

Title: PSI 17/18 - Quantum Theory - Lecture 10

Date: Sep 18, 2017 10:45 AM

URL: <http://pirsa.org/17090043>

Abstract:

Generalized
Measurement: Mathematical
Structure

P.O.V.M: (Discrete Case)

Set $\{E_v\}$ st

i) $E_v \geq 0 \quad \forall v$

ii) $\sum_v E_v = \mathbb{1}$

Postulate 2 (Discrete Case)

Any measure

Generalized Measurement: Mathematical Structure

P.O.V.M.: (Discrete Case)

Set $\{E_v\}$ s.t

i) $E_v \geq 0 \quad \forall v$

ii) $\sum_v E_v = \mathbb{1}$

v is a discrete index.

Postulate 2 (Discrete)

MEASUREMENT

can be described

a P.O.V.M. $\{E_v\}$

$\forall p \in \mathcal{D}$

$\Pr(v)$

Postulate 2 (Discrete Case)

Any measurement procedure (with discrete outcomes)
can be described by

a P.O.V.M. $\{E_v\}$ st.

$$\forall \rho \in \mathcal{D}(\mathcal{H}),$$

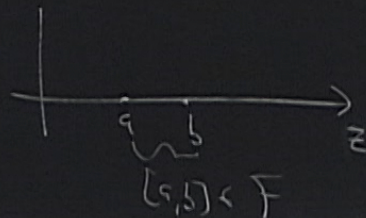
$$\Pr(v) = \text{Tr}(\hat{\rho} \hat{E}_v)$$

DoVM: continuous case

(with discrete outcomes)

Set Ω of continuous outcomes,
& a σ -algebra $\mathcal{F}$ of subsets of Ω
 $\Rightarrow (\Omega, \mathcal{F})$ define a measure space

Example: closed or open intervals on $\mathbb{R}$
 $[a, b]$ (a, b)



are very typical examples of elements $X \in \mathcal{F}$

eg

$$Pr(z \in X) = \int_a^b dz \, \rho(z)$$

↑
Probability
density

$$= \int_a^b d\mu(z)$$

where μ is a (normalized) measure

$$\hookrightarrow \int_{\mathbb{R}} d\mu(z) = 1$$

eg

$$P(z \in X) = \int_a^b dz \, \rho(z)$$

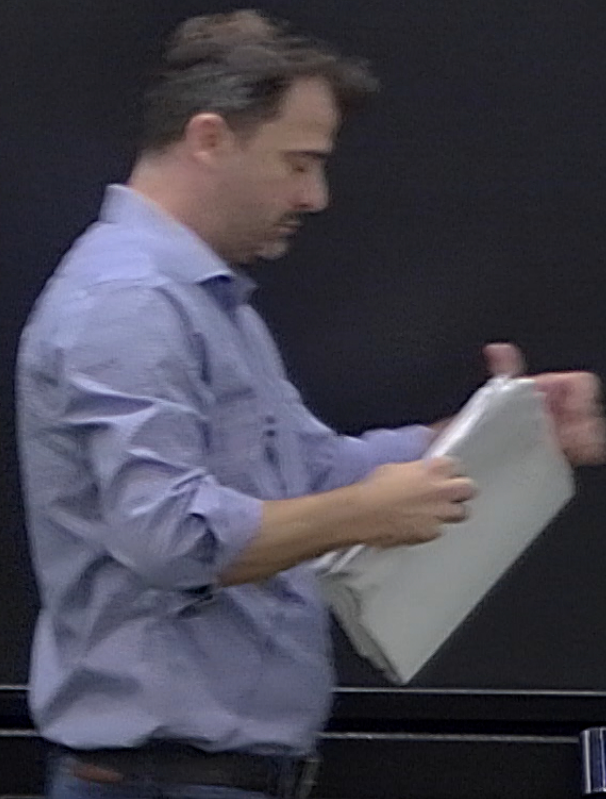
$\nearrow$ Probability density

$$= \int_a^b d\mu(z)$$

where μ is a (normalized) measure

$$\hookrightarrow \int_{\mathbb{R}} d\mu(z) = 1$$

Remark
 $0 \leq \rho \leq \infty$
 (units)



