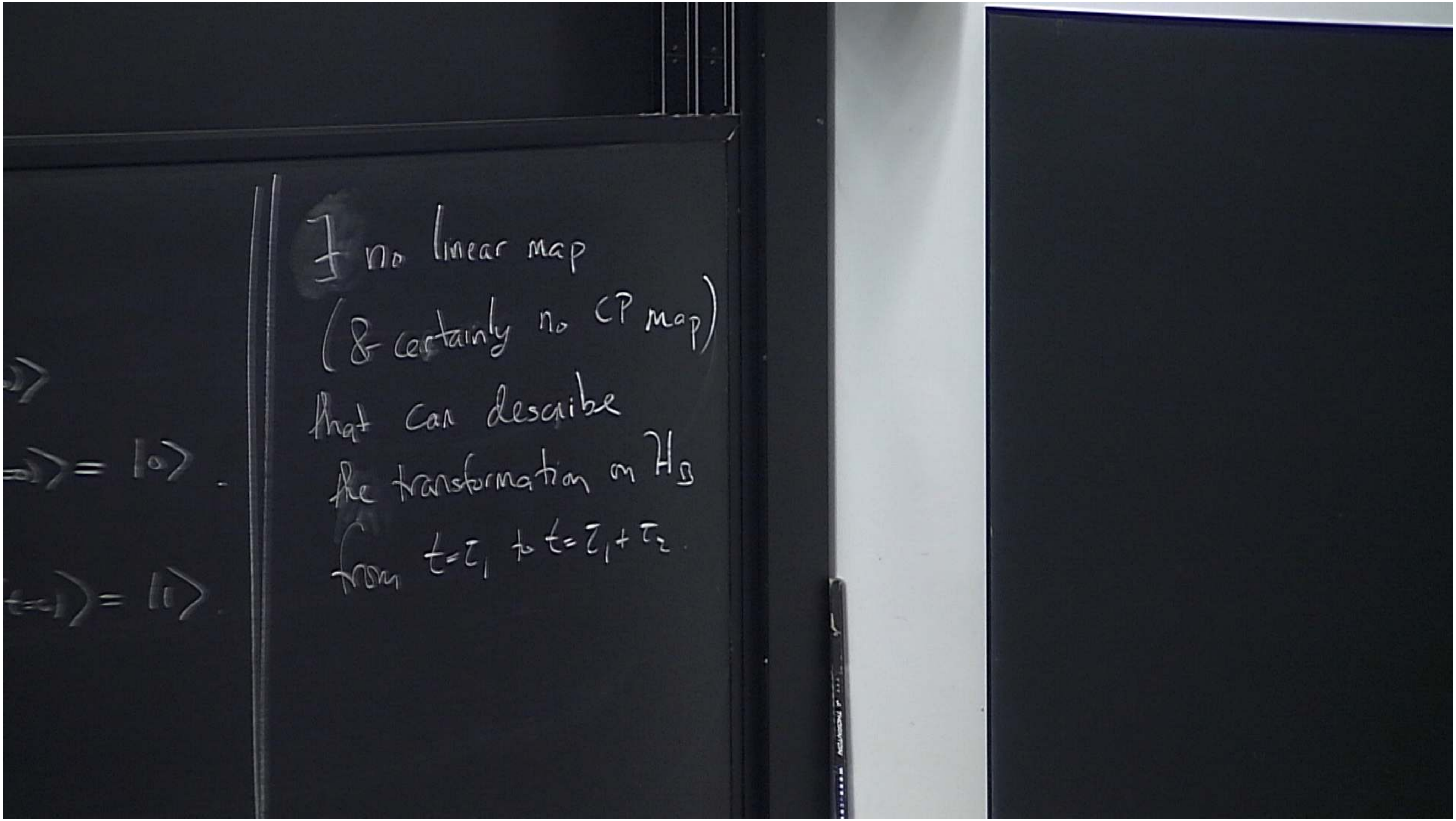

alized Postulate 3
dynamics
from $t \in (\tau_1, \tau_1 + \tau_2)$
Choose
$U_{\tau_1} = U_{\tau_2} = U_{CNOT}$
Remark $U_{CNOT}^2 = U_{CNOT} \cdot U_{CNOT} = \mathbb{1}_{AB}$
$\mathbb{1}_{AB}$ is a $\dim(\mathcal{H}) \times \dim(\mathcal{H})$
identity operator.
$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$


U_CNOT
U²_CNOT = U_CNOT · U_CNOT = 𝟙_AB
𝟙_AB is a dim(H) x
dim(H)
identity operator
H = H_A ⊗ H_B
Choose
|ψ(t=0)⟩_A = (|0⟩ + |1⟩)/√2
Consider two choices
of ancilla state
|φ(t=0)⟩ (i) = |0⟩
(ii) = |1⟩


Choose
|ψ(t=0)>_A = (|0> + |1>)/√2
Consider two choices
of ancilla state
|φ(t=0)> = { |0> (i)
|1> (ii)
At time t=τ₁ , |χ(τ₁)> ∈ H_A ⊗ H_B
(i) |χ(τ₁)> = (|00> + |11>)/√2
(ii) |χ(τ₁) = (|01> + |10>)/√2
For both case
ρ_A(t=τ₁) = Tr_B [|χ(τ₁)><χ(τ₁)|] = 1̂/2


Choose
$|\psi(t=0)\rangle_A = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$
Consider two choices of ancilla state
$|\phi(t=0)\rangle = \begin{cases} |0\rangle & (i) \\ |1\rangle & (ii) \end{cases}$
At time $t=\tau_1$, $|\chi(\tau_1)\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$
(i) $|\chi(\tau_1)\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$
(ii) $|\chi(\tau_1)\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}}$
For both cases
$\rho_A(t=\tau_1) = \mathrm{Tr}_B\left[|\chi(\tau_1)\rangle\langle\chi(\tau_1)|\right] = \frac{\mathbb{1}_A}{2}$
For both cases,
$|\chi(t=\tau_1+\tau_2)\rangle \equiv |\chi'\rangle$


For both cases,
$|\chi(t=\tau_1+\tau_2)\rangle \equiv |\chi'\rangle = |\psi(t=0)\rangle \otimes |\phi(t=0)\rangle$
But in case (i) $|\phi(t=\tau_1+\tau_2)\rangle = |\phi(t=0)\rangle = |0\rangle$.
(ii) $|\phi(t=\tau_1+\tau_2)\rangle = |\phi(t=0)\rangle = |1\rangle$.
∄ no linear map
(& certainly no CP map)
that can describe
the transformation on $\mathcal{H}_B$
from $t=\tau_1$ to $t=\tau_1+\tau_2$


∃ no linear map
(& certainly no CP map)
that can describe
the transformation on $\mathcal{H}_B$
from $t=\tau_1$ to $t=\tau_1+\tau_2$.

In fact describing

the dynamics for <u>system B</u> alone

over time $t = \tau_1$ to $t = \tau_1 + \tau_2$

requires a one-to-many relation

$\in \mathcal{H}_B$

$\rangle\rangle \otimes |\phi(t=\infty)\rangle$

τ_2

Generalized Postulate 3 asserts we should be able to assign a CP map for the dynamics of system B from $t \in (\tau_1, \tau_1+\tau_2)$.

