Title: PSI 17/18 - Quantum Theory - Lecture 9

Date: Sep 15, 2017 10:45 AM

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Abstract:

Theory of Quantum Measurement

- Model of ideal measurement apparatus as quantum system
- Derivation of state-update rule

Consider a system $\mathcal{H}_S = \mathbb{C}^D$

and an apparatus $\mathcal{H}_a = \mathbb{C}^D$

$\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_a$

Imagine initial state for system $|\psi\rangle \in \mathcal{H}_S$

and a ready state of apparatus $|0\rangle \in \mathcal{H}_a$

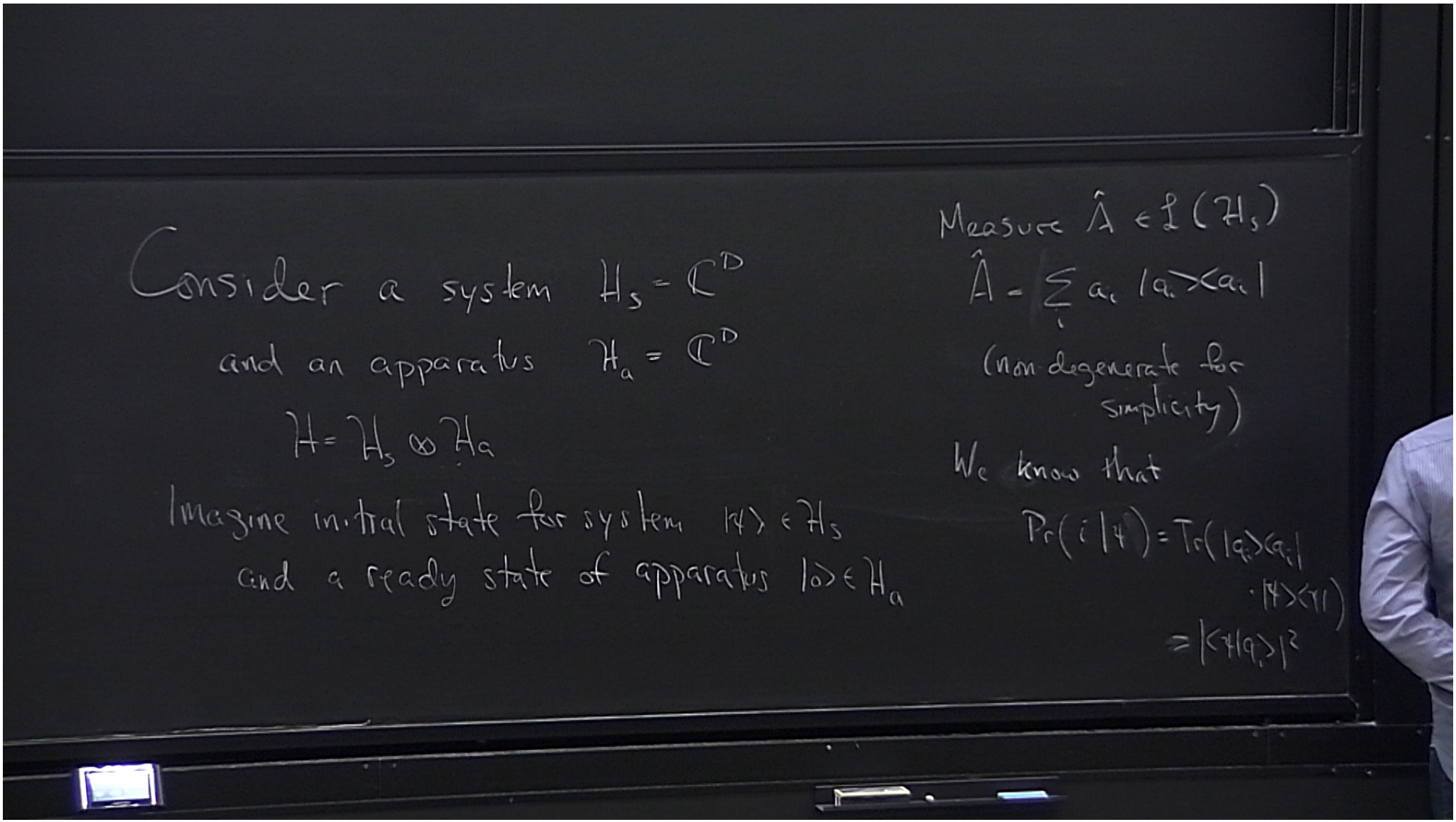
Measure $\hat{A} \in \mathcal{L}(\mathcal{H}_S)$

$\hat{A} = \sum_i a_i |a_i\rangle\langle a_i|$

(non degenerate for simplicity)

We know that

$$\Pr(i|\psi) = \mathrm{Tr}(|a_i\rangle\langle a_i| \cdot |\psi\rangle\langle\psi|) = |\langle\psi|a_i\rangle|^2$$



Update rule

- Given outcome i_0 is observed,

$$|\psi\rangle \rightarrow |a_{i_0}\rangle$$

- If outcome is not observed

$$|\psi\rangle \rightarrow \rho = \sum_i \Pr(i|\psi)\, |a_i\rangle\langle a_i|$$

Update rule

- Given outcome i_0 is observed,

$$|\psi\rangle \rightarrow |a_{i_0}\rangle$$

- If outcome is not observed

$$|\psi\rangle \rightarrow \hat{\rho} = \sum_i P_r(i|\psi)\, |a_i\rangle\langle a_i|$$

Expand $|\psi\rangle$ in basis of $\hat{A}$.

$$|\psi\rangle = \sum_i c_i |a_i\rangle$$

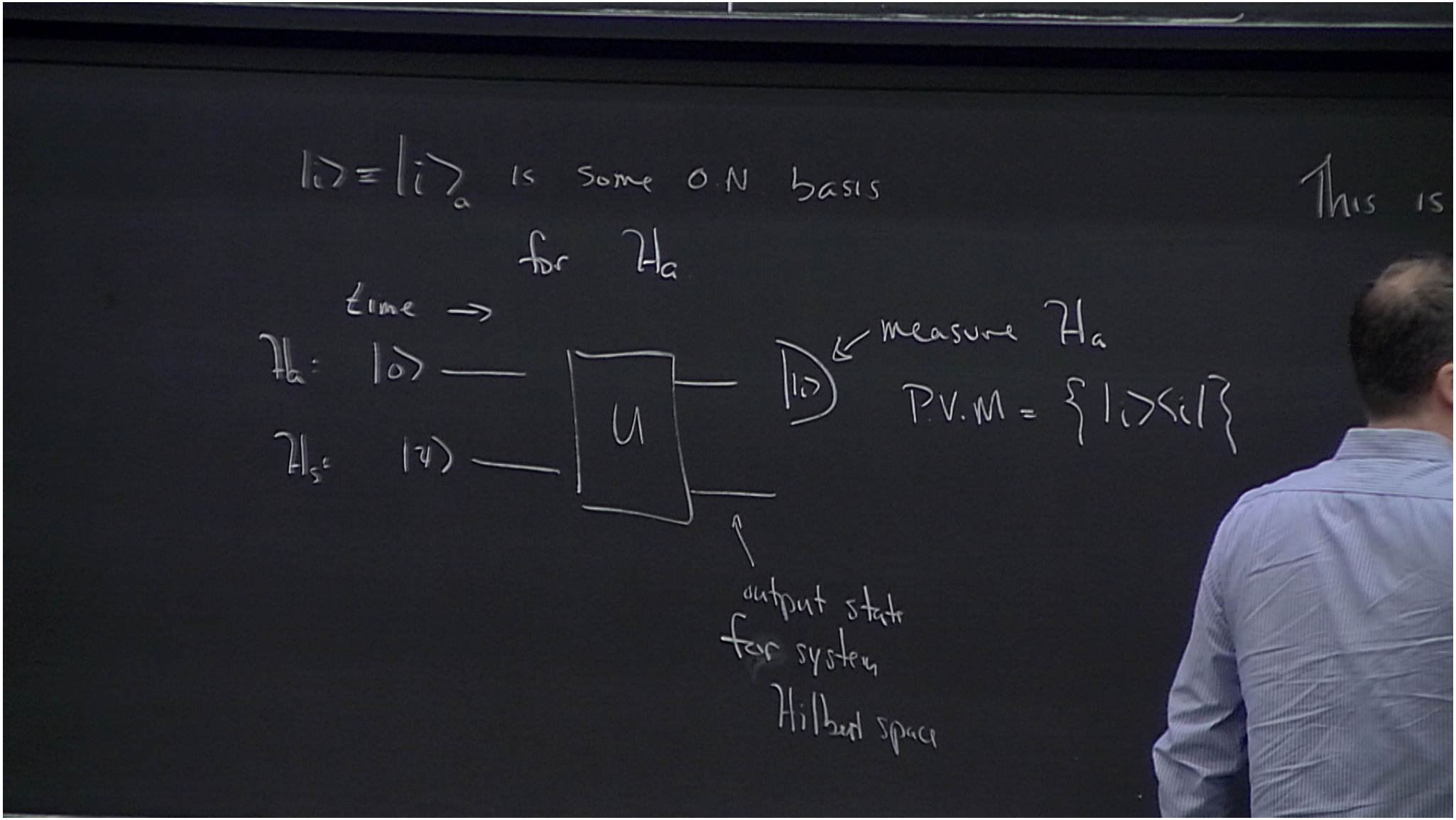
$$c_i = \langle \psi | a_i \rangle$$

Build apparatus to measure $\hat{A}$ by constructing U s.t.

$$U : |a_i\rangle \otimes |0\rangle \rightarrow |a_i\rangle \otimes |i\rangle$$

$\Rightarrow$ induces a correlation (entanglement) one-to-one b/w eigenstates of $\hat{A}$ & some O.N. basis of H_q

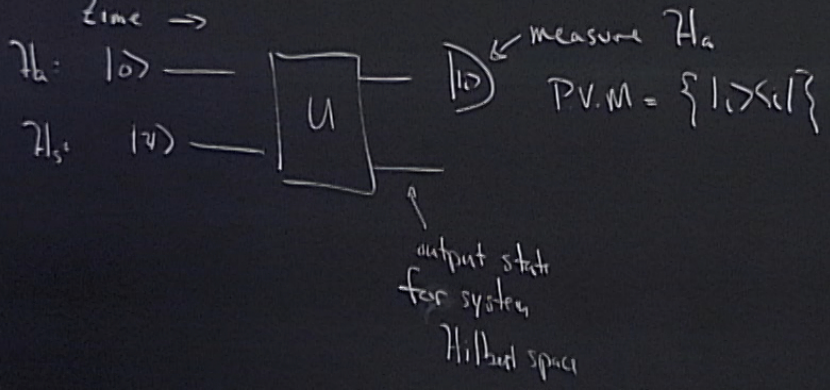
$\sum_i |a_i\rangle\langle a_i|$


$|i\rangle \equiv |i\rangle_a$ is some O.N basis
for $\mathcal{H}_a$
This is
time →
$\mathcal{H}_a$: $|0\rangle$
$\mathcal{H}_s$: $|\psi\rangle$
U
$|i\rangle$
measure $\mathcal{H}_a$
P.V.M = $\{|i\rangle\langle i|\}$
output state
for system
Hilbert space

Imagine initial state for system $|\psi\rangle \in \mathcal{H}_S$
and a ready state of apparatus $|0\rangle \in \mathcal{H}_a$

$Pr(i|\psi) = Tr(|a_i\rangle\langle a_i| \cdot |\psi\rangle\langle\psi|) = |\langle\psi|a_i\rangle|^2$

$|i\rangle = |i\rangle_a$ is some ON basis for $\mathcal{H}_a$



This is a generalization/realization of von Neumann's model of measurement (1930's)

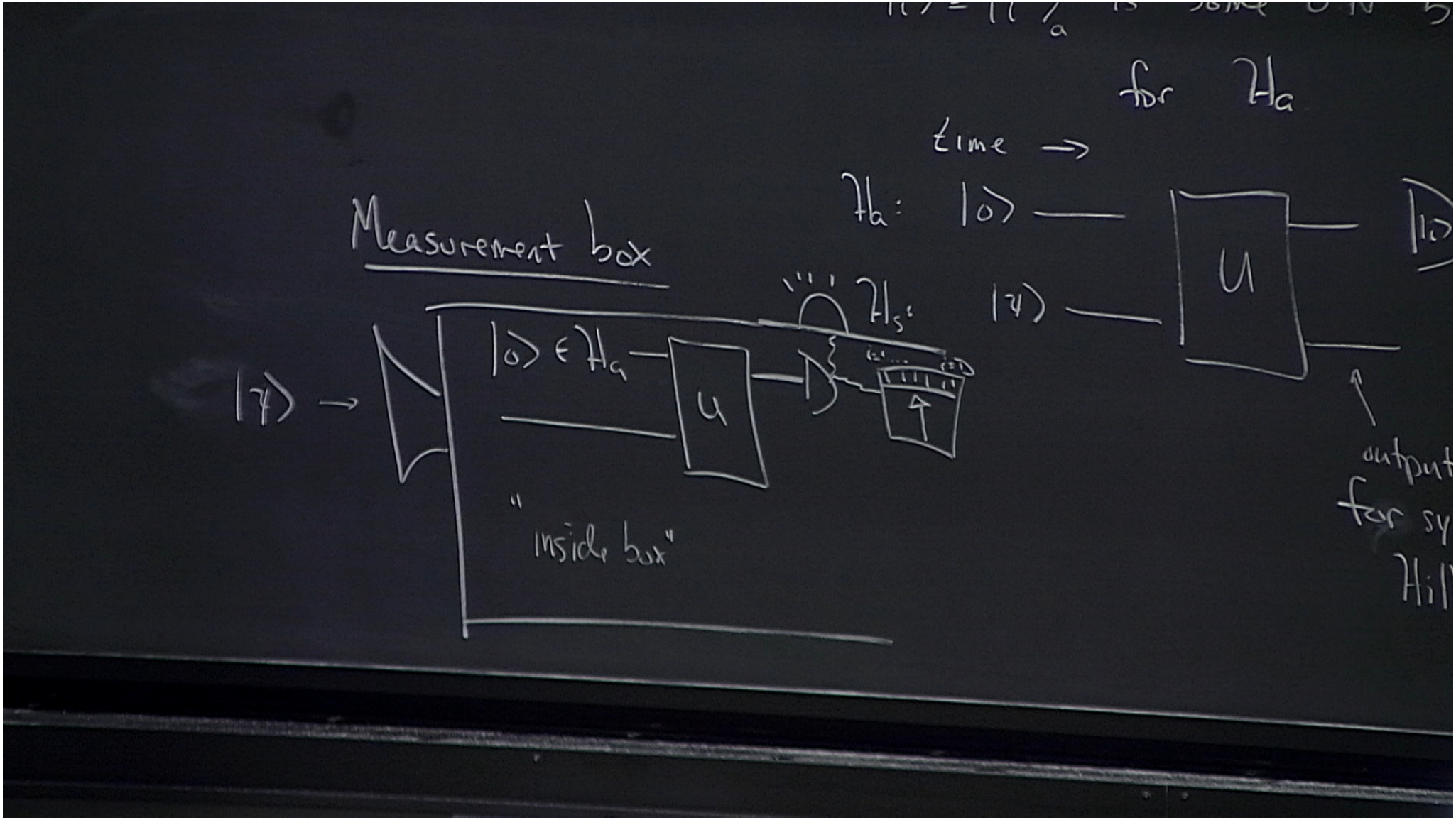
$\Leftarrow$ "von Neumann indirect measurement model"