Remark
 $1 \leq p \leq \infty$
(units)

P.O.V.M (Continuous case)

$$E : \mathcal{F} \rightarrow \mathcal{L}(\mathcal{H})$$

Let $\chi \in \mathcal{F}$

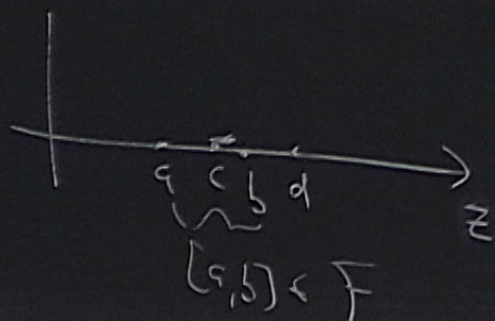
$$i) \hat{E}(\chi) \geq 0$$

$$ii) \hat{E}(\Omega) = \mathbb{1}$$

(with discrete outcomes)

Set Ω of continuous outcomes,
& a σ -algebra $\mathcal{F}$ of subsets
 $\Rightarrow (\Omega, \mathcal{F})$ define a measure space

Example: closed or open intervals on $\mathbb{R}$
 $[a, b]$ (a, b)

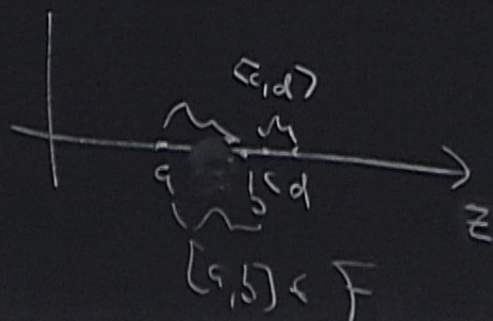


are very typical examples of $\mathcal{F}$

(with discrete outcomes)

Set Ω of continuous outcomes,
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 $\Rightarrow (\Omega, \mathcal{F})$ define a measure space

Example: closed or open intervals on $\mathbb{R}$
 $[a, b]$ (a, b)



are very typical examples of $\mathcal{F}$

Remark
 $0 \leq p \leq \infty$
(units)

P.O.V.M (Continuous case)

$$E : \mathcal{F} \rightarrow \mathcal{L}(\mathcal{H})$$

let $X \in \mathcal{F}$

$$i) \hat{E}(X) \geq 0$$

$$ii) \hat{E}(\Omega) = \hat{1}$$

$$iii) \hat{E}(\cup X_i) = \sum_i \hat{E}(X_i) \quad \text{for disjoint sequences } \{X_i\} \subset \mathcal{F}$$

Remark
 $0 \leq p \leq \infty$
(units)

P.O.V.M (Continuous case)

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for infinite sum we require only convergence
in weak topology

Postulate 2: (Continuous Case)

(...) a POVM $E: \mathcal{F} \rightarrow \mathcal{L}(\mathcal{H})$,

s.t. $\forall \hat{\rho} \in \mathcal{D}(\mathcal{H})$, the prob.
that the outcome $\in X$, is

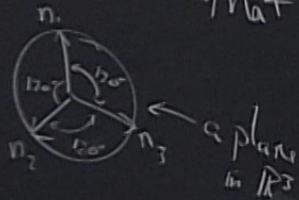
$$\Pr(\lambda \in X) = \text{Tr} [\hat{E}(X) \hat{\rho}]$$

||
Let $\lambda \in \Omega$
 $\lambda \in X$

Example d (non-trivial) discrete PDM

Time measurement
PDM

Define 3 projectors on $\mathcal{H} = \mathbb{C}^2$
that are non-orthogonal
 $\vec{n}_i \cdot \vec{n}_j \neq 0$