Choose

$U_{\tau_1} = U_{\tau_2} = U$

Remark $U_{CNOT}^2 = \mathbb{1}$

$\mathbb{1}_{AB}$ is

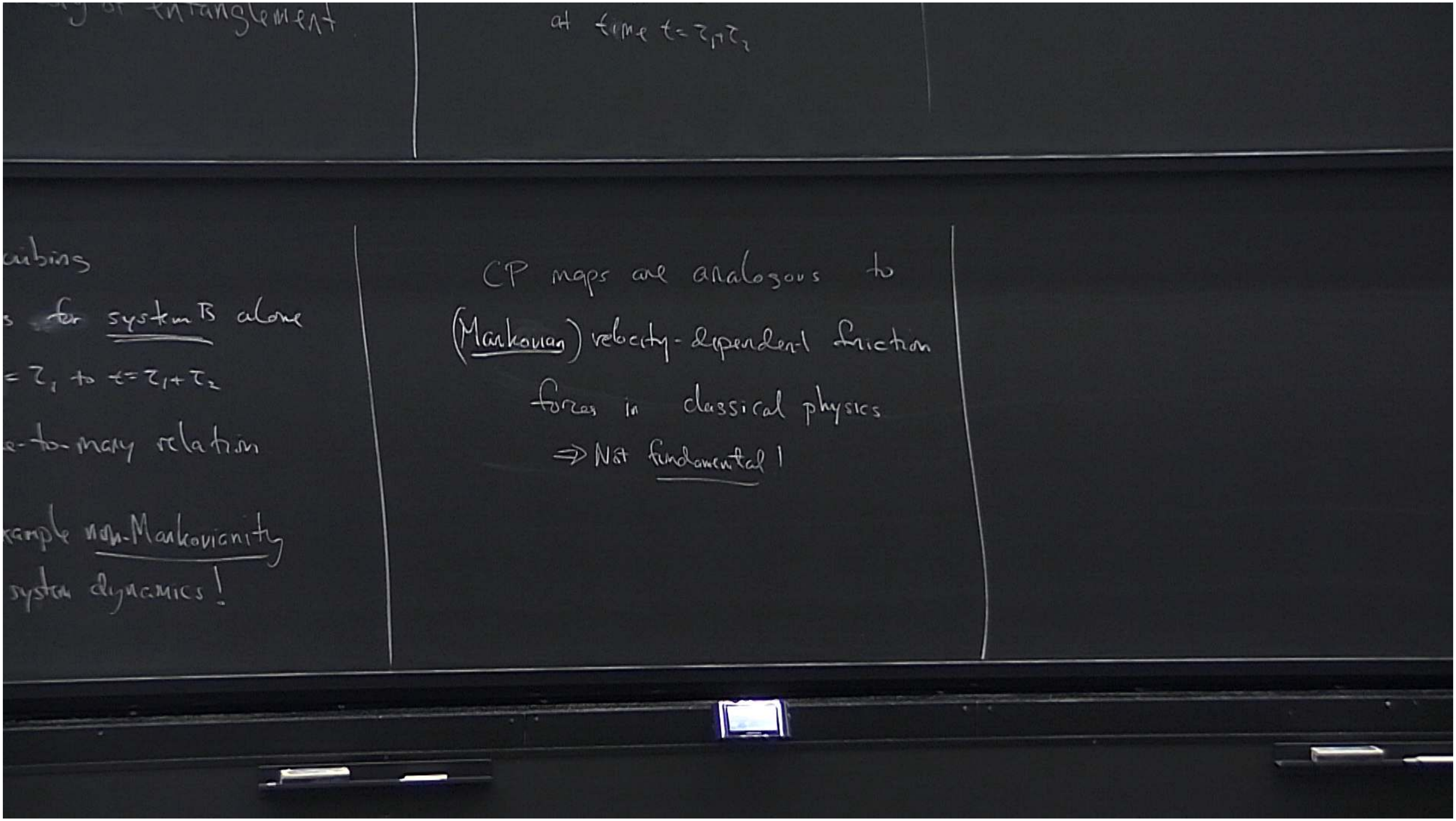
$\mathcal{H} =$

- In fact describing the dynamics for <u>system B</u> alone over time $t = \tau_1$ to $t = \tau_1 + \tau_2$ requires a one-to-many relation

- This is an example non-Markovianity in open system dynamics!

- In fact describing the dynamics for <u>system B</u> alone over time $t = \tau_1$ to $t = \tau_1 + \tau_2$ requires a one-to-many relation

- This is an example <u>non-Markovianity</u> in open system dynamics!


at time t=τ1+τ2
CP maps are analogous to
(Markovian) velocity-dependent friction
forces in classical physics
⇒ Not fundamental!
for system B alone
= τ1 to t=τ1+τ2
e-to-many relation
xample non-Markovianity
system dynamics!