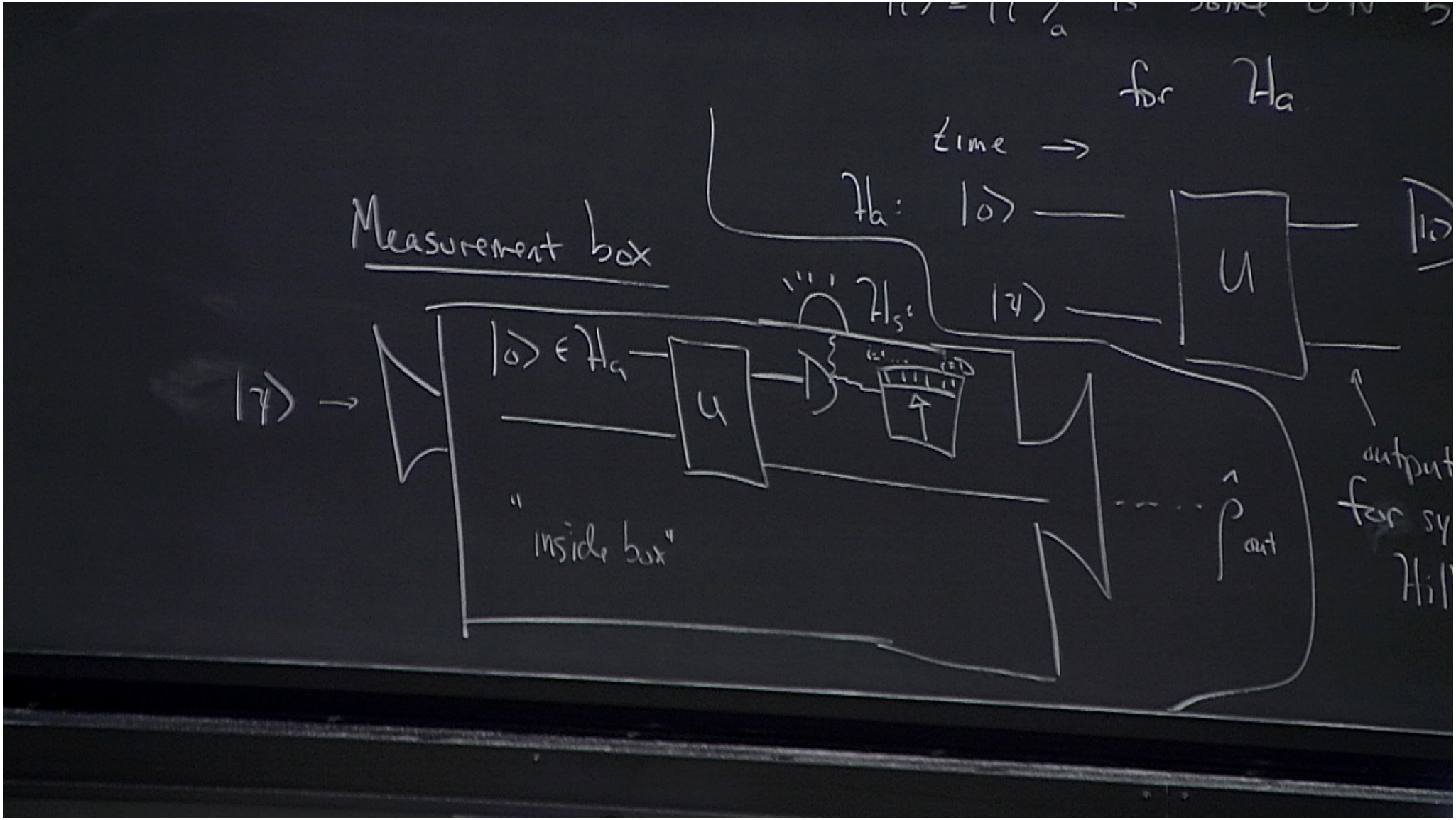
$Pr(i_a | |\psi\rangle_a |a\rangle)$

This is a generalization / realization
of von Neumann's model of measurement (1930's)

$\Leftarrow$ "von Neumann indirect measurement model"

$\Pr(i_0 \mid |\psi\rangle \otimes |\sigma\rangle)$


for $\mathcal{H}_a$
time →
$\mathcal{H}_a$: $|0\rangle$
$|\psi\rangle$
U
$\mathcal{H}_s$:
Measurement box
$|\psi\rangle$ →
$|0\rangle \in \mathcal{H}_a$
U
"inside box"
output
for sy
Hil


for $\mathcal{H}_a$
time →
$\mathcal{H}_a$: $|0\rangle$
$\mathcal{H}_s$: $|\psi\rangle$
U
Measurement box
$|0\rangle \in \mathcal{H}_a$
U
$|\psi\rangle$ →
"inside box"
$\hat{\rho}_{out}$
output
for sy
Hil

This is a generalization / realization of von Neumann's model of measurement (1930's)

$\{|i\rangle_i\}$ $\Leftarrow$ "von Neumann indirect measurement model"

inside box $\rightarrow \Pr(i_o \mid |\psi\rangle \otimes |0\rangle) = \mathrm{Tr}_S\left[\left(\mathbb{1}_S \otimes |i\rangle\langle i|_a\right) U |\psi\rangle\langle\psi| \otimes |0\rangle\langle 0| U^\dagger \right]$

$\rightarrow \Pr(i_o \mid |\psi\rangle) = |\langle \psi | q_i \rangle|^2$

outside box

This is a generalization / realization of von Neumann's model of measurement (1930's)

$\{|i\rangle\langle i|\}$ $\Leftarrow$ "von Neumann indirect measurement model"

inside box $\rightarrow \Pr(i_0 \mid |\psi\rangle\otimes|a\rangle) = \mathrm{Tr}_S\left[\underbrace{(\mathbb{1}_S \otimes |i\rangle\langle i|_a)\, U\, |\psi\rangle\langle\psi|\otimes|a\rangle\langle a|\, U^\dagger}\right]$

$= \sum_i \langle q_i| \cdots |q_i\rangle$

$\Pr(i_0 \mid |\psi\rangle) = |\langle\psi|q_i\rangle|^2$

outside box

$$\Pr(i \mid |\psi\rangle \otimes |0\rangle)$$

$$= \sum_i \langle a_i| \mathbb{1}_s \otimes |i\rangle\langle i| \; U \; |\psi\rangle\langle\psi| \otimes |0\rangle\langle 0| \; U^\dagger \; |a_i\rangle$$

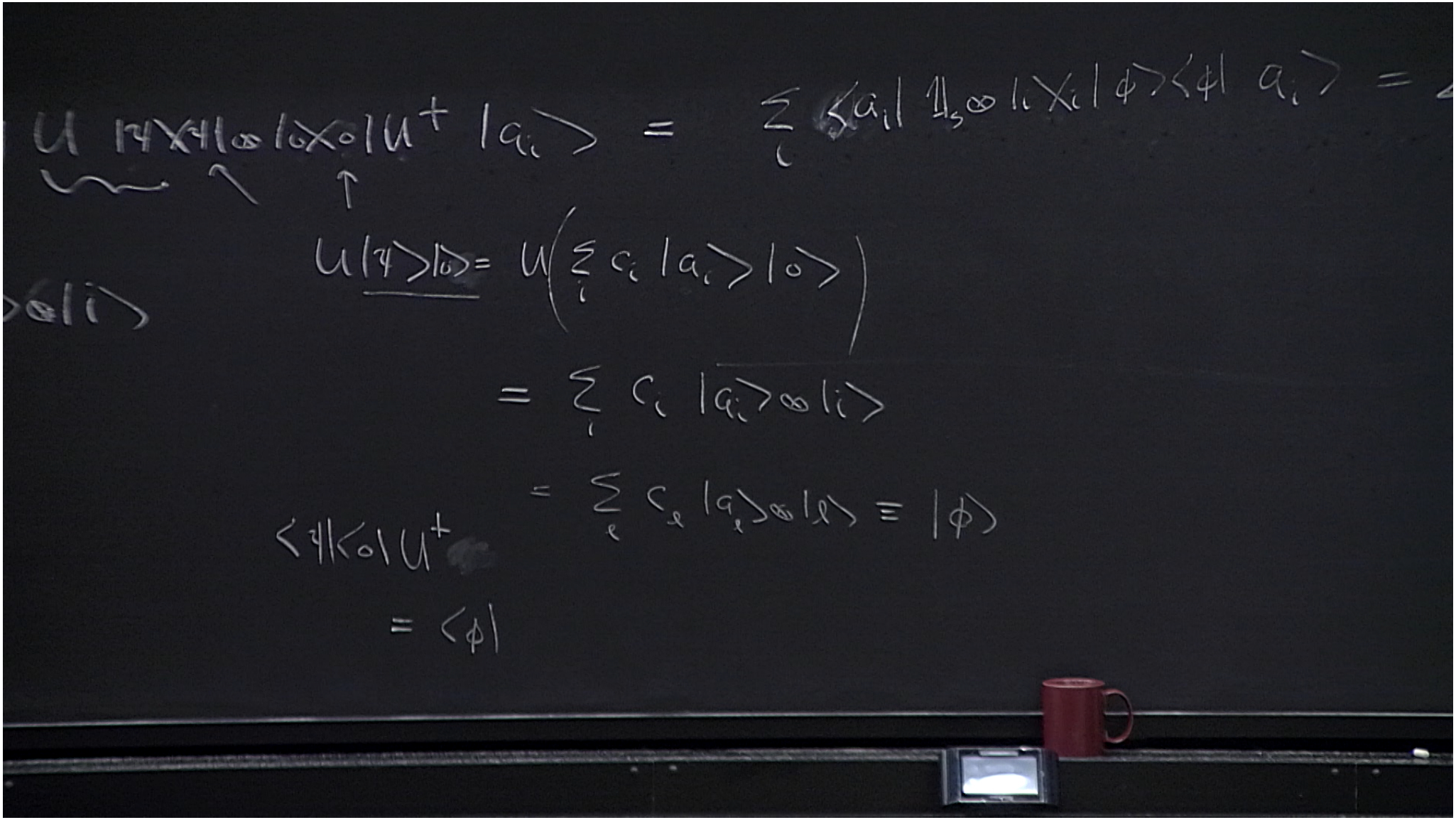
Recall $U : |a_i\rangle \otimes |0\rangle \to |a_i\rangle \otimes |i\rangle$