Projector

$$|\chi_v\rangle\langle\chi_v|$$

$$\hat{\vec{\sigma}} = (\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z)$$

$$\hat{\vec{\sigma}} \cdot \mathbf{n}_v |\chi_v\rangle = (+1) |\chi_v\rangle$$

$$\text{POVM } \{\hat{E}_v\}, v \in \{1, 2, 3\}$$

$$\hat{E}_v = \sum_3 |\chi_v\rangle\langle\chi_v|$$

Born rule

$$Pr(v) = \text{Tr}[\hat{E}_v \hat{\rho}]$$

Case)

$$F \rightarrow \mathcal{L}(\mathcal{H}),$$

, the prob

$\in X$, is

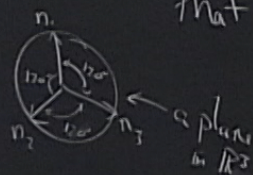
$$\text{Tr}[\hat{E}(x) \hat{\rho}]$$

Example d (non-trivial)

discrete PVM

Time measurement
PVM

Define 3 projectors on $\mathcal{H} = \mathbb{C}^2$
that are non-orthogonal



$$\vec{n}_i \cdot \vec{n}_j \neq 0$$

$$n_i \in S^2$$

$$\cos(\theta_{ij}) = 1/2$$

Projector

$$|x_v\rangle\langle x_v|$$

$$\hat{\sigma} = (\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z)$$

$$\hat{\sigma} \cdot \vec{n}_v |x_v\rangle = (+1) |x_v\rangle$$

PVM $\{E_v\}$, $v \in \{1, 2, 3\}$

$$E_v = \sum_i |x_i\rangle\langle x_i|$$

Born rule

$$Pr(v)$$

Check

i)

ii)

$$P_r(\nu) = \text{Tr}(\hat{E}_\nu \rho)$$

Check

$$i) \hat{E}_\nu \geq 0 \quad \forall \nu$$

$$ii) \sum_\nu \hat{E}_\nu = \mathbb{1}$$

Remark How do I implement this?

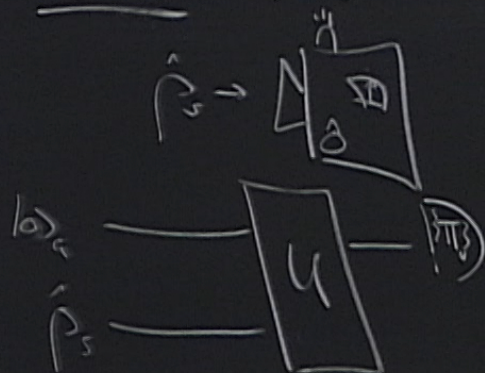
$$Pr(v) = \text{Tr}(E_v \rho)$$

check

$$i) \hat{E}_v \geq 0 \quad \forall v$$

$$ii) \sum_v \hat{E}_v = \mathbb{1}$$

Remark How do I implement this?



Design U & $\{P_\pi\}$ s.t.

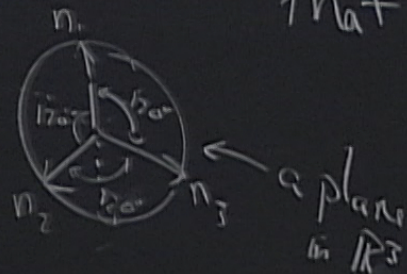
$$Pr(v) = \text{Tr}(\rho_s \otimes |\psi\rangle\langle\psi|_A \cdot U^\dagger \Pi_v U)$$