$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$

State-update rule under POVM

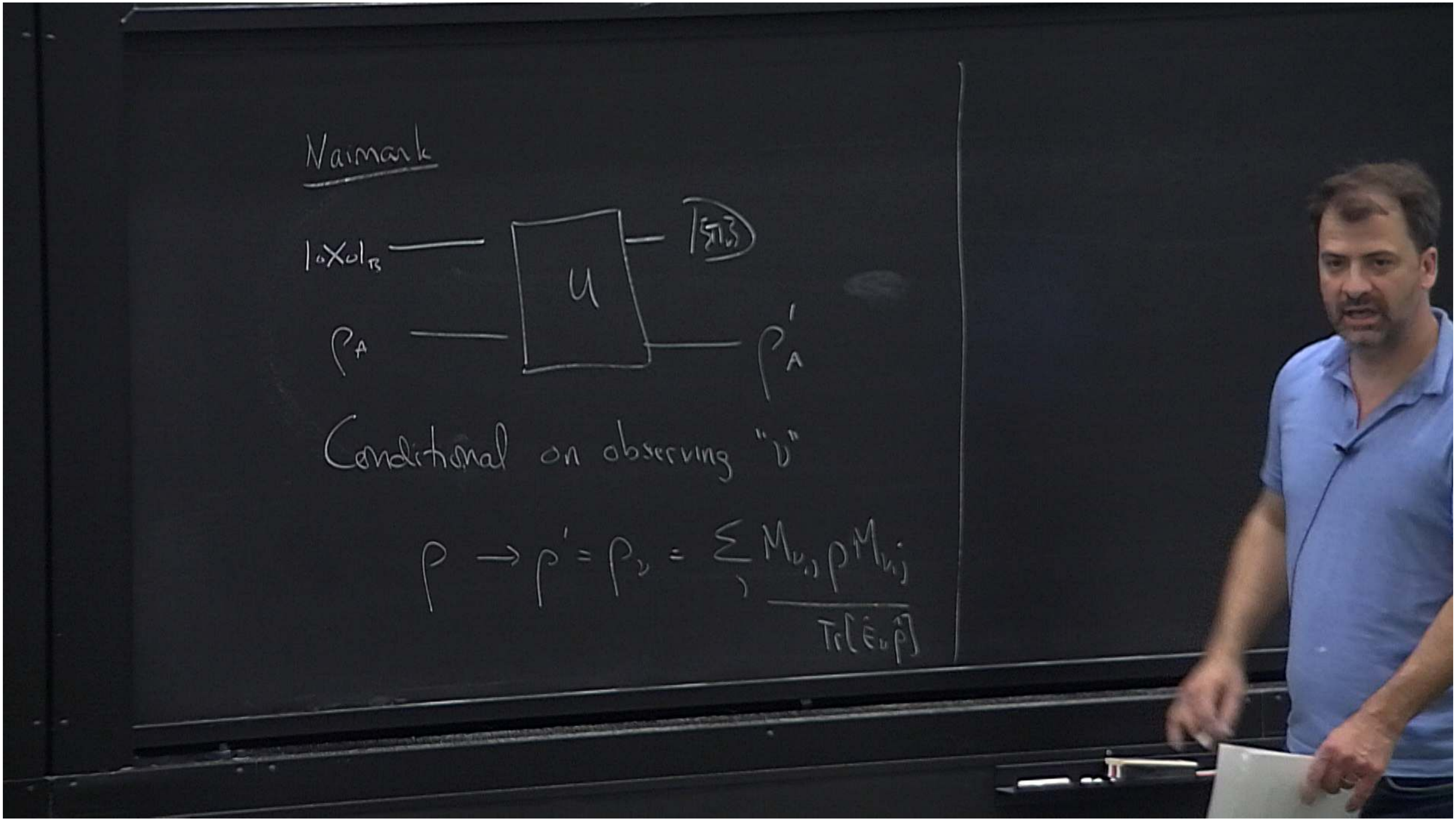
POVM $\{E_\nu\}$

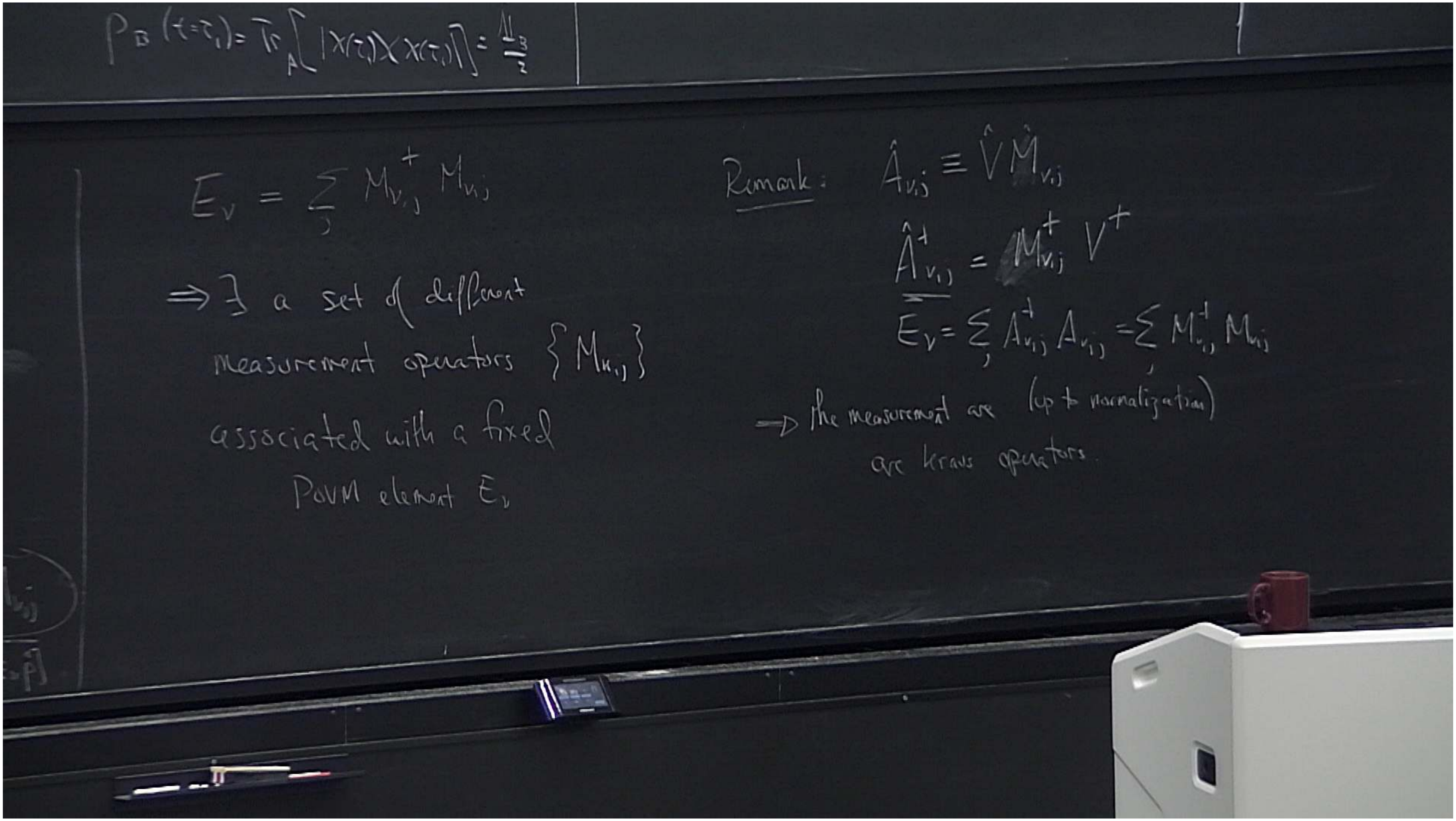
$Pr(\nu) = Tr(\hat{\rho}\hat{E}_\nu)$ given input state $\hat{\rho}$

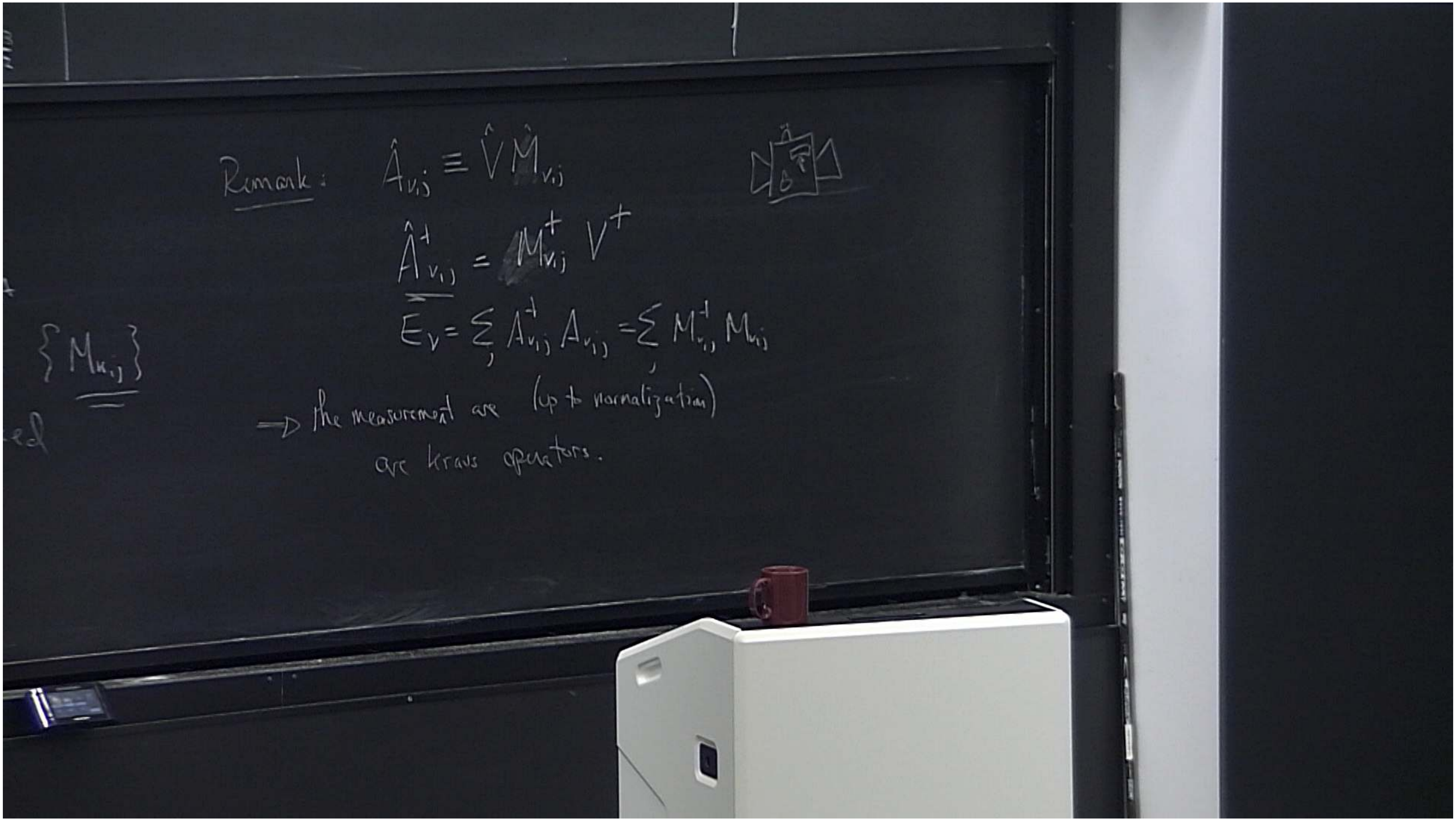
$\Rightarrow$ What is output state? It depends how you do the measurement.