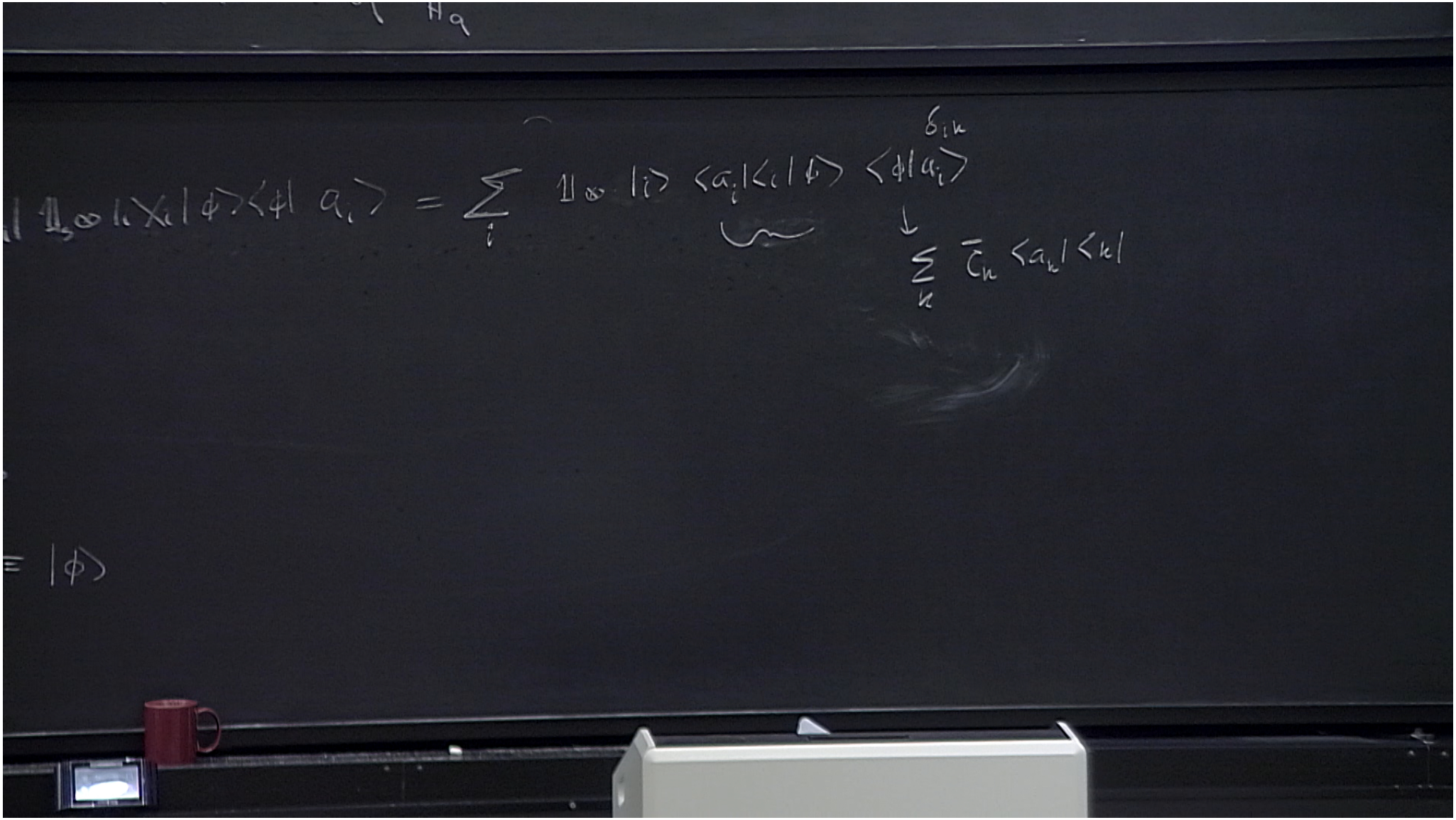
$$|\psi\rangle = \sum_i c_i |a_i\rangle$$

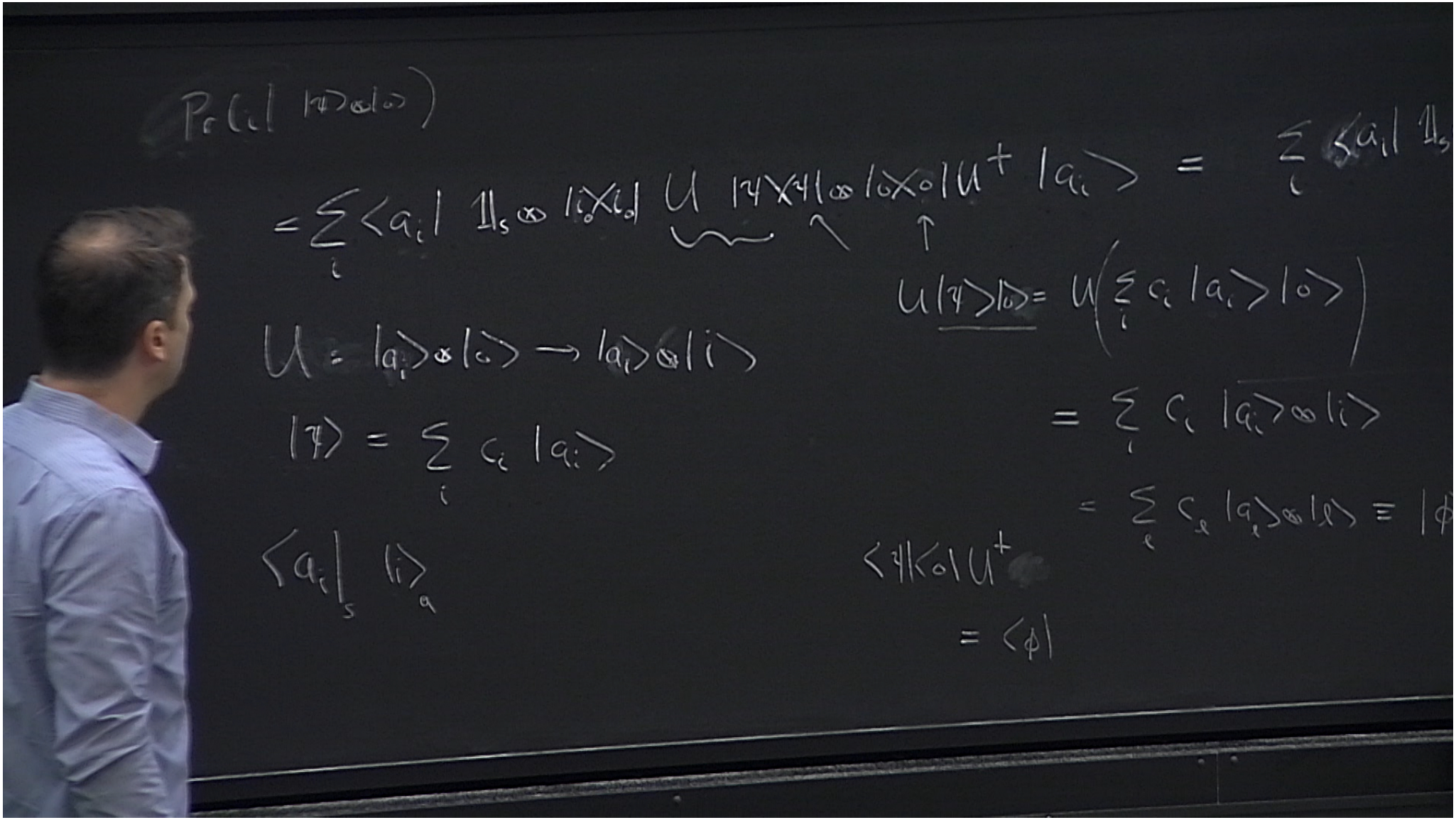
$$U|\psi\rangle|0\rangle = U\left(\sum_i c_i |a_i\rangle|0\rangle\right)$$

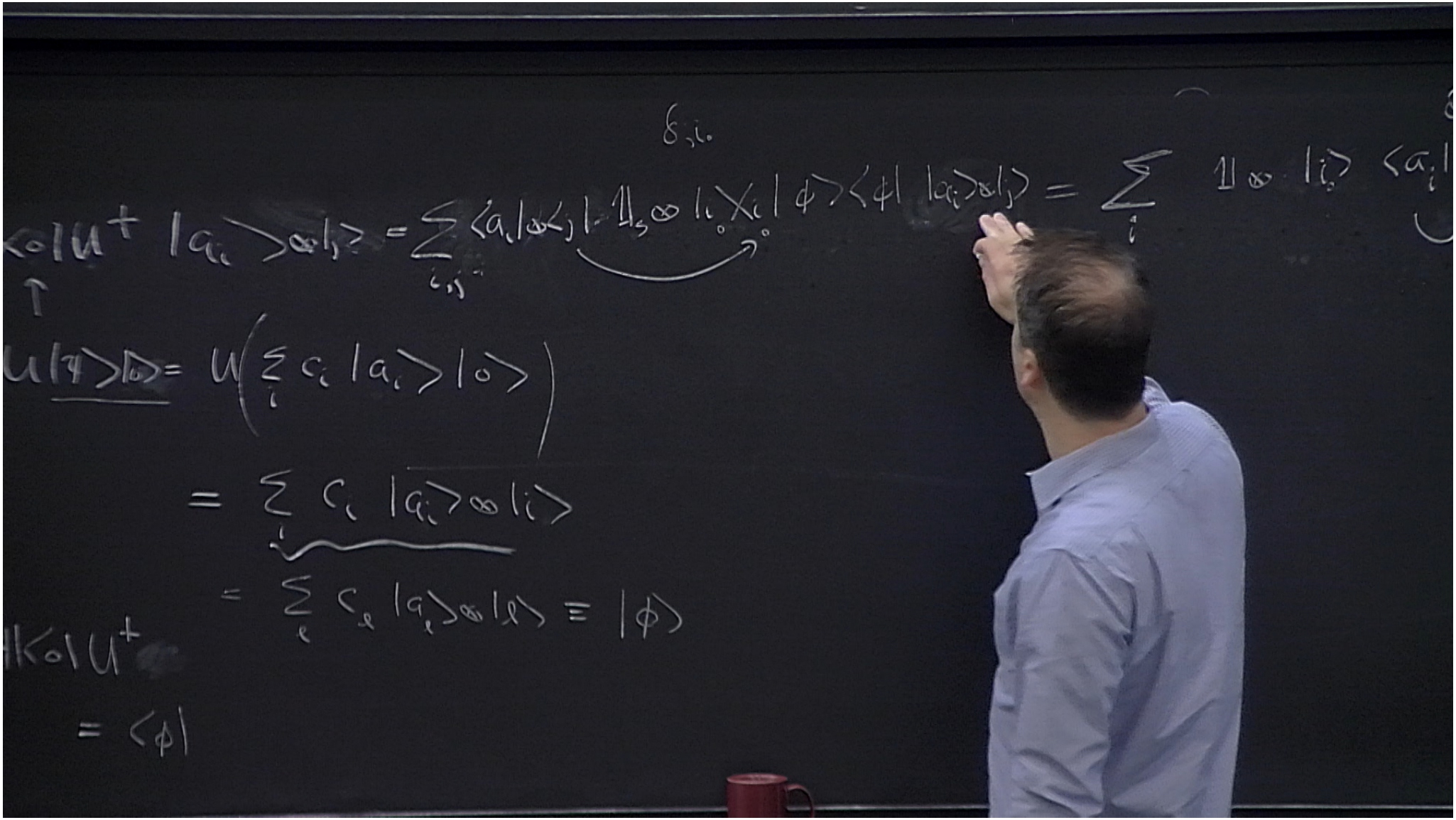
$$= \sum_i c_i |a_i\rangle \otimes |i\rangle$$

$\langle a_i|_s \; |i\rangle_a$


$U\,|\psi\rangle\otimes|0\rangle\langle 0|\otimes\langle 0|U^\dagger\,|a_i\rangle = \sum_i \langle a_i|\mathbb{1}_S\otimes|i\rangle\langle i|\phi\rangle\langle\phi|a_i\rangle$
$U|\psi\rangle|0\rangle = U\left(\sum_i c_i|a_i\rangle|0\rangle\right)$
$= \sum_i c_i\,|a_i\rangle\otimes|i\rangle$
$= \sum_i c_i\,|a_i\rangle\otimes|i\rangle \equiv |\phi\rangle$
$\langle\psi|\langle 0|U^\dagger$
$= \langle\phi|$


$\mathbb{1}_S \otimes |i\rangle\langle i| \phi\rangle\langle\phi| a_i\rangle = \sum_i \mathbb{1} \otimes |i\rangle \langle a_i|\langle i|\phi\rangle \langle\phi|a_i\rangle$
δ_{ik}
$\sum_k \bar{c}_k \langle a_k|\langle k|$
$= |\phi\rangle$


$P_E(i|\psi\rangle\otimes|0\rangle)$
$= \sum_i \langle a_i| \mathbb{1}_S \otimes |i\rangle\langle i| \, U \, |\psi\rangle\langle\psi| \otimes |0\rangle\langle 0| \, U^\dagger |a_i\rangle$
$U : |a_i\rangle \otimes |0\rangle \to |a_i\rangle \otimes |i\rangle$
$|\psi\rangle = \sum_i c_i |a_i\rangle$
$\langle a_i|_S \; |i\rangle_a$
$U|\psi\rangle|0\rangle = U\left(\sum_i c_i |a_i\rangle|0\rangle\right)$
$= \sum_i c_i |a_i\rangle \otimes |i\rangle$
$= \sum_i c_i |a_i\rangle \otimes |i\rangle \equiv |\phi$
$\langle\psi|\langle 0| U^\dagger$
$= \langle\phi|$
$= \sum_i \langle a_i| \mathbb{1}_S$


δ_{ij}
$\langle 0|U^\dagger |a_i\rangle\otimes|j\rangle = \sum_{i,j} \langle a_i|\otimes\langle j| \mathbb{1}_s\otimes|i\rangle\langle i|X_i|\phi\rangle\langle\phi| |a_j\rangle\otimes|j\rangle = \sum_i \mathbb{1}\otimes|i\rangle\langle a_i|$
$U|\psi\rangle|0\rangle = U\left(\sum_i c_i|a_i\rangle|0\rangle\right)$
$= \sum_i c_i |a_i\rangle\otimes|i\rangle$
$= \sum_i c_i |a_i\rangle\otimes|i\rangle \equiv |\phi\rangle$
$\langle 0|U^\dagger$
$= \langle\phi|$