discrete PVM

Time measurement
PVM

Define 3 projectors on $\mathcal{H} = \mathbb{C}^2$
that are non-orthogonal

Definition n_i



$$\vec{n}_i \cdot \vec{n}_j \neq 0$$

$$n_i \in S^2$$

$$\arccos(n_i \cdot n_j) = 120^\circ$$

$$|n_i| = 1$$

Projector

$$|x_i\rangle\langle x_i|$$

$$\hat{\sigma} = (\hat{\sigma}_x, \hat{\sigma}_y)$$

$$\hat{\sigma} \cdot \vec{n}$$

Po

$$v \in \{1, 2, 3\}$$

$$|x_v\rangle\langle x_v|$$

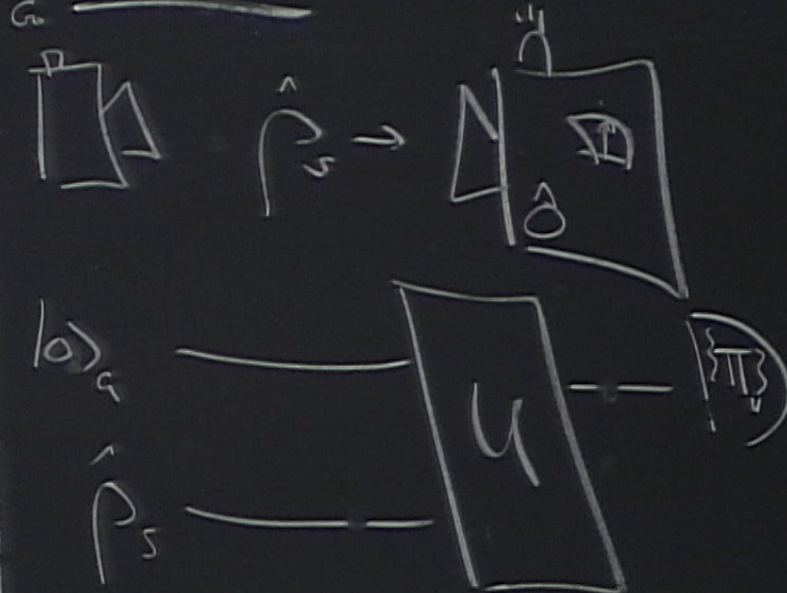
Tensor Product

$$H_A \otimes H_B$$

Tensor Sum

$$H_A \oplus H_B$$

Remark. How do I implement



Design

$$Pr(v) =$$

Question?

Can any POVM (mathematical structure)
be realized physically via the ideal
postulates?

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Naimark / Neumark's Dilation theorem

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Naimark / Neumark's Dilation theorem

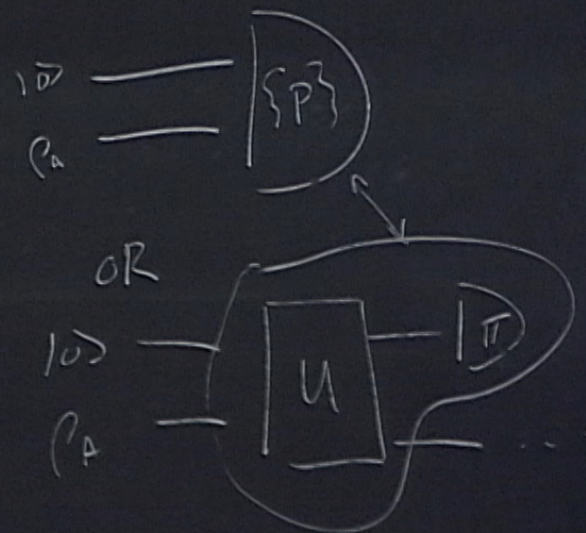
For any POVM $E: \mathcal{F} \rightarrow \mathcal{L}(\mathcal{H}_A)$, $\exists$ a PM $P \in \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B)$
s.t. $\text{Tr}(E(x) \rho) = \text{Tr}[(\hat{\rho} \otimes |0\rangle\langle 0|) P(x)]$

$\forall \hat{\rho} \in \mathcal{D}(\mathcal{H}_A)$ & $\forall \hat{E}(x) \in \text{POVM}$.

Corollary $\hat{P}(x)$ can always be
chosen in the form $\hat{1} \otimes \hat{\Pi}(x)$