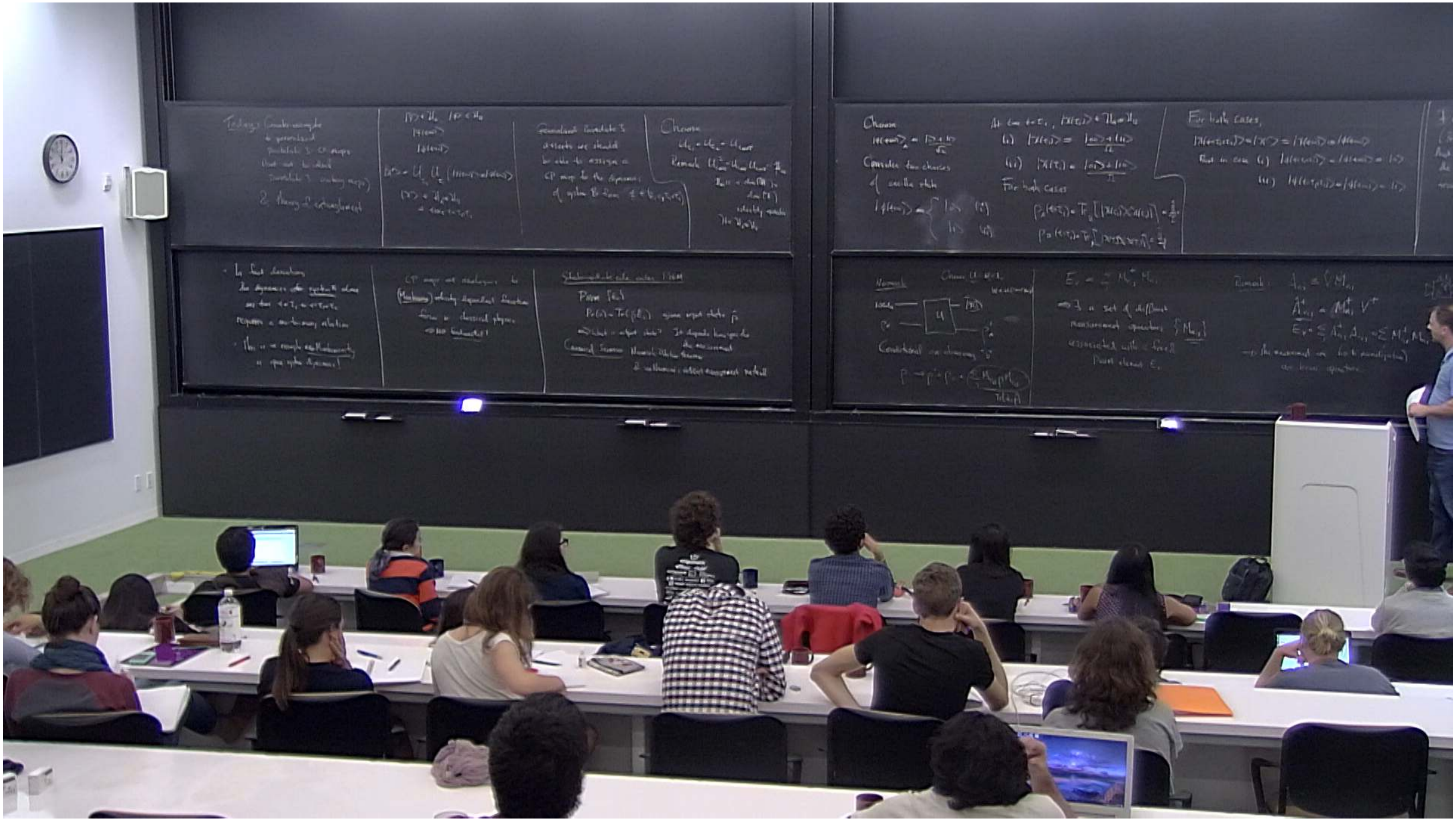
Canonical Scenario: Naimark Dilation theorem

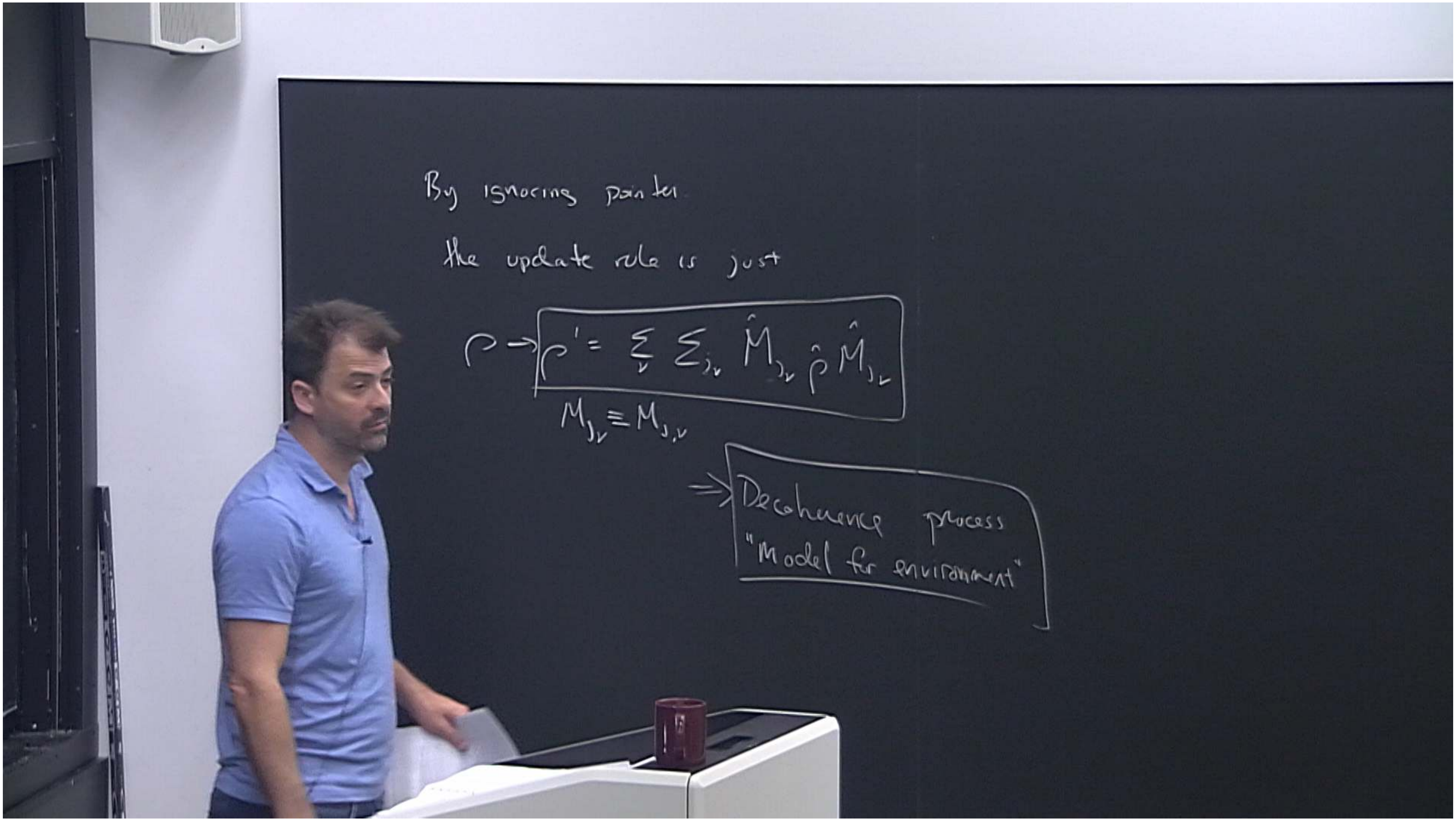
& von Neumann's indirect measurement method


Naimark
$|0\rangle\langle 0|_B$
U
$\{\Pi_\nu\}$
ρ_A
ρ'_A
Conditional on observing "ν"
$\rho \rightarrow \rho' = \rho_\nu = \frac{\sum_j M_{\nu,j} \rho M^\dagger_{\nu,j}}{Tr[\hat{E}_\nu \hat{\rho}]}$


$\rho_B(t=\tau_1) = Tr_A[\,|\chi(\tau_1)\rangle\langle\chi(\tau_1)|\,] = \frac{\mathbb{1}}{2}$
$E_\nu = \sum_j M^\dagger_{\nu,j} M_{\nu,j}$
$\Rightarrow \exists$ a set of different
measurement operators $\{M_{\nu,j}\}$
associated with a fixed
POVM element E_ν
Remark: $\hat{A}_{\nu,j} \equiv \hat{V}\hat{M}_{\nu,j}$
$\hat{A}^\dagger_{\nu,j} = M^\dagger_{\nu,j} V^\dagger$
$E_\nu = \sum_j A^\dagger_{\nu,j} A_{\nu,j} = \sum_j M^\dagger_{\nu,j} M_{\nu,j}$
$\rightarrow$ the measurement are (up to normalization)
are kraus operators.