$|\psi\rangle \to \hat{\rho} = \sum_i \Pr(i|\psi)\, |a_i\rangle\langle a_i|$

one-to-one b/w eigenstates of & some O.N. basis of H_q

$\Pr(i, |\psi\rangle_q |0\rangle_c)$

$\delta_{ij} = \langle i|i\rangle$

$= \sum_{ij} \langle a_i| \otimes \langle j| \mathbb{1}_s \otimes |i\rangle_c\langle j|_c \; U \; |\psi\rangle\langle\psi|_q \otimes |0\rangle_c\langle 0| \; U^\dagger \; |a_i\rangle \otimes |j\rangle = \sum_{ij} \langle a_i|\otimes\langle j| \mathbb{1}_s \otimes |i\rangle_c\langle j|_c \; |\phi\rangle\langle\phi| \; |a_i\rangle\otimes|j\rangle =$

call $U : |a_i\rangle \otimes |0\rangle \to |a_i\rangle \otimes |i\rangle$

$|\psi\rangle = \sum_i c_i |a_i\rangle$

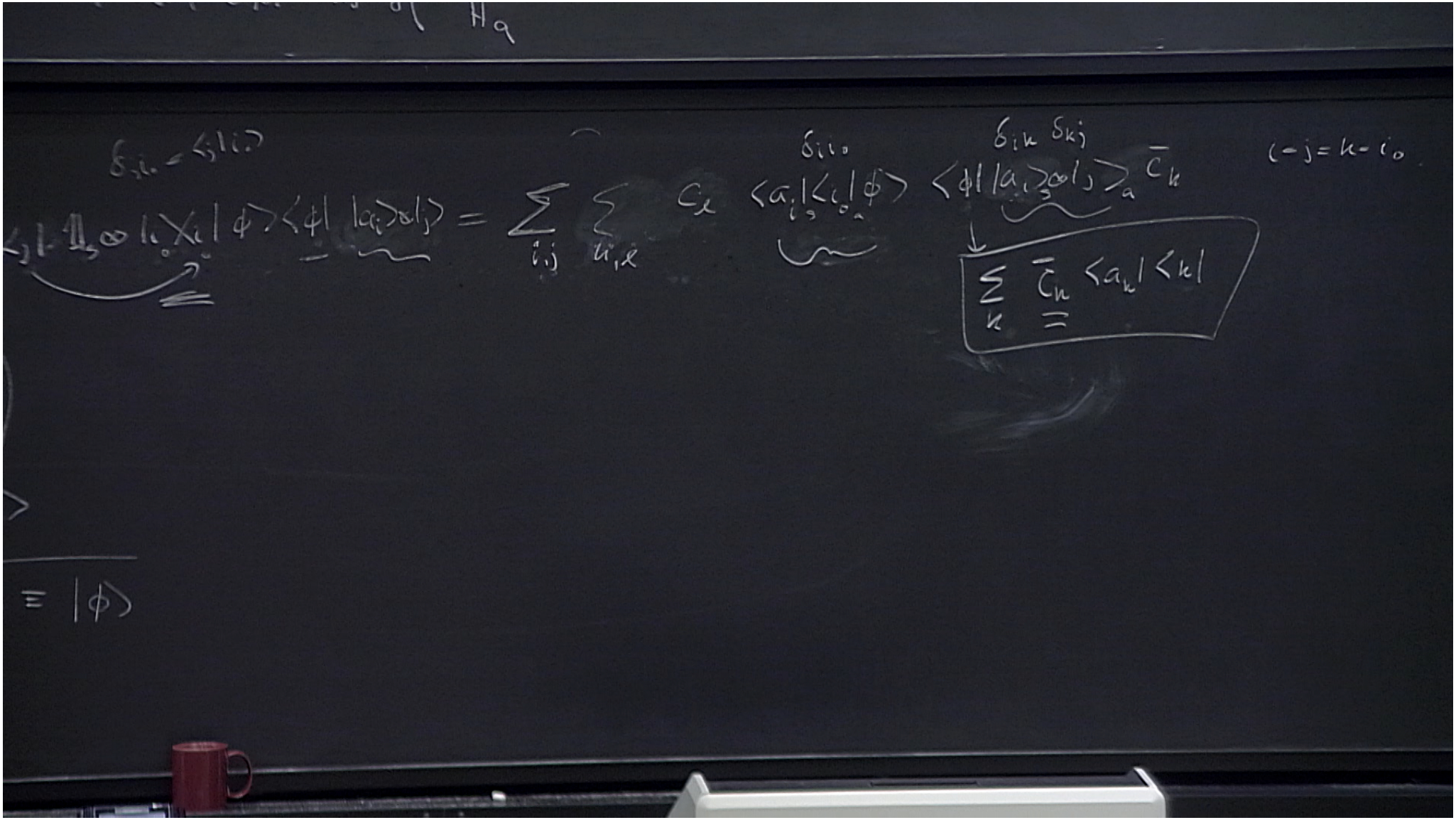
$U|\psi\rangle|0\rangle = U\left(\sum_i c_i |a_i\rangle|0\rangle\right)$

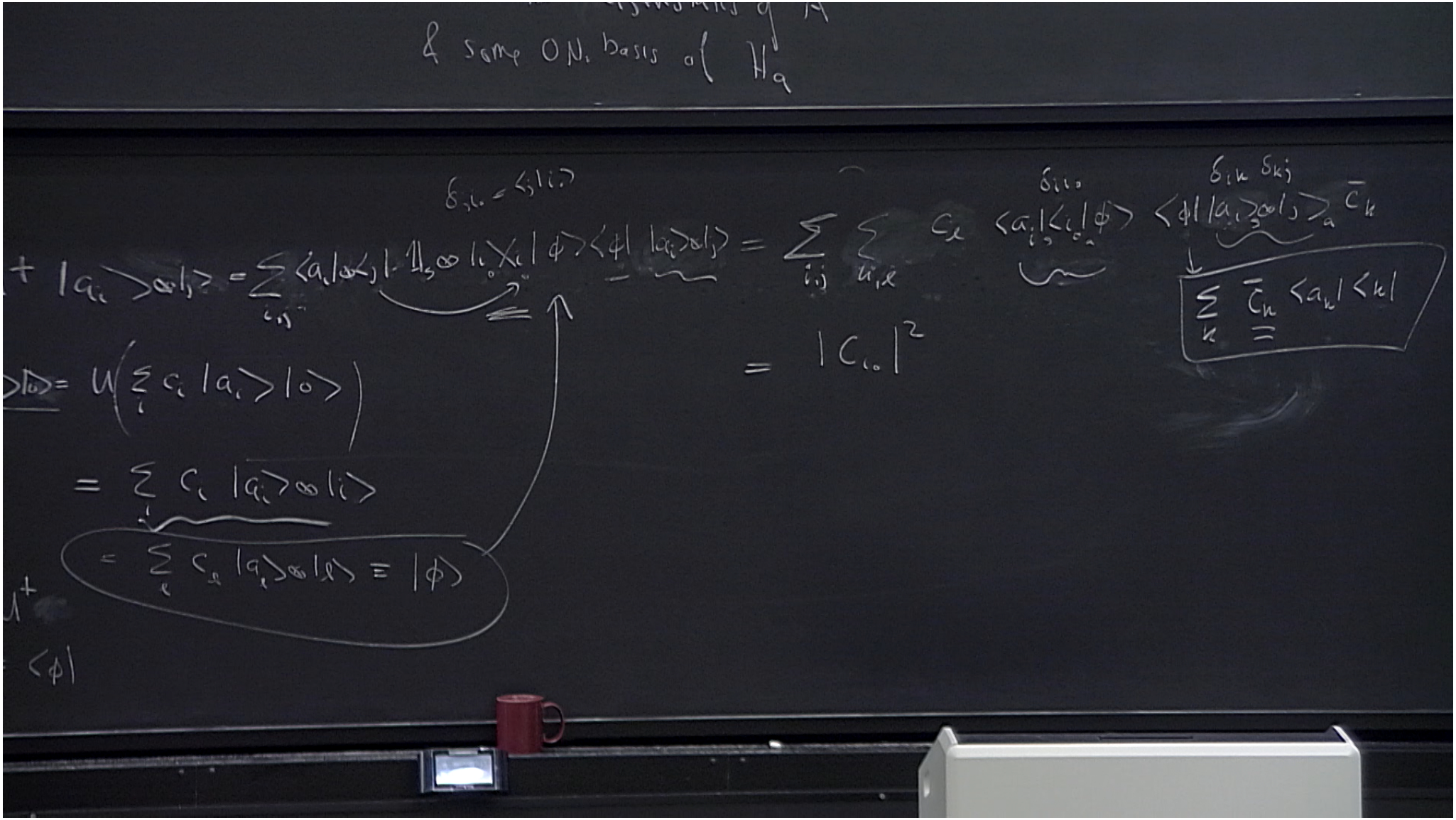
$= \sum_i c_i |a_i\rangle \otimes |i\rangle$

$= \sum_i c_i |a_i\rangle \otimes |i\rangle \equiv |\phi\rangle$

$\langle\psi|\langle 0| U^\dagger$

$= \langle\phi|$


$\delta_{ik} = \langle i | k \rangle$
$\delta_{ik}\,\delta_{kj}$
δ_{ik}
$i - j = k - i_0$
$\equiv |\phi\rangle$


& some ON. basis of $\mathcal{H}_q$
$\delta_{ik} = \langle i|i\rangle$
$\delta_{ik}\,\delta_{kj}$
δ_{ik}
$+ |a_i\rangle\otimes|j\rangle = \sum_{i,j} \langle a_i|\alpha_j\rangle\, \mathbb{1}_s\otimes|i_o\rangle\langle i_o|\, X_i\,|\phi\rangle\langle\phi|\,|a_i\rangle\otimes|j\rangle = \sum_{i,j}\sum_{k,\ell} c_\ell \langle a_i|\langle i_o|\phi\rangle \langle\phi|\,|a_i\rangle_s\otimes|j\rangle_a\, \bar{c}_k$
$= |c_{i_o}|^2$
$\sum_k \bar{c}_k \langle a_k|\langle k|$
$|0\rangle = U\left(\sum_i c_i |a_i\rangle|0\rangle\right)$
$= \sum_i c_i |a_i\rangle\otimes|i\rangle$
$= \sum_i c_i |a_i\rangle\otimes|\phi\rangle \equiv |\phi\rangle$
U^+
$\langle\phi|$

$= |\langle \psi | q_i \rangle|^2$

ON basis

$\mathcal{H}_a$

$|0\rangle$

measure $\mathcal{H}_a$

P.V.M. $= \{|i\rangle\langle i|\}$

output state for system Hilbert space

This is a generalization / realization of von Neumann's model of measurement (1930's)

$\Leftarrow$ "von Neumann indirect measurement model"

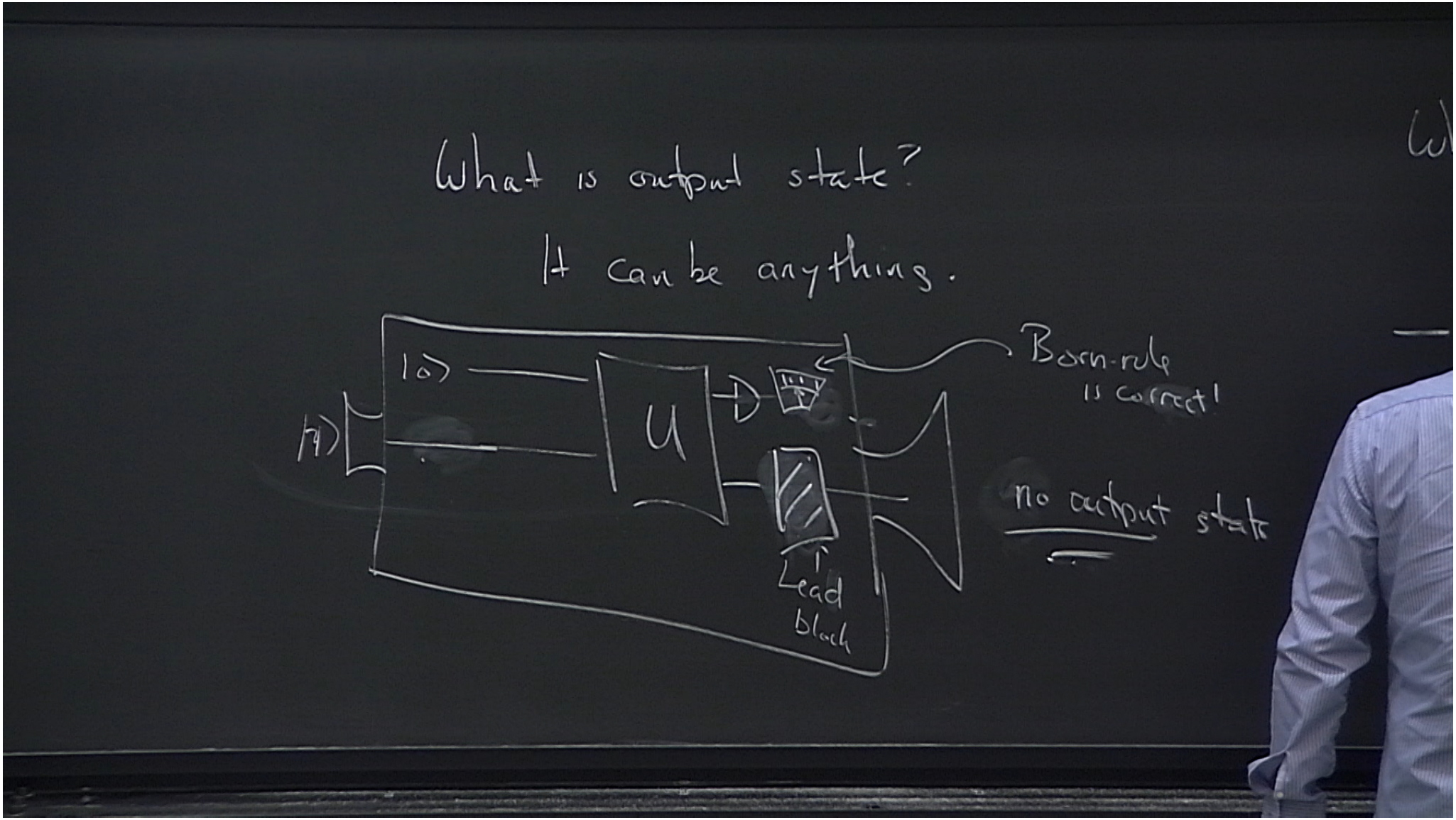
inside box $\rightarrow \Pr(i_0 \mid |\psi\rangle \otimes |0\rangle) = \mathrm{Tr}\left[\left(\mathbb{1}_s \otimes |i\rangle\langle i|_a \right) U \, |\psi\rangle\langle\psi| \otimes |0\rangle\langle 0| \, U^\dagger \right]$