where $\hat{\Pi}(x) \in \mathcal{L}(\mathcal{H}_B)$

e.g. $P(x) = U^\dagger \hat{1} \otimes \hat{\Pi}(x) U$

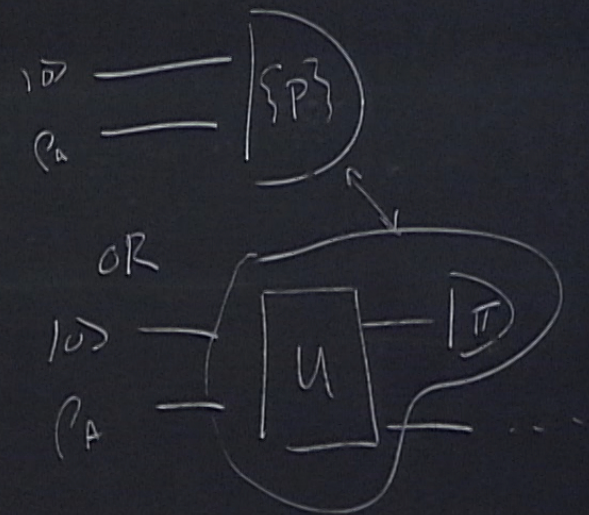


$$\forall \hat{\rho} \in \mathcal{D}(\mathcal{H}_A) \text{ \& } \forall \hat{E}(x) \in \text{POVM}.$$

Corollary $\hat{P}(x)$ can always be realized
(via U) as a ^{measurement} $\hat{\Pi}$ in the form $\hat{\Pi} \otimes \hat{\Pi}(x)$

$$\text{where } \hat{\Pi}(x) \in \mathcal{L}(\mathcal{H}_B)$$

$$\text{eg. } P(x) = U^\dagger \mathbb{I} \otimes \hat{\Pi}(x) U$$

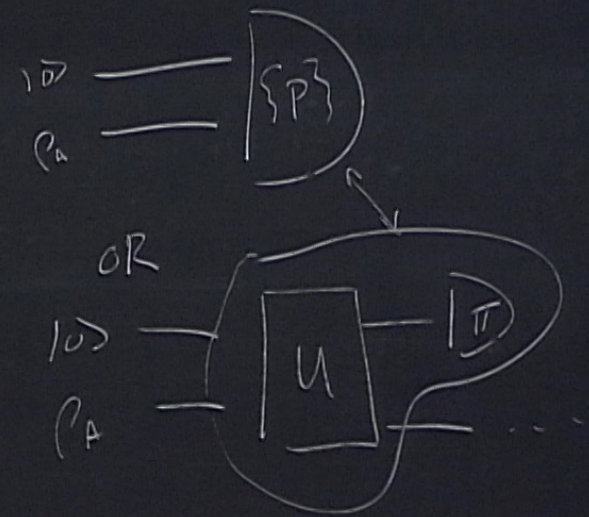


$$\forall \hat{\rho} \in \mathcal{D}(\mathcal{H}_A) \text{ \& } \forall \hat{E}(x) \in \text{POVM}.$$

Corollary $\hat{P}(x)$ can always be realized
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$$\text{where } \hat{\Pi}(x) \in \mathcal{L}(\mathcal{H}_B)$$

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Corollary $\hat{P}(x)$ can always be realized
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where $\hat{\Pi}(x) \in \mathcal{L}(\mathcal{H}_B)$

eg. $P(x) = U^\dagger \mathbb{I} \otimes \hat{\Pi}(x) U$

$A \geq 0$

$\langle \psi | A | \psi \rangle \geq 0 \quad \forall |\psi\rangle \in \mathcal{H}$

Corollary $\hat{P}(x)$ can always be realized
 (via U) as a ^{measurement} $\hat{1} \otimes \hat{\Pi}(x)$

where $\hat{\Pi}(x) \in \mathcal{L}(\mathcal{H}_B)$

eg. $P(x) = U^\dagger \hat{1} \otimes \hat{\Pi}(x) U$

$$A = A^\dagger$$

$\uparrow$

$$A \geq 0$$

$$\langle \psi | A | \psi \rangle \geq 0 \quad \forall |\psi\rangle \in \mathcal{H}$$

Generalized Transformations

Two generalizations over
Postulate 3 (Unitary operators).