Remark: $\hat{A}_{v,j} \equiv \hat{V}\hat{M}_{v,j}$
$\hat{A}^{\dagger}_{v,j} = M^{\dagger}_{v,j} V^{\dagger}$
$E_v = \sum_j A^{\dagger}_{v,j} A_{v,j} = \sum_j M^{\dagger}_{v,j} M_{v,j}$
$\{M_{v,j}\}$
→ the measurement are (up to normalization)
are kraus operators.




By ignoring pointer
the update rule is just
$\rho \to \rho' = \sum_{\nu} \sum_{j,\nu} \hat{M}_{j\nu} \hat{\rho} \hat{M}_{j\nu}$
$M_{j\nu} \equiv M_{j,\nu}$
$\Rightarrow$ Decoherence process
"Model for environment"

Entanglement & non-locality

$|\Psi_{AB}\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$

$\{|a_j\rangle\}$ O.N. basis for $\mathcal{H}_A$

$\{|b_k\rangle\}$ " " " $\mathcal{H}_B$

In general

$$|\Psi_{AB}\rangle = \sum_{j,k} \alpha_{j,k} \, |a_j\rangle \otimes |b_k\rangle$$

$$\alpha_{j,k} = \left(\langle a_j| \otimes \langle b_k|\right) |\Psi_{AB}\rangle \in \mathbb{C}$$

$\in \mathbb{C}$

Any $\hat{\rho} \in D(\mathcal{H}_A \otimes \mathcal{H}_B)$

$$\hat{\rho} = \sum_{l,k,l',k'} \rho_{ll',kk'} \, |a_l\rangle\langle a_{l'}| \otimes |b_k\rangle\langle b_{k'}|$$

$$\rho_{ll',kk'} = \langle l|\langle l'| \, \hat{\rho} \, |k\rangle|k'\rangle \in \mathbb{C}$$

<u>Def^n</u>: A pure state $|\phi\rangle$ is unentangled iff $\exists \; |\alpha\rangle \in \mathcal{H}_A$ & $|\beta\rangle \in \mathcal{H}_B$ s.t. $|\phi\rangle = |\alpha\rangle \otimes |\beta\rangle$,

otherwise it is entangled

<u>Def^n</u>: A density

$\hat{\rho} \in D($

is called

or <u>separa</u>

$\exists \; \{ p_{A,i}$

& $\{ \rho$

Def^{n}: A density operator

$$\hat{\rho} \in D(\mathcal{H}_A \otimes \mathcal{H}_B)$$

is called <u>unentangled</u>

or <u>separable</u> iff

$$\exists \{\rho_{A,i} \in D(\mathcal{H}_A)\}$$

$$\& \ \{\rho_{B,i} \in D(\mathcal{H}_B)\}$$

thod

$$\rho \rightarrow \rho' = \rho_\nu = \frac{\sum_j M_{\nu j}\, \rho\, M_{\nu j}^\dagger}{\mathrm{Tr}[\hat{E}_\nu \hat{\rho}]}$$

s.t. $\rho = \sum_i p_i\, \hat{\rho}_{A,i} \otimes \hat{\rho}_{B,i}$

where $\{p_i\}$ is a prob distr^n,

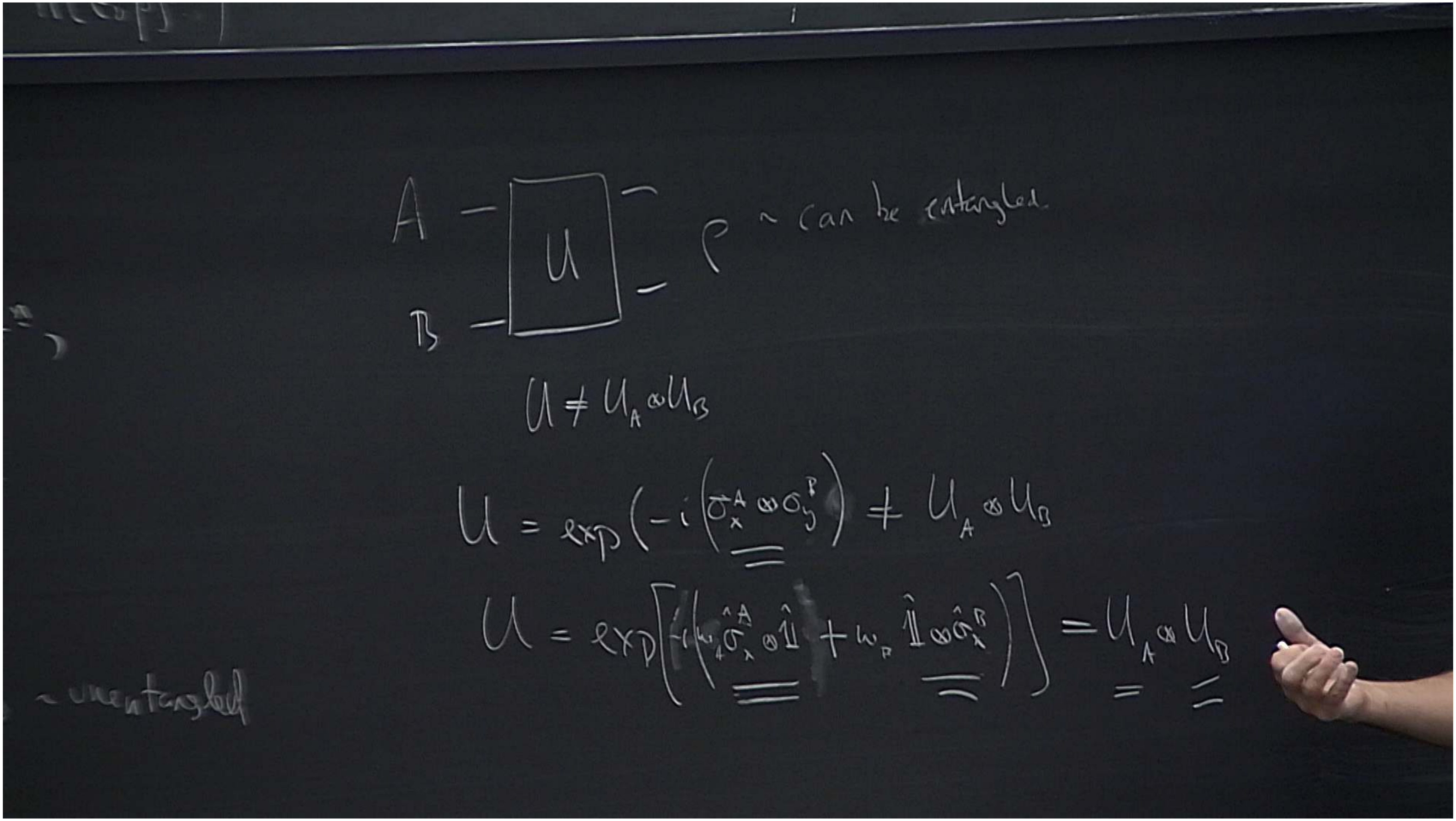
otherwise ρ is entangled.