$\Pr(i_0 \mid |\psi\rangle) = |\langle \psi | q_{i_0} \rangle|^2 = \sum_i \langle q_i | \cdots | q_i \rangle$

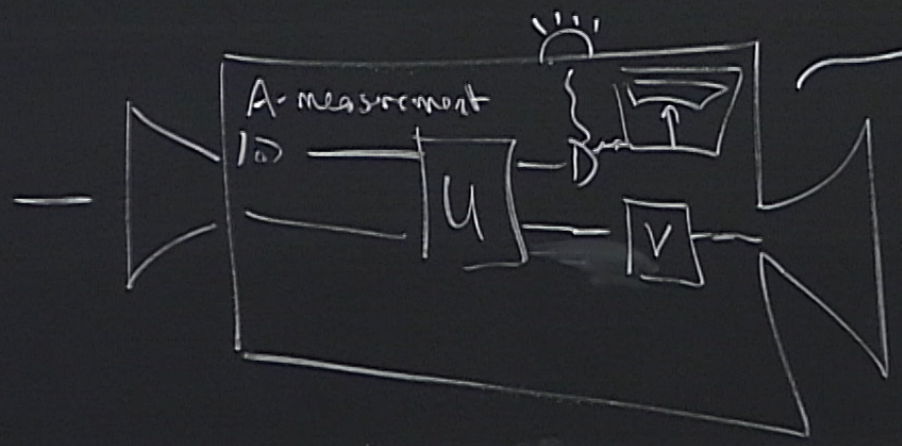
outside box

$= |c_{i_0}|^2$


If $|\psi\rangle = |i=3\rangle$
then $Pr(i=3) = 1$
$|i\rangle \equiv |i\rangle_a$ is some
for $\mathcal{H}_a$
time →
$\mathcal{H}_a$: $|0\rangle$
$|\psi\rangle$
U
$\mathcal{H}_S$
$|0\rangle \in \mathcal{H}_a$
U
surement box
"inside box"
$\hat{\rho}_{out}$


What is output state?
It can be anything.
|0⟩
|ψ⟩
U
Born-rule is correct!
no output state
Lead block

What else could output be?

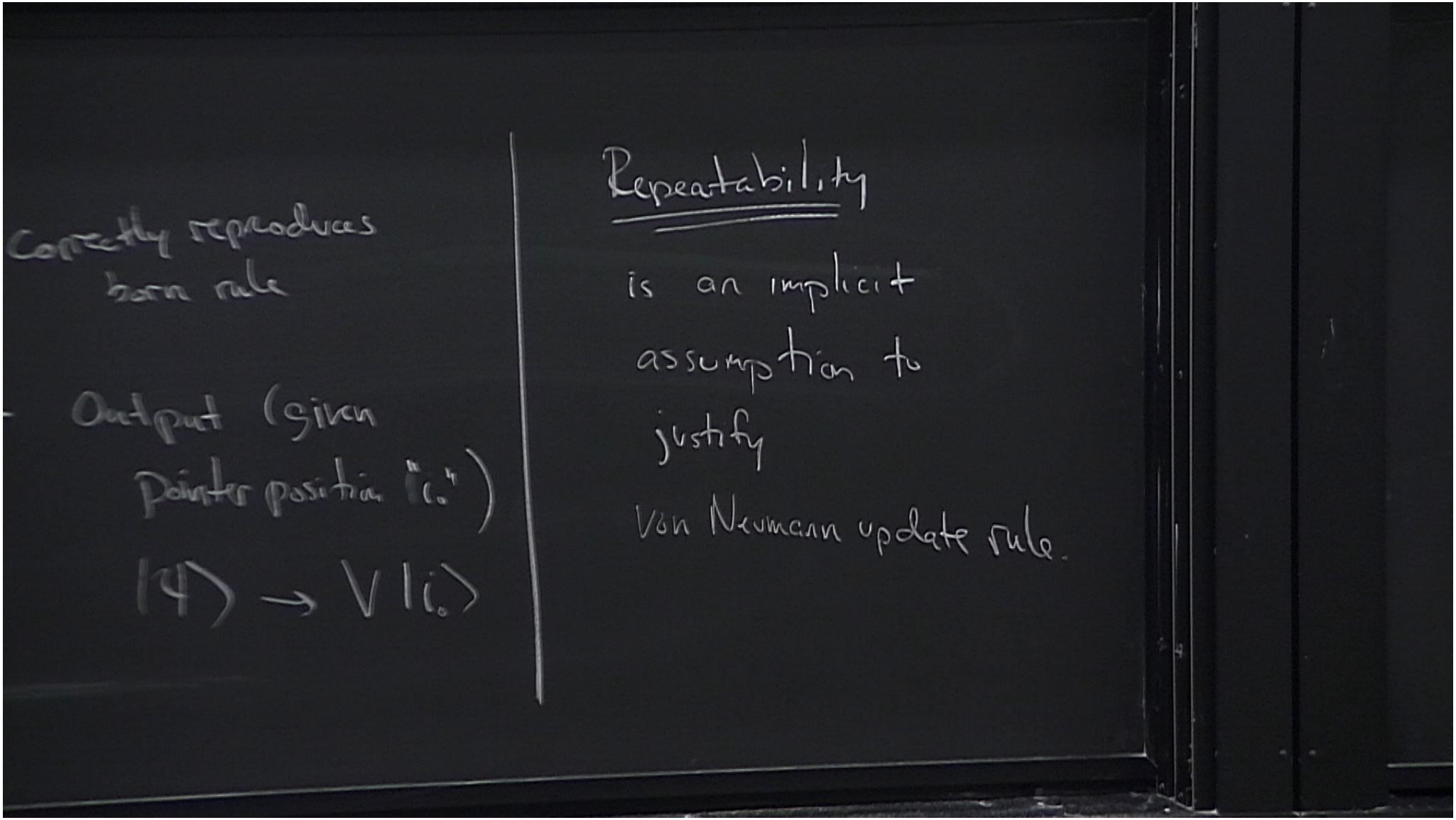


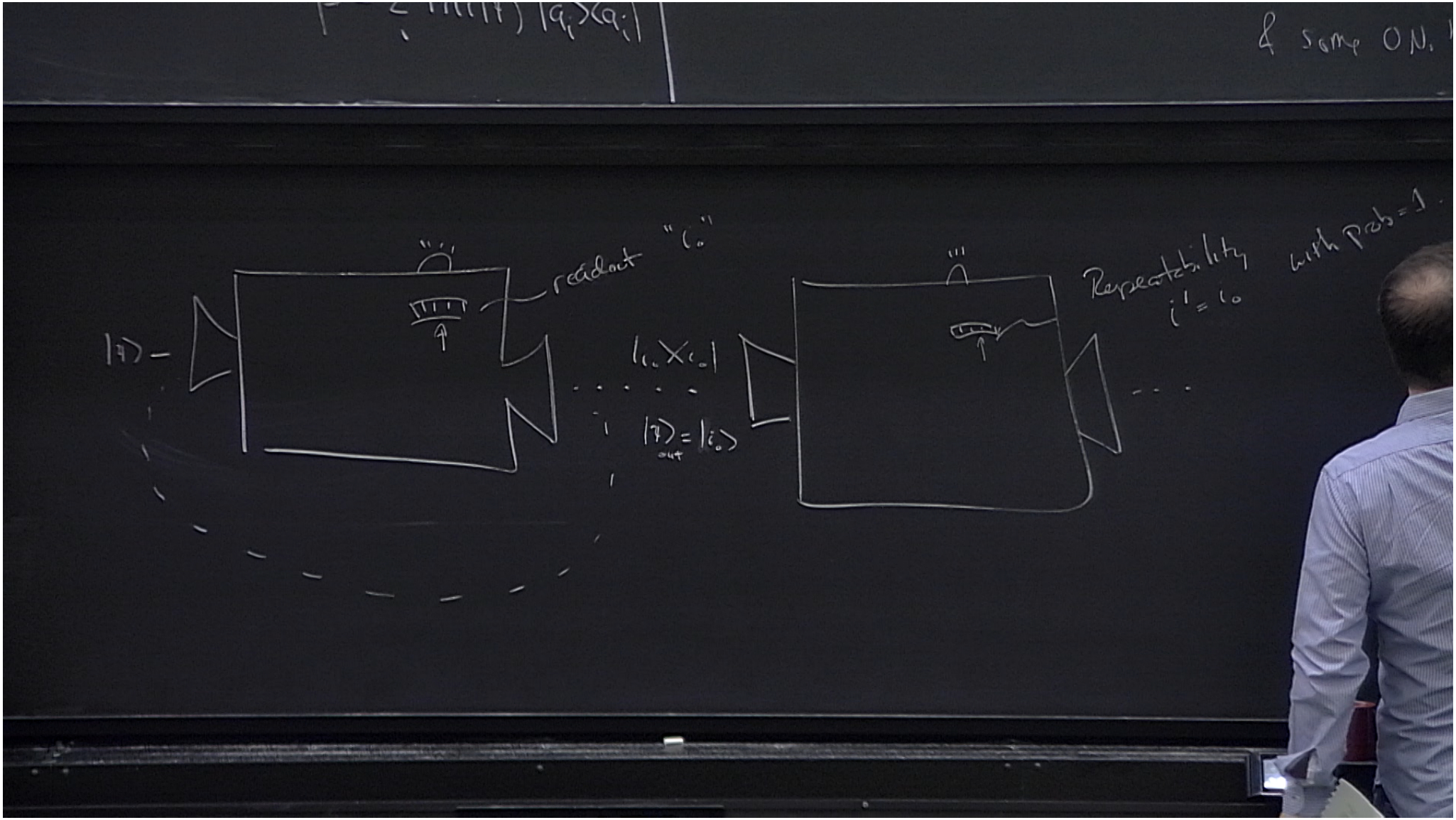
→ Correctly reproduces born rule

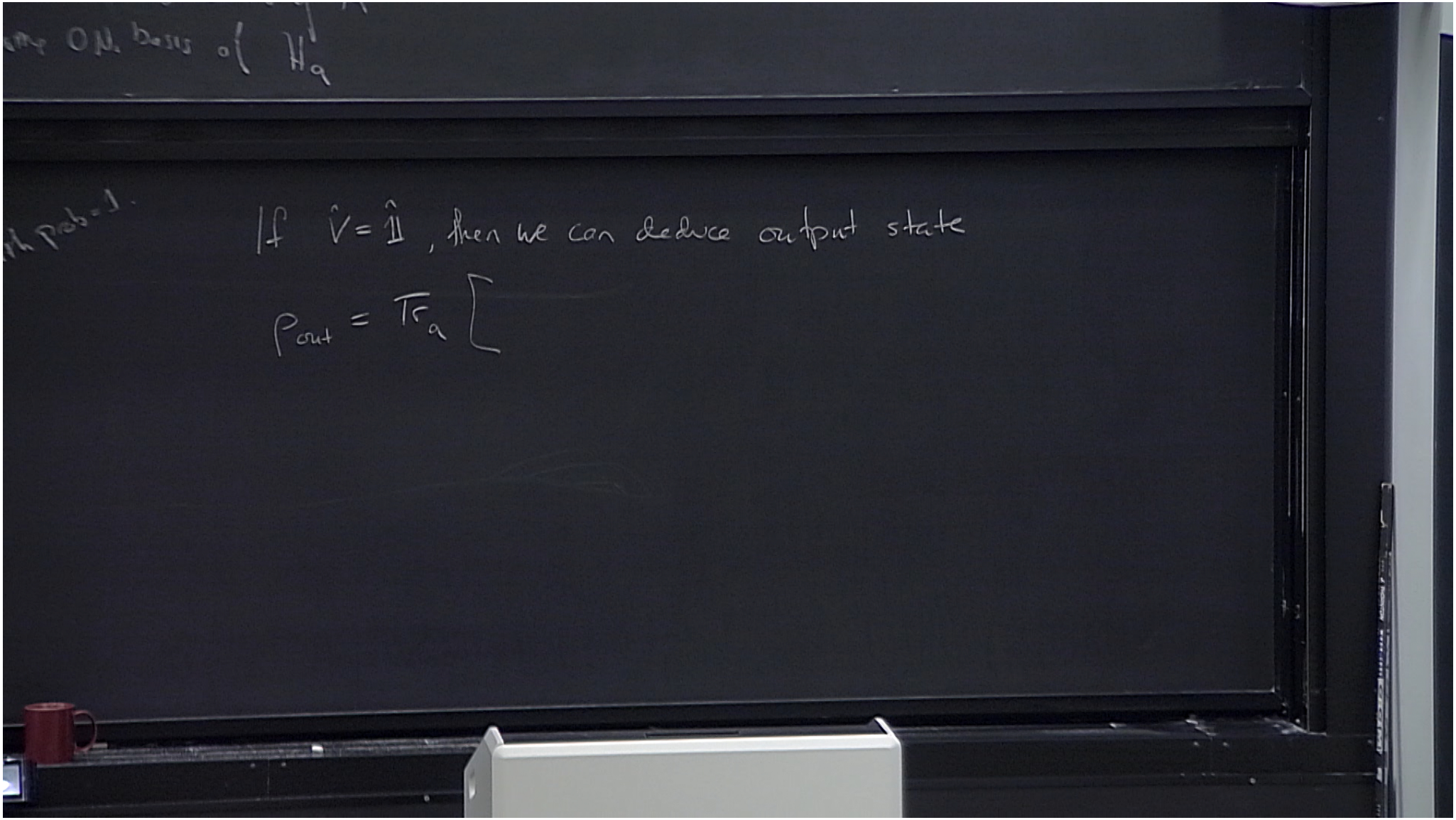
··· Output (given pointer position "i_0")