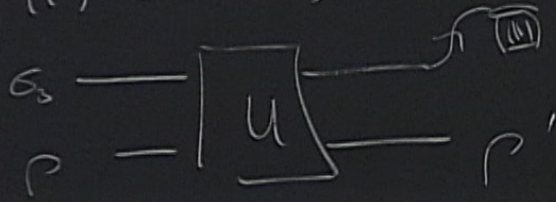
i) Convex-combinations of unitary operators

$$\Lambda: \rho \rightarrow \rho' \quad \Lambda(\rho) = \sum_i p_i U_i \rho U_i^\dagger$$

$p_i \geq 0, \sum_i p_i = 1$

ii) $\Lambda(\rho) = \text{Tr}_B \left[U \rho \otimes \sigma_B U^\dagger \right], \quad U \in \mathcal{L}(\mathcal{H} \otimes \mathcal{H}_B), \sigma_B \in \mathcal{D}(\mathcal{H}_B)$

ii) (continued)



In both cases

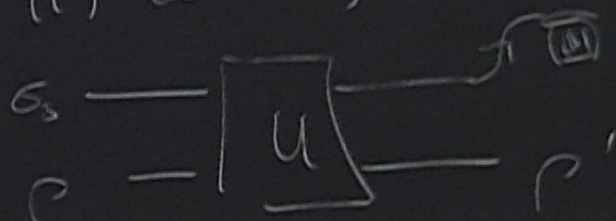
$$\Lambda: D(\mathcal{H}) \rightarrow D(\mathcal{H}) \quad (\text{positive})$$

$$\text{Tr}(\rho') = 1 \quad \text{if} \quad \text{tr}(\rho) = 1$$

ρ'

$\sigma_s = I(\mathcal{H}_s)$

ii) (continued)



In both cases

$$\Lambda: D(\mathcal{H}) \rightarrow D(\mathcal{H}) \quad (\text{positive})$$

$$\text{Tr}(\rho') = 1 \quad \text{if} \quad \text{tr}(\rho) = 1 \quad (\text{trace-preserving})$$

Another constraint
these generalized
to be guaranteed
physically realizable

$\Rightarrow$ "Complete"

$\rightarrow \rho'$

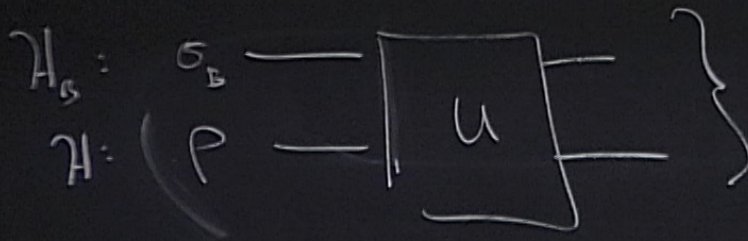
$\rightarrow D(\mathcal{H})$ (positive)

$\underline{1}$ if $\text{tr}(\rho) = 1$ (trace-preserving)

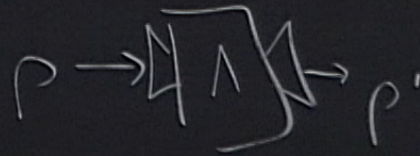
Another constraint for
 these generalized transformations
 to be guaranteed to be
 physically realizable

$\Rightarrow$ "Complete positivity"

Given

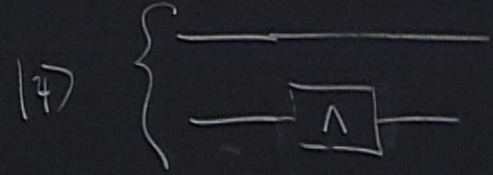


$$\rho' = \Lambda(\rho)$$



Suppose $\mathcal{H} = \mathcal{H} \otimes \mathcal{H}_C$

$$|4\rangle \in \mathcal{H} \otimes \mathcal{H}_C$$



$$|\psi\rangle \in \mathcal{H} \otimes \mathcal{H}_c$$

$$|\psi\rangle \begin{matrix} \mathcal{H}_c \\ \mathcal{H} \end{matrix} \left\{ \begin{array}{c} \text{---} \\ \text{---} \boxed{\Lambda} \text{---} \end{array} \right\} \hat{\omega} \in \mathcal{L}(\mathcal{H} \otimes \mathcal{H}_c)$$

Λ is completely-positive if

ω is positive for any
 $|\psi\rangle$ and choice of $\mathcal{H}_c$

where $\hat{\omega} = \underbrace{\Lambda \otimes I_c}_{(1 \times N)}$