Compare

A — U_A —
B — U_B — } ρ ~ unentangled

A — U —
B — U — ρ ~ can be entangled

$U \neq U_A \otimes U_B$


A
B
U
ρ ~ can be entangled
U ≠ U_A ⊗ U_B
U = exp(−i(σ̂_x^A ⊗ σ_y^B)) ≠ U_A ⊗ U_B
U = exp[−i(ω_A σ̂_x^A ⊗ 𝟙̂ + ω_B 𝟙̂ ⊗ σ̂_x^B)] = U_A ⊗ U_B
~ unentangled

By ignoring pointer.

the update rule is just

$$\rho \to \rho' = \sum_{\nu} \sum_{j_\nu} \hat{M}_{j_\nu} \hat{\rho} \hat{M}_{j_\nu}$$

$$M_{j_\nu} \equiv M_{j,\nu}$$

$\Rightarrow$ Decoherence process "Model for environment"

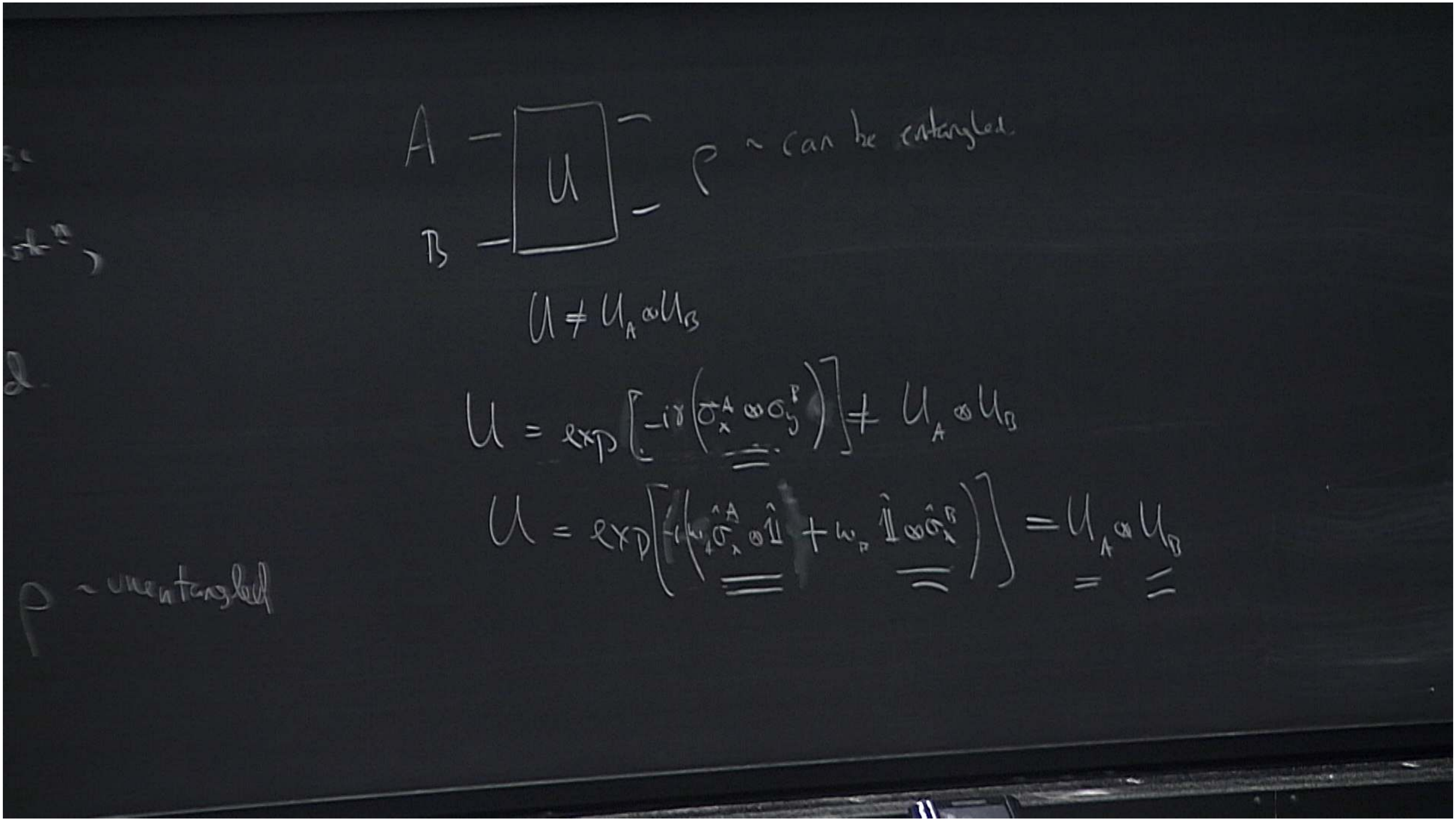
By ignoring pointer.

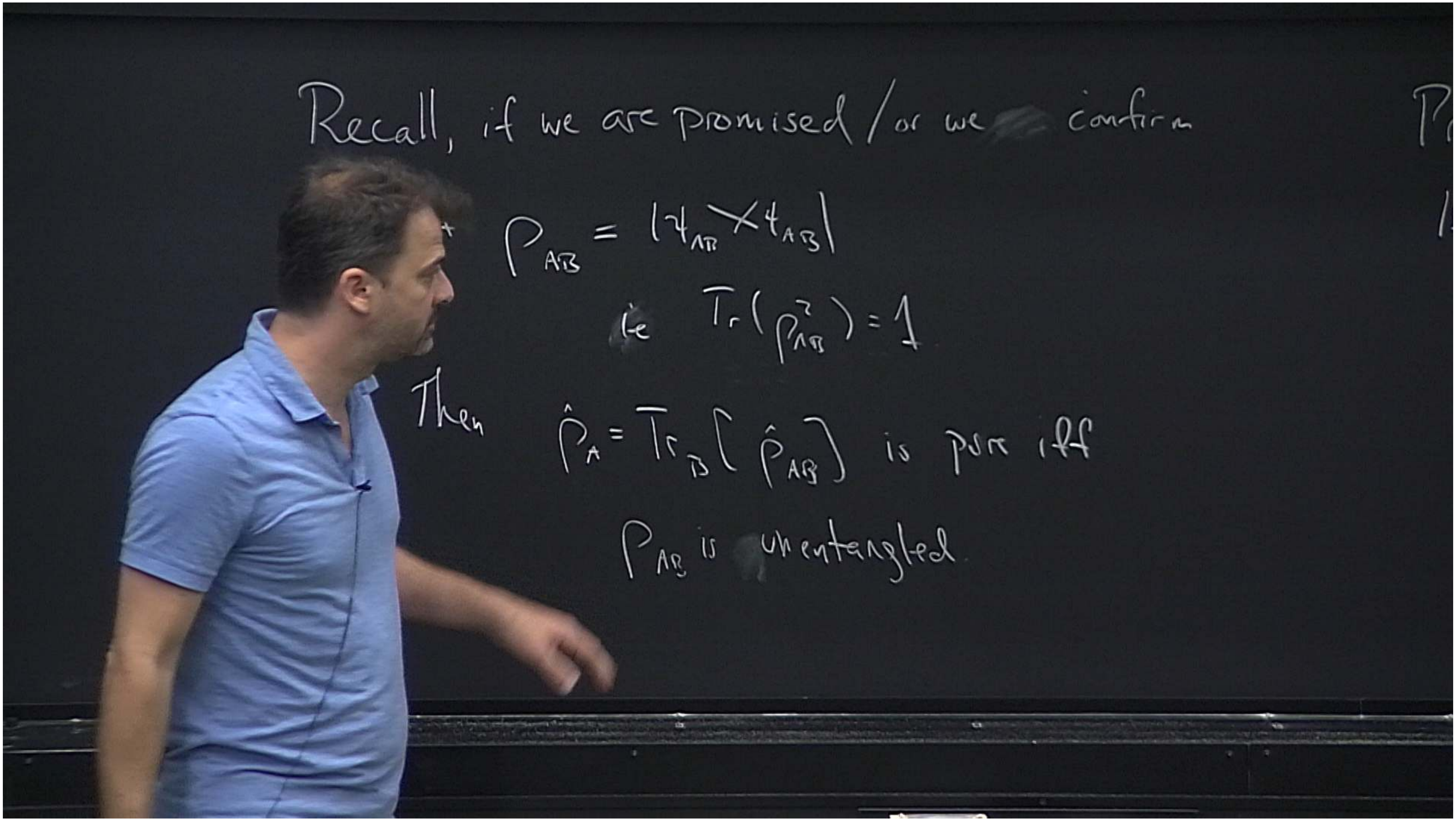
the update rule is just

$$\rho \rightarrow \boxed{\rho' = \sum_{\nu} \sum_{j_\nu} \hat{M}_{j_\nu} \hat{\rho} \hat{M}_{j_\nu}}$$