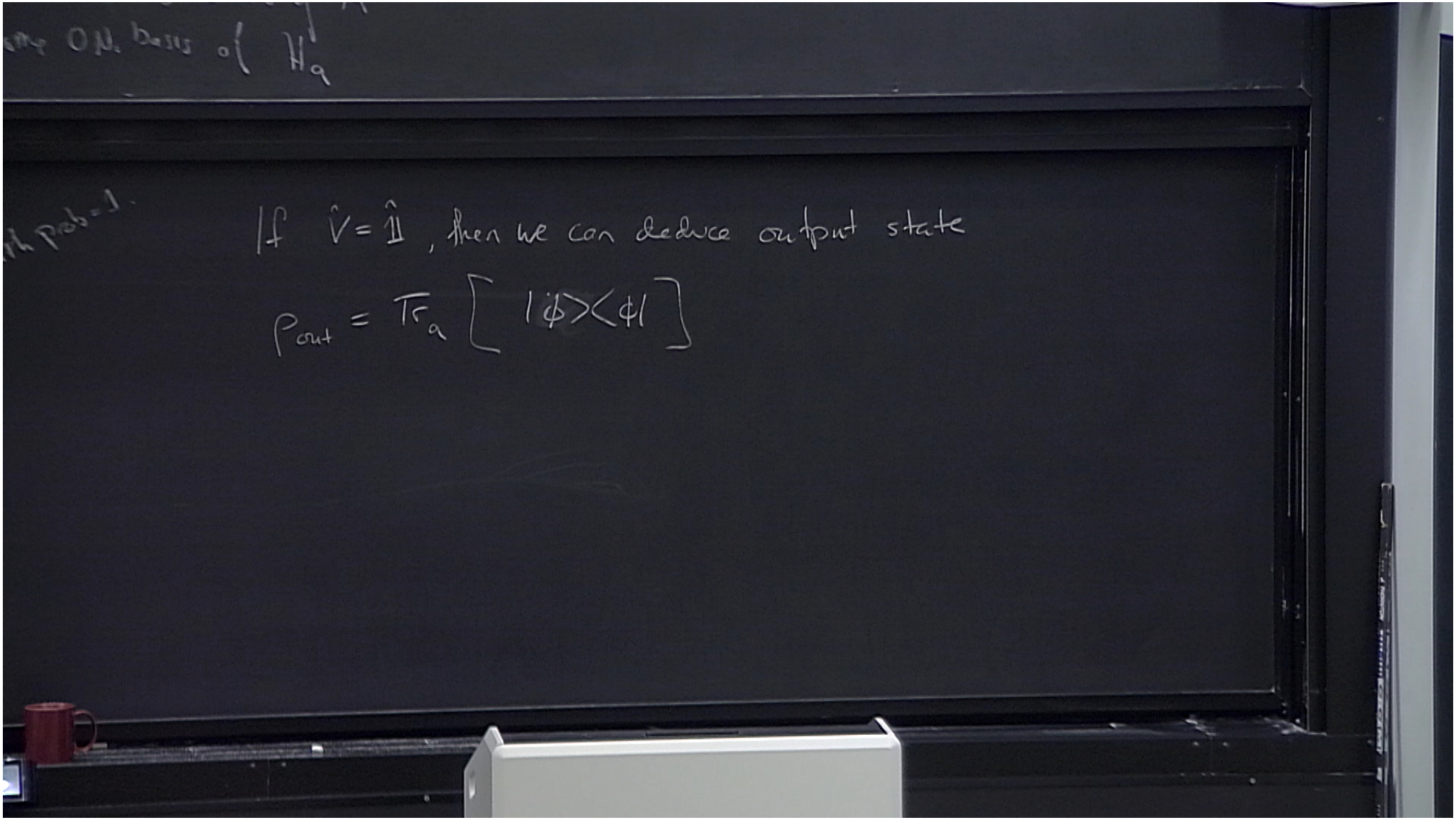
$|4\rangle \to V|i_0\rangle$

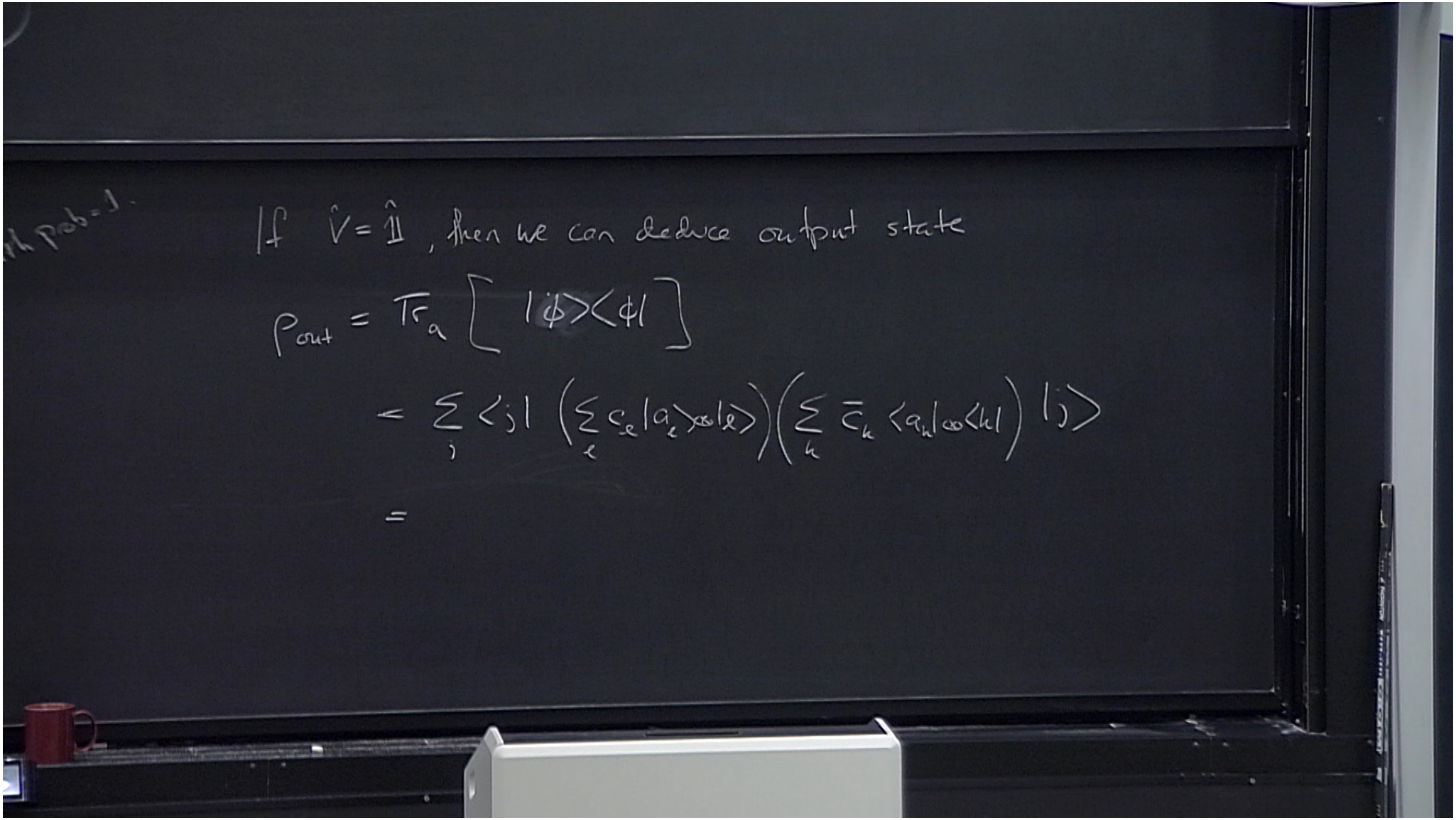
$V \in U(D)$ for system

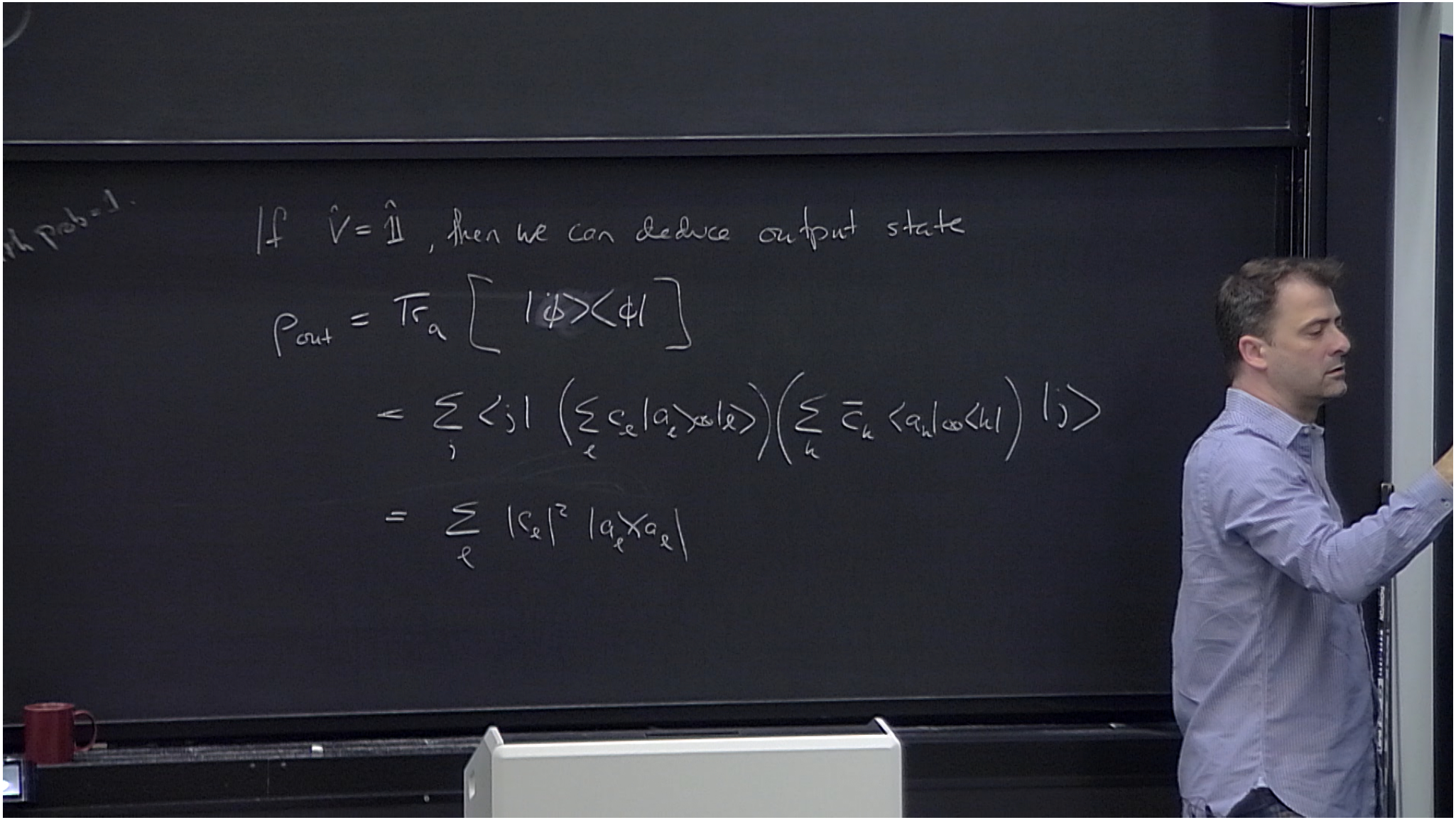

Correctly reproduces born rule
- Output (given pointer position "i.")
|4⟩ → V|i.⟩
Repeatability
is an implicit assumption to justify
Von Neumann update rule.


& some ON.
readout "i_0"
$|i_0\rangle\langle i_0|$
$|\psi\rangle = |i_0\rangle$
Repeatability
$i' = i_0$
with Prob = 1.