$M_{j_\nu} \equiv M_{j,\nu}$

$\Rightarrow$ Decoherence process
"Model for environment"


A
U
ρ ~ can be entangled
B
$U \neq U_A \otimes U_B$
$U = \exp\left[-i\gamma\left(\sigma_x^A \otimes \sigma_y^B\right)\right] \neq U_A \otimes U_B$
$U = \exp\left[-i\left(\omega_A \hat{\sigma}_x^A \otimes \hat{\mathbb{1}} + \omega_B \mathbb{1} \otimes \hat{\sigma}_x^B\right)\right] = U_A \otimes U_B$
ρ ~ unentangled


Recall, if we are promised / or we confirm
$\rho_{AB} = |\psi_{AB}\rangle\langle\psi_{AB}|$
ie $Tr(\rho_{AB}^2) = 1$
Then $\hat{\rho}_A = Tr_B[\hat{\rho}_{AB}]$ is pure iff
ρ_{AB} is unentangled

Recall, if we are promised / or we confirm

that $\rho_{AB} = |\psi_{AB}\rangle\langle\psi_{AB}|$

i.e. $\mathrm{Tr}(\rho_{AB}^2) = 1$

Then $\hat{\rho}_A = \mathrm{Tr}_B[\hat{\rho}_{AB}]$ is pure iff

ρ_{AB} is unentangled.

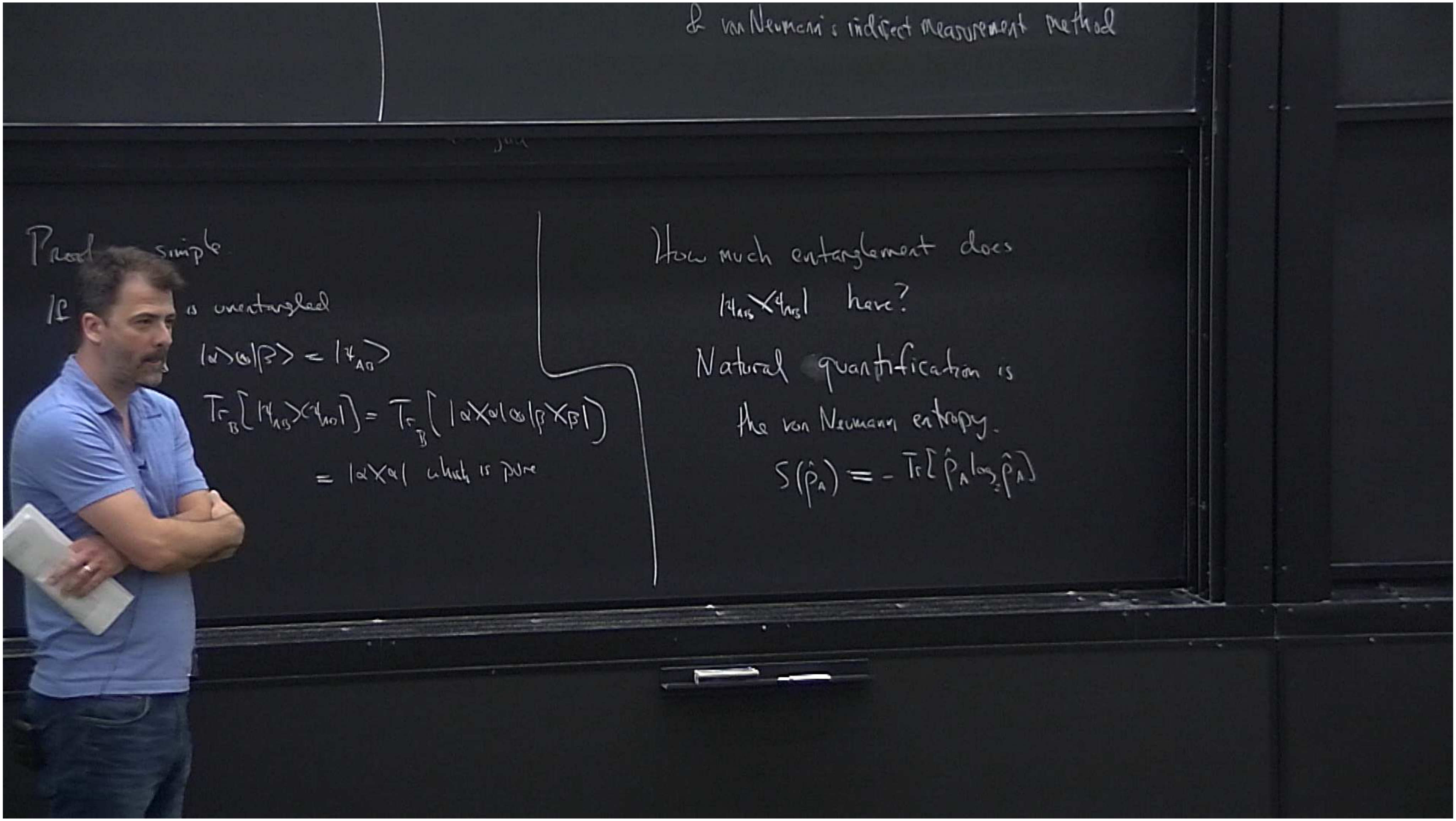
Proof is simple.

If $|\psi_{AB}\rangle$ is unentangled

then $\exists\ |\alpha\rangle \otimes |\beta\rangle = |\psi_{AB}\rangle$

then $Tr_B\left[|\psi_{AB}\rangle\langle\psi_{AB}|\right] = Tr_B\left(|\alpha\rangle\langle\alpha| \otimes |\beta\rangle\langle\beta|\right)$

$= |\alpha\rangle\langle\alpha|$ which is pure


& von Neumann's indirect measurement method
simple
is unentangled
= |α⟩⟨α| which is pure
How much entanglement does
here?
Natural quantification is
the von Neumann entropy.
S(ρ̂_A) = - Tr[ρ̂_A log_2 ρ̂_A]

Recall $F(\hat{A}) := \sum_i F(a_i) |a_i\rangle\langle a_i|$

where $\hat{A} = \sum_i a_i |a_i\rangle\langle a_i|$

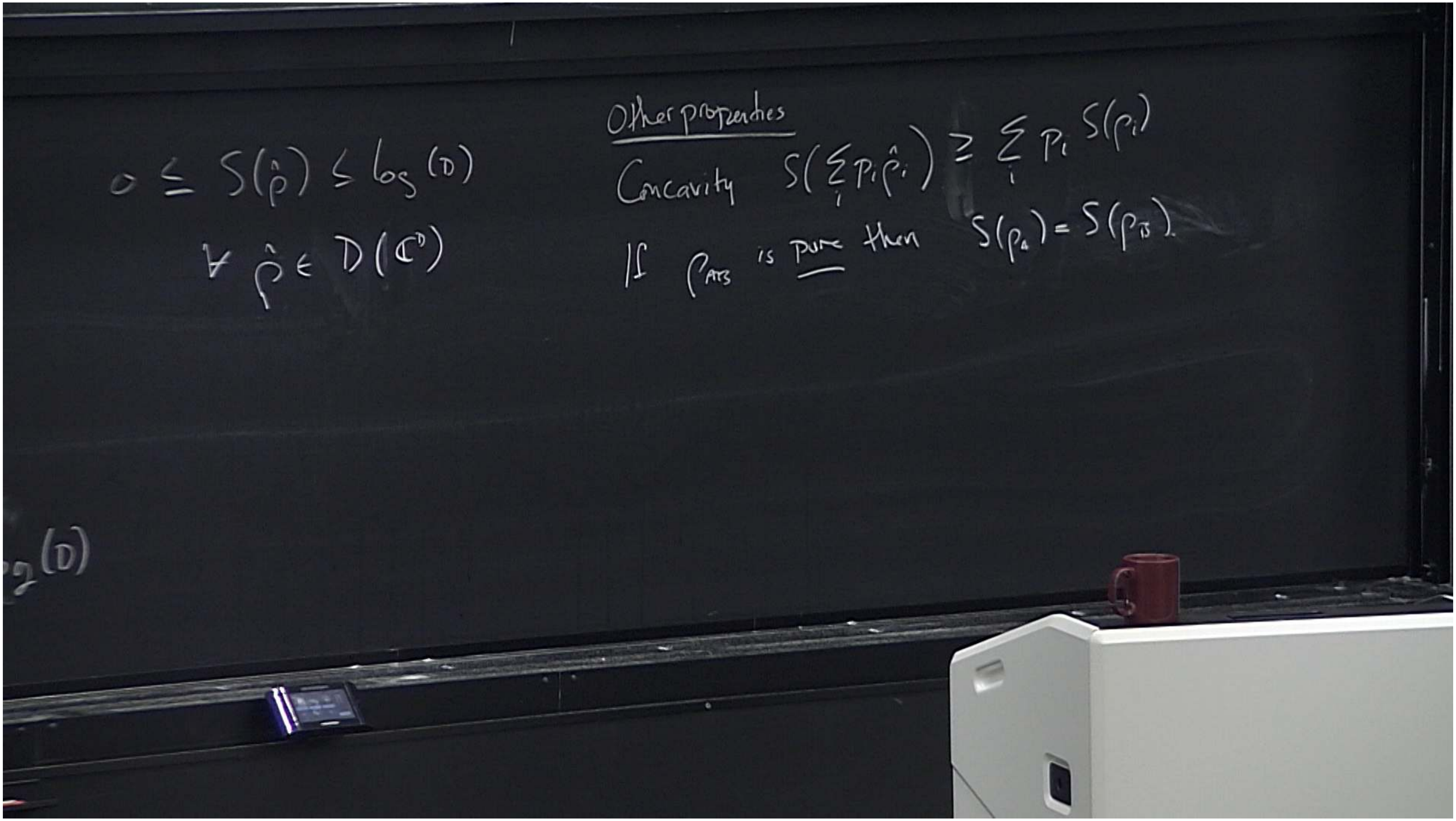
$0 \leq S(\hat{\rho})$

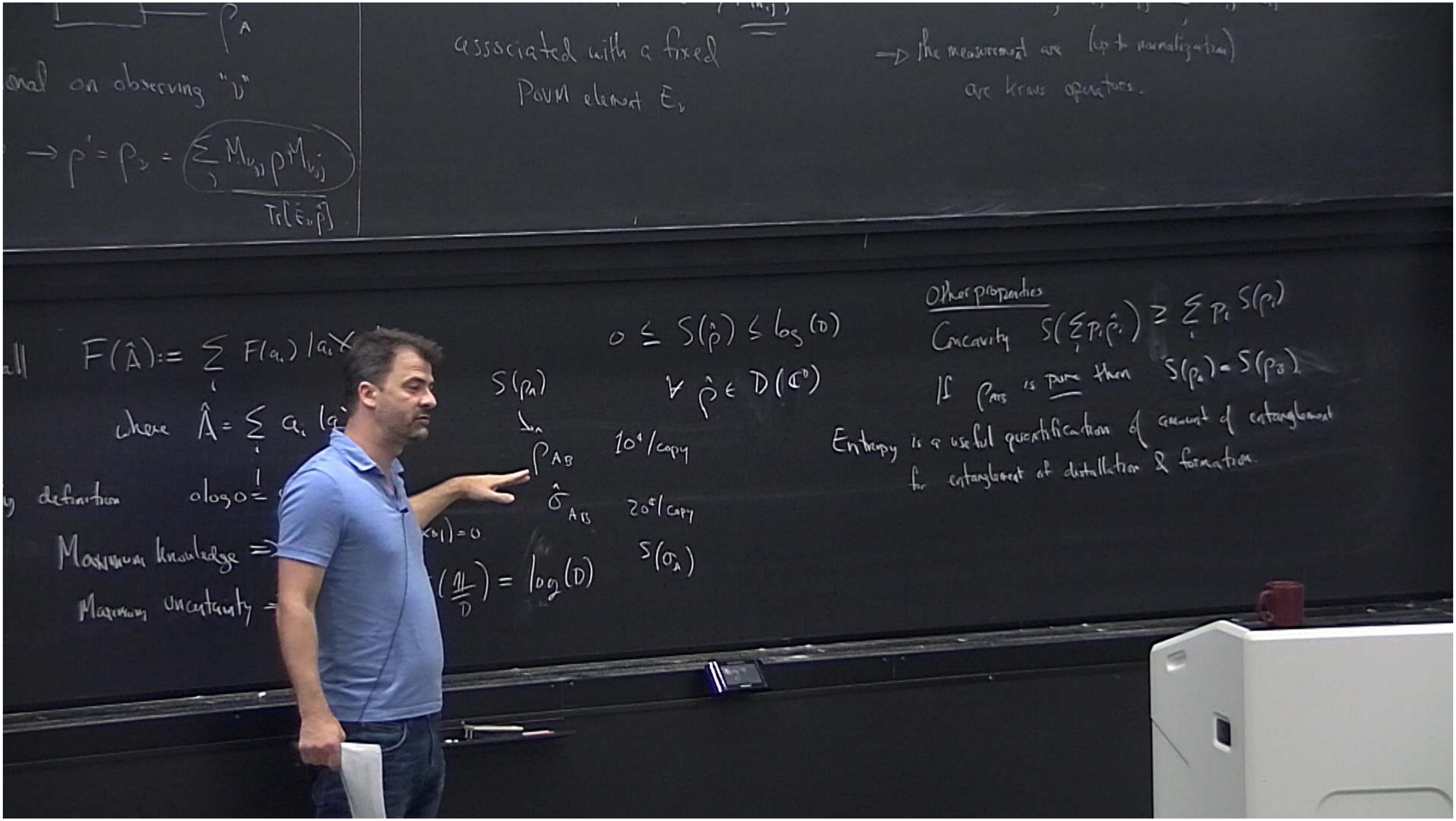
$\forall\ \hat{\rho} \in$

By definition $0 \log 0 \overset{!}{=} 0$

Maximum knowledge $\Rightarrow |\alpha\rangle\langle\alpha| \Rightarrow S(|\alpha\rangle\langle\alpha|) = 0$

Maximum uncertainty $\Rightarrow \rho_A = \frac{\hat{\mathbb{1}}}{D} \Rightarrow S\left(\frac{\mathbb{1}}{D}\right) = \log(D)$


$0 \leq S(\hat{\rho}) \leq \log(D)$
$\forall \hat{\rho} \in D(\mathbb{C}^D)$
Other properties
Concavity $S(\sum_i p_i \hat{\rho}_i) \geq \sum_i p_i S(\rho_i)$
If ρ_{AB} is pure then $S(\rho_A) = S(\rho_B)$.


associated with a fixed
POVM element E_ν
→ the measurement are (up to normalization)
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onal on observing "ν"
$\rightarrow \rho' = \rho_\nu = \frac{\sum_j M_{\nu,j} \rho M_{\nu,j}^\dagger}{Tr[\hat{E}_\nu \hat{\rho}]}$
$F(\hat{A}) := \sum_i F(a_i) |a_i\rangle\langle$
where $\hat{A} = \sum_i a_i |a_i\rangle$
definition $0 \log 0 = 0$
Maximum knowledge ⇒
Maximum uncertainty ⇒
$0 \leq S(\hat{\rho}) \leq \log(D)$
$\forall \hat{\rho} \in D(\mathbb{C}^D)$
$S(\rho_A)$
ρ_{AB} 10¢/copy
$\hat{\sigma}_{AB}$ 20¢/copy
$S(\sigma_A)$
$(\frac{\mathbb{1}}{D}) = \log(D)$
Other properties
Concavity $S(\sum_i p_i \hat{\rho}_i) \geq \sum_i p_i S(\rho_i)$
If ρ_{AB} is pure then $S(\rho_A) = S(\rho_B)$
Entropy is a useful quantification of amount of entanglement
for entanglement of distillation & formation