ON. basis $\{ H_q \}$
with prob = 1.
If $\hat{V} = \hat{\mathbb{1}}$, then we can deduce output state
$\rho_{out} = \mathrm{Tr}_q [$


ON basis of H_q'
with prob = 1.
If $\hat{V} = \hat{\mathbb{1}}$, then we can deduce output state
$\rho_{out} = Tr_q [\, |\phi\rangle\langle\phi| \,]$


If $\hat{V} = \hat{\mathbb{1}}$, then we can deduce output state
$\rho_{out} = Tr_a \left[|\phi\rangle\langle\phi| \right]$
$= \sum_j \langle j| \left(\sum_\ell c_\ell |a_\ell\rangle \otimes |\ell\rangle \right) \left(\sum_k \bar{c}_k \langle a_k| \otimes \langle k| \right) |j\rangle$
$=$


If $\hat{V} = \hat{\mathbb{1}}$, then we can deduce output state
$\rho_{out} = Tr_a \left[|\phi\rangle\langle\phi| \right]$
$= \sum_j \langle j| \left(\sum_\ell c_\ell |a_\ell\rangle \otimes |\ell\rangle \right) \left(\sum_k \bar{c}_k \langle a_k| \otimes \langle k| \right) |j\rangle$
$= \sum_\ell |c_\ell|^2 |a_\ell\rangle\langle a_\ell|$

w/o conditioning on observed output,

$$\rho_{out} = \sum_x |c_x|^2 |a_x\rangle\langle a_x|$$

for input $|\psi\rangle = \sum_x c_x |a_x\rangle$.

$$\sigma = |\psi\rangle\langle\psi| = \left(\sum_x c_x |a_x\rangle\right)\left(\sum_n c_n \langle a_n|\right)$$

$$= \sum_x |c_x|^2 |a_x\rangle\langle a_x| + \sum_{x \neq n} c_x \bar{c}_n |a_x\rangle\langle a_n|$$

"Coherences"



- By Schrodinger evolution

$$\rho_{out} = Tr_q \left[|\phi\rangle\langle\phi| \right] = \sum_c |c_c|^2 |a_c\rangle\langle a_c|$$

which is a mixed state

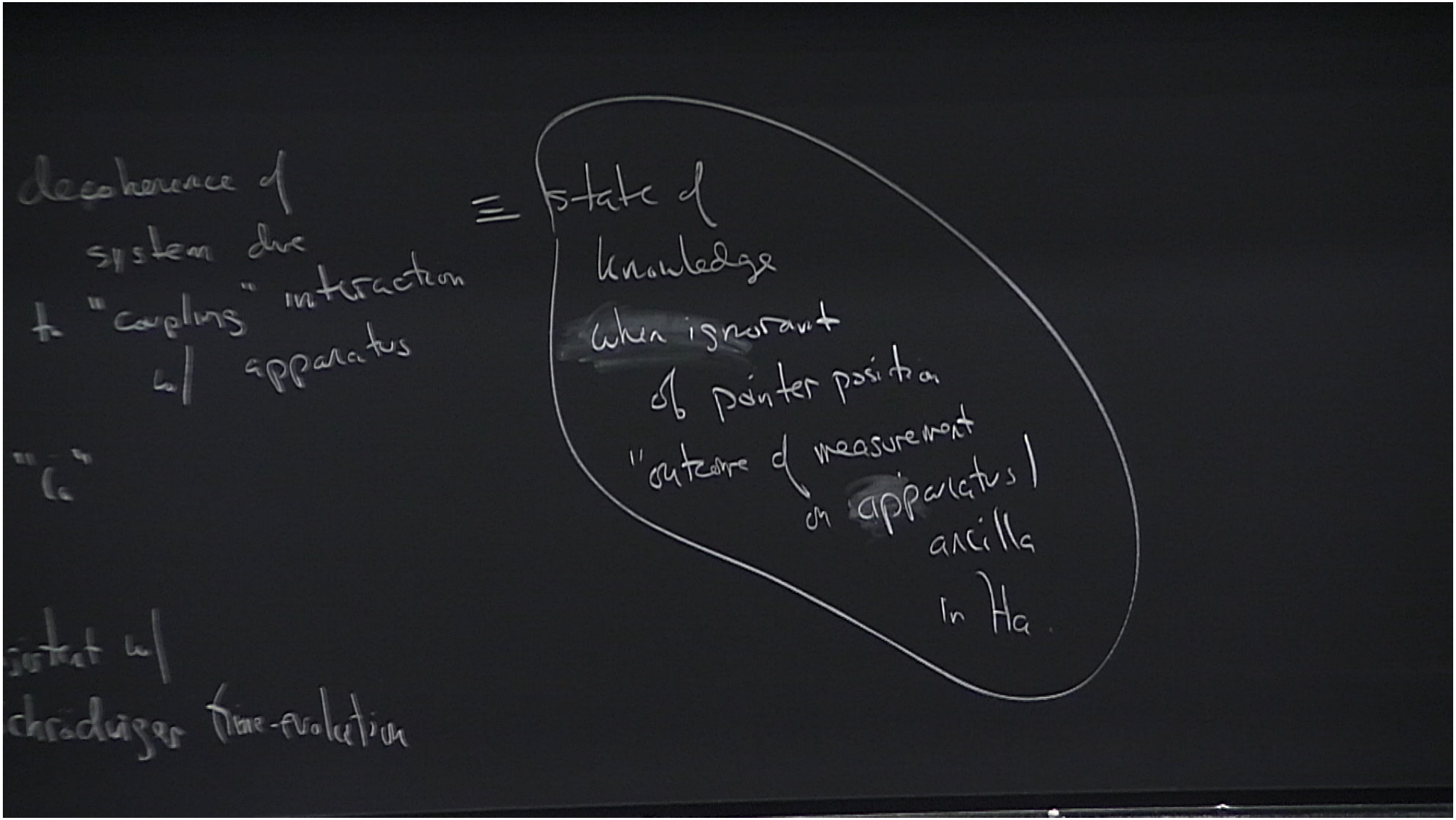
} decoherence of system due to "coupling" interaction w/ apparatus ≡ state of knowled... when is... of ... "outcom...

- By repeatability & conditioning on observed output "c_0"

$$\Rightarrow \rho_{out} = |c_0\rangle\langle c_0| \quad \text{– pure state}$$

"collapse of wavefn"

} inconsistent w/ Schrodinger time-evolution


decoherence of
system due
to "coupling" interaction
w/ apparatus
≡
state of
knowledge
when ignorant
of pointer position
"outcome of measurement
on apparatus /
ancilla
in Ha
"Ua"
subset w/
Schrodinger time-evolution

$$\rho_{out} = Tr_a \left[|\psi\rangle\langle\psi| \right] = \sum_i |c_i|^2$$

which is a mixed state

· By repeatability & conditioning on obs

state of knowledge if you do know

$\Rightarrow$ $\rho_{out} = |i\rangle\langle i|$ — pure st

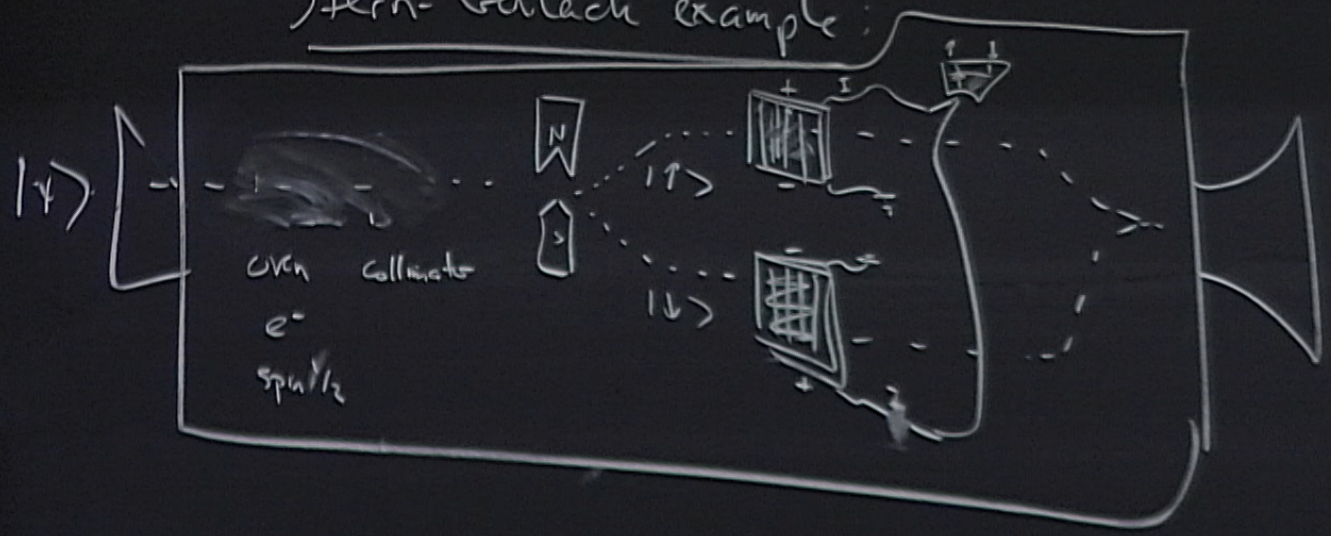
(i) output "i"

(ii) repeatable

" collapse of wavefn

$|\psi\rangle \in \mathbb{C}^2$

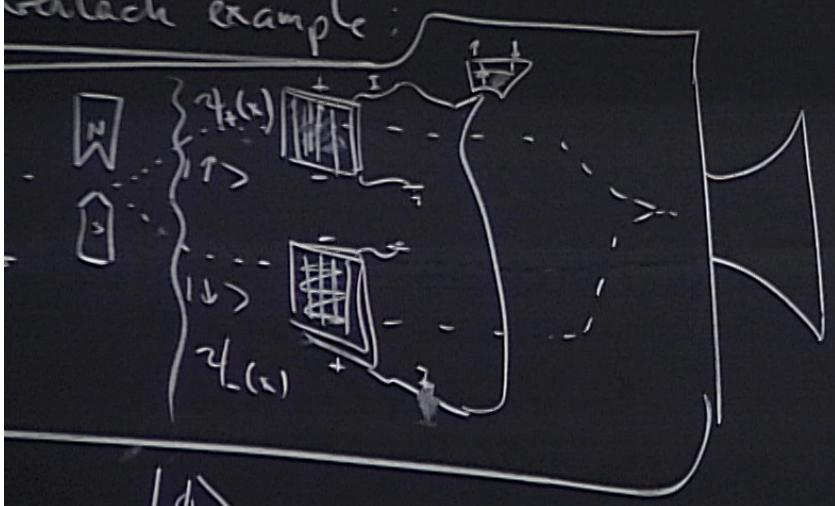
Stern-Gerlach example:



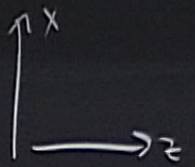
$= \langle \phi |$

$\mathcal{H}_s = \mathbb{C}^2$, $\mathcal{H}_a = L^2(\mathbb{R})$, $\psi(x) \in L^2(\mathbb{R})$

Gerlach example:

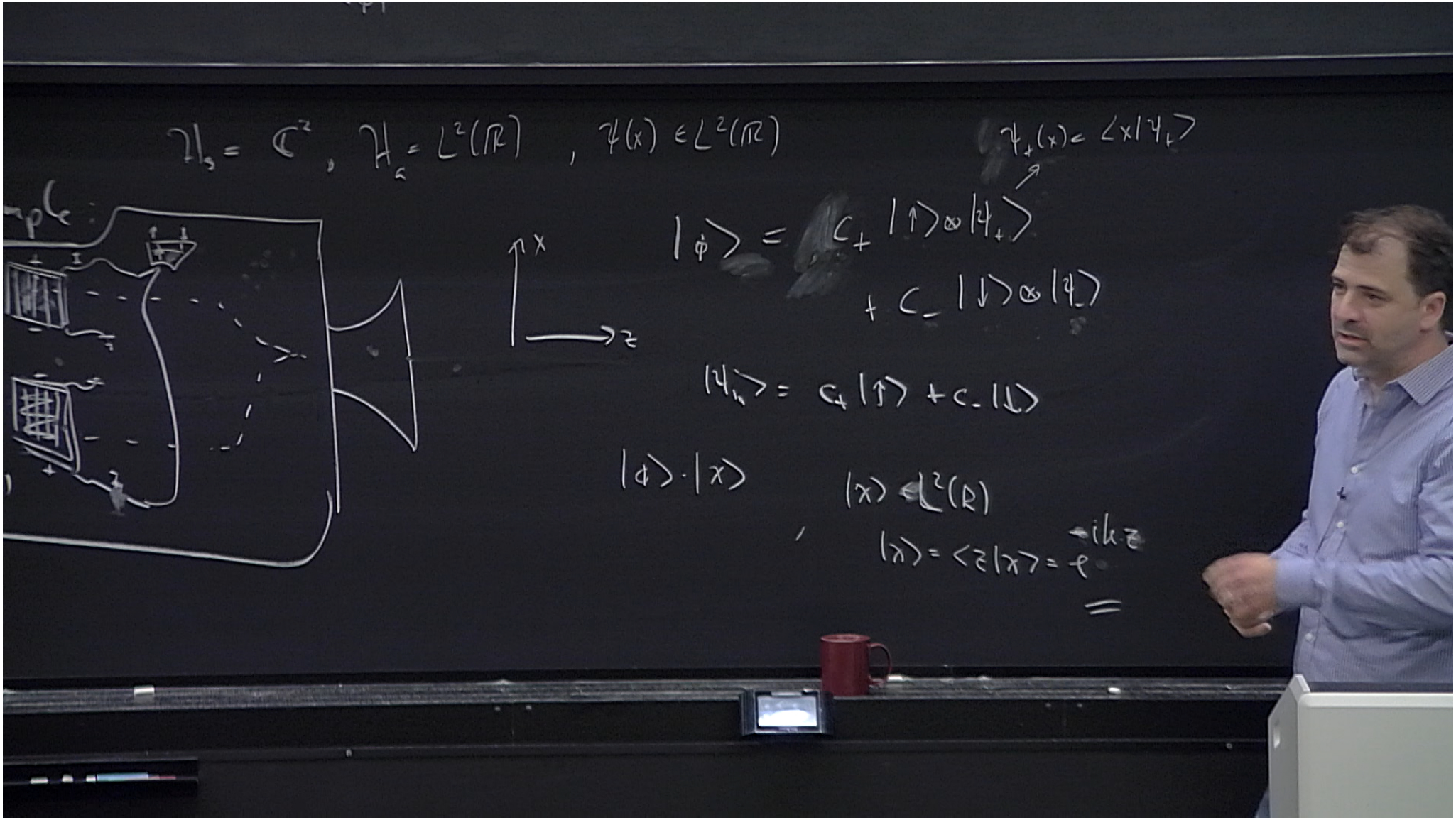


$|\phi\rangle$



$|\phi\rangle = c_+ |\uparrow\rangle \otimes |\psi_+\rangle$
$+ c_- |\downarrow\rangle \otimes |\psi_-\rangle$

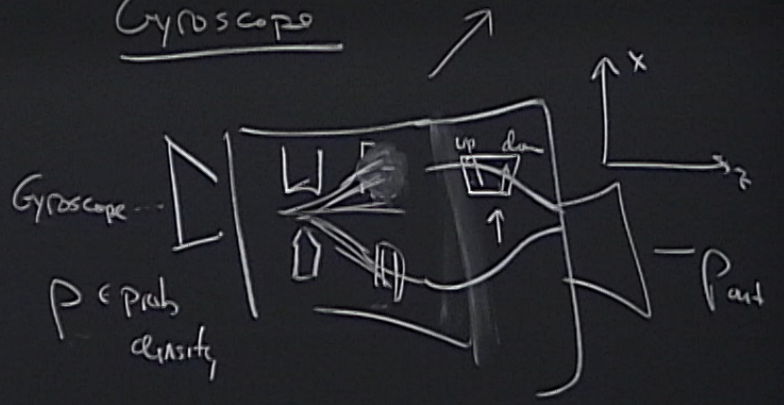
$|\psi_{in}\rangle = c_+ |\uparrow\rangle + c_- |\downarrow\rangle$


$\mathcal{H}_s = \mathbb{C}^2$, $\mathcal{H}_c = L^2(\mathbb{R})$, $\psi(x) \in L^2(\mathbb{R})$
$\psi_+(x) = \langle x | \psi_+ \rangle$
$|\phi\rangle = c_+ |\uparrow\rangle \otimes |\psi_+\rangle$
$+ c_- |\downarrow\rangle \otimes |\psi_-\rangle$
$|\psi_+\rangle = c_+ |\uparrow\rangle + c_- |\downarrow\rangle$
$|\phi\rangle \cdot |x\rangle$
$|x\rangle \in L^2(\mathbb{R})$
$|x\rangle = \langle z | x \rangle = e^{ikz}$
x
z

outside box $= |c_{i_0}|^2$

Classical analog

Gyroscope



$\rho \in$ Prob density

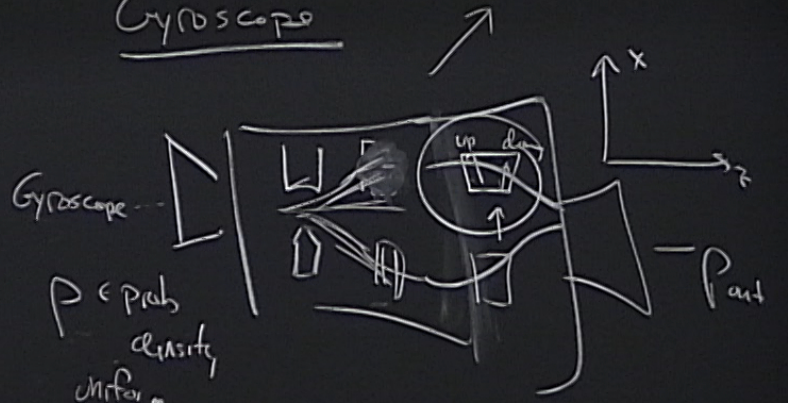
$|\psi\rangle \in \mathbb{C}^2$

$|\psi\rangle$

outside box $= |c_{i_0}|^2$

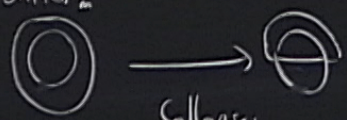
Classical analog

Gyroscope



$\rho \in$ prob density

uniform



Collapses to ρ_{out}

ρ_{out} = hemisphere of uncertainty

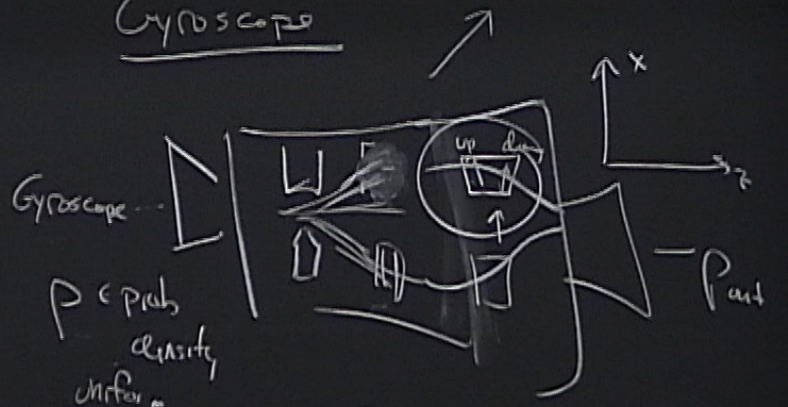
$|\psi\rangle \in \mathbb{C}^2$

$|\psi\rangle$

outside box $= |c_{i_0}|^2$

Classical analog

Gyroscope



$\rho \in$ prob density uniform

ρ_{out} = hemisphere of uncertainty

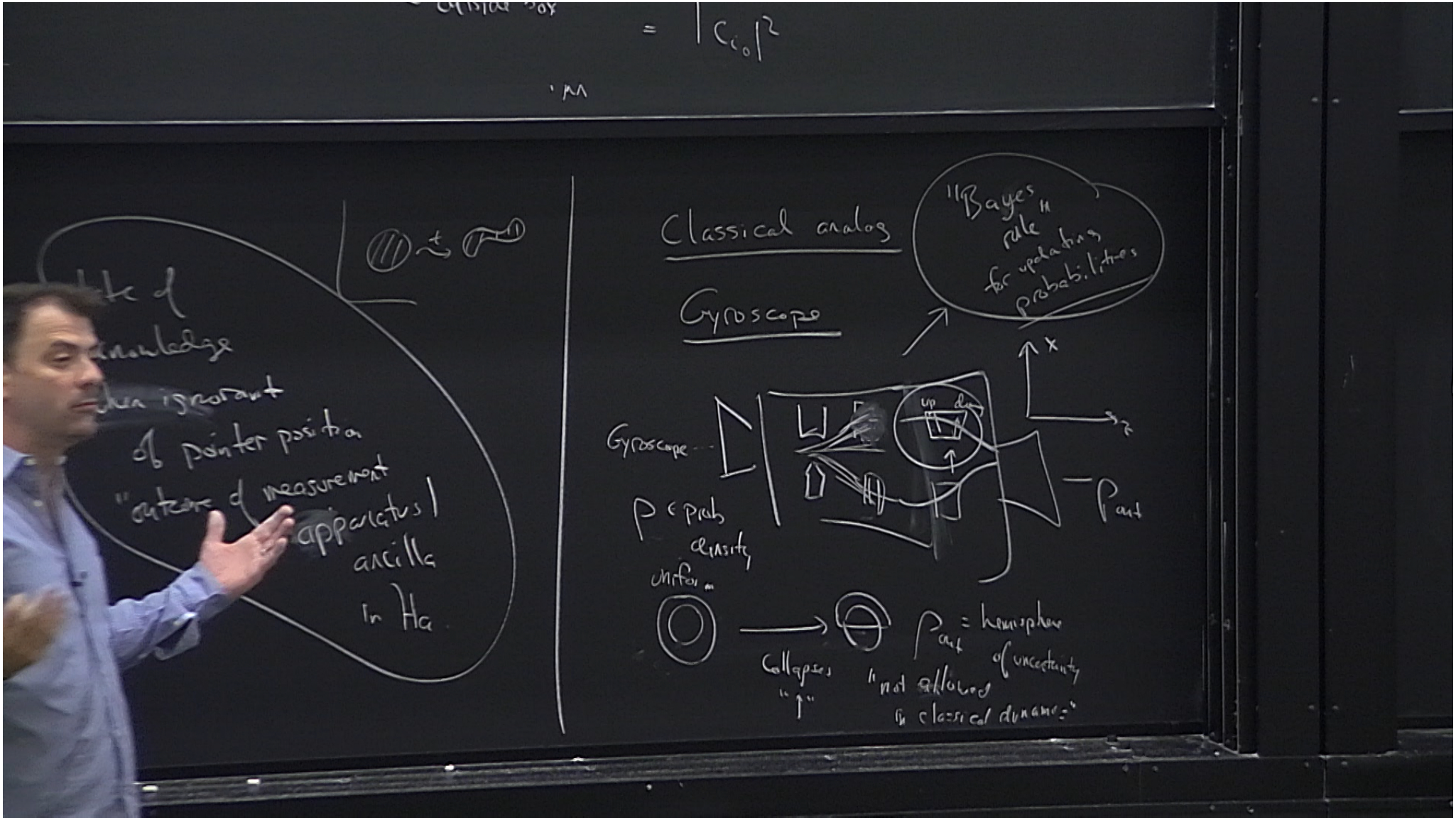
Collapses "↑"

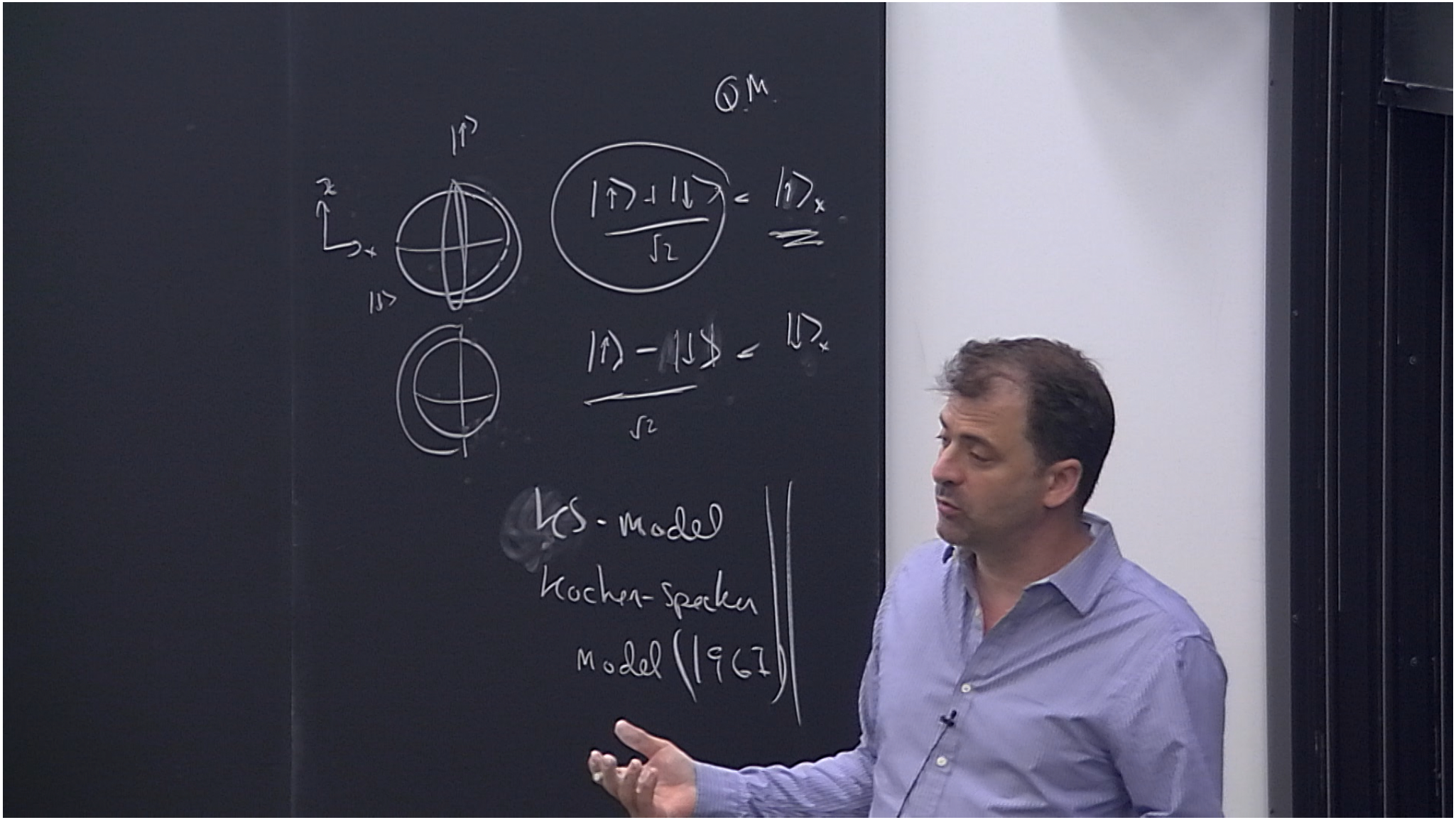
"not following classical dynamics"

pointer position
apparatus
ancilla
in $\mathcal{H}_a$

$|\psi\rangle \in \mathbb{C}^2$

$|\psi\rangle$ [ ]


Classical analog
Gyroscope
"Bayes" rule for updating probabilities
Gyroscope
up
down
x
z
ρ ∈ Prob density
uniform
collapses
"not allowed in classical dynamics"
of uncertainty


QM
$|\uparrow\rangle$
z
x
$|\downarrow\rangle$
$\frac{|\uparrow\rangle + |\downarrow\rangle}{\sqrt{2}} = |\rightarrow\rangle_x$
$\frac{|\uparrow\rangle - |\downarrow\rangle}{\sqrt{2}} = |\leftarrow\rangle_x$
KS-model
Kochen-Specker
model (1967)